AIAA Modeling and Simulation Technologies Conference and Exhibit
20 - 23 August 2007, Hilton Head, South Carolina

AIAA 2007-6703

A General Solution to the Aircraft Trim Problem

Agostino De Marco*
University of Naples Federico II, Department of Aerospace Engineering
Via Claudio, 21 80125 Naples, Italy

Eugene L. Duke†
Rain Mountain Systems Incorporated, Glasgow, Virginia, 24555-2509, USA

Jon S. Berndt‡
JSBSim Project Team Lead, League City, TX, USA

Trim defines conditions for both design and analysis based on aircraft models. In fact, we often define these *analysis points* more broadly than the conditions normally associated with *trim conditions* to facilitate that analysis or design. In simulations, these analysis points establish initial conditions comparable to flight conditions. Based on aerodynamic and propulsion systems models of an aircraft, trim analysis can be used to provide the data needed to define the operating envelope or the performance characteristics. Linear models are typically derived at trim points. Control systems are designed and evaluated at points defined by trim conditions. And these trim conditions provide us a starting point for comparing one model against another, one implementation of a model against another implementation of the same model, and the model to flight-derived data.

In this paper we define what we mean by trim, examine a variety of trim conditions that have proved useful and derive the equations defining those trim conditions. Finally we present a general approach to trim through constrained minimization of a cost function based on the nonlinear, six-degree-of freedom state equations coupled with the aerodynamic and propulsion system models. We provide an example of how a trim algorithm is used with a simulation by showing an example from JSBSim.

List of Symbols

$[\,\cdot\,]$	a generic 3×3 matrix.
$[\,\tilde{(\cdot)}\,]_{\mathrm{F}}$	the anti-symmetric matrix that expresses the cross product by vector $\vec{(\cdot)}$ in frame F, i.e. $\vec{\Omega} \times \vec{V}$ becomes $[\tilde{\Omega}]_{\mathrm{W}} \cdot \{V\}_{\mathrm{W}}$ in the wind frame (*tilde operator*).
$[M] \cdot \{c\}$	standard matrix product, row-by-column, of matrix $[M]$ times column matrix $\{c\}$.
$[C_{\mathrm{F}_2 \leftarrow \mathrm{F}_1}]$	transformation matrix from frame F_1 to frame F_2, i.e. $\{V\}_{\mathrm{F}_2} = [C_{\mathrm{F}_2 \leftarrow \mathrm{F}_1}] \cdot \{V\}_{\mathrm{F}_1}$.
$[I]_{\mathrm{B}}$	aircraft inertia tensor in the body-axis frame (mass·length^2).
$[I^r]_{\mathrm{B}}$	inertia tensor of a generic rotating subpart, e.g. engine rotor or propeller, in a reference frame with axes parallel to the main body axes and origin somewhere on the part's axis of rotation (mass·length^2).
δ_e, δ_a, δ_r	angular deflections of elevator, ailerons, and rudder (radians).
δ_T	throttle setting (mapped in the range $[0,1]$).
γ	flight path angle (radians).
$\boldsymbol{g}$, $\boldsymbol{f}$	vector valued functions representing implicit and explicit, respectively, aircraft state equations.
$\boldsymbol{u}$	n_c-dimensional vector of aircraft control inputs.

*Assistant Professor, University of Naples Federico II, Department of Aerospace Engineering, agostino.demarco@unina.it.
†Chief Engineer, duke@rainmountainsystems.com, Member AIAA.
‡Aerospace Engineer, JSBSim Development Coordinator, jon@jsbsim.org, Senior Member AIAA.

$\boldsymbol{x}$	aircraft n_s-dimensional state vector.
$\mathcal{J}$	trim cost function.
$\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$	components of the resultant moment $\Sigma\vec{M}$ in the body-axis frame.
ω	angular velocity (radians/sec).
ϕ, θ, ψ	aircraft Euler angles (radians).
$\vec{\imath}^{r}$	rotor spin axis unit vector.
$\vec{\omega}^{r}$	angular rotation vector relative to the main body of a generic rotating subpart, e.g. engine rotor or propeller (radians/sec).
$\vec{F}_{\mathrm{G}}$, $\vec{F}_{\mathrm{A}}$, $\vec{F}_{\mathrm{T}}$	resultant forces on aircraft due to gravity, aerodynamic and thrust forces, respectively (independent on reference frame).
$\xi_1, \ldots, \xi_{n_{\mathrm{tc}}}$	generic cost function undependent variables.
$\{(\cdot)\}_{\mathrm{F}}$	a 1×3 column matrix representing vector $(\vec{\cdot})$ projected onto the reference frame F.
D, C, L	aerodynamic forces, drag, cross-force and lift (force).
F_c	centripetal force in a turn (force).
g	gravitational acceleration (length/sec^2).
h	altitude, equal to $-z_{\mathrm{E}}$ (length).
I^r	moment of inertia of a generic rotating subpart, e.g. engine rotor or propeller (mass·length^2).
I_x, I_y, I_z	aircraft moments of inertia about body axes (mass·length^2).
I_1, I_2, …, I_6	entries of matrix $\big[I\big]_{\mathrm{B}}^{-1}$ (mass·length^2).
I_{xy}, I_{xz}, I_{yz}	aircraft products of inertia about body axes (mass·length^2).
m	aircraft mass (mass).
n	load factor.
n_c	number of aircraft control inputs.
n_{tc}	number of trim algorithm control variables.
p, q, r	components of the aircraft angular velocity $\vec{\Omega}$ in the body-fixed reference frame (radians/sec).
R	radius of steady-state turn (length).
t	time.
u, v, w	components of the aircraft center of mass velocity $\vec{V}_C$ in the body-fixed reference frame (length/sec).
X, Y, Z	components of the resultant force $\Sigma\vec{F}$ in the body-axis frame.
x_C, y_C, z_C	coordinates of aircraft center of mass C into a fixed reference frame (length).
x_{E}, y_{E}, z_{E}	coordinates of airplane's center of mass in the Earth frame in flat-Earth hypothesis (length).

Subscripts

$(\,)_{\mathrm{G}}$, $(\,)_{\mathrm{A}}$, $(\,)_{\mathrm{T}}$	due, respectively, to gravity, aerodynamic and thrust (propulsive) actions.
$(\cdot)_d$, $(\cdot)_k$	dynamic and kinematic part of state vector $\boldsymbol{x}$ or of function $\boldsymbol{g}$.
$(\cdot)_i$	i-th element of state vector $\boldsymbol{x}$ or of function $\boldsymbol{g}$, or initial value of a simulation variable.
$(\cdot)_{\mathrm{B}}$, $(\cdot)_{\mathrm{W}}$	in body axes and in wind axes.
$(\cdot)_{\mathrm{V}}$, $(\cdot)_{\mathrm{E}}$	in local vertical and in Earth frame.
$(\cdot)_{\mathrm{eq}}$	equilibrium value or value at trim.

Superscripts

$(\,)^{\mathsf{T}}$	transpose of a matrix.
$(\cdot)^r$	rotating subpart of the aircraft.
$\dot{(\,)}$	time derivative.
$(\vec{\cdot})$	A vector of the three-dimensional Euclidean space; a quantity that does not depend upon the reference frame where it is represented.

I. Introduction

Determining aircraft steady-state flight conditions, or also trimmed states, is of primary importance in a variety of engineering studies. Trim defines conditions for both design and analysis based on aircraft models. In fact, we often define these *analysis points* more broadly than the conditions normally associated with *trim conditions* to facilitate that analysis or design. In simulations, these analysis points establish initial conditions comparable to flight conditions. Based on aerodynamic and propulsion systems models of an aircraft, trim analysis can be used to provide the data needed to define the operating envelope or the performance characteristics. Linear models are typically derived at trim points. Control systems are designed and evaluated at points defined by trim conditions. And these trim conditions provide us a starting point for comparing one model against another, one implementation of a model against another implementation of the same model, and the model to flight-derived data.

The task of trimming a vehicle model in symmetric and asymmetric flight conditions represents, however, a nontrivial job. This paper discusses the definition of trim conditions and derivation of constraint equations for a steady-state flight applicable to conventional and unconventional aircraft shapes of symmetric and asymmetric layout.

Classical textbooks often follow a simplified approach to discuss the aircraft trim problem and pose with special emphasis on static stability and control, treating the longitudinal and lateral-directional axes as two separate topics, assuming that they are uncoupled. This is an engineering correct approach in many cases and is supported by practical evidence, but it does not address the general problem, i.e. is not valid for some real cases or unconventional configurations. Many example exist in fact showing that geometrical and/or aerodynamic asymmetries imply asymmetric trimmed attitudes. In these cases the longitudinal and lateral-directional axes are coupled naturally from the start, even if the desired trimmed state is a steady straight flight at a constant altitude.

In addition, most of the methods of solution of the aircraft trim problem—both classical ones[4,5] and even more sophisticated, up-to-date, analytical frameworks with the capability to trim the aircraft in six degrees-of-freedom[11,12] —are presented and implemented by making use of static stability derivatives, of damping derivatives and of control derivatives (control effectiveness) concepts, resulting in a set of nonlinear algebraic equations for the unknown attitude angles, aerodynamic surface deflections and thrust output ensuring the trim. Those treatments are valid in regimes of flight where aerodynamic coefficients vary linearly and are not applicable to situations in which the aerodynamic derivatives are not constant with angle of attack, angle of sideslip or Mach number. Even more general situations, for example airplanes for which unconventional configuration variations may occur in flight—such as thruster tilt—have to be addressed by a more general approach.

The general treatment of aircraft trim proposed here starts from the standard equations of motion of an airplane in atmospheric flight but does not make any limiting assumptions on the geometrical properties of the aircraft nor on the aerodynamic coefficient curves. Regarding the latter, in general, all that one really expects is an aerodynamic model that provides nondimensional aerodynamic coefficients, no matter where those parameters come from or how they are derived. In practice, a convenient aerodynamic model should be available in the form of tabulated data for the widest possible ranges of aerodynamic angles and velocities and for all possible aircraft configurations.

The combination of attitude angles, aerodynamic surface deflections and thrust output/direction for the desired trim condition is obtained numerically by minimizing a conveniently defined cost function. This general approach naturally includes the classical results (symmetric aircraft and linear aerodynamics) as a particular case.

The generality of the proposed approach stems both from the general trim states embraced by all possible minima of the cost function and from the generic model chosen. We show how these trim functions are implemented in JSBSim, an open-source, nonlinear dynamic simulation model, and include results from various aircraft models in a variety of trim conditions.

The research primary aim is the development of a practical tool with the capability to trim a generic aircraft model in six degrees-of-freedom. Such a tool is meant to be used as a building block for the linearization of an airplane aerodynamic model around some given sensible trim points within the flight envelope.

The ultimate objective is to assist in the development of performance-optimal stability and control design solutions for advanced conventional and unconventional aircraft configuration shapes.

II. The trim problem in a general state-space form

A. General equations, unknowns and inputs

In general terms, the aircraft state equations can be expressed by the following *implicit system*[4]

$$\boldsymbol{g}\big(\,\dot{\boldsymbol{x}},\,\boldsymbol{x},\,\boldsymbol{u}\,\big)=0 \tag{1}$$

In the above equation $\boldsymbol{x}$ is the aircraft *state* vector; $\boldsymbol{g}$ is a vector of n_s scalar nonlinear functions g_i resulting from aircraft nonlinear, six-degree-of freedom state equations projected onto a convenient reference frame; and $\boldsymbol{u}$ is the column of n_c control variables. The system of equations (1) is generally implicit when, in some of the n_s scalar equations, the aerodynamic model is such that one component of the vector $\dot{\boldsymbol{x}}$ of state variable time derivatives cannot be expressed explicitly in terms of the remaining quantities. This happens, for instance, when the model of aerodynamic forces presents a general dependence on the angle of attack time derivative. The latter is function of one of the components of $\dot{\boldsymbol{x}}$.

The most widespread choice of the state vector $\boldsymbol{x}$, which is not absolutely the only one possible and generally depends on the adopted set of equations of motions, is represented by the following arrangement[5]

$$\boldsymbol{x}=\big[\,\boldsymbol{x}_d^{\mathsf{T}}\,,\,\boldsymbol{x}_k^{\mathsf{T}}\,\big]^{\mathsf{T}} \tag{2}$$

i.e. a column vector including a *dynamic* and a *kinematic* part. When the aircraft equations of motion are projected onto the standard body-fixed reference frame we have for example

$$\boldsymbol{x}_d=\big[\,u\,,\,v\,,\,w\,,\,p\,,\,q\,,\,r\,\big]^{\mathsf{T}}\,,\qquad \boldsymbol{x}_k=\big[\,x_C\,,\,y_C\,,\,x_C\,,\,\phi\,,\,\theta\,,\,\psi\,\big]^{\mathsf{T}} \tag{3}$$

with (u,v,w) and (p,q,r) the body-axis frame components of aircraft center of mass velocity $\vec{V}_C$ and of the aircraft angular velocity $\vec{\Omega}$, respectively; (x_C,y_C,z_C) the evolving coordinates of the center of mass C into the chosen inertial reference frame; (ϕ,θ,ψ) the standard aircraft Euler angles representing the evolving body attitude with respect to the inertial reference frame.

In modern flight simulation practice the representation of the airplane's rotational degrees of freedom by the Euler angles and the related kinematic formulation has been subsided by the approach based on quaternions and quaternion algebra. When this is the case the last three entries in $\boldsymbol{x}_k$ are replaced by the four components of a *quaternion*.[4,6] Consequently, the last three components of the vector-valued function $\boldsymbol{g}$ in (1) are replaced by four other functions, which are defined according to the quaternion kinematics. For the purposes of this paper we will not go into the details of quaternion formulations, rather we will use the classical Euler angle kinematic equations to derive the desired trim conditions.

Regarding the vector $\boldsymbol{u}$ of control inputs, its entries may depend in number by the type of aircraft. For a conventional configuration aircraft the minimum arrangement of the inputs is usually given by

$$\boldsymbol{u}=\big[\,\delta_T\,,\,\delta_e\,,\,\delta_a\,,\,\delta_r\,\big]^{\mathsf{T}} \tag{4}$$

where δ_T is the throttle setting and δ_e, δ_a, δ_r are the angular deflections of elevator, ailerons, and rudder, respectively. These quantities have standard signs and their range depend on the particular aircraft under consideration. In flight simulation practice their variation is associated with the normalized setting of a corresponding control in the cockpit. Usually the range of throttle setting goes from 0 (idle) to +1 (maximum power), while the stick excursions are all mapped to a range that goes from −1 to +1. These mappings often depend on the presence of control laws that may alter the final effect of pilot action on the actual effector deflections and thrust output. When talking about trim, we will always refer to the inputs as some required combination of aerodynamic surface deflections and thrust output. In this context δ_T may be considered as the current fraction of the maximum thrust output available at the actual flight speed and altitude.

In mathematical terms, whether the actual aerosurface deflections and thrust output or the normalized command ranges are considered, they are seen as a set of bounds for the control variables in the vector $\boldsymbol{u}$. When we use these quantities as independent variables to implement a trim algorithm, checking that their values are within bounds becomes an important step in the assessment of physical significance of the final result.

B. Trimmed states

A classical concept introduced in the theory of nonlinear systems, in our case the airplane whose model is given by (1), is the *equilibrium point* or *trim point*. For an autonomous (no external control inputs), time-invariant system the equilibrium point is denoted as $\boldsymbol{x}_{\text{eq}}$ and is defined as the particular $\boldsymbol{x}$ vector which satisfies

$$\boldsymbol{g}\big(0,\, \boldsymbol{x}_{\text{eq}},\, \boldsymbol{u}_{\text{eq}}\big) = 0 \quad \text{with } \dot{\boldsymbol{x}}_{\text{eq}} \equiv 0 \text{ and } \boldsymbol{u}_{\text{eq}} = 0 \text{ or } = \boldsymbol{u}_0 \tag{5}$$

Condition (5) also defines a set of control settings $\boldsymbol{u}_{\text{eq}}$ that make the steady state possible. The state of the system defined by (5) corresponds to the generalized idea of *rest*, i.e. the condition when all the derivatives are identically zero. In our case this concept may apply only to the dynamic part $\boldsymbol{x}_d$ of the aircraft state vector, i.e. to those independent variables ruled by Newton's laws. As will be shown in the rest of the paper, the kinematic variables in $\boldsymbol{x}_k$ may or may not have zero time derivatives in trimmed flight. For example, the derivatives $\dot{x}_C$ and $\dot{y}_C$ will never be zero unless the aircraft is at rest on the ground or is flying along a vertical trajectory. In some trim conditions the time derivatives of Euler angles may be all zero, in some other one of them is equal to a prescribed nonzero value the rest being zero.

Steady-state flight is defined as a condition in which all of the aircraft *motion* variables are constant or zero. That is, the linear and angular velocities are constant or zero and all the acceleration components are zero. This definition, which is quite general, is usually made more restrictive by making some simplifying assumptions. The first typical assumption considers the aircraft mass as a constant. A further simplification is related to the choice of inertial frame to which the flight is referenced. When the aircraft motion is modelled by the round-Earth equations, because of the Earth's angular velocity, only *minor-circle*, constant latitude[4] flight around the Earth is actually a true steady-state condition. When the Earth's oblateness is taken into account, minor circles are the only trajectories along which gravity remains strictly constant in magnitude.

Usually the *flat-Earth equations* are considered satisfactory for all control system design purposes, consequently those equations are satisfactory also for the derivation of trim conditions. Then, in the flat-Earth hypothesis, the definition of equilibrium state certainly allows *wings-level horizontal flight*, in any direction, and *constant altitude turning flight* to be two candidate for a steady state condition. Furthermore, if the change in atmospheric density with altitude is neglected, then also *wings-level climb* and *climbing turn* are permitted as steady state flight conditions.

In the above hypotheses, given definitions (2)-(3), the standard NED (North-East-Down) *position equations* for (x_C, y_C, z_C) in system (1), do not couple back into the rest of equations of motion and need not be used in finding a steady-state condition.[4] When we assume a flat-Earth we call the aircraft position $(x_{\text{E}}, y_{\text{E}}, z_{\text{E}})$. In the case of flat-Earth only the position equation for z_{E}, i.e. the altitude equation, is relevant to the development of a flight dynamics model trimming capability.

The general steady-state flight condition resulting from the above discussion is given as follows:

$$\begin{aligned} \text{accelerations} &\Rightarrow \dot{u}, \dot{v}, \dot{w} \left(\text{or } \dot{V}, \dot{\alpha}, \dot{\beta}\right) \equiv 0\,, \quad \dot{p}, \dot{q}, \dot{r} \left(\text{or } \dot{p}_{\text{W}}, \dot{q}_{\text{W}}, \dot{r}_{\text{W}}\right) \equiv 0\,, \\ \text{linear velocities} &\Rightarrow u, v, w \,(\text{ or } V, \alpha, \beta) = \text{prescribed constant values}\,, \\ \text{angular velocities} &\Rightarrow p, q, r \,(\text{ or } p_{\text{W}}, q_{\text{W}}, r_{\text{W}}) = \text{prescribed constant values} \\ \text{aircraft controls} &\Rightarrow \delta_T, \delta_e, \delta_a, \delta_r = \text{appropriate constant values} \end{aligned} \tag{6}$$

The steady-state conditions $\dot{p}, \dot{q}, \dot{r} \equiv 0$ require the angular rates to be zero or constant (as in steady turns), and therefore the aerodynamic and thrust *moments* must be zero or constant. On the other hand, the steady-state conditions $\dot{u}, \dot{v}, \dot{w} \equiv 0$ require the airspeed, angle of attack, and sideslip angle to be constant, and hence the aerodynamic *forces* must be zero or constant.

While an actual pilot may not find it very difficult to put an aircraft into a steady-state flight condition, trimming an aircraft mathematical model requires the solution of the simultaneous nonlinear equations (5), simplified according to the chosen formulation (flat-Earth in our case). In general, because of the nonlinearity, a steady-state solution can only be found by using a numerical method on a digital computer. Multiple solutions may exist, and a feasible solution will emerge only when practical constraints are placed on the variables.

In the next section we go further in the details of trim constraint derivation by making the system (1) explicit. We will present a complete formulation of the trim problem (5) by introducing the equations of motion written in an appropriate reference frame.

III. Constraint Equation Derivation

In this section, we define the general steady-state, dynamic equations for an aircraft. First, we define some basic translational and rotational equations, then we use these basic equations for the definition of the constraint equations defining the specific types of trim. Even though these trim types may represent only transitory conditions, we can describe those conditions in such a way that we can use these results to implement a trimming algorithm in a simulation or linearization program.

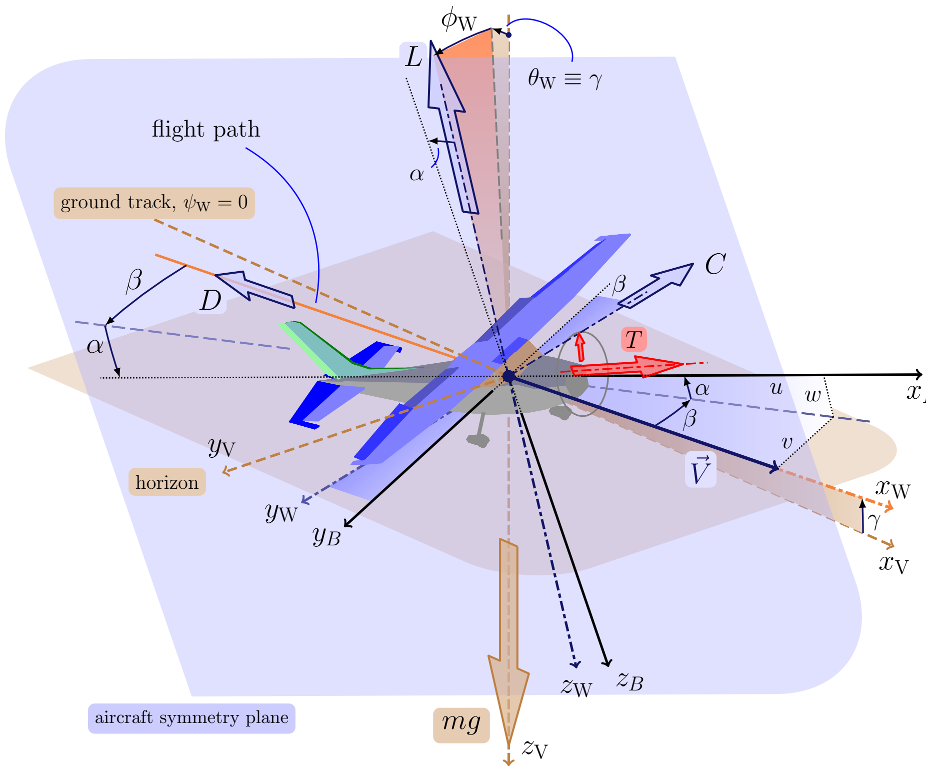

Figure 1. The standard *body*-, *wind*-, and *local vertical*-axis reference frames. The aircraft motion represented is a steady *crabbing* flight towards North with nonzero climb rate.

A. General equations of unsteady motion

The reader may refer to a number of well known textbooks for a detailed derivation of the general equations of motion of a system of rigid bodies or of the equations of unsteady motion of an aircraft. Examples of such textbooks are those by Goldstein[1] and Marion[2] and those by Etkin,[3] Stevens and Lewis,[4] Stengel[5] and Phillips.[6] Here we will recall the equations of motion of a rigid, constant mass airplane in proximity of a flat-Earth without making any particular assumption on body shape, inertias or aerodynamics and propulsion models.

1. *Translational Equations*

Using the results from the above cited textbooks we can write the equation defining the translational acceleration of the aircraft in the inertial reference system as

$$m \frac{\mathrm{d}\vec{V}}{\mathrm{d}t} = \Sigma\vec{F} \tag{7}$$

where $\Sigma\vec{F}$ is the resultant force acting on the aircraft at time t. The general accepted expression of the resultant force, when no particular external action is present, such as towing by another aircraft or gear contact forces, is the following

$$\Sigma\vec{F} = \vec{F}_{\mathrm{G}} + \vec{F}_{\mathrm{A}} + \vec{F}_{\mathrm{T}} \tag{8}$$

where $\vec{F}_{\mathrm{G}}$, $\vec{F}_{\mathrm{A}}$ and $\vec{F}_{\mathrm{T}}$ are the resultant forces due to gravity, aerodynamics and thrust, respectively.

If we choose, for example, to express vector equation (7) in the standard wind-axis frame, being the latter a rotating frame, the translational acceleration vector becomes

$$m\left(\{\dot{V}\}_{\mathrm{W}} + [\tilde{\Omega}]_{\mathrm{W}} \cdot \{V\}_{\mathrm{W}}\right) = \Sigma\{F\}_{\mathrm{W}} \tag{9}$$

where the term $[\tilde{\Omega}]_{\mathrm{W}} \cdot \{V\}_{\mathrm{W}}$, a skew-symmetric, square matrix times a column, is the representation of the cross product $\vec{\Omega} \times \vec{V}$ in the chosen frame. The same formulation (9) will apply, with subscripts $(\cdot)_{\mathrm{B}}$ in place of $(\cdot)_{\mathrm{W}}$ when the general vector equation (7) is projected onto the standard aircraft body-axis frame. In the wind-axis and body-axis reference frames we will have

$$\{V\}_{\mathrm{W}} = \left\{\begin{matrix} V \\ 0 \\ 0 \end{matrix}\right\}, \quad \{\Omega\}_{\mathrm{W}} = \left\{\begin{matrix} p_{\mathrm{W}} \\ q_{\mathrm{W}} \\ r_{\mathrm{W}} \end{matrix}\right\} \quad \text{and} \quad \{V\}_{\mathrm{B}} = \left\{\begin{matrix} u \\ v \\ w \end{matrix}\right\}, \quad \{\Omega\}_{\mathrm{B}} = \left\{\begin{matrix} p \\ q \\ r \end{matrix}\right\} \tag{10}$$

and the cross product matrices $[\tilde{\Omega}]_{\mathrm{W}}$ and $[\tilde{\Omega}]_{\mathrm{B}}$ will be defined as

$$[\tilde{\Omega}]_{\mathrm{W}} = \begin{bmatrix} 0 & -r_{\mathrm{W}} & q_{\mathrm{W}} \\ r_{\mathrm{W}} & 0 & -p_{\mathrm{W}} \\ -q_{\mathrm{W}} & p_{\mathrm{W}} & 0 \end{bmatrix}, \quad [\tilde{\Omega}]_{\mathrm{B}} = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \tag{11}$$

The choice of the suitable reference frame to expand the necessary equations is a matter of convenience. The aerodynamic force, because aerodynamic data are often referred to the relative wind direction, is conveniently defined in the wind-axis reference frame. The thrust is typically defined in the body-axis reference frame. The action of gravity is, instead, simply defined in the local vertical frame. Therefore we have

$$\{F_{\mathrm{A}}\}_{\mathrm{W}} = \left\{\begin{matrix} -D \\ C \\ -L \end{matrix}\right\}, \quad \{F_{\mathrm{T}}\}_{\mathrm{B}} = \left\{\begin{matrix} X_{\mathrm{T}} \\ Y_{\mathrm{T}} \\ Z_{\mathrm{T}} \end{matrix}\right\}, \quad \{F_{\mathrm{G}}\}_{\mathrm{V}} = \left\{\begin{matrix} 0 \\ 0 \\ m\,g \end{matrix}\right\} \tag{12}$$

When the body frame is chosen, a matrix equation similar to (9) expands further to the following

$$m\left(\{\dot{V}\}_{\mathrm{B}} + [\tilde{\Omega}]_{\mathrm{B}} \cdot \{V\}_{\mathrm{B}}\right) = [C_{\mathrm{B}\leftarrow\mathrm{W}}] \cdot \{F_{\mathrm{A}}\}_{\mathrm{W}} + \{F_{\mathrm{T}}\}_{\mathrm{B}} + [C_{\mathrm{B}\leftarrow\mathrm{V}}] \cdot \{F_{\mathrm{G}}\}_{\mathrm{V}} \tag{13}$$

where $[C_{\mathrm{B}\leftarrow\mathrm{W}}]$ and $[C_{\mathrm{B}\leftarrow\mathrm{V}}]$ are transformation matrices from the wind and from the local vertical frame, respectively, to the body frame. These matrices, representing rotations that bring one frame onto another, are orthogonal matrices, i.e. their inverse coincides with the transpose. They are defined as follows

$$[C_{\mathrm{W}\leftarrow\mathrm{B}}] = \begin{bmatrix} \cos\alpha\cos\beta & \sin\beta & \sin\alpha\cos\beta \\ -\cos\alpha\sin\beta & \cos\beta & -\sin\alpha\sin\beta \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}, \quad [C_{\mathrm{B}\leftarrow\mathrm{W}}] = [C_{\mathrm{W}\leftarrow\mathrm{B}}]^{\mathsf{T}} \tag{14}$$

$$[C_{\mathrm{V}\leftarrow\mathrm{B}}] = \begin{bmatrix} \cos\theta\cos\psi & \begin{pmatrix}\sin\phi\sin\theta\cos\psi \\ -\cos\phi\sin\psi\end{pmatrix} & \begin{pmatrix}\cos\phi\sin\theta\cos\psi \\ +\sin\phi\sin\psi\end{pmatrix} \\ \cos\theta\sin\psi & \begin{pmatrix}\sin\phi\sin\theta\sin\psi \\ +\cos\phi\cos\psi\end{pmatrix} & \begin{pmatrix}\cos\phi\sin\theta\sin\psi \\ -\sin\phi\cos\psi\end{pmatrix} \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix}, \quad [C_{\mathrm{B}\leftarrow\mathrm{V}}] = [C_{\mathrm{V}\leftarrow\mathrm{B}}]^{\mathsf{T}} \tag{15}$$

The classical system of scalar force equations in the body-axis reference arising from (13) is the following

$$\boxed{\begin{cases} \dot{u} = \dfrac{1}{m}\left(X_{\mathrm{A}} + X_{\mathrm{T}}\right) - wq + vr - g\sin\theta \\ \dot{v} = \dfrac{1}{m}\left(Y_{\mathrm{A}} + Y_{\mathrm{T}}\right) - ur + wp + g\cos\theta\sin\phi \\ \dot{w} = \dfrac{1}{m}\left(Z_{\mathrm{A}} + Z_{\mathrm{T}}\right) - vp + uq + g\cos\theta\cos\phi \end{cases}} \tag{16}$$

We can also derive translational equations, which are equivalent to the system (16), but are written in terms of V, α and β and their time derivatives. By using the translational equations from Duke *et alii.*[9] we have

$$
\begin{cases}
\dot{V} = \dfrac{1}{m}\Big[-D\cos\beta + C\sin\beta + X_\mathrm{T}\cos\alpha\cos\beta + Y_\mathrm{T}\sin\beta + Z_\mathrm{T}\sin\alpha\cos\beta \\
\qquad\qquad - mg\big(\sin\theta\cos\alpha\cos\beta - \cos\theta\sin\phi\sin\beta - \cos\theta\cos\phi\sin\alpha\cos\beta\big)\Big] \\
\dot{\alpha} = q - \tan\beta\,\big(p\cos\alpha + r\sin\alpha\big) \\
\qquad + \dfrac{1}{Vm\cos\beta}\Big[-L + Z_\mathrm{T}\cos\alpha - X_\mathrm{T}\sin\alpha + mg\big(\cos\theta\cos\phi\cos\alpha + \sin\theta\sin\alpha\big)\Big] \\
\dot{\beta} = +p\sin\alpha - r\cos\alpha \\
\qquad + \dfrac{1}{Vm}\Big[D\sin\beta + C\cos\beta - X_\mathrm{T}\cos\alpha\sin\beta + Y_\mathrm{T}\cos\beta - Z_\mathrm{T}\sin\alpha\sin\beta \\
\qquad\qquad + m\,g\,\big(\sin\theta\cos\alpha\sin\beta + \cos\theta\sin\phi\cos\beta - \cos\theta\cos\phi\sin\alpha\sin\beta\,\big)\Big]
\end{cases}
\tag{17}
$$

where the aerodynamic force components in the wind-axis frame appear explicitly and

$$
u = V\cos\beta\cos\alpha\,, \quad v = V\sin\beta\,, \quad w = V\cos\beta\sin\alpha \tag{18}
$$

The equations (17) are more general than those found in Etkin[3] or Stevens and Lewis.[4]

2. *Rotational Equations*

Again, using the results from classical textbooks we can write the equation defining the rotational acceleration of the aircraft in the chosen inertial reference system as

$$
\frac{\mathrm{d}\vec{H}}{\mathrm{d}t} = \Sigma\vec{M} \tag{19}
$$

where $\Sigma\vec{M}$ is the sum of all the moments acting upon the aircraft and $\vec{H}$ is the total angular momentum about the center of gravity. For a rotating reference frame such as those used in our analysis, e.g. the body-axis system, equation (19) is written in matrix form as

$$
\big\{\dot{H}\big\}_\mathrm{B} + \big[\tilde{\Omega}\big]_\mathrm{B}\cdot\big\{H\big\}_\mathrm{B} = \Sigma\big\{M\big\}_\mathrm{B} \tag{20}
$$

As pointed out by Etkin,[3] the total angular momentum projected in the body-axis system can be generally expressed as the following sum

$$
\big\{H\big\}_\mathrm{B} = \big[I\big]_\mathrm{B}\big\{\Omega\big\}_\mathrm{B} + \big\{H^*\big\}_\mathrm{B} \tag{21}
$$

of a "rigid-body component", $[I]_\mathrm{B}\cdot\{\Omega\}_\mathrm{B}$, and of a "deformation component," $\{H^*\}_\mathrm{B}$. Matrices

$$
\big[I\big]_\mathrm{B} = \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{bmatrix}\,, \quad \big[I\big]_\mathrm{B}^{-1} = \frac{1}{\det\big[I\big]_\mathrm{B}}\begin{bmatrix} I_1 & I_2 & I_3 \\ I_2 & I_4 & I_5 \\ I_3 & I_5 & I_6 \end{bmatrix} \tag{22}
$$

represent the aircraft inertia tensor in the body-axis frame and its inverse, with

$$
\begin{aligned}
\det\big[I\big]_\mathrm{B} &= I_x I_y I_z - I_x I_{yz}^2 - I_z I_{xy}^2 - I_y I_{xz}^2 - 2 I_{yz} I_{xz} I_{xy} \\
I_1 &= I_y I_z - I_{yz}^2\,, \quad I_2 = I_{xy} I_{zz} + I_{yz} I_{xz}\,, \quad I_3 = I_{xy} I_{yz} + I_{yy} I_{xz} \\
I_4 &= I_x I_z - I_{xz}^2\,, \quad I_5 = I_{xx} I_{yz} + I_{xy} I_{xz} \\
I_6 &= I_x I_y - I_{xy}^2
\end{aligned} \tag{23}
$$

Even when the effects of aircraft aeroelastic shape modification are neglected, the deformation component of the angular momentum $\vec{H}$, i.e. the starred term $\{H^*\}_\mathrm{B}$ in (21), might have a relevant role. This is the case when the rotation of those rigid aircraft sybsystems like engine rotors or propellers has to be taken into account. Ordinarily the mass centers of spinning bodies lie on their own axis of rotation. When one of these parts, say the k-th, rotates with angular velocity $\vec{\omega}_k^r$ relative to the main body it is easily shown that its contribution to $\{H^*\}_\mathrm{B}$ is

$$\{H^*\}_{\mathrm{B},k} = [I^r]_{\mathrm{B},k} \cdot \{\omega^r\}_{\mathrm{B},k} \tag{24}$$

where $[I^r]_{\mathrm{B},k}$ is the inertia matrix with respect to a reference frame with axes parallel to the main body axes and origin somewhere on the part's axis of rotation.

From (24), the equation (20) for a rigid aircraft with a number n_r of spinning rotors, expands to the following form

$$[I]_\mathrm{B}\{\dot{\Omega}\}_\mathrm{B} + \sum_{k=1}^{n_r} [I^r]_{\mathrm{B},k} \cdot \{\dot{\omega}^r\}_{\mathrm{B},k} + [\tilde{\Omega}]_\mathrm{B} \cdot \Big([I]_\mathrm{B}\{\Omega\}_\mathrm{B} + \sum_{k=1}^{n_r} [I^r]_{\mathrm{B},k} \cdot \{\omega^r\}_{\mathrm{B},k} \Big) = \Sigma\{M\}_\mathrm{B} \tag{25}$$

Moreover, the spin axis is, typically, also a principal axis of inertia of the rotor. Therefore the vector $\vec{H}_k^*$ is collinear with $\vec{\omega}_k^r$ and has magnitude $I_k^r\,\omega_k^r$, where I_k^r is the moment of inertia of the rotor about the spin axis. Equation (24) is then rewritten as

$$\{H^*\}_{\mathrm{B},k} = I_k^r\,\omega_k^r\,\{i^r\}_{\mathrm{B},k} \tag{26}$$

with $\{i^r\}_{\mathrm{B},k}$ the rotation axis unit vector represented in the body-axis reference. In equation (25), assuming that rotors have a constant inertia, we have, besides the typical (p, q, r), the additional scalar unknowns ω_k^r. In simulation these are found by coupling to the system (25) a suitable engine model that calculates $\dot{\omega}_k^r$ and updates ω_k^r. When a numerical, iterative trim algorithm is applied to the equations of motion, it is important to make sure that, given the throttle setting $\delta_{T,k}$ at each step, the engine model is driven to a steady state ($\dot{\omega}_k^r = 0$).

The resultant moment $\Sigma\vec{M}$ on the body-axis frame, coming from aerodynamic, propulsive and control actions, will have components $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$ in the body-axis system. Usually the control moments are included in the aerodynamic actions so that

$$\mathcal{L} = \mathcal{L}_\mathrm{A} + \mathcal{L}_\mathrm{T}\,, \quad \mathcal{M} = \mathcal{M}_\mathrm{A} + \mathcal{M}_\mathrm{T}\,, \quad \mathcal{N} = \mathcal{N}_\mathrm{A} + \mathcal{N}_\mathrm{T} \tag{27}$$

Considering (24) and (27), the equation (25) will expand further to the final matrix equation

$$\begin{aligned} \begin{Bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{Bmatrix} = \frac{1}{\det [I]_\mathrm{B}} \begin{bmatrix} I_1 & I_2 & I_3 \\ I_2 & I_4 & I_5 \\ I_3 & I_5 & I_6 \end{bmatrix} \cdot \Bigg(& \begin{Bmatrix} \mathcal{L}_\mathrm{A} + \mathcal{L}_\mathrm{T} \\ \mathcal{M}_\mathrm{A} + \mathcal{M}_\mathrm{T} \\ \mathcal{N}_\mathrm{A} + \mathcal{N}_\mathrm{T} \end{Bmatrix} - \sum_k I_k^r\,\dot{\omega}_k^r \begin{Bmatrix} i_x^r \\ i_y^r \\ i_y^r \end{Bmatrix}_k \\ & - \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \cdot \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{bmatrix} \cdot \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} \\ & - \sum_k I_k^r\,\omega_k^r \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \cdot \begin{Bmatrix} i_x^r \\ i_y^r \\ i_z^r \end{Bmatrix}_k \Bigg) \end{aligned} \tag{28}$$

3. Auxiliary kinematic equations

The force and moment equations, when projected onto a moving reference frame, do not form a complete system until they are coupled with some convenient auxiliary equations. The latter are kinematic in nature and, when the body-axis formulation is considered, they relate the components of airplane's velocity and angular rate vectors in the moving frame, respectively, to the velocity vector components in the fixed frame and to the rates of change of Euler angles. The first relationship is obviously the orthogonal transformation $\{V\}_{\mathrm{E}} \equiv \{V\}_{\mathrm{V}} = [C_{\mathrm{V}\leftarrow\mathrm{B}}] \cdot \{V\}_{\mathrm{B}}$, which expands to the following

$$\begin{Bmatrix} \dot{x}_{\mathrm{E}} \\ \dot{y}_{\mathrm{E}} \\ \dot{z}_{\mathrm{E}} \end{Bmatrix} = \begin{bmatrix} \cos\theta\cos\psi & \begin{pmatrix}\sin\phi\sin\theta\cos\psi \\ -\cos\phi\sin\psi\end{pmatrix} & \begin{pmatrix}\cos\phi\sin\theta\cos\psi \\ +\sin\phi\sin\psi\end{pmatrix} \\ \cos\theta\sin\psi & \begin{pmatrix}\sin\phi\sin\theta\sin\psi \\ +\cos\phi\cos\psi\end{pmatrix} & \begin{pmatrix}\cos\phi\sin\theta\sin\psi \\ -\sin\phi\cos\psi\end{pmatrix} \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix} \cdot \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \tag{29}$$

also known as the system of *navigation* or *position equations*. As we have pointed out in section B, for the flat-Earth formulation only the third of equations (29) has to be considered to obtain a trim constraint. The second kinematic relationship is the following well known system of *gimbal equations*[3]

$$\begin{Bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix} = \begin{bmatrix} 1 & \dfrac{\sin\phi\sin\theta}{\cos\theta} & \dfrac{\cos\phi\sin\theta}{\cos\theta} \\ 0 & \cos\phi & -\sin\phi \\ 0 & \dfrac{\sin\phi}{\cos\theta} & \dfrac{\cos\phi}{\cos\theta} \end{bmatrix} \cdot \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} \tag{30}$$

Even if in modern flight simulation codes the above system is replaced by a singularity free system of quaternion updates, equations (30) are still conveniently used when trim constrain equations are derived. Trim conditions for particular cases, when $\theta = \pm\pi/2$, are to be found with a defferent convention on the definition of Euler angles or with a quaternion formulation.

B. Equilibrium equations

The general translational equilibrium equations are found from (17) by imposing $\dot{V} = \dot{\alpha} = \dot{\beta} = 0$. We obtain the system

$$\begin{aligned} &-D\cos\beta + C\sin\beta + X_{\mathrm{T}}\cos\alpha\cos\beta + Y_{\mathrm{T}}\sin\beta + Z_{\mathrm{T}}\sin\alpha\cos\beta \\ &\qquad = mg\big(\sin\theta\cos\alpha\cos\beta - \cos\theta\sin\phi\sin\beta - \cos\theta\cos\phi\sin\alpha\cos\beta\big) \end{aligned} \tag{31}$$

$$L - Z_{\mathrm{T}}\cos\alpha + X_{\mathrm{T}}\sin\alpha = mg\big(\cos\theta\cos\phi\cos\alpha + \sin\theta\sin\alpha\big) + Vm\Big[q\cos\beta - \sin\beta\big(p\cos\alpha + r\sin\alpha\big)\Big] \tag{32}$$

$$\begin{aligned} &D\sin\beta + C\cos\beta - X_{\mathrm{T}}\cos\alpha\sin\beta - Y_{\mathrm{T}}\cos\beta - Z_{\mathrm{T}}\sin\alpha\sin\beta \\ &\qquad = -mg\,\big(\sin\theta\cos\alpha\sin\beta + \cos\theta\sin\phi\cos\beta - \cos\theta\cos\phi\sin\alpha\sin\beta\big) - Vm\,\big(p\sin\alpha - r\cos\alpha\big) \end{aligned} \tag{33}$$

The general rotational equilibrium equations are found from (28) imposing $\dot{p} = \dot{q} = \dot{r} = 0$. We obtain the system

$$\begin{aligned} \begin{Bmatrix} \mathcal{L}_{\mathrm{A}} + \mathcal{L}_{\mathrm{T}} \\ \mathcal{M}_{\mathrm{A}} + \mathcal{M}_{\mathrm{T}} \\ \mathcal{N}_{\mathrm{A}} + \mathcal{N}_{\mathrm{T}} \end{Bmatrix} &= \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \cdot \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{bmatrix} \cdot \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} \\ &\quad + \sum_k I_k^r\,\omega_k^r \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \cdot \begin{Bmatrix} i_x^r \\ i_y^r \\ i_z^r \end{Bmatrix}_k \end{aligned} \tag{34}$$

Any algorithm devised to find a required steady-state flight condition will assume some aircraft state and control variables as given and some others as unknown. As we will see in section D these unknown are seen as the trim algorithm control variables ξ_j ($j = 1, \ldots, n_{\mathrm{tc}}$). Equations (31)-(34) play the role of a set of constraint onto the trim controls ξ_j, upon which the aerodynamics and propulsion terms may depend.

C. Other Equations

We define the normal load factor in the wind-axis frame, n, as

$$n = \frac{L}{mg} \tag{35}$$

This equation is used to define several of the maneuvers in this paper.

Flight path angle, γ, can be defined in terms of altitude rate, $\dot{h} = -\dot{z}_{\mathrm{E}}$, as

$$\gamma = \sin^{-1}\left(\frac{\dot{h}}{V}\right) \tag{36}$$

From the expression (15) of $\left[C_{\mathrm{V}\leftarrow\mathrm{B}}\right]$ and knowing from (14) that $\left[C_{\mathrm{B}\leftarrow\mathrm{W}}\right] = \left[C_{\mathrm{W}\leftarrow\mathrm{B}}\right]^{\mathsf{T}}$ we have the altitude rate defined as

$$\begin{aligned} \dot{h} =& -V_{z_{\mathrm{V}}} = \begin{bmatrix}0 & 0 & -1\end{bmatrix} \cdot \left[C_{\mathrm{V}\leftarrow\mathrm{B}}\right] \cdot \left[C_{\mathrm{B}\leftarrow\mathrm{W}}\right] \cdot \left\{V\right\}_{\mathrm{W}} \\ &= V\left(\cos\alpha\cos\beta\sin\theta - \sin\beta\sin\phi\cos\theta - \sin\alpha\cos\beta\cos\phi\cos\theta\right) \end{aligned} \tag{37}$$

so that when

$$\gamma = 0 \qquad (\textit{constant altitude}\ \text{condition}) \tag{38}$$

and for the wings-level case

$$\phi = 0 \qquad (\textit{wings-level}\ \text{condition}) \tag{39}$$

when we combine equations (36) and (37), we get

$$\sin\gamma = \cos\beta\left(\cos\alpha\sin\theta - \sin\alpha\cos\theta\right) \tag{40}$$

and when we use equation (38) we get

$$\theta = \alpha \tag{41}$$

If we use equations (41) and (39), equations (31), (32), and (33) become

$$-D\cos\beta + C\sin\beta + X_{\mathrm{T}}\cos\alpha\cos\beta + Y_{\mathrm{T}}\sin\beta + Z_{\mathrm{T}}\sin\alpha\cos\beta = 0 \tag{42}$$

$$L - Z_{\mathrm{T}}\cos\alpha + X_{\mathrm{T}}\sin\alpha = mg - Vm\Big[q\cos\beta - \sin\beta\big(p\cos\alpha + r\sin\alpha\big)\Big] \tag{43}$$

$$D\sin\beta + C\cos\beta - X_{\mathrm{T}}\cos\alpha\sin\beta - Y_{\mathrm{T}}\cos\beta - Z_{\mathrm{T}}\sin\alpha\sin\beta = -Vm\big(p\sin\alpha - r\cos\alpha\big) \tag{44}$$

expressing the *translational constraint equations for a horizontal wings-level flight.*

D. Discussion of the equations

Not considering for simplicity the cases of *stick-free* and *reversible* flight controls, we have that the system formed by the dynamic equations, i.e. (16)—or the equivalent (17)—and (28), and by the kinematic auxiliary equations updating airplane's position and attitude, i.e. (29) and (30), is a closed set of twelve differential, ordinary, nonlinear equations. The unknowns are the entries of the state vector $\boldsymbol{x}$

$$\boldsymbol{x} = \left[\boldsymbol{x}_d^{\mathsf{T}}, \boldsymbol{x}_k^{\mathsf{T}}\right]^{\mathsf{T}} = \Big[\left[u, v, w, p, q, r\right]^{\mathsf{T}}, \left[x_{\mathrm{E}}, y_{\mathrm{E}}, x_{\mathrm{E}}, \phi, \theta, \psi\right]^{\mathsf{T}}\Big]^{\mathsf{T}} \tag{45}$$

defined by (2)-(3), and the inputs, for a traditional configuration, are

$$\boldsymbol{u} = \left[\delta_T, \delta_e, \delta_a, \delta_r\right]^{\mathsf{T}} \tag{4}$$

In the system defined above a crucial role is played by the accuracy and completeness of the aerodynamics and propulsion models. These models consist formally in a set of laws expressing the dependence of terms like $(\cdot)_{\mathrm{A}}$ and $(\cdot)_{\mathrm{T}}$ from the state, the control variables and their time derivatives. The experience has shown that the force and moment components involved in equations (16)-(28) depend to some extent from all or some of the variables listed in Table 1.

Table 1. Dependency of force and moment components upon aircraft state and control variables

	Component	State Variables												
		h	V	α	β	p	q	r	$\dot{\alpha}$	$\dot{\beta}$	δ_e	δ_a	δ_r	δ_T
1	$X_{\mathrm{A}} + X_{\mathrm{T}}$	○[d]/●[e]	●	●	○	∼	∼	∼	∼	∼	○	∼	∼	●
2	$Y_{\mathrm{A}} + Y_{\mathrm{T}}$	○[d]/●[e]	●	○[a,b]	●	○	∼	●	∼	●/○	∼	○	●	○[c]
3	$Z_{\mathrm{A}} + Z_{\mathrm{T}}$	●[d,e]	●	●	○	○	●	∼	●	∼	●	∼	∼	○
4	$\mathcal{L}_{\mathrm{A}} + \mathcal{L}_{\mathrm{T}}$	○[d]/●[e]	●	○[a]	●	●	∼	●	∼	∼	∼	●	●	○
5	$\mathcal{M}_{\mathrm{A}} + \mathcal{M}_{\mathrm{T}}$	●[d,e]	●	●	○	∼	●	∼	●	∼	●	∼	∼	●/○
6	$\mathcal{N}_{\mathrm{A}} + \mathcal{N}_{\mathrm{T}}$	○[d]/●[e]	●	○	●	○	∼	●	∼	●/○	∼	○	●	○[c]

symbols: ● = dependent on; ○ = could depend on, may vary with configuration; ∼ = almost always not dependent on

[a] When β is not zero.

[b] When thrusters work in nonaxial flow.

[c] When a nonsymmetric thrust is applied.

[d] When ground effect is modelled.

[e] When engine state is altitude dependent.

In the dependency summary reported in Table 1 we have the altitude, $h = -z_{\mathrm{E}}$, but not the other two coordinates, $(x_{\mathrm{E}}, y_{\mathrm{E}})$. Furthermore we have in some cases a significant dependence on the aerodynamic angle rates $\dot{\alpha}$ and $\dot{\beta}$. The table also reports the cases of models that might present a particular dependence or might be influenced by the particular aircraft configuration.

When the dependence onto the rates $\dot{\alpha}$ and $\dot{\beta}$ is completely general, the equations of unsteady motion (16), (28), (29) and (30) indeed form a system of implicit equations like (1). Often the unsteady aerodynamic effects are well approximated by means of constant multipliers, e.g. the stability derivatives $C_{L_{\dot{\alpha}}}$ and $C_{m_{\dot{\alpha}}}$, and the system of equations is such that all the state variable rates are explicitly expressed in terms of the remaining quantities. Then we have the following explicit state equation

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \tag{46}$$

with $\boldsymbol{f} = [f_1, f_2, \ldots, f_{12}]^{\mathsf{T}}$. Since trim conditions must derive from equilibrium, steady-state flight, for which $\dot{\alpha} = \dot{\beta} = 0$, or equivalently $\dot{u} = \dot{v} = \dot{w} = 0$, a state equation in the form (46) is sufficient for the purpose of finding aircraft trimmed states.

According to the general definition of trim (6), given the models of aircraft aerodynamics and propulsion, an equilibrium state will be found by:

(i) declaring the desired trajectory of the airplane's center of mass, assigning the values of some state variables (e.g. V, h), of some kinematic variable rates (e.g. $\dot{h}$ for a climbing trimmed flight) or of some related rates (e.g. $\dot{\psi}_{\mathrm{W}}$ in a turn), and possibly of some aircraft control (e.g. when a trimmed state with aileron failure is wanted, with $\delta_a = 0$);

(ii) constraining the values of some other state variables according to the kinematic equations (e.g. deriving the values of p, q, r); and

(iii) solving the system

$$f_k(\boldsymbol{x}, \boldsymbol{u}) = 0\,, \quad \text{for } k = 1, \ldots, 6 \tag{47}$$

of six nonlinear, transcendent, algebraic equations stemming from the dynamic part of (46), i.e. from the coupled constraint equations (31)-(34). The system (47) will generally provide information on the unknown control settings and the remaining state variable values ensuring the desired steady-state flight.

The above steps still describe a generic situation. From (47) we can only expect that, after assigning a first set of state variables and controls $\boldsymbol{\xi}_1$ and constraining a second set of state variables $\boldsymbol{\xi}_2$, the number of unknowns to be found, i.e. the entries of a column vector $\boldsymbol{\xi}$ defined as

$$\boldsymbol{\xi}_1,\, \boldsymbol{\xi}_2 = \text{two distinct subsets of } \left[\boldsymbol{x}^\mathsf{T}, \boldsymbol{u}^\mathsf{T}\right]^\mathsf{T} \Rightarrow \boldsymbol{\xi} = \text{the remaining subset of } \left[\boldsymbol{x}^\mathsf{T}, \boldsymbol{u}^\mathsf{T}\right]^\mathsf{T} \tag{48}$$

is not greater than six. The dimension of $\boldsymbol{\xi}$ may depend on the desired trim condition and, in the general case, on the aircraft configuration and number of controls. In this context we call a *trim algorithm* one that is capable of solving the problem presented above. Then we define the n_{tc} unknowns $\boldsymbol{\xi}$ as the *trim controls*. Finally equations (47) become

$$f_k\big(\boldsymbol{\xi}\big) = 0\,, \quad \text{for } k = 1, \ldots, 6 \tag{49}$$

McFarland[7] defines the trim problem as *overdetermined* if a system like (49) has more equations than trim controls, *determined* if the number of equations equals the number of controls, *underdetermined* if the number of equations is less than the number of controls.

To clarify with an example the type of problem we have to face with, let us consider a particular case. Suppose that conditions for a steady-state flight along a rectilinear trajectory are desired, for a traditional configuration aircraft. Speed V, altitude h, flight path angle γ and flight path heading ψ_{W} are assigned. No particular requirements are prescribed to the aerodynamic control surface deflections or to the attitude. In this case we have a set of initial values

$$\boldsymbol{\xi}_1 = \big[\, V\,,\, z_{\text{E}}\,,\, \gamma \Rightarrow \dot{z}_{\text{E}}\,,\, \psi_{\text{W}}\,,\, q_{\text{W}} = 0 \,\big] \tag{50}$$

and a set of derived values

$$\boldsymbol{\xi}_2 = \big[\, p = 0\,,\, q = 0\,,\, r = 0 \,\big] \tag{51}$$

We have assigned the velocity vector $\vec{V}$ of airplane's mass center, in magnitude and direction with respect to the Earth, and we have deducted for the type of flight condition required that the angular rates have to be zero. To ensure that the flight dynamics model is able to accomplish the desired trimmed state we have to find the correct aircraft attitude in space *and* the correct set of control surface deflections and thrust output. Then we have the following set of trim controls

$$\boldsymbol{\xi} = \big[\, \phi\,,\, \theta\,,\, \psi\,,\, \delta_T\,,\, \delta_e\,,\, \delta_a\,,\, \delta_r \,\big] \tag{52}$$

Note that a given aircraft attitude in terms of Euler angles corresponds to a precise attitude with respect to $\vec{V}$, that is, to a particular couple (α, β). According to (52) our trim problem is underdetermined, being $\boldsymbol{\xi}$ a set of seven trim controls as opposed to the six equilibrium equations (49). This means that there is the chance that more than one set of $\boldsymbol{\xi}$ will satisfy the posed problem. One such situation is depicted in Figure 1. There is also the chance that the assigned velocity and flight path angle are such that no possible combination of trim controls will give a trimmed state.

If we restate the problem by adding the requirement of wings being level, $\phi = 0$, we lower by one the dimension of $\boldsymbol{\xi}$ and make (49) a determined system. Even in this case more than one possible solution to the problem may exist. For example if a nonzero sideslip condition turns out to satisfy trim equations (49), possibly with nonzero rudder and aileron deflections, also the condition with opposite sign of beta will be a candidate trim condition.

In his paper McFarland presents a solution to the trim problem based on a variational approach. The method of solution proposed here is based instead on the derivation of one single algebraic equation from the system (49), which is solved in the space of $\boldsymbol{\xi}$'s, subject to some bounds. This method is presented in section E for a completely general case. Examples of application of this solution method to some simplified aircraft models are found in the textbooks by Stevens and Lewis[4] and Stengel.[5]

E. Generation of trim conditions using minimization

One of the most widely accepted numerical method to find a generic steady-state flight condition is based on the minimization of a *cost function*. This function is typically defined as a general scalar dependence

$\mathcal{J}\big(\xi_1,\ldots,\xi_{n_{\rm tc}}\big)$ upon $n_{\rm tc}$ parameters. The independent variables include the unknown aircraft control settings belonging to vector $\boldsymbol{u}$. The cost function is, by definition, always non negative and evaluates to zero when the aircraft is in a steady-state flight. A minimization algorithm finds the values of control inputs and of some state variables that make the cost function zero.

Typically the cost function is derived from the dynamic equations of motion. A good choice is the quadratic function

$$\mathcal{J} = \dot{u}^2 + \dot{v}^2 + \dot{w}^2 + \dot{p}^2 + \dot{q}^2 + \dot{r}^2 \tag{53}$$

or more generally

$$\mathcal{J} = \{\dot{x}_d\}^{\mathsf{T}} \cdot [W] \cdot \{\dot{x}_d\} \tag{54}$$

with $[W]$ a symmetric, positive-definite, square matrix of weights yielding a non negative $\mathcal{J}$. The role of $[W]$ in the sum (54), as opposed to the simple sum (53), is mainly that of making the single addends of the same order of magnitude, accounting for differences in units of measure and providing control of the minimization. It has to be noted that choosing $\dot{V}$, $\dot{\alpha}$ and $\dot{\beta}$ in place of $\dot{u}$, $\dot{v}$ and $\dot{w}$ in the above definition is perfectly equivalent.

The scalar function $\mathcal{J}$ defined above depends generally on both dynamic and kinematic state variables, and on control settings. Therefore the cost function is formally a $\mathcal{J}\big(\boldsymbol{x}_d, \boldsymbol{x}_k, \boldsymbol{u}\big)$. For the purpose of trimming this dependence has to be restricted to a subset $\boldsymbol{\xi}$ of all the independent variables, the remaining ones being thought of as fixed parameters. When function $\mathcal{J}\big(\boldsymbol{\xi}\big)$ is minimized according to control variable bounds and to flight-path constraints then the trim condition is met.

As discussed in section IV, there are different possible steady-state flight conditions and, according to the desired one, some state variables may be prescribed or constrained. Each desired trim condition is associated to a different set $\boldsymbol{\xi}$ of independent variables. For all trim conditions the velocity V and altitude h are invariably specified. Moreover, the initial geographical position does not influence the trim. Hence V, $x_{\rm E}$, $y_{\rm E}$ and $z_{\rm E}$ never appear among the variables ξ_j.

IV. Trim conditions

In this section we present some of the most interesting trim conditions from the standpoint of engineering design and of flight dynamics model evaluation. For each of them we clarify the set of independent variables to be adjusted in a cost function minimization algorithm. Details on the algorithm that we have chosen for the application of the concepts discussed in this paper will be given in a subsequent section.

The conditions presented here are not comprehensive of all possible steady-state flight situations. Nevertheless the results of this paper are easily extended to special cases.

A. Straight flight

1. *General straight flight (any γ, possibly asymmetric attitude)*

When a steady-state flight along a straight path is desired one may wish to specify, besides velocity and altitude, the flight-path angle γ and the wind frame heading $\psi_{\rm W}$. The latter quantity is the direction in the plane $x_{\rm E}y_{\rm E}$ towards which the aircraft velocity vector must point. It turns out that the angle $\psi_{\rm W}$ does not influence the trim strategy and can be set to zero without loss of generality. We refer to the trimmed flight of an aircraft along a straight path, with a possibly nonzero γ and possibly nonzero ϕ, θ and ψ, as the *general straight flight*, see Figure 1.

A general straight flight, achieved with attitude asymmetries with respect to the vertical plane containing the velocity vector, see Figure 2, is such that the angles $\psi - \psi_{\rm W}$, θ and ψ are not zero and their *trimmed* values result from the equilibrium conditions. The possible asymmetries are due to one or some of the following situations: (*i*) the aircraft configuration is not symmetric with respect to the body-fixed longitudinal plane, (*ii*) some lateral or directional aerodynamic actions do not vanish with zero sideslip (this situation is generally coupled with the previous one, for example: different incidence settings of the two wings, opening failure of one of the two landing gears, a prominent probe on one of the two wings, etc.), (*iii*) the propulsive actions are asymmetric (for example: a rolling moment due to propeller, the failure of one of the engines).

In general the heading difference $\psi - \psi_{\rm W}$ is related to the sideslip angle β required to achieve the overall balance of lateral-directional actions on the aircraft. In the particular case when a flight with zero sideslip angle is required, the difference $\psi - \psi_{\mathrm{W}}$ will be zero, i.e. $\psi = 0$, and and the two remaining Euler angles, elevation θ and bank ϕ, are to be set properly for the achievement of equilibrium.

Other particular trim conditions are those that require the wings to be leveled. In these cases ϕ will be set to zero and the two Euler angles ψ and θ will remain as free parameters.

When the cost function is minimized for a straight trimmed flight, all the angular rates p, q and r are set to zero and are left out of $\boldsymbol{\xi}$. As discussed above, depending on aircraft configuration and possible asymmetries, the Euler angles ψ, θ and ϕ may not be zero and they are treated as independent variables. They are incorporated in the vector $\boldsymbol{\xi}$ and made part of the set of adjustable parameters controlled by the cost function numerical minimization algorithm. Therefore, assuming an airplane configuration with the traditional four controls, the minimization control vector is

$$\boldsymbol{\xi} = \big[\, \phi \,,\, \theta \,,\, \psi \,,\, \delta_T \,,\, \delta_e \,,\, \delta_a \,,\, \delta_r \,\big]^{\mathsf{T}} \tag{55}$$

The presence in (55) of all the attitude angles permits the intrinsic adjustment of the two aerodynamic angles α and β. In general, for a given aircraft aerodynamics and propulsion model, and given a flight-path angle γ to be maintained at the assigned velocity V, there is no *a priori* knowledge of the airplane attitude and control settings that guarantee a steady-state flight. A trim algorithm based on cost function minimization finds the combination of parameters (55) that makes (54) zero within a given tolerance.

When the airplane has symmetrical characteristics, both geometric and aerodynamic, and the thrust is symmetric, the steady-state flight along a straight path can be certainly achieved with zero sideslip and bank angles, $\beta = \phi = 0$, and lateral-directional controls at their neutral positions, $\delta_a = \delta_r = 0$. Then, conceptually, the trimmed angle of attack α is deducible from the equilibrium of longitudinal forces and moments and the attitude angle θ is found by the sum $\gamma + \alpha$. When this is the case, it is often more practical considering the zero bank as a requirement, putting $\phi = \psi_{\mathrm{W}} = \psi = 0$, assigning $\delta_a = \delta_r = 0$, and letting the trim control variable vector (55) become

$$\boldsymbol{\xi}_{\mathrm{Long}} = \big[\, \theta \,,\, \delta_T \,,\, \delta_e \,\big]^{\mathsf{T}} \tag{56}$$

In this context we refer to the adjustment of trim control parameters (56) as the *longitudinal trim algorithm*.

When there are asymmetries of any type in the flight dynamics model, it is a suitable combination of all the ξ_j's in (55) that assures the steady-state straight flight. The sketch reported in fig. 2 illustrates this general situation. To reach the desired trimmed flight along a straight line the minimization control parameters are adjusted until all the equilibrium relations given by (5) are met for the desired type of steady-state flight.

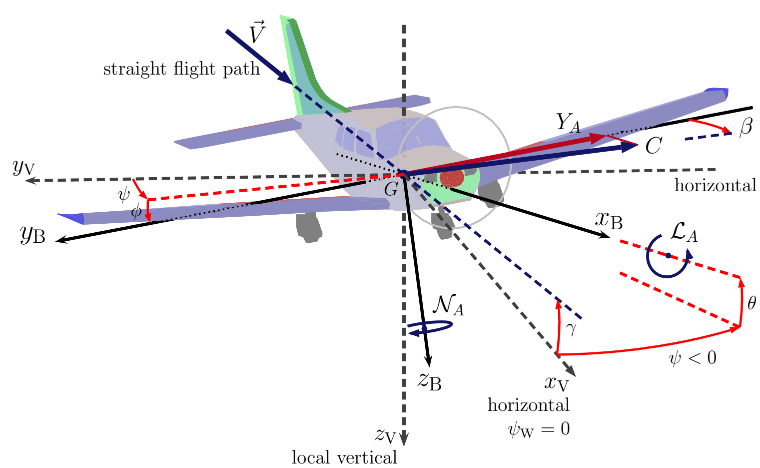

Figure 2. Possible attitude angles of a trimmed translational flight in a generic asymmetric condition. The desired velocity V and flight path angle γ are attained by trimming both longitudinal and lateral-directional actions on the airplane. In asymmetric conditions the aerodynamic actions $(\cdot)_{\mathbf{A}}$ are balanced by gravity and propulsive actions only at nonzero roll ϕ and sideslip β, i.e. heading deviation $\psi - \psi_{\mathrm{W}}$. The cost function minimization should proceed by adjusting all parameters in (55) until the consequent perturbations in terms of forces and moments result in a set of reasonably balanced actions.

Table 2 summarizes the equations used for straight-and-level trim.

Table 2. Initial conditions, constraint equations, and computed parameters for steady-state, straight flight

Type Parameters	Definitions
Initial Conditions	$V = V_0$, $h = h_0$ $\gamma = \gamma_0$, $\psi_{\rm W} = 0$
Derived parameters	$p = q = r = 0$
Constraint Equations	see (31)-(32)-(33), (34), with the above conditions
Trim Control Parameters	$\boldsymbol{\xi} = \left[\, \phi \,,\, \theta \,,\, \psi \,,\, \delta_T \,,\, \delta_e \,,\, \delta_a \,,\, \delta_r \,\right]^{\mathsf{T}}$
Computed Parameters	dynamic state variables: β and α

2. *Straight-and-level flight*

What is termed generically *straight-and-level flight* may not mean the aircraft is flying straight (with symmetric attitude and with wings kept horizontal) or level (with zero climb rate). Here we will mean for straight-and-level flight the case in which

$$\phi = 0 \tag{57}$$

$$\gamma = 0 \tag{58}$$

that is, wings level and constant altitude, with no particular requirement on the sideslip angle.

According to the discussion reported in the previous section, when we have a symmetric aircraft model about its xz body-axis plane, in mass distribution, aerodynamics, and thrust, and when the aircraft is flying with a zero flight-path angle *then* we may have certainly a straight-and-level flight with zero sideslip, i.e. with $\psi = \psi_{\rm W}$.

For a generic straight-and-level flight, we have, besides the basic requirements stated above, the following trivial derived conditions

$$p = q = r = 0 \tag{59}$$

and a vector of trim controls

$$\boldsymbol{\xi} = \left[\, \theta \,,\, \psi \,,\, \delta_T \,,\, \delta_e \,,\, \delta_a \,,\, \delta_r \,\right]^{\mathsf{T}} \tag{60}$$

From conditions (59) we can rewrite the constraint equilibrium equations (43) and (44) as

$$L - Z_{\rm T} \cos\alpha + X_{\rm T} \sin\alpha = mg \tag{61}$$

$$D \sin\beta + C \cos\beta - X_{\rm T} \cos\alpha \sin\beta - Y_{\rm T} \cos\beta - Z_{\rm T} \sin\alpha \sin\beta = 0 \tag{62}$$

where α and β will be determined by the presence in $\boldsymbol{\xi}$ of the Euler angles θ and ψ.

Table 3 summarizes the equations used for straight-and-level trim.

Table 3. Initial conditions, constraint equations, and computed parameters for steady-state, straight-and-level flight

Type Parameters	Definitions
Initial Conditions	$V = V_0$, $h = h_0$ $\phi = \gamma = \psi_\mathrm{W} = 0$
Derived parameters	$p = q = r = 0$
Constraint Equations see (42)-(43)-(44), (34)	$D\cos\beta - C\sin\beta = X_\mathrm{T}\cos\alpha\cos\beta + Y_\mathrm{T}\sin\beta + Z_\mathrm{T}\sin\alpha\cos\beta$ $L - mg = Z_\mathrm{T}\cos\alpha - X_\mathrm{T}\sin\alpha$ $D\sin\beta + C\cos\beta = X_\mathrm{T}\cos\alpha\sin\beta + Y_\mathrm{T}\cos\beta + Z_\mathrm{T}\sin\alpha\sin\beta$ $\mathcal{L}_\mathrm{A} + \mathcal{L}_\mathrm{T} = 0$, $\mathcal{M}_\mathrm{A} + \mathcal{M}_\mathrm{T} = 0$, $\mathcal{N}_\mathrm{A} + \mathcal{N}_\mathrm{T} = 0$
Trim Control Parameters	$\boldsymbol{\xi} = \big[\,\theta\,,\,\psi\,,\,\delta_T\,,\,\delta_e\,,\,\delta_a\,,\,\delta_r\,\big]^\mathsf{T}$
Computed Parameters	dynamic state variables: β (and $\alpha = \theta$)

B. Push-Over/Pull-Up

A push-over or a pull-up maneuver can be performed with the wings level, zero sideslip, or—in the case of an aircraft symmetric in mass distribution, aerodynamics, and thrust—both. We will examine the wings-level case with a zero flight path angle:

$$\phi = \gamma = 0 \tag{63}$$

By definition, a load factor $n > 1$ is associated with a pull-up maneuver and load factor $n < 1$ is associated with a push-over maneuver. In addition to these definitions, we have the following derived constraints:

$$p = r = 0 \tag{64}$$

If we begin with equation (43) and the defining characteristics of the push-over and pull-up maneuvers we get

$$L - Z_\mathrm{T}\cos\alpha + X_\mathrm{T}\sin\alpha = mg + qVm\cos\beta \tag{65}$$

Which, using the definition of load factor from equation (35), gives us

$$q = \frac{1}{Vm\cos\beta}\Big[mg\,(n-1) + Z_\mathrm{T}\cos\alpha - X_\mathrm{T}\sin\alpha\Big] \tag{66}$$

We summarize these results in Table 4 showing the initial conditions, the derived parameters, the constraint equations, and the computed parameters.

Table 4. Initial conditions, derived parameters, constraint equations, and computed parameters for push-over or pull-up condition

Type Parameters	**Definitions**
Initial Conditions[a]	$V = V_0,\ h = h_0,\ \psi_{\mathrm{W}} = 0$ $n = n_0$ $\phi = \gamma = 0$
Derived Parameters see (66)	$q = \dfrac{1}{V m \cos\beta}\Big[mg\,(n-1) + Z_{\mathrm{T}}\cos\alpha - X_{\mathrm{T}}\sin\alpha\Big],\ p = r = 0$
Constraint Equations see (42)-(43)-(44), (34)	$D\cos\beta - C\sin\beta = X_{\mathrm{T}}\cos\alpha\cos\beta + Y_{\mathrm{T}}\sin\beta + Z_{\mathrm{T}}\sin\alpha\cos\beta$ $D\sin\beta + C\cos\beta = X_{\mathrm{T}}\cos\alpha\sin\beta + Y_{\mathrm{T}}\cos\beta + Z_{\mathrm{T}}\sin\alpha\sin\beta$ $\mathcal{L}_{\mathrm{A}} + \mathcal{L}_{\mathrm{T}} = -I_{yz}q^2 + \sum_k I_k^r\,\omega_k^r\,i_{zk}^r\,q$ $\mathcal{M}_{\mathrm{A}} + \mathcal{M}_{\mathrm{T}} = 0$ $\mathcal{N}_{\mathrm{A}} + \mathcal{N}_{\mathrm{T}} = I_{xy}q^2 - \sum_k I_k^r\,\omega_k^r\,i_{xk}^r\,q$
Trim Control Parameters	$\boldsymbol{\xi} = \big[\,\theta\,,\,\psi\,,\,\delta_T\,,\,\delta_e\,,\,\delta_a\,,\,\delta_r\,\big]^{\mathsf{T}}$
Computed Parameters	dynamic state variables: β (and $\alpha = \theta$)

[a] Initial values are referred to the condition reported in Figure 3, i.e. at time t_0 when the plane $x_{\mathrm{W}}y_{\mathrm{W}}$ is horizontal.

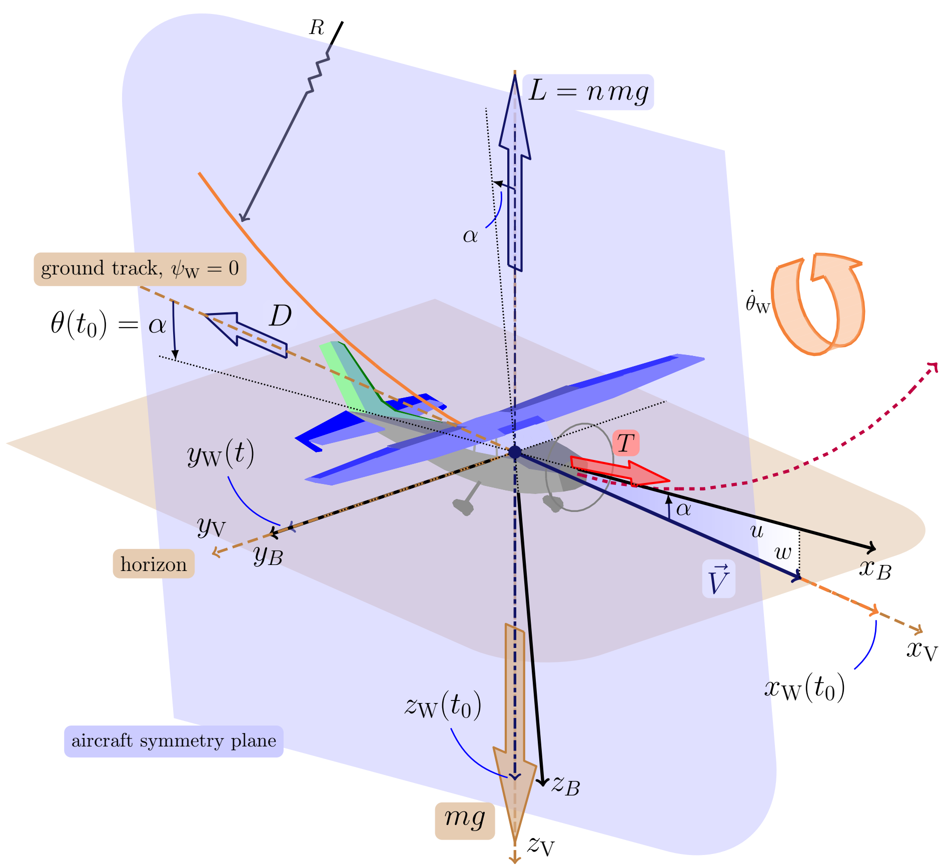

Figure 3. Forces in the wings-level pull-up condition.

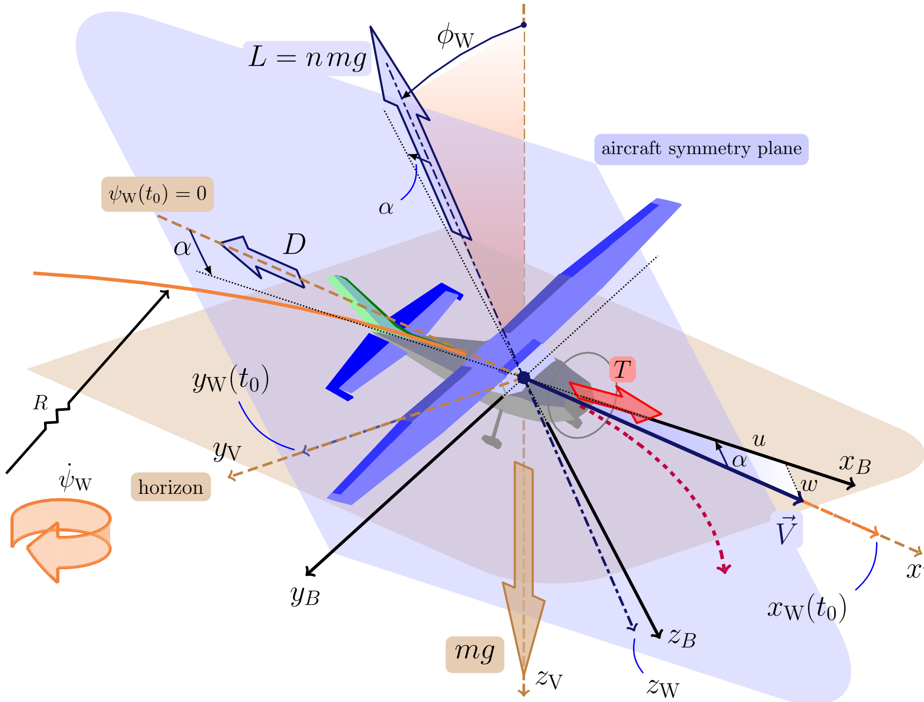

Figure 4. Three dimensional sketch of forces in steady-state, level, coordinated turn

C. Steady-State turn

For the steady-state turn let us start examining Figure 4, depicting a flight situation known as *level turn* (with load factor, n, greater than one). This scheme of forces assumes that the turn is, in fact, *level*, i.e. with zero flight path angle, $\gamma = 0$. Moreover, the turn represented in figure is called *coordinated turn*, being a maneuver with zero sideslip angle, $\beta = 0$.

A steady-state, level turn is a particular case of a more general steady-state turn. A generic steady turn will be performed with a nonzero climb rate and possibly with nonzero sideslip angle, the airplane center of mass following a helical path with constant speed. For our mathematical derivations ee will discuss the general case first and we will consider the level, coordinated turn as a particular case.

From the kinematic standpoint we can start deriving some nontrivial trigonometric expressions relating the angles γ, ϕ_{W}, ψ, θ ϕ, α and β. For this purpose we have reported Figure 5. In a generic steady turn, assuming the instant of time when the velocity heading angle is zero, the wind frame plane $x_{\mathrm{W}}z_{\mathrm{W}}$ is banked about the velocity vector $\vec{V}$, and the wind frame attitude with respect to the local vertical frame is given by the Euler angle triad: $(\phi_{\mathrm{W}}, \theta_{\mathrm{W}} = \gamma, \psi_{\mathrm{W}} = 0)$. In the plane $x_{\mathrm{W}}z_{\mathrm{W}}$ so placed in the three-dimensional space, we have the axis z_{W}, which also belongs to the aircraft body plane $x_{\mathrm{B}}z_{\mathrm{B}}$. The two planes $x_{\mathrm{B}}z_{\mathrm{B}}$ and $x_{\mathrm{W}}z_{\mathrm{W}}$ have the axis z_{W} as their intersection line and form what in Geometry is called a *dihedral angle* given by β.

According to the above observations, a trim algorithm considers the variables γ and ϕ_{W} as given quantities, and takes at least one of the three aircraft Euler angles, for instance ψ, as a free, adjustable parameter at the generic step. The adjustment of ψ will depend on the specific minimization algorithm chosen. At each step, once this parameter is set, the remaining two attitude angles θ and ϕ have to be assigned for the evaluation of the cost function.

If the turn is coordinated, β is zero and the two planes $x_{\mathrm{W}}z_{\mathrm{W}}$ and $x_{\mathrm{B}}z_{\mathrm{B}}$ will coincide, that is, from Figure 5, the points A and A' will coincide. When this is the case, the aircraft elevation θ will be necessarily equal to the angle $\bar{\theta}$ represented in figure. As we will later in this section, the remaining aircraft Euler angle ϕ, known γ, ϕ_{W}, ψ and θ, will be determined from one of the available transformations between the frames $(\cdot)_{\mathrm{V}}$, $(\cdot)_{\mathrm{W}}$ and $(\cdot)_{\mathrm{B}}$. Therefore, even if the sideslip angle is zero, the body-axis reference frame will have a nonzero Euler angle ψ and obviously nonzero values of θ and ϕ. The latter is in general *not* equal to ϕ_{W}.

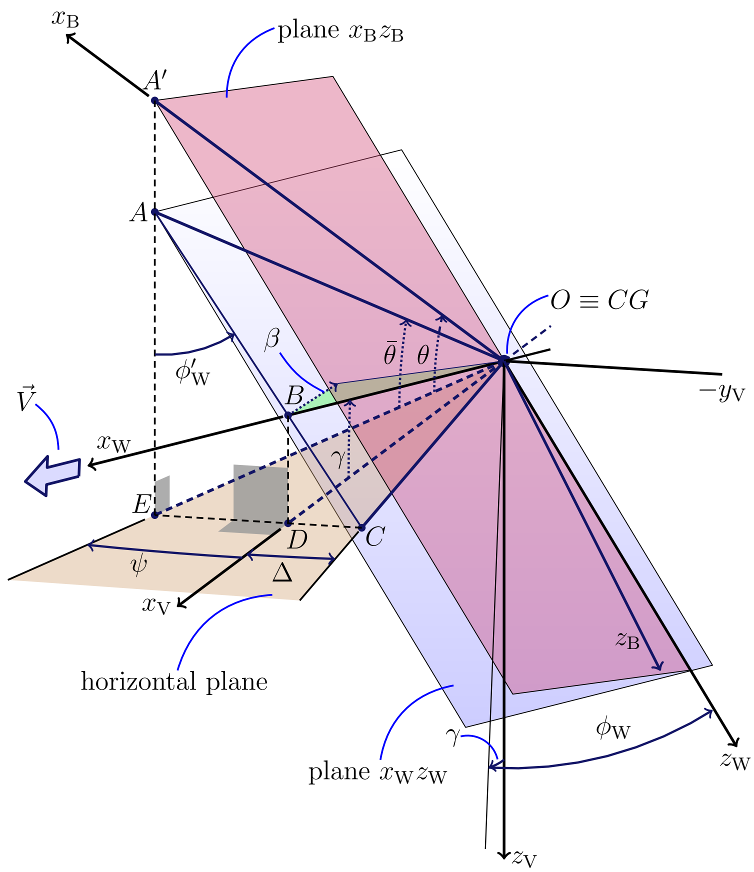

Figure 5. Wind frame and body fixed frame Euler angles in a turn. When $\beta = 0$ the turn is said to be coordinated and the axis x_B belongs to plane $x_W z_W$. In coordinated turns, given γ, ϕ_W and ψ, the elevation θ is constrained to be equal to $\bar{\theta}$.

When the turn maneuver is noncoordinated, then also the elevation θ has to be considered as a trim control variable, together with ψ. This is definitely the situation depicted in Figure 5, where $\beta \neq 0$ and $\theta \neq \bar{\theta}$. Finally, from the same figure we note that both z_W and z_B belong to the aircraft longitudinal plane $x_B z_B$ and that the axis z_B is obtained by rotating z_W about the transversal, nonhorizontal axis y_B of the angle α.

Let us now derive some important equations that relate aircraft Euler angles and flow angles in a turn. By definition of transformation matrices, we have that the unit vectors of the various reference frames considered in this paper are formally expressed by the following equations

$$\left\{ \begin{matrix} \vec{\imath}_B \\ \vec{\jmath}_B \\ \vec{k}_B \end{matrix} \right\} = \left[C_{B\leftarrow V}\right] \cdot \left\{ \begin{matrix} \vec{\imath}_V \\ \vec{\jmath}_V \\ \vec{k}_V \end{matrix} \right\} , \quad \left\{ \begin{matrix} \vec{\imath}_B \\ \vec{\jmath}_B \\ \vec{k}_B \end{matrix} \right\} = \left[C_{B\leftarrow W}\right] \cdot \left\{ \begin{matrix} \vec{\imath}_W \\ \vec{\jmath}_W \\ \vec{k}_W \end{matrix} \right\} , \quad \left\{ \begin{matrix} \vec{\imath}_W \\ \vec{\jmath}_W \\ \vec{k}_W \end{matrix} \right\} = \left[C_{W\leftarrow V}\right] \cdot \left\{ \begin{matrix} \vec{\imath}_V \\ \vec{\jmath}_V \\ \vec{k}_V \end{matrix} \right\} \tag{67}$$

Consequently we have the following matrix identity

$$\left[C_{B\leftarrow V}\right] = \left[C_{B\leftarrow W}\right] \cdot \left[C_{W\leftarrow V}\right] \tag{68}$$

that gives us nine possible relationships between the attitude angles.

In the general case of steady-state turn we can only substitute in equations (68) $\psi_W = 0$. For a coordinated, level turn, being also $\theta_W \equiv \gamma = 0$, and $\beta = 0$, matrices $[C_{W\leftarrow V}]$ and $[C_{B\leftarrow W}]$ have much simpler expressions.

Knowing that the unit vectors $\vec{\imath}_{y_W}$ and $\vec{\imath}_{y_B}$ are the normals to the planes $x_W z_W$ and $x_B z_B$, respectively, we have that the cosine of their dihedral angle is given by

$$\cos\beta = \vec{\imath}_{y_W} \cdot \vec{\imath}_{y_B} \tag{69}$$

When the two planes are described by their canonical equations in local vertical frame coordinates

$$x_{\mathrm{W}} z_{\mathrm{W}} \; : \quad a\,x + b\,y + c\,z + d = 0\,, \qquad x_{\mathrm{B}} z_{\mathrm{B}} \; : \quad a'\,x + b'\,y + c'\,z + d' = 0 \tag{70}$$

we have

$$\cos\beta = \frac{a\,a' + b\,b' + c\,c'}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}} \tag{71}$$

Expressing the coefficients (a,b,c,d) and (a',b',c',d') according to equation (67), it easily shown that the angle β is given by the following equation

$$\cos\beta = \frac{C_1\,\sin\phi_{\mathrm{W}}\cos\gamma + C_2\,\cos\phi_{\mathrm{W}} + C_3\,\sin\phi_{\mathrm{W}}\sin\gamma}{\sqrt{C_1^2 + C_2^2 + C_3^2}} \tag{72}$$

derived from (71), with

$$\begin{aligned} C_1 &= \cos\phi_{\mathrm{W}}\sin\gamma\cos\theta\sin\psi + \sin\phi_{\mathrm{W}}\cos\theta\cos\psi \\ C_2 &= \cos\phi_{\mathrm{W}}\cos\gamma\cos\theta\cos\psi + \cos\phi_{\mathrm{W}}\sin\gamma\sin\theta \\ C_3 &= \sin\phi_{\mathrm{W}}\sin\theta + \cos\phi_{\mathrm{W}}\cos\gamma\cos\theta\sin\psi \end{aligned} \tag{73}$$

Given γ, ϕ_{W}, ψ and θ the angle β is determined by the inverse cosine of the right-hand side of (72).

Moreover, considering the entry $(2,3)$ of both sides of equation (68) we obtain the identity

$$\sin\phi = \frac{\cos\beta\sin\phi_{\mathrm{W}}\cos\gamma - \sin\beta\sin\gamma}{\cos\theta} \tag{74}$$

giving ϕ in terms of the remaining quantities. At this point, having set ψ and θ, and having determined ϕ, we have *placed* the aircraft in the space. In particular we have placed the aircraft with respect to the velocity vector, the aerodynamic angles β and α being given, respectively, by (72) and by the following equation

$$\cos\alpha = \sin\left(\frac{\pi}{2} + \alpha\right) = \vec{\imath}_{z_{\mathrm{W}}} \times \vec{\imath}_{x_{\mathrm{B}}} \tag{75}$$

The trim control vector for the research of a general steady-state turn condition will be given by

$$\boldsymbol{\xi} = \left[\,\theta\,,\,\psi\,,\,\delta_T\,,\,\delta_e\,,\,\delta_a\,,\,\delta_r\,\right]^{\mathsf{T}} \tag{76}$$

In terms of attitude angles, at each step of the cost function numerical minimization algorithm the set of quantities (76) together with the given values of ψ_{W} and γ will enable the determination of ϕ, β and α.

The vector of trim controls in the case of a coordinated, level turn is easily derived from the above discussion by assuming $\gamma = \beta = 0$

$$\boldsymbol{\xi} = \left[\,\psi\,,\,\delta_T\,,\,\delta_e\,,\,\delta_a\,,\,\delta_r\,\right]^{\mathsf{T}} \tag{77}$$

As we have pointed out above, when sideslip angle is zero the sole Euler angle to be considered as free adjustable parameter is ψ while

$$\theta = \bar{\theta} = \tan^{-1}\left(\frac{\sin\psi\cos\gamma + \cos\psi\sin\gamma}{\cos\gamma}\right) \tag{78}$$

In steady-state turning flight the heading will be changing at a constant prescribed rate $\dot{\psi}_{\mathrm{W}}$. In the general case the difference $\psi - \psi_{\mathrm{W}}$ may not be zero but will be constant.

Given the turn heading rate $\dot{\psi}_{\mathrm{W}}$, the wind-axis frame angular rates are constrained from the following Euler angle rate equation

$$\left\{\begin{array}{c} p_{\mathrm{W}} \\ q_{\mathrm{W}} \\ r_{\mathrm{W}} \end{array}\right\} = \left[\begin{array}{ccc} 1 & 0 & -\sin\gamma \\ 0 & \cos\phi_{\mathrm{W}} & \cos\gamma\sin\phi_{\mathrm{W}} \\ 0 & -\sin\phi_{\mathrm{W}} & \cos\gamma\cos\phi_{\mathrm{W}} \end{array}\right] \cdot \left\{\begin{array}{c} \dot{\phi}_{\mathrm{W}} \\ \dot{\gamma} \\ \dot{\psi}_{\mathrm{W}} \end{array}\right\} \tag{79}$$

And, because for the steady-state turn,

$$\dot{\gamma} = \dot{\phi}_{\mathrm{W}} = 0 \tag{80}$$

we have the following equations for the general case of a climbing turn

$$\begin{aligned} p_{\mathrm{W}} &= -\dot{\psi}_{\mathrm{W}} \sin\gamma \\ q_{\mathrm{W}} &= \quad \dot{\psi}_{\mathrm{W}} \cos\gamma \sin\phi_{\mathrm{W}} \\ r_{\mathrm{W}} &= \quad \dot{\psi}_{\mathrm{W}} \cos\gamma \cos\phi_{\mathrm{W}} \end{aligned} \tag{81}$$

Using the transformation from the wind-axis reference system to the body axis system, $[C_{\mathrm{B}\leftarrow\mathrm{W}}]$ given by (14), we can get the body axis rates

$$\left\{\begin{array}{c} p \\ q \\ r \end{array}\right\} = \left[C_{\mathrm{B}\leftarrow\mathrm{W}}\right] \cdot \left\{\begin{array}{c} p_{\mathrm{W}} \\ q_{\mathrm{W}} \\ r_{\mathrm{W}} \end{array}\right\} = \left\{\begin{array}{c} p_{\mathrm{W}} \cos\alpha\cos\beta - q_{\mathrm{W}} \cos\alpha\sin\beta - r_{\mathrm{W}}\sin\alpha \\ p_{\mathrm{W}}\sin\beta + q_{\mathrm{W}}\cos\beta \\ p_{\mathrm{W}}\sin\alpha\cos\beta - q_{\mathrm{W}}\sin\alpha\sin\beta + r_{\mathrm{W}}\cos\alpha \end{array}\right\} \tag{82}$$

And using the results from equations (81), we can write the equations for the body axis rates in equation (82) as

$$\left\{\begin{array}{c} p \\ q \\ r \end{array}\right\} = -\dot{\psi}_{\mathrm{W}} \left\{\begin{array}{c} \sin\gamma\cos\alpha\cos\beta + \cos\gamma\sin\phi_{\mathrm{W}}\cos\alpha\sin\beta + \cos\gamma\cos\phi_{\mathrm{W}}\sin\alpha \\ \sin\gamma\sin\beta - \cos\gamma\sin\phi_{\mathrm{W}}\cos\beta \\ \sin\gamma\sin\alpha\cos\beta + \cos\gamma\sin\phi_{\mathrm{W}}\sin\alpha\sin\beta - \cos\gamma\cos\phi_{\mathrm{W}}\cos\alpha \end{array}\right\} \tag{83}$$

Then substituting the results from equations (110)

$$\left\{\begin{array}{c} p \\ q \\ r \end{array}\right\} = -\frac{g}{V}\tan\phi_{\mathrm{W}} \left\{\begin{array}{c} \sin\gamma\cos\alpha\cos\beta + \cos\gamma\sin\phi_{\mathrm{W}}\cos\alpha\sin\beta + \cos\gamma\cos\phi_{\mathrm{W}}\sin\alpha \\ \sin\gamma\sin\beta - \cos\gamma\sin\phi_{\mathrm{W}}\cos\beta \\ \sin\gamma\sin\alpha\cos\beta + \cos\gamma\sin\phi_{\mathrm{W}}\sin\alpha\sin\beta - \cos\gamma\cos\phi_{\mathrm{W}}\cos\alpha \end{array}\right\} \tag{84}$$

Multiplying through each equation by the term $\tan\phi_{\mathrm{W}}$ and simplifying yields

$$p = -\frac{g}{V}\left[\sin\gamma\tan\phi_{\mathrm{W}}\cos\alpha\cos\beta + \frac{\cos\gamma\sin^2\phi_{\mathrm{W}}\cos\alpha\sin\beta}{\cos\phi_{\mathrm{W}}} + \cos\gamma\sin\phi_{\mathrm{W}}\sin\alpha\right] \tag{85}$$

$$q = -\frac{g}{V}\left[\sin\gamma\tan\phi_{\mathrm{W}}\sin\beta - \frac{\cos\gamma\sin^2\phi_{\mathrm{W}}\cos\beta}{\cos\phi_{\mathrm{W}}}\right] \tag{86}$$

$$r = -\frac{g}{V}\left[\sin\gamma\tan\phi_{\mathrm{W}}\sin\alpha\cos\beta + \frac{\cos\gamma\sin^2\phi_{\mathrm{W}}\sin\alpha\sin\beta}{\cos\phi_{\mathrm{W}}} - \cos\gamma\sin\phi_{\mathrm{W}}\cos\alpha\right] \tag{87}$$

These results for the body-axis rates are different from those obtained by Chen and Jeske[10] and used by Duke *et al.*[9] Chen and Jeske assume that the steady-state turn is a "coordinated turn" and that the angle of sideslip, β, is zero. In our approach, we make no assumption on the angle of sideslip.

Our final task is to determine the Euler angles for the steady-state turn. To accomplish this, we need to first examine the equations defining the body-axis rates Euler angle rates given by the following equations from Stevens and Lewis:[4]

$$\dot{\phi} = p + q\sin\phi\tan\theta + r\cos\phi\tan\theta \tag{88}$$

$$\dot{\theta} = q\cos\phi - r\sin\phi \tag{89}$$

$$\dot{\psi} = q\sin\theta + r\cos\phi\sec\theta \tag{90}$$

For a steady-state turn

$$\dot{\phi} = \dot{\theta} = 0 \tag{91}$$

From equation (89), we can get

$$\phi = \pm\tan^{-1}\left(\frac{q}{r}\right) \tag{92}$$

and from equation (88), we can get

$$\theta = \tan^{-1}\left(\frac{-p}{q\sin\phi + r\cos\phi}\right) \tag{93}$$

In equations (92) as in equation (100), the plus sign indicates a right-hand turn and the minus sign a left-hand turn.

We can use the force equations to solve for two key terms that define the maneuver: the wind-axis bank attitude, ϕ_{W}, and the time derivative of wind-axis heading (sometimes refered to as wind-axis heading rate), $\dot{\psi}_{\mathrm{W}}$. From these parameters, we can determine the constraint equations for the maneuver.

Noting from Figure 4 the vertical forces and assuming that the entire wind-reference systems z-axis force results from lift, for a steady-state condition we must have

$$L\,\cos\phi_{\mathrm{W}} = m\,g \tag{94}$$

If we use the definition of load factor from equation (35), we can write equation (94) as

$$n\,m\,g\,\cos\phi_{\mathrm{W}} = m\,g \tag{95}$$

or

$$\cos\phi_{\mathrm{W}} = \frac{1}{n} \tag{96}$$

with

$$\sin\phi_{\mathrm{W}} = \frac{\left(n^2-1\right)^{\frac{1}{2}}}{n} \tag{97}$$

$$\tan\phi_{\mathrm{W}} = \left(n^2-1\right)^{\frac{1}{2}} \tag{98}$$

If we are only interested in level turns, equation (98) gives us the results we need.

However, we can extend these results to the more general case of nonzero γ by recognizing that in Figure 4, instead of equation (94), we would have the equation

$$L\cos\phi_{\mathrm{W}}\cos\theta_{\mathrm{W}} = mg \tag{99}$$

and—by replacing θ_{W} with γ—that would result in the following equation

$$\tan\phi_{\mathrm{W}} = \frac{\left(n^2-\cos^2\gamma\right)^{\frac{1}{2}}}{\cos\gamma} \tag{100}$$

From this equation, we get the equation for the wind-axis bank attitude

$$\boxed{\pm\phi_{\mathrm{W}} = \tan^{-1}\left[\frac{\left(n^2-\cos^2\gamma\right)^{\frac{1}{2}}}{\cos\gamma}\right]} \tag{101}$$

where the plus sign indicates a right-hand turn and the minus sign a left-hand turn.

Noting from Figure 4 the horizontal forces, we can write the centripetal force equation

$$F_c = m\frac{V^2}{R} \tag{102}$$

for a vehicle moving along a circular path of radius R with angular velocity, ω,

$$\omega = \frac{V}{R} \tag{103}$$

Force F_C is also given by

$$F_c = mV\omega \tag{104}$$

Then using the horizontal component of the lift vector, we can write

$$L\,\sin\phi_{\mathrm{W}} = m\,V\,\omega \tag{105}$$

Recognizing that

$$\omega = \dot{\psi}_{\mathrm{W}} \tag{106}$$

we can write the centripetal force equation as

$$L \sin\phi_{\mathrm{W}} = m\, V\, \dot{\psi}_{\mathrm{W}} \tag{107}$$

Then, using the equation for load factor [equation (35)] we have

$$n\, m\, g\, \sin\phi_{\mathrm{W}} = m\, V\, \dot{\psi}_{\mathrm{W}} \tag{108}$$

or

$$\dot{\psi}_{\mathrm{W}} = \frac{g}{V}\left(n\, \sin\phi_{\mathrm{W}} \right) \tag{109}$$

which can be written [using equations (97) and (98)] as

$$\dot{\psi}_{\mathrm{W}} = \frac{g}{V}\, \tan\phi_{\mathrm{W}} \tag{110}$$

Table 5. Initial conditions and constraint equations for steady-state turn

Type Parameters	Definitions
Initial Conditions	$V = V_0$, $h = h_0$, $\gamma = \gamma_0$ $\psi_{\mathrm{W}} = 0$ $n = n_0$
Derived Parameters see (101), (85), (86), (87), (92), (93)	$\phi_{\mathrm{W}} = \pm\tan^{-1}\left[\dfrac{\left(n^2 - \cos^2\gamma\right)^{\frac{1}{2}}}{\cos\gamma}\right]$ $p = -\dfrac{g}{V}\left[\sin\gamma\tan\phi_{\mathrm{W}}\cos\alpha\cos\beta + \dfrac{\cos\gamma\sin^2\phi_{\mathrm{W}}\cos\alpha\sin\beta}{\cos\phi_{\mathrm{W}}} + \cos\gamma\sin\phi_{\mathrm{W}}\sin\alpha\right]$ $q = -\dfrac{g}{V}\left[\sin\gamma\tan\phi_{\mathrm{W}}\sin\beta - \dfrac{\cos\gamma\sin^2\phi_{\mathrm{W}}\cos\beta}{\cos\phi_{\mathrm{W}}}\right]$ $r = -\dfrac{g}{V}\left[\sin\gamma\sin\tan\phi_{\mathrm{W}}\alpha\cos\beta + \dfrac{\cos\gamma\sin^2\phi_{\mathrm{W}}\sin\alpha\sin\beta}{\cos\phi_{\mathrm{W}}} - \cos\gamma\sin\phi_{\mathrm{W}}\cos\alpha\right]$
Trim Control Parameters	$\boldsymbol{\xi} = \left[\,\theta\,,\,\psi\,,\,\delta_T\,,\,\delta_e\,,\,\delta_a\,,\,\delta_r\,\right]^{\mathsf{T}}$
Computed Parameters	dynamic state variables: β, α kinematic variables: ϕ

Table 5 provides a summary of the initial conditions and the derived parameters in a steady-state turn. This table also list the parameters computed by the optimization function.

V. Cost function minimization methods

A. Gradient-based minimization methods

The problem of multidimensional minimization requires finding a point $\boldsymbol{\xi}$ such that the scalar function $\mathcal{J}(\xi_1, \ldots, \xi_n)$ takes a value which is lower than at any neighboring point. Assuming that $\mathcal{J}$ is a smooth function, the gradient $\nabla\mathcal{J}$ should vanish at the minimum. In general there are no bracketing methods available for the minimization of n-dimensional functions. The algorithms proceed from an initial guess using a search algorithm which attempts to move in a downhill direction. Algorithms making use of the gradient of the function perform a one-dimensional line minimization along this direction until the lowest point is found to a suitable tolerance. The search direction is then updated with local information from the function and its derivatives, and the whole process repeated until the true n-dimensional minimum is found. When such methods are selected there is the need to know the vector-valued function

$$\nabla\mathcal{J} \equiv \left\{\frac{\partial\mathcal{J}}{\partial\boldsymbol{\xi}}\right\} = \left[\frac{\partial\mathcal{J}}{\partial\phi}, \frac{\partial\mathcal{J}}{\partial\theta}, \ldots, \frac{\partial\mathcal{J}}{\partial\delta_T}, \frac{\partial\mathcal{J}}{\partial\delta_e}, \ldots\right]^{\mathsf{T}} \tag{111}$$

For the sake of clarification, let us consider the derivation of (53) with respect to one of the variables upon which the cost function depends, for example p. The application of the chain rule gives

$$\frac{\partial\mathcal{J}}{\partial\phi} = 2\,\dot{u}\,\frac{\partial f_{\dot{u}}}{\partial\phi} + 2\,\dot{v}\,\frac{\partial f_{\dot{v}}}{\partial\phi} + 2\,\dot{w}\,\frac{\partial f_{\dot{w}}}{\partial\phi} + 2\,\dot{p}\,\frac{\partial f_{\dot{p}}}{\partial\phi} + 2\,\dot{q}\,\frac{\partial f_{\dot{q}}}{\partial\phi} + 2\,\dot{r}\,\frac{\partial f_{\dot{r}}}{\partial\phi} \tag{112}$$

where the functions that explicitly give the rates $\dot{V}$, $\dot{\alpha}$, $\dot{\beta}$, $\dot{p}$, $\dot{q}$, $\dot{r}$ are denoted as $f_{\dot{V}}$, $f_{\dot{\alpha}}$, $\ldots$, $f_{\dot{r}}$ and derived by rearranging the equations of motion (1). Hence the equation (112) is the particularization of the general rule

$$\boxed{\frac{\partial\mathcal{J}}{\partial\xi_j} = 2\,\dot{u}\,\frac{\partial f_{\dot{u}}}{\partial\xi_j} + 2\,\dot{v}\,\frac{\partial f_{\dot{v}}}{\partial\xi_j} + 2\,\dot{w}\,\frac{\partial f_{\dot{w}}}{\partial\xi_j} + 2\,\dot{p}\,\frac{\partial f_{\dot{p}}}{\partial\xi_j} + 2\,\dot{q}\,\frac{\partial f_{\dot{q}}}{\partial\xi_j} + 2\,\dot{r}\,\frac{\partial f_{\dot{r}}}{\partial\xi_j}} \tag{113}$$

that gives the generic component of vector $\nabla\mathcal{J} = \{\partial\mathcal{J}/\partial\xi_j\}$. When V, α and β are considered in place of state variables u, v and w the above rule is equivalent to

$$\frac{\partial\mathcal{J}}{\partial\xi_j} = 2\,\dot{u}\,\frac{\partial f_{\dot{V}}}{\partial\xi_j} + 2\,\dot{v}\,\frac{\partial f_{\dot{\alpha}}}{\partial\xi_j} + 2\,\dot{w}\,\frac{\partial f_{\dot{\beta}}}{\partial\xi_j} + 2\,\dot{p}\,\frac{\partial f_{\dot{p}}}{\partial\xi_j} + 2\,\dot{q}\,\frac{\partial f_{\dot{q}}}{\partial\xi_j} + 2\,\dot{r}\,\frac{\partial f_{\dot{r}}}{\partial\xi_j} \tag{114}$$

The partial derivatives appearing in (111), in particular in equation (114), are determined following the noteworthy NASA paper by Duke *et alii*.[8] The necessary formulas are reported next.

$$\frac{\partial f_{\dot{V}}}{\partial\phi} = g\Big(\cos\theta_0\,\cos\phi_0\,\sin\beta_0 - \cos\theta_0\,\sin\phi_0\,\sin\alpha_0\,\cos\beta_0\Big) \tag{115}$$

$$\frac{\partial f_{\dot{V}}}{\partial\theta} = g\Big(-\cos\theta_0\,\cos\alpha_0\,\cos\beta_0 - \sin\theta_0\,\sin\phi_0\,\sin\beta_0 - \sin\theta_0\,\cos\phi_0\,\sin\alpha_0\,\cos\beta_0\Big) \tag{116}$$

$$\frac{\partial f_{\dot{V}}}{\partial\psi} = 0 \tag{117}$$

$$\begin{aligned}\frac{\partial f_{\dot{V}}}{\partial\delta_j} = {} & \frac{\bar{q}S}{m}\Big(-\cos\beta_0\,C_{D_{\delta_j}} + \sin\beta_0\,C_{Y_{\delta_j}}\Big)\\ & + \frac{1}{m}\Big(+\cos\alpha_0\,\cos\beta_0\,\frac{\partial X_{\mathrm{T}}}{\partial\delta_j} + \sin\beta_0\,\frac{\partial Y_{\mathrm{T}}}{\partial\delta_j} + \sin\alpha_0\,\cos\beta_0\,\frac{\partial Z_{\mathrm{T}}}{\partial\delta_j}\Big)\end{aligned} \tag{118}$$

$$\frac{\partial f_{\dot{\alpha}}}{\partial\phi} = \frac{g}{V_0\,\cos\beta_0}\,\cos\theta_0\,\sin\phi_0\,\cos\alpha_0 \tag{119}$$

$$\frac{\partial f_{\dot{\alpha}}}{\partial\theta} = \frac{g}{V_0\,\cos\beta_0}\big(\sin\theta_0\,\cos\phi_0\,\cos\alpha_0 - \cos\theta_0\,\sin\alpha_0\big) \tag{120}$$

$$\frac{\partial f_{\dot{\alpha}}}{\partial\psi} = 0 \tag{121}$$

$$\frac{\partial f_{\dot{\alpha}}}{\partial\delta_j} = \frac{1}{m\,V_0\,\cos\beta_0}\left(-\bar{q}\,S\,C_{L_{\delta_j}} + \cos\alpha_0\,\frac{\partial Z_{\mathrm{T}}}{\partial\delta_j} - \sin\alpha_0\,\frac{\partial X_{\mathrm{T}}}{\partial\delta_j}\right) \tag{122}$$

$$\frac{\partial f_{\dot{\beta}}}{\partial\phi} = \frac{g}{V_0}\Big(\cos\theta_0\,\cos\phi_0\,\cos\beta_0 + \cos\theta_0\,\sin\phi_0\,\sin\alpha_0\,\sin\beta_0\Big) \tag{123}$$

$$\frac{\partial f_{\dot\beta}}{\partial \theta} = \frac{g}{V_0}\Big(\cos\theta_0\, \cos\alpha_0\, \sin\beta_0 - \sin\theta_0\, \sin\phi_0\, \cos\beta_0 + \sin\theta_0\, \cos\phi_0\, \sin\alpha_0\, \sin\beta_0 \Big) \tag{124}$$

$$\frac{\partial f_{\dot\beta}}{\partial \psi} = 0 \tag{125}$$

$$\begin{aligned} \frac{\partial f_{\dot\beta}}{\partial \delta_j} = \frac{1}{m\,V_0}\Bigg[&\bar{q}\,S\Big(\sin\beta_0\, C_{D_{\delta_j}} + \cos\beta_0\, C_{Y_{\delta_j}}\Big) \\ &- \cos\alpha_0\, \sin\beta_0\, \frac{\partial X_{\mathrm{T}}}{\partial \delta_j} + \sin\beta_0\, \frac{\partial Y_{\mathrm{T}}}{\partial \delta_j} - \sin\alpha_0\, \sin\beta_0\, \frac{\partial Z_{\mathrm{T}}}{\partial \delta_j}\Bigg] \end{aligned} \tag{126}$$

$$\frac{\partial f_{\dot p}}{\partial \phi} = \frac{\partial f_{\dot p}}{\partial \theta} = \frac{\partial f_{\dot p}}{\partial \psi} = 0 \tag{127}$$

$$\frac{\partial f_{\dot p}}{\partial \delta_j} = \frac{\bar{q}\,S}{\det[I]_{\mathrm{B}}}\Big(I_1\, b\, C_{\ell_{\delta_j}} + I_2\, \bar{c}\, C_{m_{\delta_j}} + I_3\, b\, C_{n_{\delta_j}}\Big) \tag{128}$$

$$\frac{\partial f_{\dot q}}{\partial \phi} = \frac{\partial f_{\dot q}}{\partial \theta} = \frac{\partial f_{\dot q}}{\partial \psi} = 0 \tag{129}$$

$$\frac{\partial f_{\dot q}}{\partial \delta_j} = \frac{\bar{q}\,S}{\det[I]_{\mathrm{B}}}\Big(I_2\, b\, C_{\ell_{\delta_j}} + I_4\, \bar{c}\, C_{m_{\delta_j}} + I_5\, b\, C_{n_{\delta_j}}\Big) \tag{130}$$

$$\frac{\partial f_{\dot r}}{\partial \phi} = \frac{\partial f_{\dot r}}{\partial \theta} = \frac{\partial f_{\dot r}}{\partial \psi} = 0 \tag{131}$$

$$\frac{\partial f_{\dot r}}{\partial \delta_j} = \frac{\bar{q}\,S}{\det[I]_{\mathrm{B}}}\Big(I_3\, b\, C_{\ell_{\delta_j}} + I_5\, \bar{c}\, C_{m_{\delta_j}} + I_6\, b\, C_{n_{\delta_j}}\Big) \tag{132}$$

where δ_j is the generic control input.

The implementation of the above functions requires the determination of the derivatives of some aerodynamic coefficients and of propulsive force components. Depending on the aircraft model these derivatives might not be directly available and have to be reconstructed numerically. This circumstance makes more attractive a minimization a technique that does not use derivatives. This approach is outlined in next subsection.

B. Gradient-free minimization methods: *Direct Search*

Among the methods that try to minimize a multi-variate scalar function $f(\boldsymbol{\xi})$ there are those named *direct search methods*, also known as *optimization techniques that do not explicitly use derivatives*. Direct search methods were formally proposed and widely applied in the 1960s but fell out of favor with the mathematical optimization community by the early 1970s because they lacked coherent mathematical analysis. Nonetheless, users remained loyal to these methods, most of which were easy to program, some of which were reliable.

Being straightforward to implement and not requiring derivatives are not necessarily two compelling features today. Sophisticated implementations of derivative-based methods, with line search or trust region globalization strategies and options to generate approximations to the gradient and/or the Hessian, are widely available and relatively easy to use. Furthermore, today there are automatic differentiation tools as well as modeling languages that compute derivatives automatically. Thus, a user only needs to provide a procedure that calculates the function values. Today, most people's first recommendation to solve an unconstrained problem for which accurate first derivatives can be obtained would not be a direct search method, but rather a gradient-based method. If second derivatives were also available, the top choice would be a Newton-based method.

But this does not mean that direct search methods are no longer needed. They still have their niche. In particular, the maturation of simulation-based optimization has led to optimization problems with features that make it difficult to apply methods that require derivative information. There are also optimization problems where methods based on derivatives cannot be used because the objective function being optimized is not numerical in nature.

The term *simulation-based optimization* is currently applied to the methodology in which complex physical systems are designed, analyzed, and controlled by optimizing the results of computer simulations. In the simulation-based optimization setting, a computer simulation must be run, repeatedly, in order to compute the various quantities needed by the optimization algorithm. Furthermore, the resulting simulation output must then be postprocessed to arrive finally at values of the objective and constraint functions. These complications can make obtaining derivatives for gradient-based methods at the very least difficult, even when the underlying objective and constraint functions are smooth (i.e., continuously differentiable).

A comprehensive review of direct search methods is given by Kolda, Lewis and Torczon.[17] The reader is referred to that article and to the works by Nelder and Mead,[16] Gurson,[18] and Dolan[19] for more details on some classical and modern methods and on the available algorithmic options. The results presented here were obtained by using the *DirectSearch* C++ library developed by Torczon *et alii*.[20,21]

The aircraft trim problem falls right into the context of simulation-based optimization, where the term *optimization* equals to determine the combination of flight control settings and other state variables that make the steady-state flight possible. This paper proposes a practical solution of the trim problem by means of the JSBSim flight dynamics model library.[14,15] In JSBSim, like in any other flight simulation software, when it comes to solving the problem of trim one has to choose necessarily the cost function minimization technique. The choice could fall obviously on a gradient-based technique or on a gradient-free technique. The first option entails the difficulty of implementing all the functions (115)-(132), taking into account the overall structure of the chosen simulation code and its design philosophy. On the contrary, the direct search class of methods implemented in the above mentioned *DirectSearch* library appeared to be a suitable choice for the solution of the aircraft trim problem based on JSBSim. *DirectSearch* implements a fairly satisfying treatment of trim control variable bounds based on a *penalty approach*.[17] This feature allows obtaining trim results that are physically correct (cost function minima corrensponding to feasible aerosurface deflections and thrust settings), and that reflect the limitations of the available aerodynamic model (angles of attack and sideslip within the range of available data).

We want to emphasize here that the advanced flight simulation libraries tend to have a complex structure. In JSBSim this is due to the advanced capability of managing a completely data-driven aircraft model and to its general, extensible physics/math model. Therefore, the choice of a direct search method as opposed to a gradient-based method was not only more suitable but, with the aim of spending the minimum effort in additional coding, it was *really* our only choice.

VI. Trim Algorithm Implementation in JSBSim

A trimming capability based on the approach presented in this paper has been coded into JSBSim. A pre-existing trimming capability in JSBSim was developed by Tony Peden in the class `FGTrim`. Currently this class and the related one, `FGTrimAxis`, are still usable and left unmodified in their own place in the JSBSim source tree. The aim of the new trim capability is to provide the flight dynamics model library with a default, accurate algorithm for the determination of aircraft equilibrium states.

A. Essential implementation details

The new trim algorithm has been implemented in a trim class named `FGTrimAnalysis`, whose user interface has been adapted from the pre-existing one to allow a smooth transition to the new functionality. This class uses a related class named `FGTrimAnalysisControl`. The new trim classes provide the user with a number of pre-programmed *types of trim conditions*, each one associated with some given *constraint equations* and a given set of *trim controls* ($\boldsymbol{\xi}$). For each trim condition, a dedicated algorithm will vary the appropriate parameters to find the combination of values that minimizes the related cost function. The minimum represents the desired trimmed state.

All trim conditions are identified with a C++ enumerated type in JSBSim's namespace, (in the first code listings reported below, see the use of `JSBSim::FGTrimAnalysisMode`), and are exposed to the user as the following set of integers:

`ID` .... *Description* ....

- `0`, for the longitudinal trim (only longitudinal quantities involved, i.e. δ_T, δ_e, θ), flight along a straight path;
- `1`, for full trim (all main controls and Euler angles involved), flight along a straight path;
- `2`, for full trim, wings-level ($\phi = 0$) flight along a straight path;
- `3`, for coordinated turn trim (all main controls and Euler angles involved and $\beta = 0$ enforced), flight along a helical path;
- `4`, for turn trim (all main controls and Euler angles involved, $\beta = 0$ not enforced), flight along a helical path;
- `5`, for wings-level pull-up/push-over trim (all main controls and Euler angles involved, $\psi = \phi = 0$ enforced), flight along a circular path in a vertical plane.

Low level coding of new trim conditions is possible and relatively easy through the structure and interfaces of the two classes `FGTrimAnalysis` and `FGTrimAnalysisControl`.

In Listing 1 we report an example of high level coding that shows how the user of JSBSim can set up a trim problem for a given aircraft model (in the particular case we have chosen the default `c172p` model).

Listing 1. Simple code in C++. An example of how JSBSim library is used to set up a trim problem.

```
#include <FGFDMExec.h>
#include <FGTrimAnalysis.h>
#include <iostream>
using namespace std;

int main(int argc, char* argv[])
{
    // Set up JSBSim
    string aircraftName = "c172p";          // hardcoded A/C name
    string initFileName = "c172p_init_001"; // hardcoded i.c. file name
    string aircraftPath = "aircraft";
    string enginePath   = "engine";
    // The sim executive
    JSBSim::FGFDMExec* fdmExec = new JSBSim::FGFDMExec();
    fdmExec->SetAircraftPath( aircraftPath );
    fdmExec->SetEnginePath( enginePath );

    // Loading A/C data
    if ( ! fdmExec->LoadModel( aircraftPath, enginePath, aircraftName ) ) {
      cerr << "  JSBSim could not be started" << endl << endl;
      exit(-1);
    }
    // Loading initial conditions
    JSBSim::FGInitialCondition *ic = fdmExec->GetIC();
    if ( ! ic->Load( initFileName ) ) {
      cerr << "Initialization unsuccessful" << endl;
      exit(-1);
    }

    // The trim object, full trim mode
    JSBSim::FGTrimAnalysis fgta( fdmExec, (JSBSim::TrimAnalysisMode)1 );
    // 0: Longitudinal, 1: Full, 2: Full,Wings-Level,
    // 3: Coordinated Turn, 4: Turn, 5: Pull-up/Push-over

    // The ic cfg file contains trim directives as well
    fgta.Load( initFileName );
    // Optimize cost function
    if ( ! fgta.DoTrim() )
       cout << "Trim Failed" << endl;
    fgta.Report();

    return 0;
}
```

A dedicated initialization class, `FGInitialCondition`, is used to retrieve the initial state of the simulation, see use of pointer `ic` in Listing 1. Some initial quantities are specified by the user through an initialization file in XML format. In all cases the user should supply the flight speed, the flight path angle, the ground track angle, and the altitude. Regarding the trim, for a desired trimmed state, the user is required to incorporate in the initialization file a set of configuration parameters and initial values delimited by the *tags*: <trim_config> ...</trim_config>. This section of the initialization file has been specifically added to configure the trim algorithm. In Listing 2 we report an example of a possible initialization file `c172p_init_001.xml` (see `initFileName` in Listing 1).

Listing 2. A typical JSBSim initialization file in XML format.

```
<?xml version="1.0"?>
<initialize name="init_for_full_trim">
    <latitude  unit="DEG">    0.0  </latitude>
    <longitude unit="DEG">    0.0  </longitude>
    <altitude  unit="FT">  5500.0  </altitude>
    <psi       unit="DEG">    0.0  </psi>
    <vc        unit="KTS">   88.0  </vc>
    <gamma     unit="DEG">    0.0  </gamma>
    <theta     unit="DEG">    5.0  </theta>
    <alpha     unit="DEG">    5.0  </alpha>
    <phi       unit="DEG">    0.0  </phi>
    <running value="1"/>
    <!-- Now set up the trim parameters -->
    <trim_config name="trim01" type="FULL">
        <search type="Nelder-Mead"/>
        <initial_values>
            <phi   action="From-IC"/>
            <theta action="From-IC"/>
            <psi   action="From-IC"/>
            <throttle_cmd>  0.9  </throttle_cmd>
            <elevator_cmd> -0.1  </elevator_cmd>
            <rudder_cmd>    0.0  </rudder_cmd>
            <aileron_cmd>  -0.2  </aileron_cmd>
        </initial_values>
        <steps>
            <phi   unit="DEG"> 0.3 </phi>
            <theta unit="DEG"> 0.3 </theta>
            <psi   unit="DEG"> 0.3 </psi>
            <throttle_cmd>     0.2 </throttle_cmd>
            <elevator_cmd>     0.2 </elevator_cmd>
            <rudder_cmd>       0.2 </rudder_cmd>
            <aileron_cmd>      0.2 </aileron_cmd>
        </steps>
        <output_file name="trim-log.txt"/>
    </trim_config>
</initialize>
```

The cost function minimization algorithm used by the new trim class is implemented by the `DirectSearch` class and its derived concrete classes by Torczon *et alii*.[20] The details of this class and its mathematical foundations are found in the documentation webpage.[21]

Some detailed examples of use of the `DirectSearch` class and an annotated presentation of the member function `FGTrimAnalysis::DoTrim` (see Listing 1) are found in a white paper by De Marco.[a]

B. Examples of trim with JSBSim

We report in Figures 6-23 some examples of trim analysis for the default Cessna 172 flight dynamics model distributed with JSBSim (`c172p`).[b] Each example includes the convergence histories of commands, of Euler angles (as appropriate) and of cost function. The last case, in particular, is an example of nonconverged trim algorithm for a steady turn.

Figure 10 and Figure 23 are examples of simulated trajectories. They are obtained by assigning the results of the trim algorithm to the initial states.

A summary of trim results for the chosen flight dynamics model is finally reported in Table 6.

[a] De Marco, A.: "The Aircraft Trim Problem in Flight Simultion. Some Ideas for a New Trim Algorithtm in JSBSim." A white paper available at: `http://www.dpa.unina.it/demarco/work/trim_doc.pdf`.

[b] The Cessna `c172p` model was created by volunteers and does not originate from Cessna.

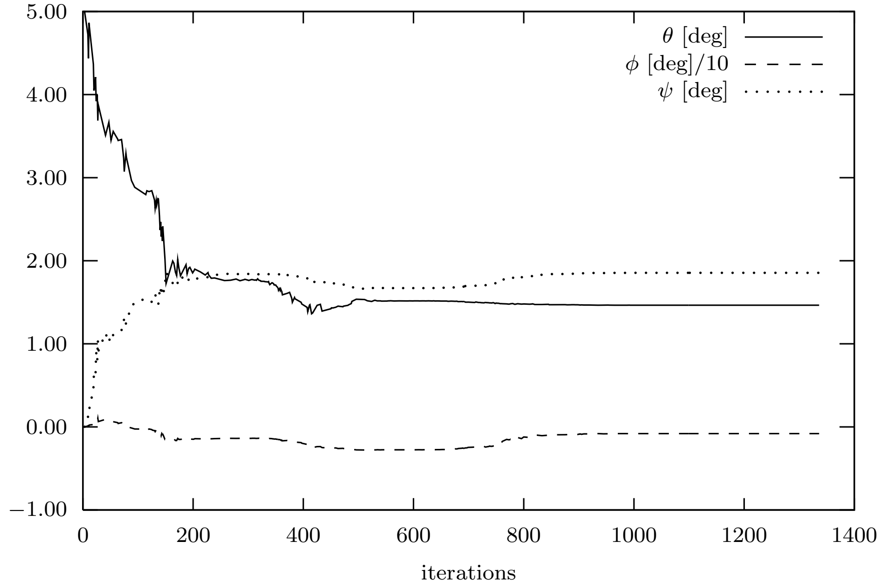

Figure 6. Trim algorithm results for the default Cessna 172 flight dynamics model distributed with JSBSim. Horizontal flight required; $V_{\mathbf{TAS}} = 88$ knots, $h_{\mathbf{ASL}} = 5500$ ft. Convergence history of Euler angles.

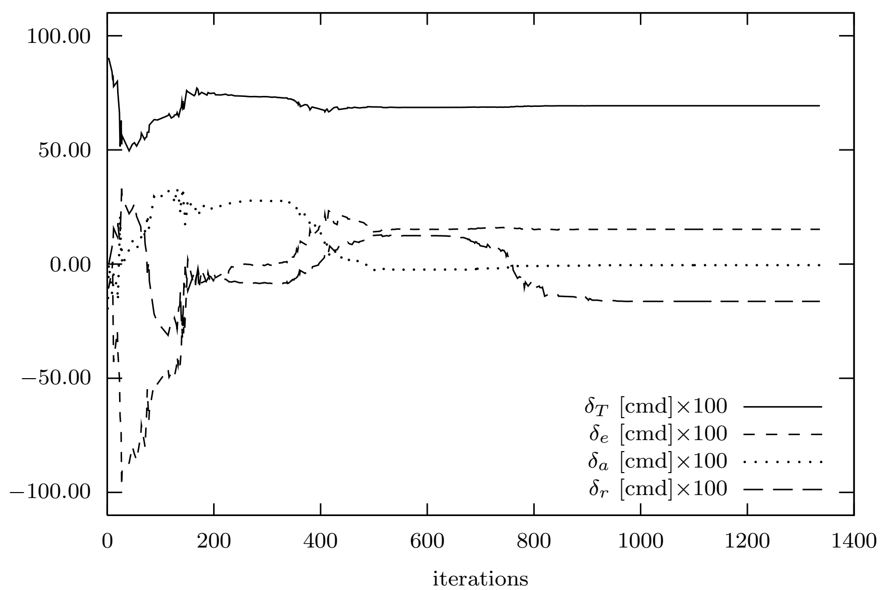

Figure 7. Trim algorithm results for the default Cessna 172 flight dynamics model distributed with JSBSim. Horizontal flight required; $V_{\mathbf{TAS}} = 88$ knots, $h_{\mathbf{ASL}} = 5500$ ft. Convergence history of normalized control positions.

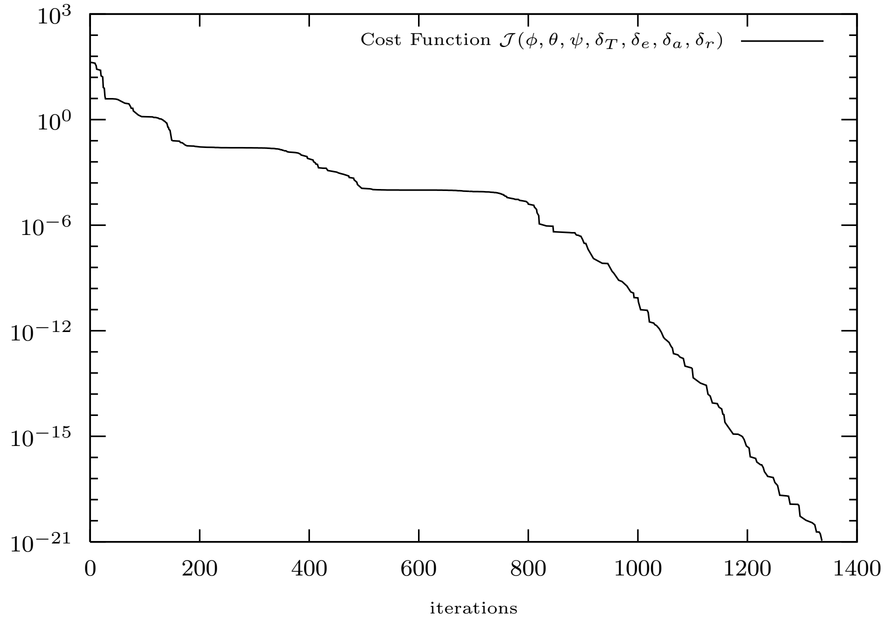

Figure 8. Trim algorithm results for the default Cessna 172 flight dynamics model distributed with JSBSim. Horizontal flight required; $V_{\mathbf{TAS}} = 88$ knots, $h_{\mathbf{ASL}} = 5500$ ft. Convergence history of cost function $\mathcal{J}$.

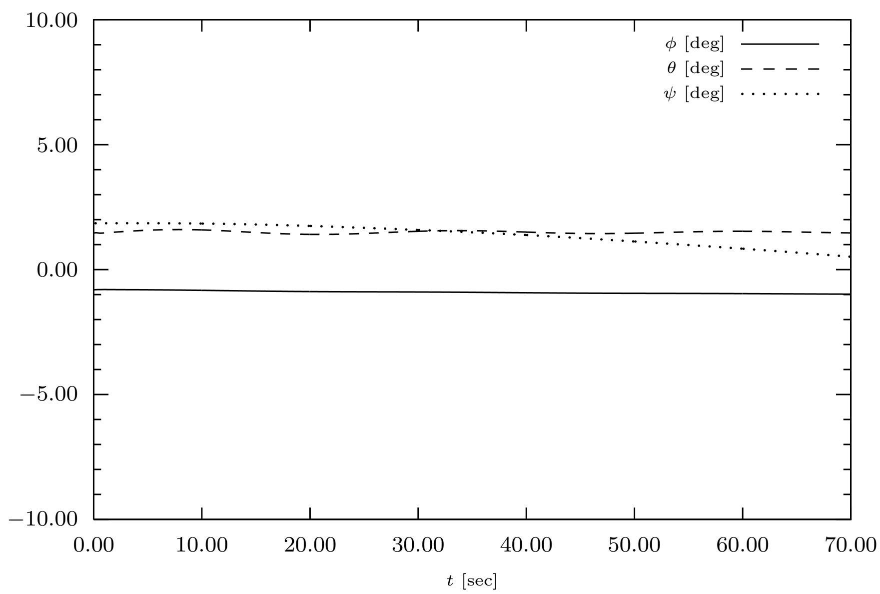

Figure 9. Time histories of Euler angles. Initial conditions are taken from the results of the trim algorithm of Figures 6-8.

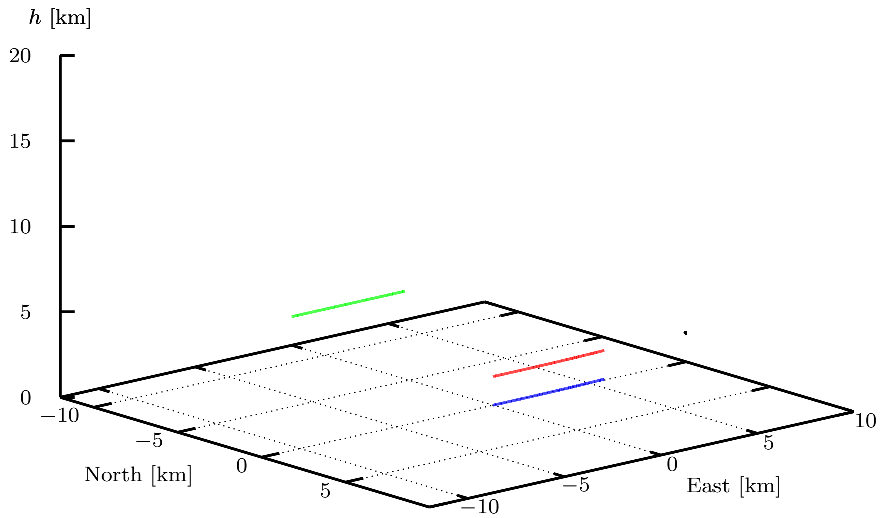

Figure 10. Aircraft trajectory. Initial conditions are taken from the results of the trim algorithm of Figures 6-8. Initial position projected on-ground: (0,0).

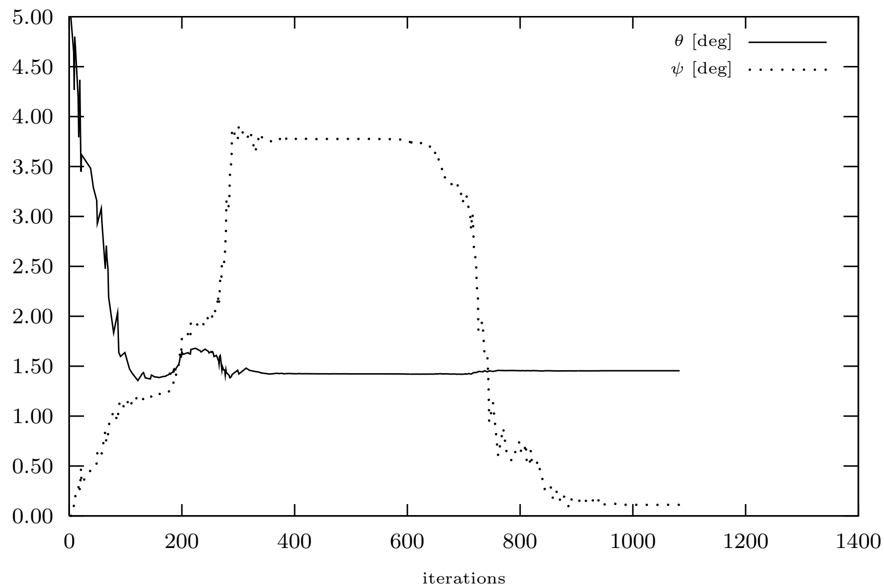

Figure 11. Trim algorithm results for the default Cessna 172 flight dynamics model distributed with JSBSim. Horizontal, wings-level flight required; $V_{\mathbf{TAS}} = 88$ knots, $h_{\mathbf{ASL}} = 5500$ ft. Convergence history of Euler angles.

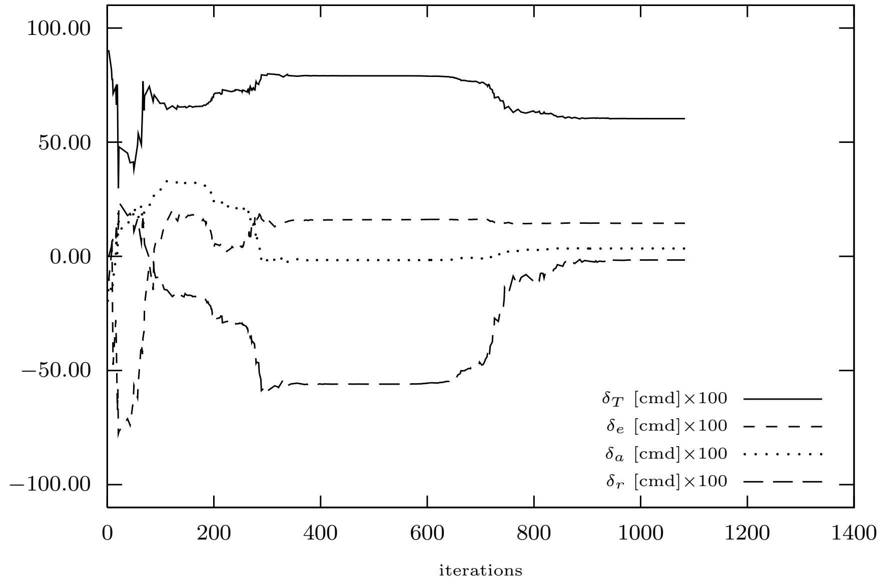

Figure 12. Trim algorithm results for the default Cessna 172 flight dynamics model distributed with JSBSim. Horizontal, wings-level flight required; $V_{\mathbf{TAS}} = 88$ knots, $h_{\mathbf{ASL}} = 5500$ ft. Convergence history of normalized control positions.

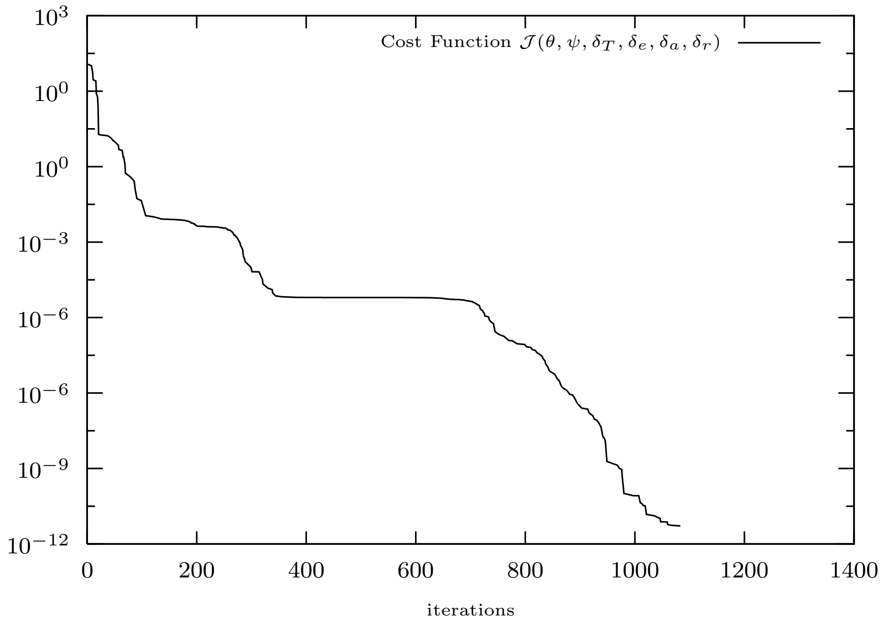

Figure 13. Trim algorithm results for the default Cessna 172 flight dynamics model distributed with JSBSim. Horizontal, wings-level flight required; $V_{\mathbf{TAS}} = 88$ knots, $h_{\mathbf{ASL}} = 5500$ ft. Convergence history of cost function $\mathcal{J}$.

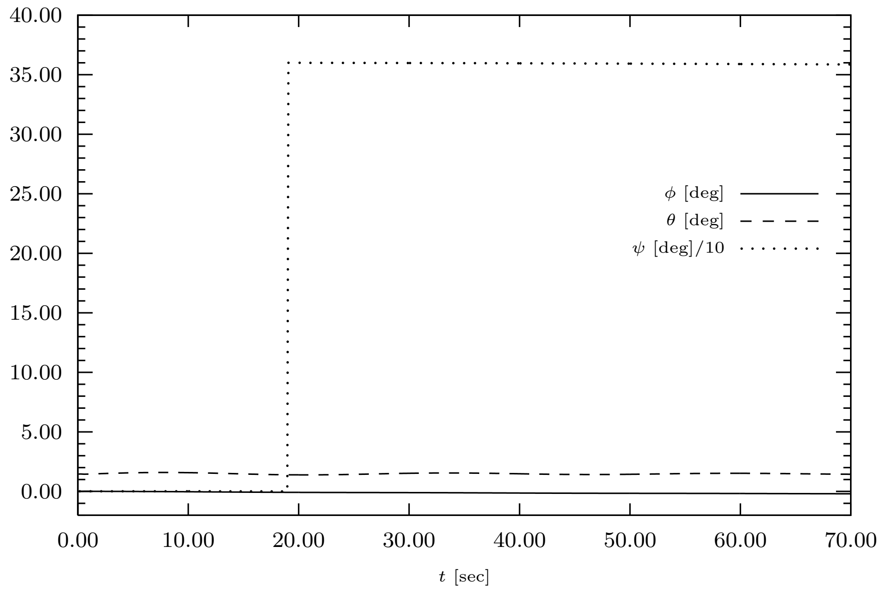

Figure 14. Time histories of Euler angles. Initial conditions are taken from the results of the trim algorithm of Figures 11-13.

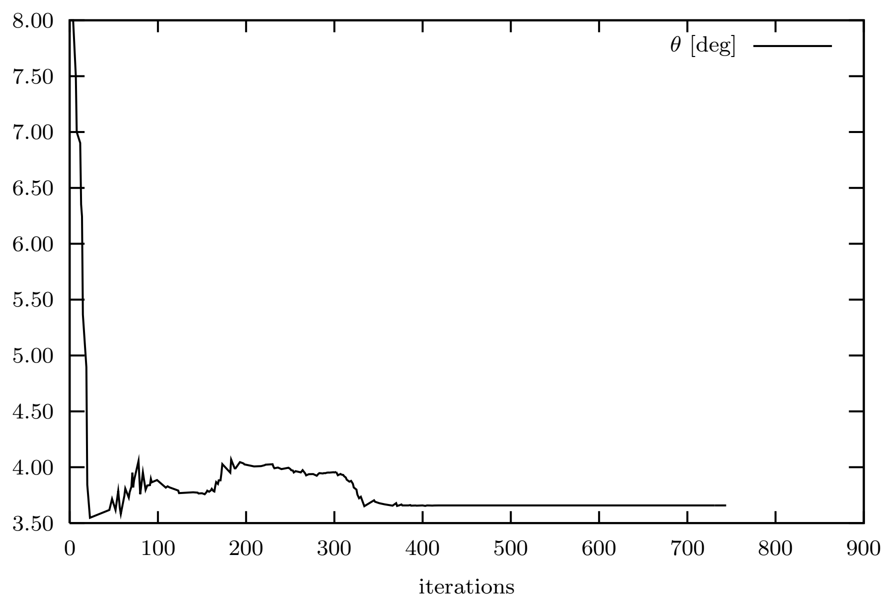

Figure 15. Trim algorithm results for the default Cessna 172 flight dynamics model distributed with JSBSim. Pull-up, wings-level flight required; $n = 1.5$, $V_{\mathbf{TAS}} = 90$ **knots,** $h_{\mathbf{ASL}} = 5500$ **ft. Convergence history of Euler angles.**

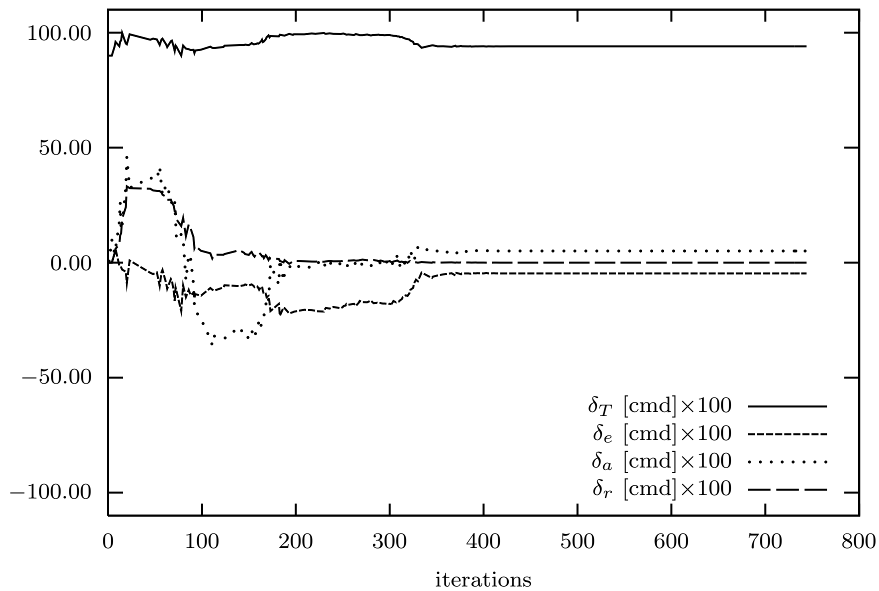

Figure 16. Trim algorithm results for the default Cessna 172 flight dynamics model distributed with JSBSim. Pull-up, wings-level flight required; $n = 1.5$, $V_{\mathbf{TAS}} = 90$ knots, $h_{\mathbf{ASL}} = 5500$ ft. Convergence history of normalized control positions.

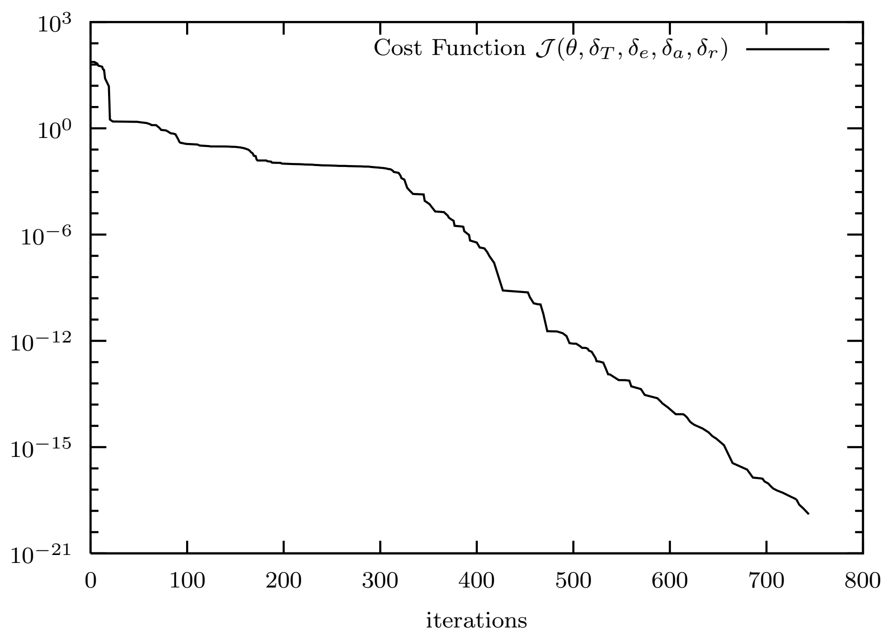

Figure 17. Trim algorithm results for the default Cessna 172 flight dynamics model distributed with JSBSim. Pull-up, wings-level flight required; $n = 1.5$, $V_{\mathbf{TAS}} = 90$ knots, $h_{\mathbf{ASL}} = 5500$ ft. Convergence history of cost function $\mathcal{J}$.

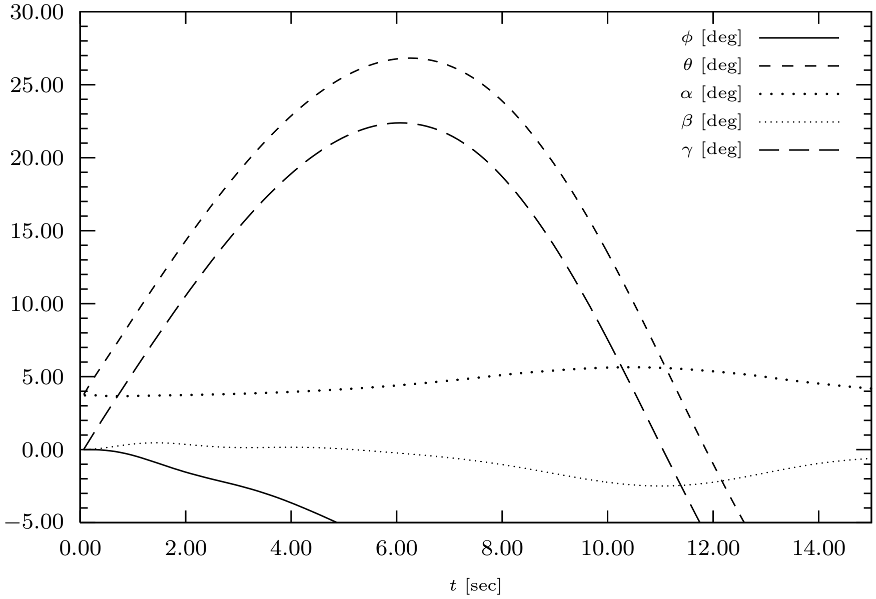

Figure 18. Time histories of Euler angles, aerodynamic and flight-path angles. Initial conditions are taken from the results of the trim algorithm of Figures 15-17.

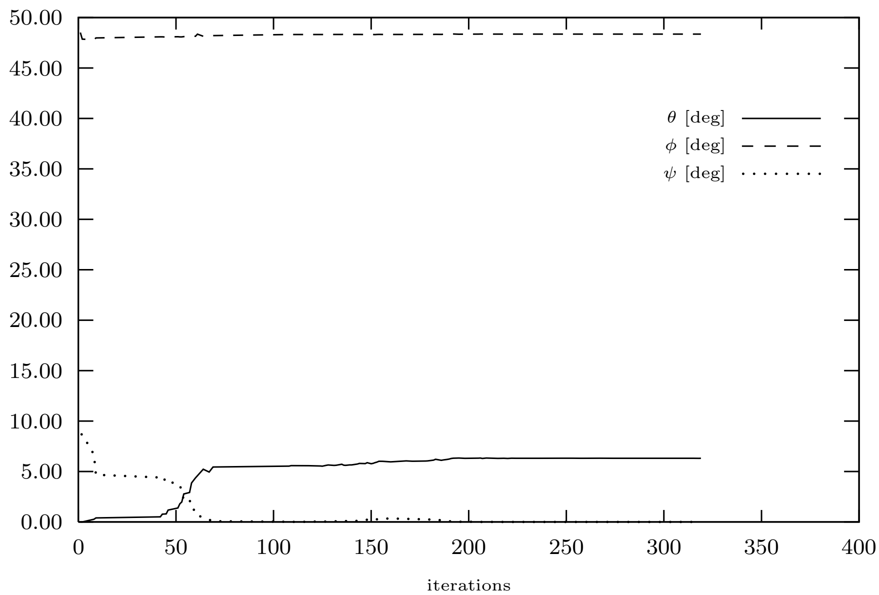

Figure 19. Trim algorithm results for the default Cessna 172 flight dynamics model distributed with JSBSim. Non-coordinated turn; $n = 1.5$, $V_{\mathbf{TAS}} = 94$ knots, $h_{\mathbf{ASL}} = 5000$ ft. Convergence history of Euler angles.

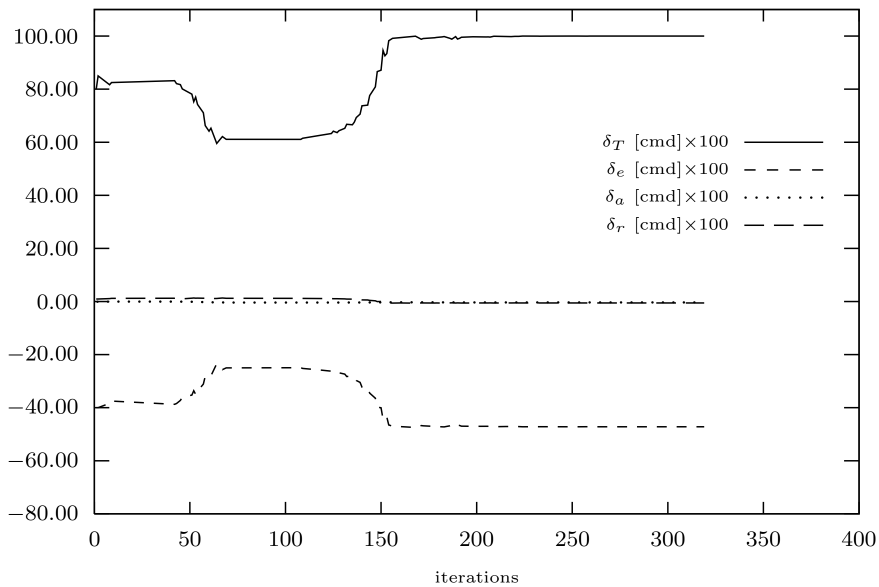

Figure 20. Trim algorithm results for the default Cessna 172 flight dynamics model distributed with JSBSim. Non-coordinated turn; $n = 1.5$, $V_{\mathbf{TAS}} = 94$ **knots,** $h_{\mathbf{ASL}} = 5000$ **ft. Convergence history of normalized control positions.**

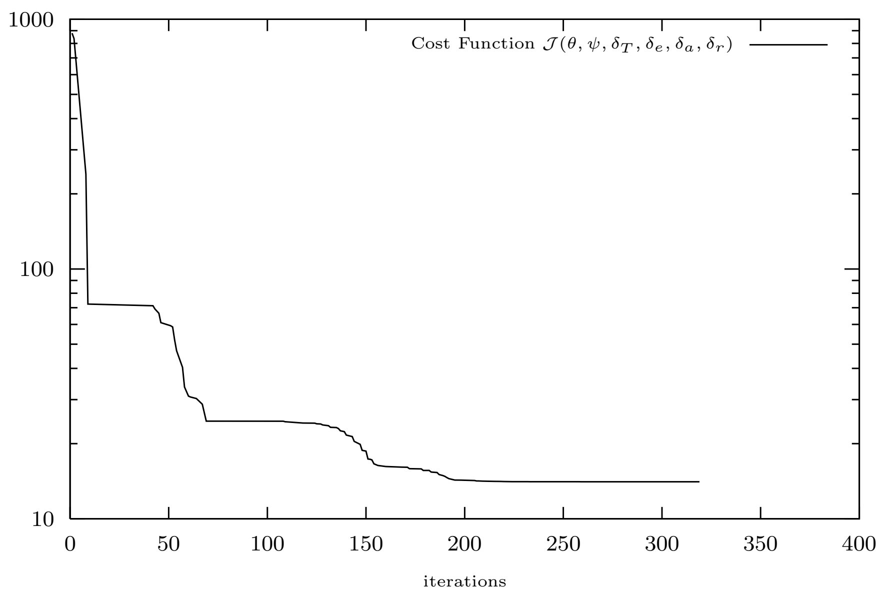

Figure 21. Trim algorithm results for the default Cessna 172 flight dynamics model distributed with JSBSim. Non-coordinated turn; $n = 1.5$, $V_{\mathbf{TAS}} = 94$ **knots,** $h_{\mathbf{ASL}} = 5000$ **ft. Convergence history of cost function** $\mathcal{J}$**.**

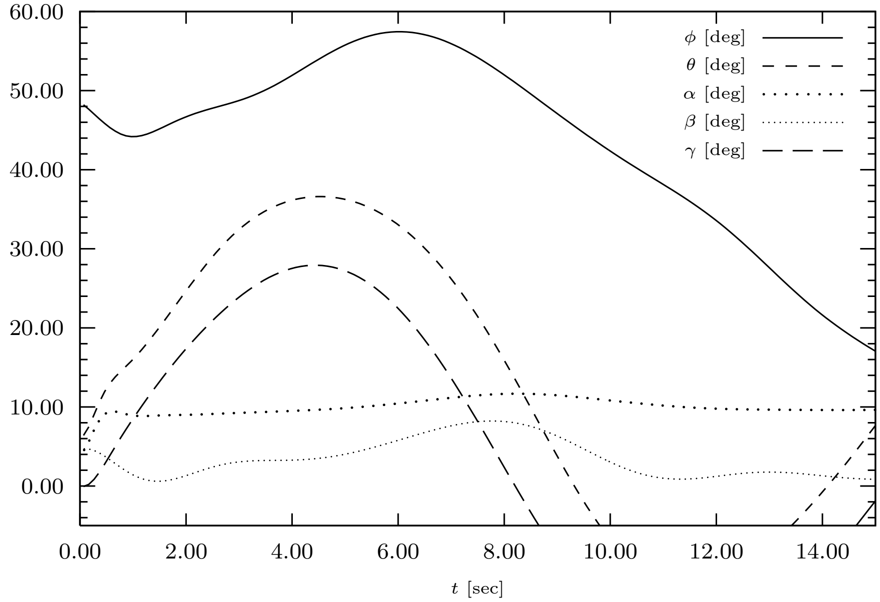

Figure 22. Time histories of Euler angles, aerodynamic and flight-path angles. Initial conditions are taken from the results of the trim algorithm of Figures 19-21.

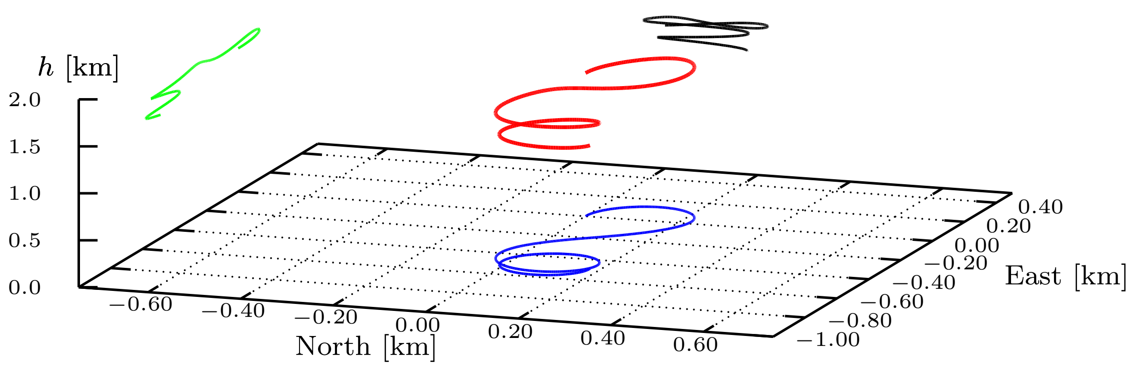

Figure 23. Trajectory for a set of initial conditions taken from Figures 19-20. Initial position projected on-ground: (0,0). Aircraft initially turns right. Later on, the uncommanded flight evolves in a left turn and in a spiral.

Table 6. **Summary of trim algorithm results for for the default Cessna 172 flight dynamics model.**

	trim targets			trim results: attitude			trim results: normalized controls position			
	V_0 [kts]	h_0 [ft]	n	ϕ [deg]	θ [deg]	ψ [deg]	δ_T	δ_e	δ_a	δ_r
longitudinal	88	5500	-	-	1.455	-	0.597	0.145	-	-
full	88	5500	-	-0.807	1.465	1.855	0.693	0.152	-0.0047	-0.163
full wings-level	88	5500	-	-	1.454	0.112	0.603	0.145	0.034	-0.016
pull-up	90	5500	1.50	-	3.658	-	0.940	-0.047	0.051	0.000
turn[a]	88	5000	1.5	48.36	6.31	8.5e-6	0.99	-0.47	-0.003	-0.005

[a] Non coordinated turn. Convergence not obtained (see Figures 19-21).

VII. Conclusions

In this paper we have examined a variety of general trim conditions and we have derived the equations defining them, regardless of the aircraft shape and aerodynamic model peculiarities. We have presented a general approach to trim through constrained minimization of a cost function. Finally we have provided an example of how a trim algorithm is used with an open source flight dynamics model like JSBSim.

VIII. Acknowledgments

The authors would like to thank Giancarlo Troise for testing the trim code with JSBsim, Tony Peden for his helpful hints on trimming strategies, and Bill Galbraith for giving his tangible moral support for this research.

References

[1]Goldstein, H: *Classical Mechanics*, Addison Wesley Publishing Company, Inc., 1959.

[2]Marion, J. B.: *Classical Dynamics of Particles and Systems*, Academic Press, 1970.

[3]Etkin, B.: *Dynamics of Atmospheric Flight*, Dover Publications, 2005.

[4]Stevens, B. L.; Lewis, F. L.: *Aircraft Control and Simulation*. John Wiley & Sons, 1992.

[5]Stengel, R. F.: *Flight Dynamics*. Princeton University Press, 2004.

[6]Phillips, W. F.: *Mechanics of Flight*, John Wiley & Sons, 2004.

[7]McFarland, R. E.: Trimming an Aircraft Model for Flight Simulation. NASA Technical Memorandum TM-89466, NASA Ames Research Center, 1987.

[8]Duke, E. L.; Antoniewicz, R. F.; Krambeer, K. D.: Derivation and Definition of a Linear Aircraft Model. Technical Report NASA Reference Publication RP-1207, Research Engineering, NASA Ames Research Center and NASA Dryden Flight Research Facility, 1988.

[9]Duke, E. L.; Patterson, B. P.; Antoniewicz, R. F.: User's Manual for LINEAR, a FORTRAN Program to Derive Linear Aircraft Models. NASA TP-2768, December 1987.

[10]Chen, Robert T.N.; Jeske, James A.: Kinematic Properties of the Helicopter in Coordinated Turns. NASA TP-1773, April 1981.

[11]Chudoba, B.; Cook M. V.: "Trim Equations of Motion For Aircraft Design: Steady State Straight-Line Flight." AIAA Atmospheric Flight Mechanics Conference and Exhibit, 11-14 August 2003, Austin, Texas.

[12]Chudoba, B.; Cook M. V.: "Trim Equations of Motion For Aircraft Design: Turning Flight, Pull-Up and Push-Over." AIAA Atmospheric Flight Mechanics Conference and Exhibit, 11-14 August 2003, Austin, Texas.

[13]Miller, G. D.; Jacques, D. R.; Pachter, M.: "Aircraft Trim Control." AIAA Guidance, Navigation, and Control Conference and Exhibit, 11-14 August 2003, Austin, Texas

[14]JSBSim home page: http://www.jsbsim.org/

[15]Berndt, J. S.: "JSBSim: An Open Source Flight Dynamics Model in C++." AIAA Modeling and Simulation Technology Conference, 16-19 August 16, 2004, Providence, Rhode Island.

[16]Nelder, J. A.;Mead, R.: "A simplex method for Function Minimization." *Comput. J.*, vol. 7, 1964, pp. 308-313.

[17]Kolda, T. G.; Lewis, R. M.; Torczon, V.: *Optimization by Direct Search: New Perspectives on Some Classical and Modern Methods*. SIAM Review, 45 (2003), pp. 385-482, Society for Industrial and Applied Mathematics, 2003.

[18]Gurson, A. P.: *Simplex Search Behavior in Nonlinear Optimization*. Honors Thesis, Department of Computer Science, College of William & Mary, Williamsburg, VA, 2000.

[19]Dolan, E. D.: *Pattern Search Behavior in Nonlinear Optimization*. Honors Thesis, Department of Computer Science, College of William & Mary, Williamsburg, VA, 1999.

[20]Torczon, V.; Dolan, L.; Gurson, A.; Shepherd, A.; Siefert, C.; Yates, A.: C++ *DirectSearch* Classes. Software available at http://www.cs.wm.edu/~va/software/DirectSearch/ .

[21]Shepherd, P. L.: Class Documentation for *DirectSearch* and its derived classes. Available at http://www.cs.wm.edu/~plshep/ .