POMDPs

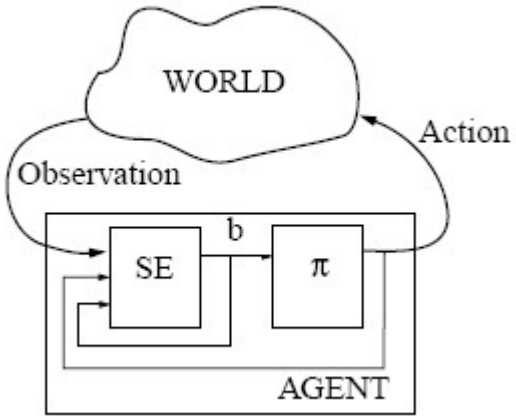
- MDPs policy: to find a mapping from states to actions
- POMDPs policy: to find a mapping from probability distributions (over states) to actions.
  - belief state: a probability distribution over states
  - belief space: the entire probability space, infinite

POMDPs

- Partially Observable MDPs

A partially observable Markov decision process can be described as a tuple $\langle \mathcal{S}, \mathcal{A}, T, R, \Omega, O \rangle$, where

- $\mathcal{S}$, $\mathcal{A}$, T, and R describe a Markov decision process;
- Ω is a finite set of observations the agent can experience of its world; and
- $O : \mathcal{S} \times \mathcal{A} \to \Pi(\Omega)$ is the *observation function*, which gives, for each action and resulting state, a probability distribution over possible observations (we write $O(s', a, o)$ for the probability of making observation o given that the agent took action a and landed in state s').



POMDPs

- In POMDPs we apply the very same idea as in MDPs.
- **Since the state is not observable**, the agent has to **make its decisions based on** the belief state which is a **posterior distribution over states**.
- Let b be the belief of the agent about the state under consideration.
- POMDPs compute a **value function over belief space**:

$$V_T(b) = \gamma \max_u \left[r(b,u) + \int V_{T-1}(b') p(b' \mid u, b) \, db' \right]$$

Problems

- Each belief is a probability distribution, thus, each value in a **POMDP is a function of an entire probability distribution**.
- **This is problematic, since probability distributions are continuous**.
- Additionally, we have to deal with the **huge complexity of belief spaces**.
- For **finite worlds** with finite state, action, and measurement spaces and finite horizons, however, we can **effectively represent the value functions by piecewise linear functions**.

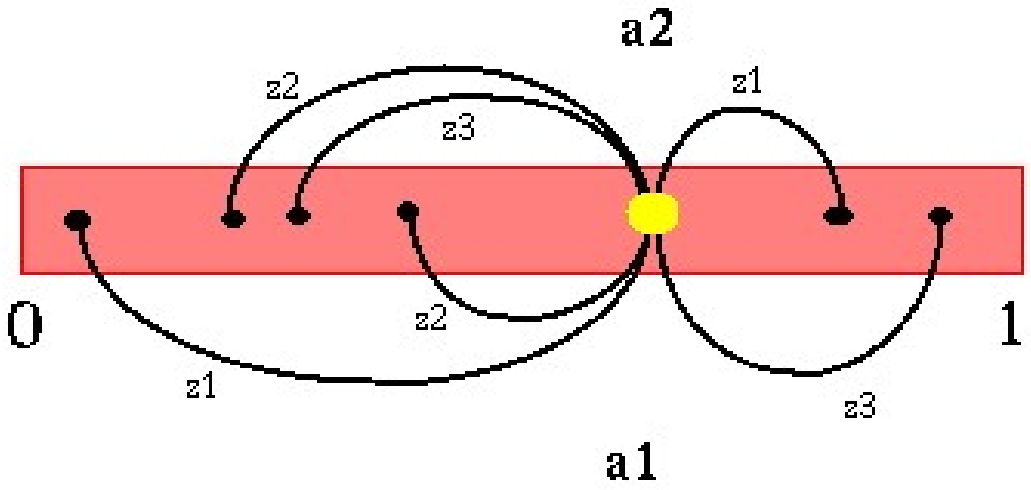
A two state POMDP

- represent the belief state with a single number *p*.
- the entire space of belief states can be represented as a line segment.

belief space for a 2 state POMDP

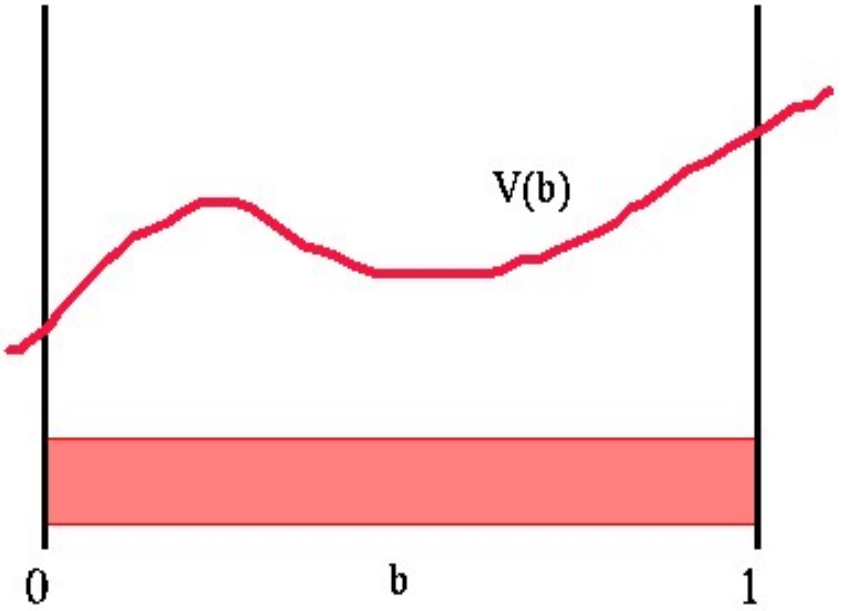
belief state updating

- finite number of possible next belief states, given a belief state
  - a finite number of actions
  - a finite number of observations
- b′ = *T*(b′| b, a, z). Given *a* and *z,* b′ is fully determined.



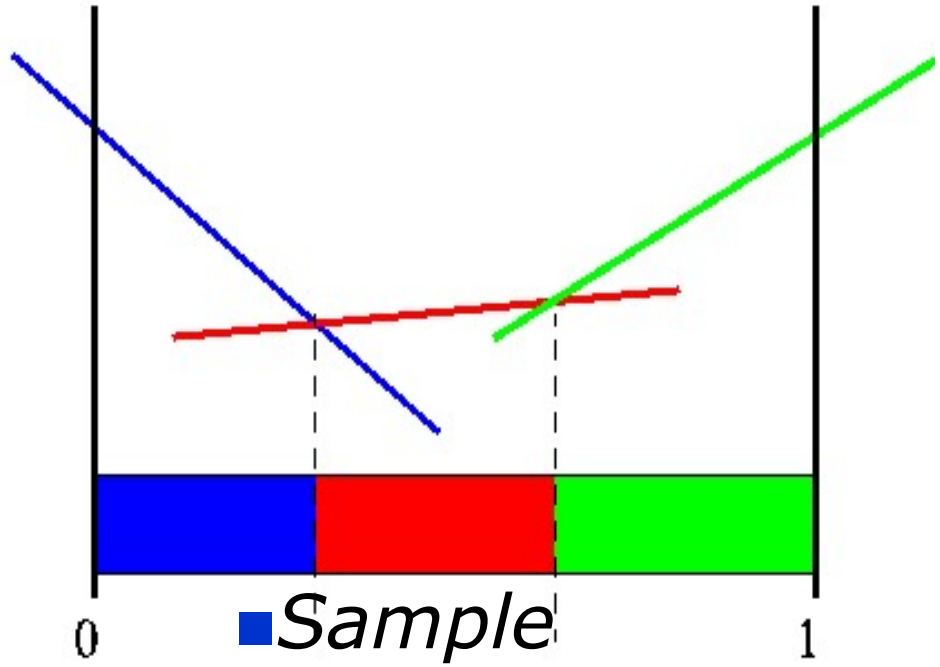
- the process of maintaining the belief state is Markovian: the next belief state depends only on the current belief state (and the current action and observation)
- we are now back to solving a MDP policy problem with some adaptations

- continuous space: value function is some arbitrary function
  - b: belief space
  - V(b): value function
- Problem: how we can easily represent this value function?



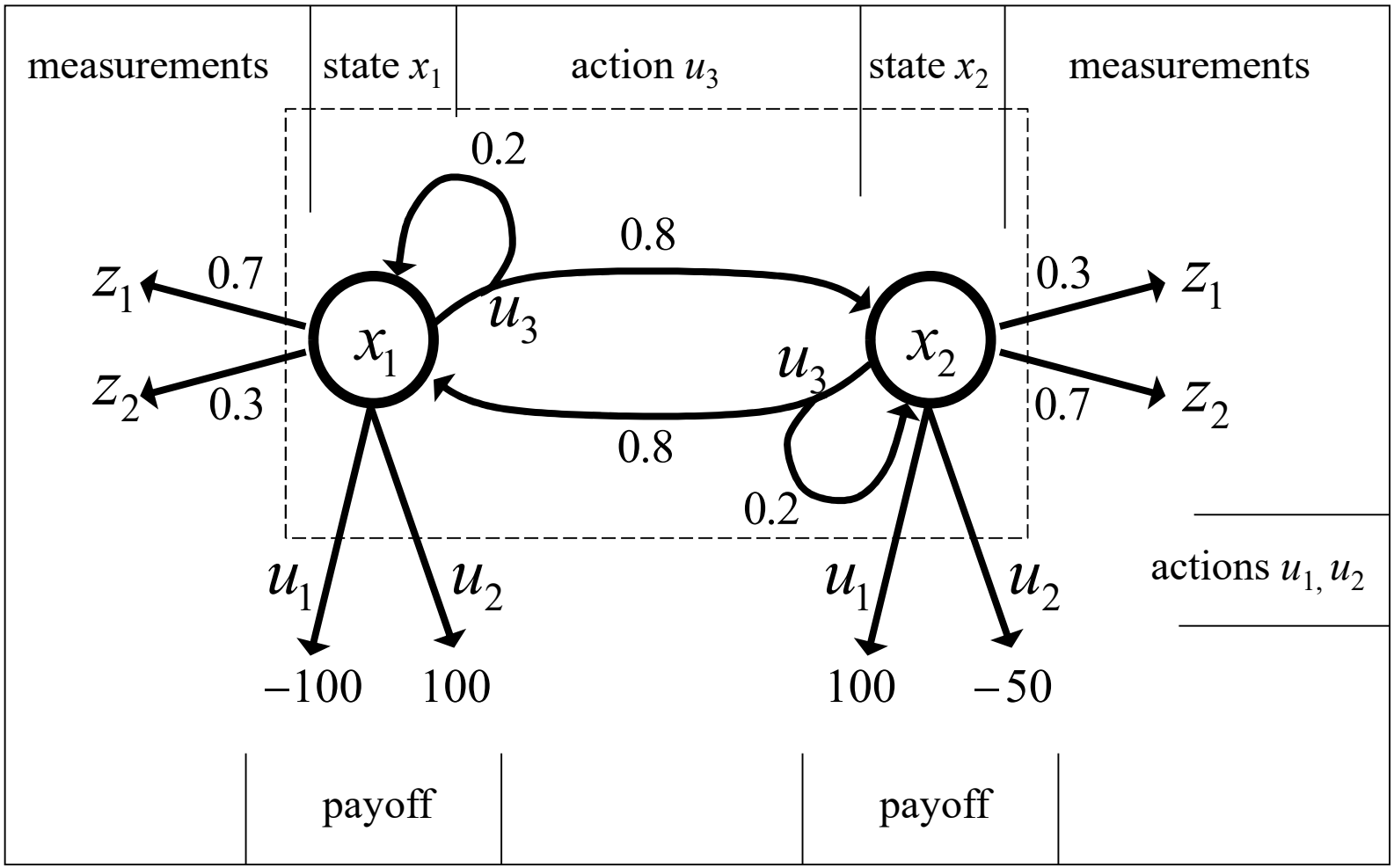
- *Value function over belief space*

Fortunately, the finite horizon value function is piecewise linear and convex (PWLC) for every horizon length.



Sample PWLC function

An Illustrative Example



The Parameters of the Example

- The actions u_1 and u_2 are terminal actions.
- The action u_3 is a sensing action that potentially leads to a state transition.
- The horizon is finite and γ=1.

$$
\begin{aligned}
r(x_1, u_1) &= -100 & \qquad r(x_2, u_1) &= +100 \\
r(x_1, u_2) &= +100 & \qquad r(x_2, u_2) &= -50 \\
r(x_1, u_3) &= -1 & \qquad r(x_2, u_3) &= -1
\end{aligned}
$$

$$
\begin{aligned}
p(x_1'|x_1, u_3) &= 0.2 & \qquad p(x_2'|x_1, u_3) &= 0.8 \\
p(x_1'|x_2, u_3) &= 0.8 & \qquad p(z_2'|x_2, u_3) &= 0.2
\end{aligned}
$$

$$
\begin{aligned}
p(z_1|x_1) &= 0.7 & \qquad p(z_2|x_1) &= 0.3 \\
p(z_1|x_2) &= 0.3 & \qquad p(z_2|x_2) &= 0.7
\end{aligned}
$$

Payoff in POMDPs

- In MDPs, the payoff (or return) depended on the state of the system.
- In POMDPs, however, the true state is not exactly known.
- Therefore, we compute the **expected payoff** by **integrating over all states**:

$$\begin{aligned} r(b,u) &= E_x[r(x,u)] \\ &= \int r(x,u)p(x)\,dx \\ &= p_1\, r(x_1,u) + p_2\, r(x_2,u) \end{aligned}$$

Payoffs in Our Example (1)

- If we are totally certain that we are in state x_1 and execute action u_1, we receive a reward of -100
- If, on the other hand, we definitely know that we are in x_2 and execute u_1, the reward is +100.
- In between it is the linear combination of the extreme values weighted by the probabilities

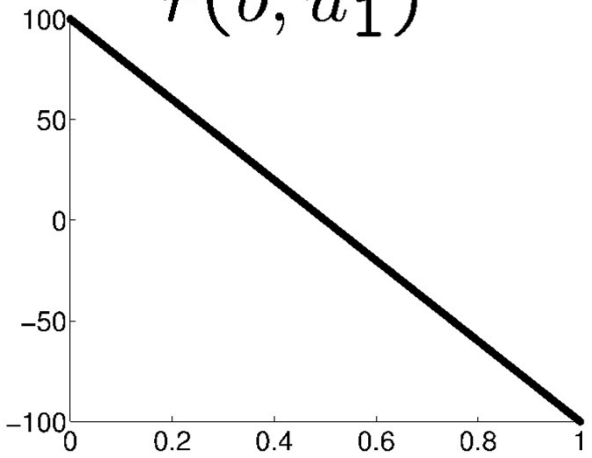
$$\begin{aligned} r(b, u_1) &= -100\, p_1 + 100\, p_2 \\ &= -100\, p_1 + 100\, (1 - p_1) \end{aligned}$$

$$r(b, u_2) = 100\, p_1 - 50\, (1 - p_1)$$

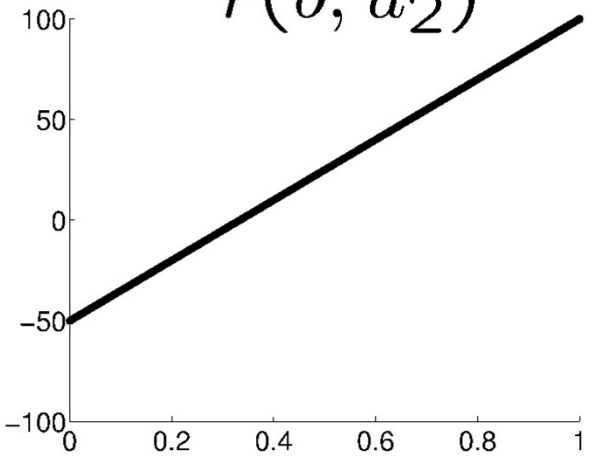
$$r(b, u_3) = -1$$

Payoffs in Our Example (2)

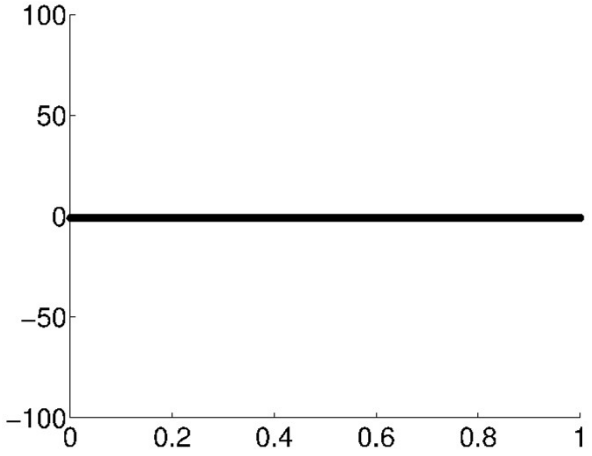
$r(b, u_1)$



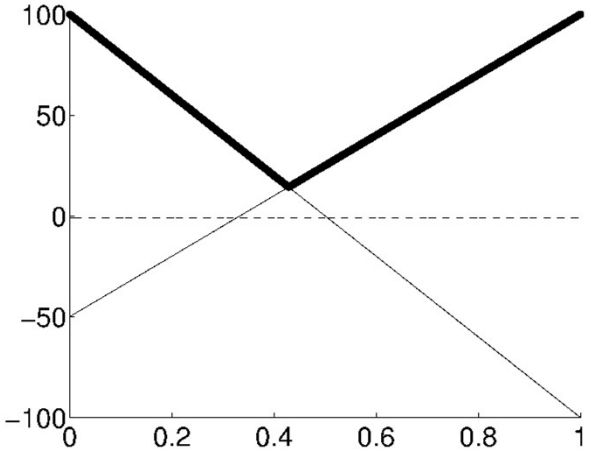
$r(b, u_2)$



$r(b, u_3)$



$V_1(b) = max_u r(b, u)$



The Resulting Policy for T=1

- Given we have a finite POMDP with T=1, we would use $V_1(b)$ to determine the optimal policy.
- In our example, the optimal policy for T=1 is

$$\pi_1(b) = \begin{cases} u_1 & \text{if } p_1 \leq \frac{3}{7} \\ u_2 & \text{if } p_1 > \frac{3}{7} \end{cases}$$

- This is the upper thick graph in the diagram.

Piecewise Linearity, Convexity

- The resulting value function $V_1(b)$ is the maximum of the three functions at each point

$$
\begin{aligned}
V_1(b) &= \max_u r(b,u) \\
&= \max \left\{ \begin{array}{rr} -100\, p_1 & +100\,(1-p_1) \\ 100\, p_1 & -50\,(1-p_1) \\ -1 & \end{array} \right\}
\end{aligned}
$$

- It is piecewise linear and convex.

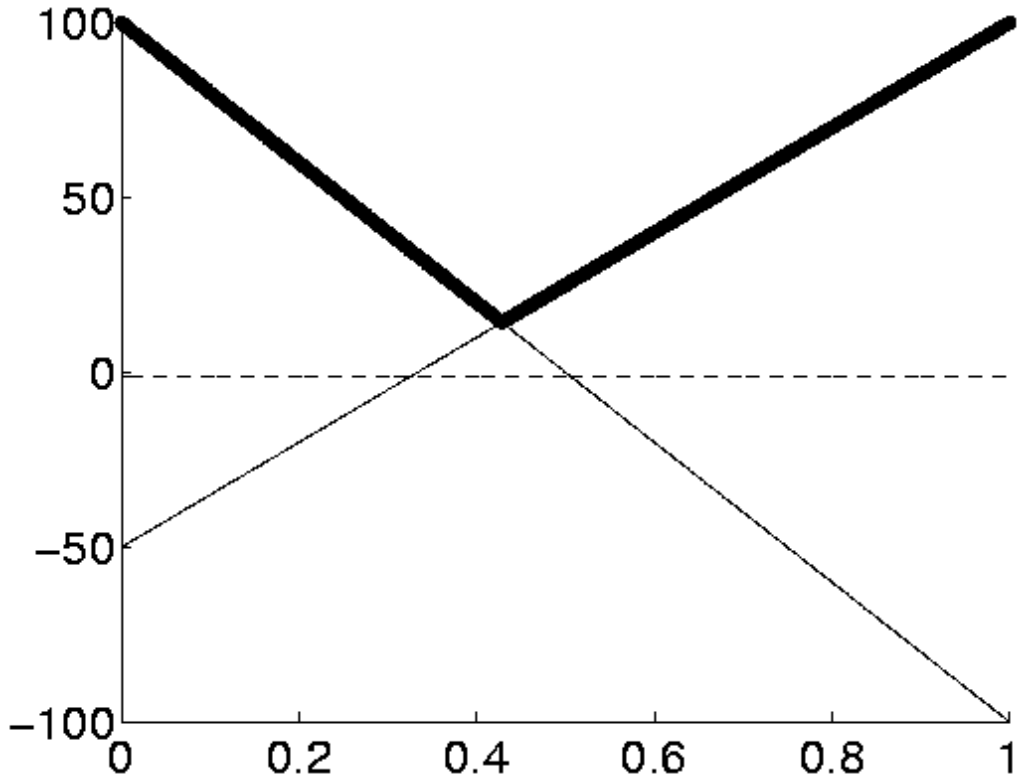
Pruning

- If we carefully consider $V_1(b)$, we see that only the first two components contribute.
- The third component can therefore safely be pruned away from $V_1(b)$.

$$V_1(b) = \max \left\{ \begin{array}{ll} -100\, p_1 & +100\,(1-p_1) \\ 100\, p_1 & -50\,(1-p_1) \end{array} \right\}$$

Increasing the Time Horizon

- Assume the robot can make an observation before deciding on an action.



$V_1(b)$

Increasing the Time Horizon

- Assume the robot can make an observation before deciding on an action.
- Suppose the robot perceives z_1 for which $p(z_1 \mid x_1)=0.7$ and $p(z_1 \mid x_2)=0.3$.
- Given the observation z_1 we update the belief using Bayes rule.

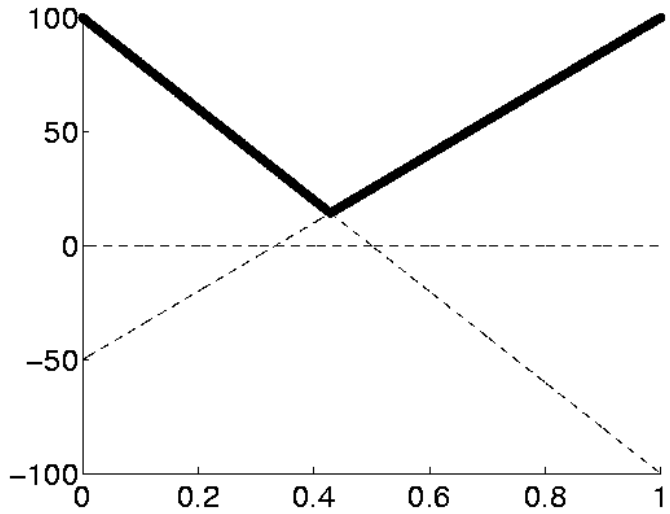
$$p'_1 = \frac{0.7\, p_1}{p(z_1)}$$

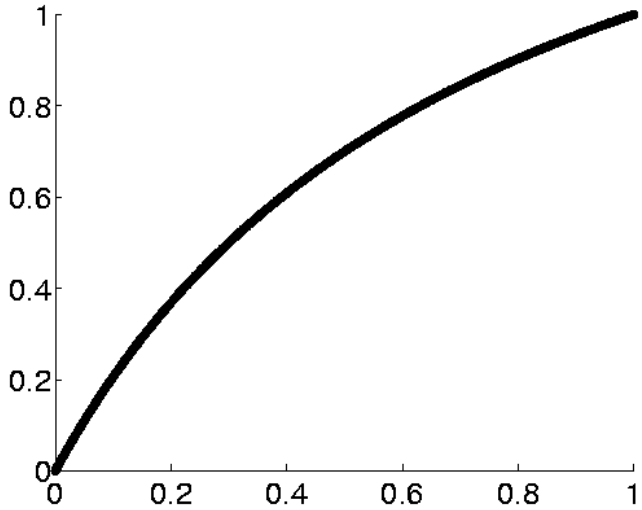
$$p'_2 = \frac{0.3(1-p_1)}{p(z_1)}$$

$$p(z_1) = 0.7\, p_1 + 0.3(1-p_1) = 0.4\, p_1 + 0.3$$

Value Function

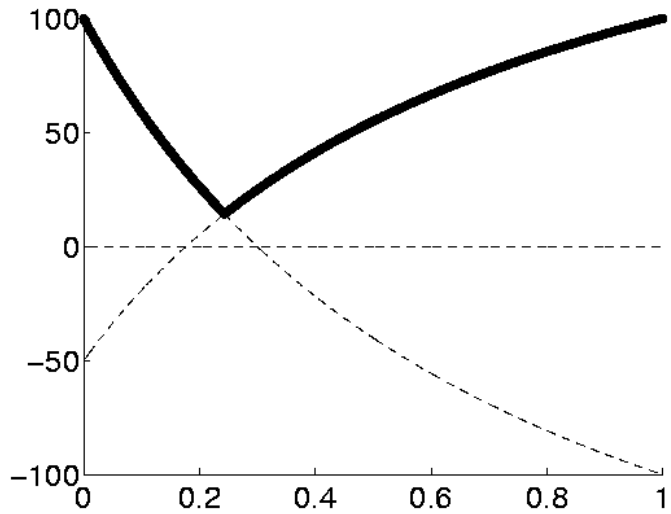
$V_1(b)$





$b'(b|z_1)$

$V_1(b|z_1)$



Increasing the Time Horizon

- Assume the robot can make an observation before deciding on an action.
- Suppose the robot perceives z_1 for which $p(z_1 \mid x_1)=0.7$ and $p(z_1 \mid x_2)=0.3$.
- Given the observation z_1 we update the belief using Bayes rule.
- Thus $V_1(b \mid z_1)$ is given by

$$V_1(b \mid z_1) = \max \left\{ \begin{array}{ll} -100 \cdot \frac{0.7\, p_1}{p(z_1)} & +100 \cdot \frac{0.3\, (1-p_1)}{p(z_1)} \\ 100 \cdot \frac{0.7\, p_1}{p(z_1)} & -50 \cdot \frac{0.3\, (1-p_1)}{p(z_1)} \end{array} \right\}$$

$$= \frac{1}{p(z_1)} \max \left\{ \begin{array}{ll} -70\, p_1 & +30\, (1-p_1) \\ 70\, p_1 & -15\, (1-p_1) \end{array} \right\}$$

Expected Value after Measuring

- Since we do not know in advance what the next measurement will be, we have to compute the expected belief

$$\overline{V}_1(b) = E_z[V_1(b \mid z)] = \sum_{i=1}^{2} p(z_i) V_1(b \mid z_i)$$

$$= \sum_{i=1}^{2} p(z_i) V_1\left(\frac{p(z_i \mid x_1) p_1}{p(z_i)} \right)$$

$$= \sum_{i=1}^{2} V_1\left(p(z_i \mid x_1) p_1 \right)$$

Expected Value after Measuring

- Since we do not know in advance what the next measurement will be, we have to compute the expected belief

$$
\begin{aligned}
\bar{V}_1(b) &= E_z[V_1(b \mid z)] \\
&= \sum_{i=1}^{2} p(z_i)\, V_1(b \mid z_i) \\
&= \max \left\{ \begin{array}{ll} -70\, p_1 & +30\, (1 - p_1) \\ 70\, p_1 & -15\, (1 - p_1) \end{array} \right\} \\
&\quad + \max \left\{ \begin{array}{ll} -30\, p_1 & +70\, (1 - p_1) \\ 30\, p_1 & -35\, (1 - p_1) \end{array} \right\}
\end{aligned}
$$

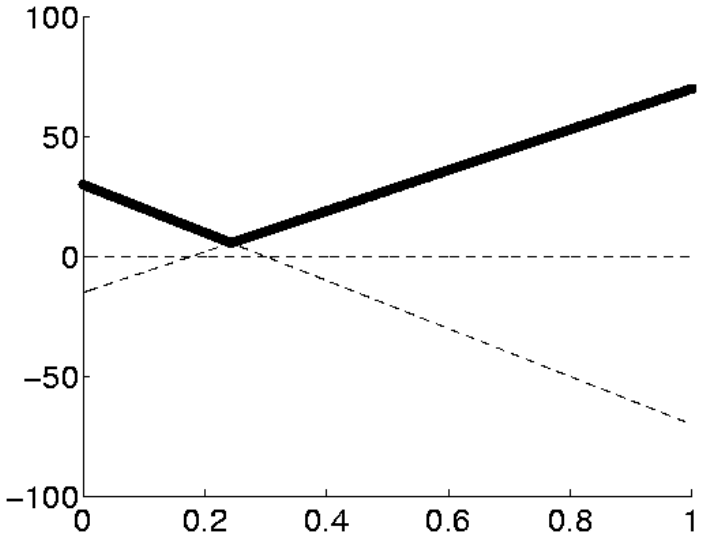
Resulting Value Function

- The four possible combinations yield the following function which then can be simplified and pruned.

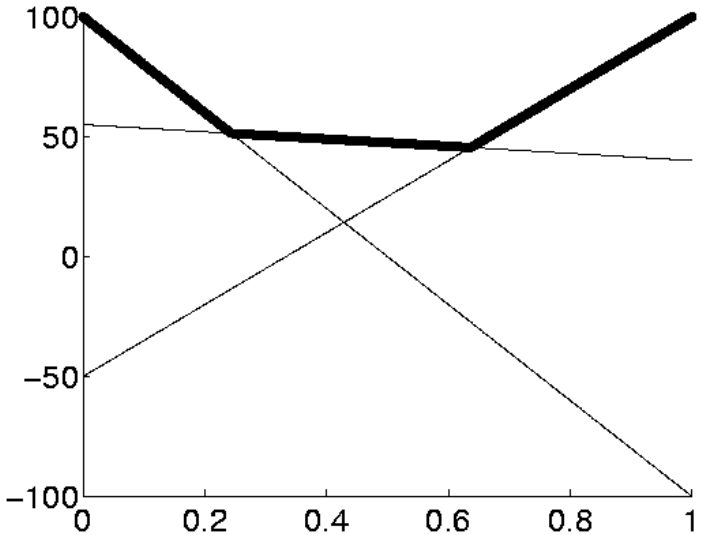
$$\bar{V}_1(b) = \max \left\{ \begin{array}{cccc} -70\,p_1 & +30\,(1-p_1) & -30\,p_1 & +70\,(1-p_1) \\ -70\,p_1 & +30\,(1-p_1) & +30\,p_1 & -35\,(1-p_1) \\ +70\,p_1 & -15\,(1-p_1) & -30\,p_1 & +70\,(1-p_1) \\ +70\,p_1 & -15\,(1-p_1) & +30\,p_1 & -35\,(1-p_1) \end{array} \right\}$$

$$= \max \left\{ \begin{array}{cc} -100\,p_1 & +100\,(1-p_1) \\ +40\,p_1 & +55\,(1-p_1) \\ +100\,p_1 & -50\,(1-p_1) \end{array} \right\}$$

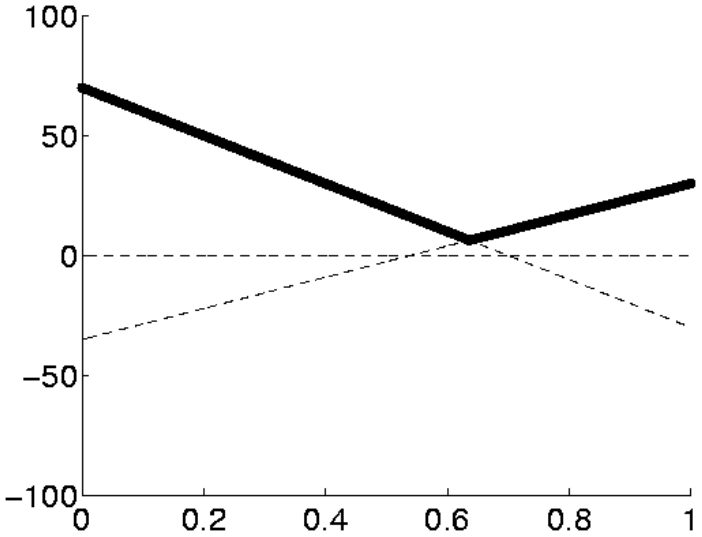
Value Function



$p(z_1)\ V_1(b|z_1)$



$\bar{V}_1(b)$



$p(z_2)\ V_2(b|z_2)$

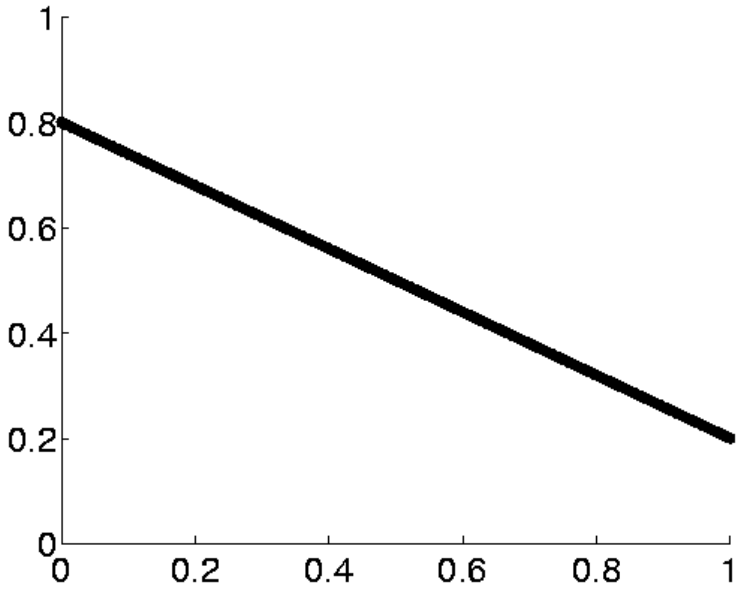
State Transitions (Prediction)

- When the agent selects u_3 its state potentially changes.
- When computing the value function, we have to take these potential state changes into account.

$$\begin{aligned} p_1' &= E_x[p(x_1 \mid x, u_3)] \\ &= \sum_{i=1}^{2} p(x_1 \mid x_i, u_3) p_i \\ &= 0.2p_1 + 0.8(1 - p_1) \\ &= 0.8 - 0.6p_1 \end{aligned}$$

State Transitions (Prediction)

$$
\begin{aligned}
p_1' &= E_x[p(x_1 \mid x, u_3)] \\
&= \sum_{i=1}^{2} p(x_1 \mid x_i, u_3) p_i \\
&= 0.2p_1 + 0.8(1 - p_1) \\
&= 0.8 - 0.6p_1
\end{aligned}
$$



Resulting Value Function after executing u_3

- Taking the state transitions into account, we finally obtain.

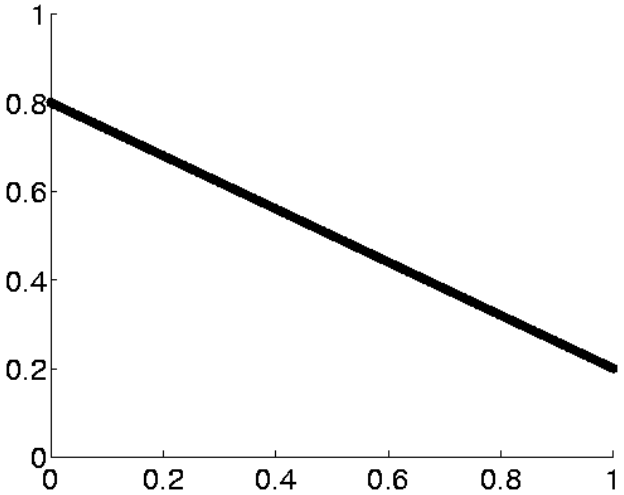
$$\bar{V}_1(b) = \max \left\{ \begin{array}{cccc} -70\,p_1 & +30\,(1-p_1) & -30\,p_1 & +70\,(1-p_1) \\ -70\,p_1 & +30\,(1-p_1) & +30\,p_1 & -35\,(1-p_1) \\ +70\,p_1 & -15\,(1-p_1) & -30\,p_1 & +70\,(1-p_1) \\ +70\,p_1 & -15\,(1-p_1) & +30\,p_1 & -35\,(1-p_1) \end{array} \right\}$$

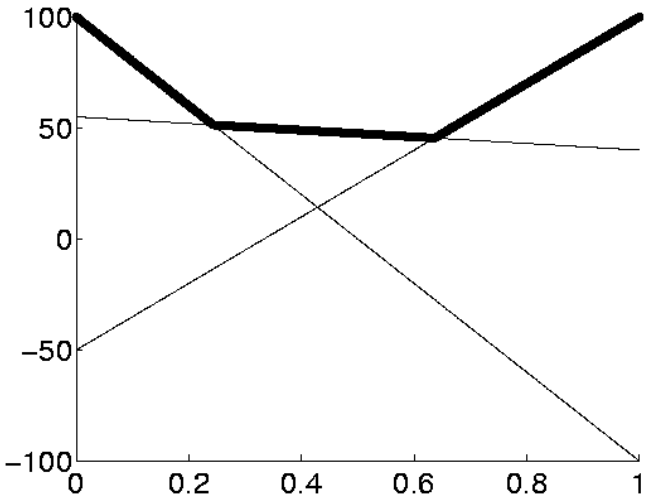
$$= \max \left\{ \begin{array}{cc} -100\,p_1 & +100\,(1-p_1) \\ +40\,p_1 & +55\,(1-p_1) \\ +100\,p_1 & -50\,(1-p_1) \end{array} \right\}$$

$$\bar{V}_1(b \mid u_3) = \max \left\{ \begin{array}{cc} 60\,p_1 & -60\,(1-p_1) \\ 52\,p_1 & +43\,(1-p_1) \\ -20\,p_1 & +70\,(1-p_1) \end{array} \right\}$$

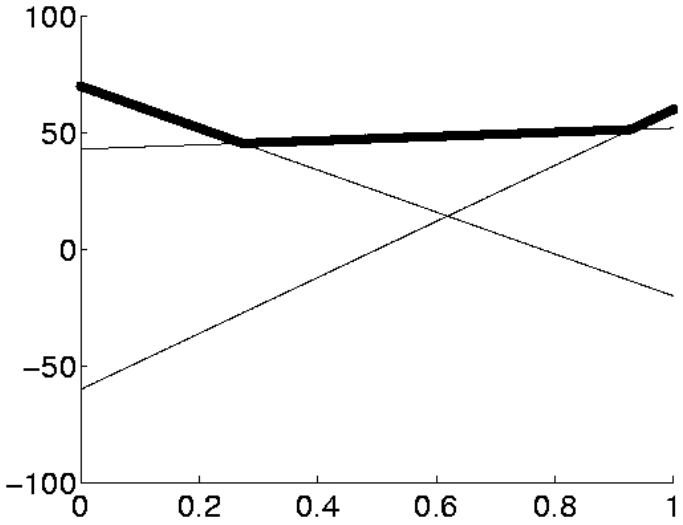
Value Function after executing u_3

$\bar{V}_1(b)$





$\bar{V}_1(b \mid u_3)$

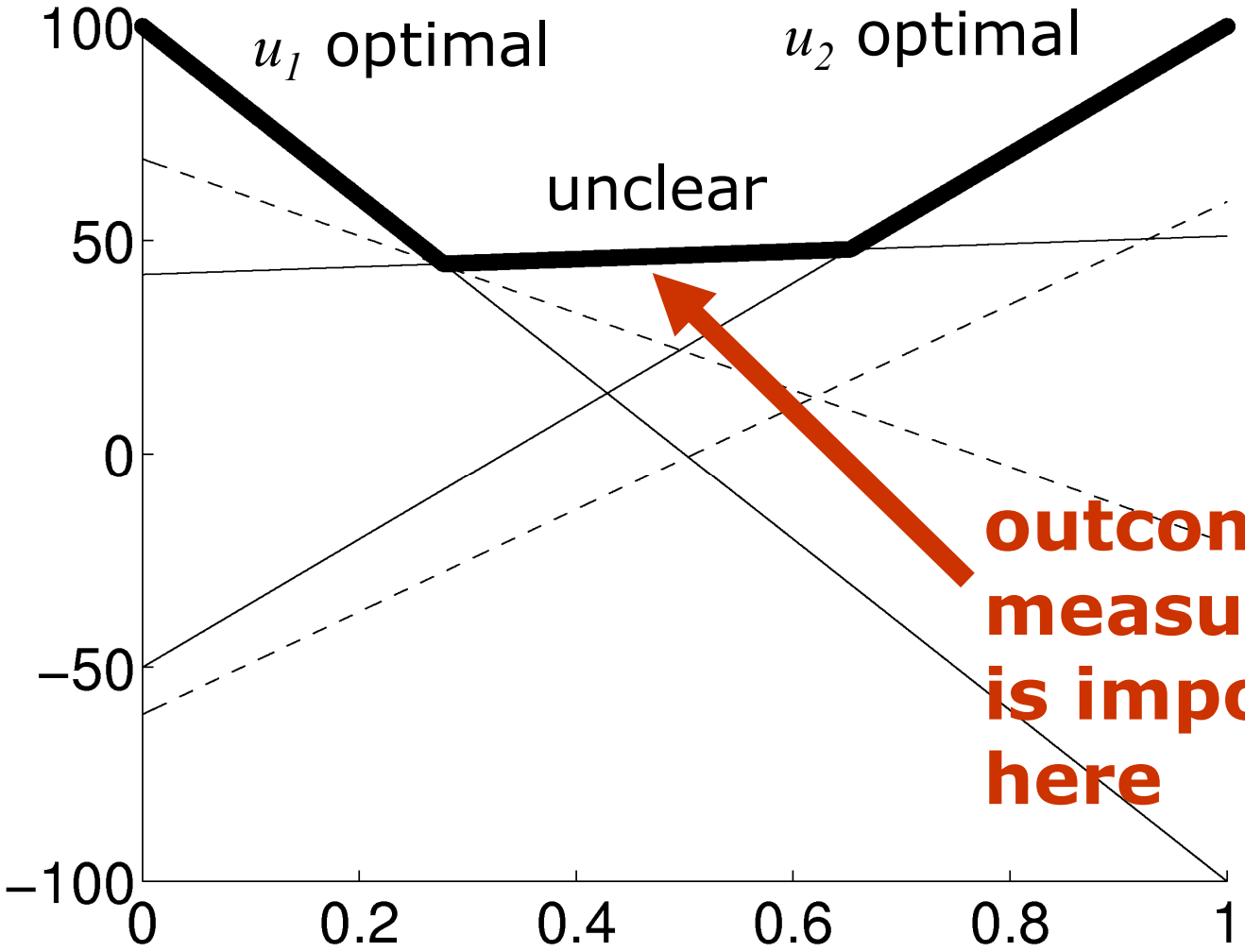


Value Function for T=2

- Taking into account that the agent can either directly perform u_1 or u_2 or first u_3 and then u_1 or u_2, we obtain (after pruning)

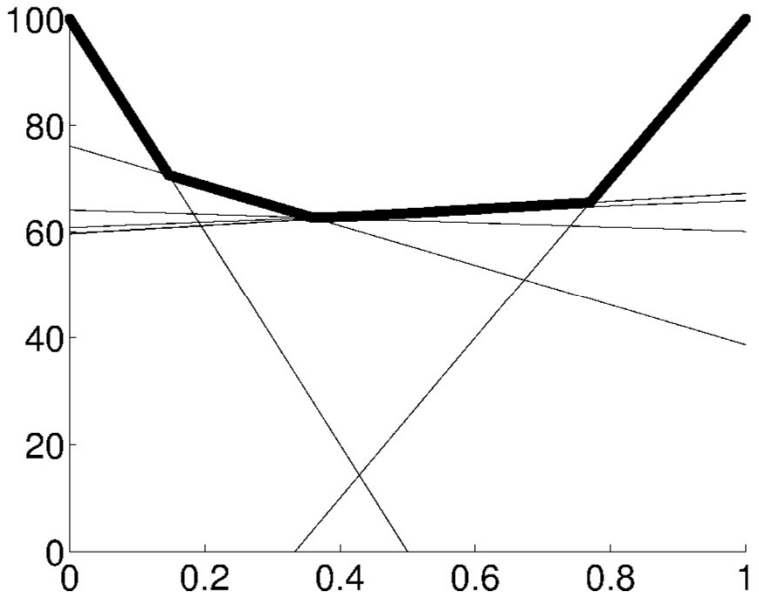
$$\bar{V}_2(b) = \max \left\{ \begin{array}{rr} -100\, p_1 & +100\,(1-p_1) \\ 100\, p_1 & -50\,(1-p_1) \\ 51\, p_1 & +42\,(1-p_1) \end{array} \right\}$$

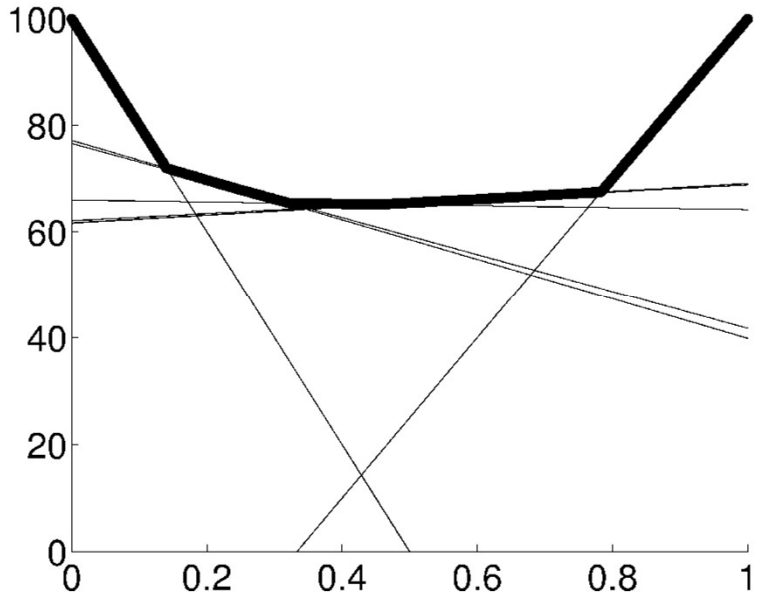
Graphical Representation of $V_2(b)$



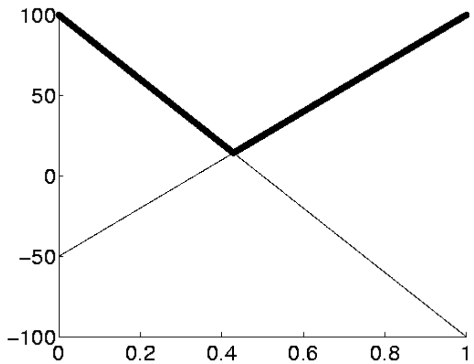
Deep Horizons and Pruning

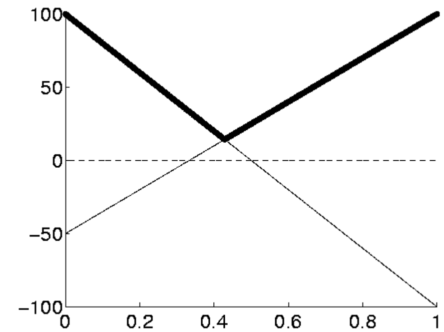
- We have now completed a full backup in belief space.
- This process can be applied recursively.
- The value functions for T=10 and T=20 are

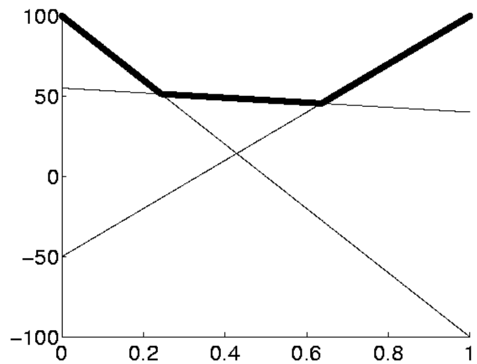


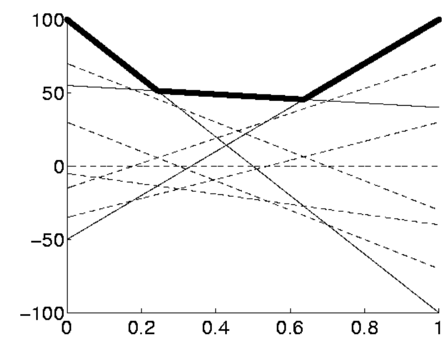


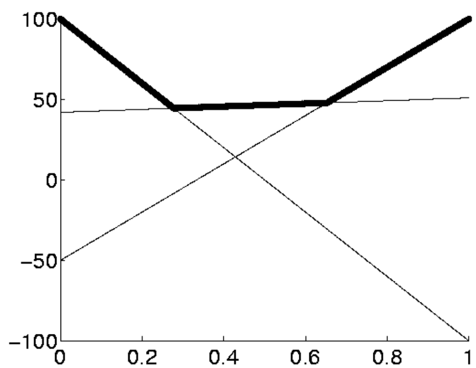
Deep Horizons and Pruning

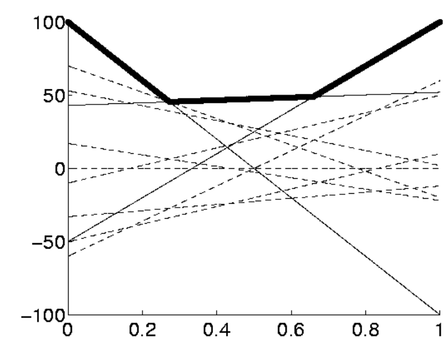


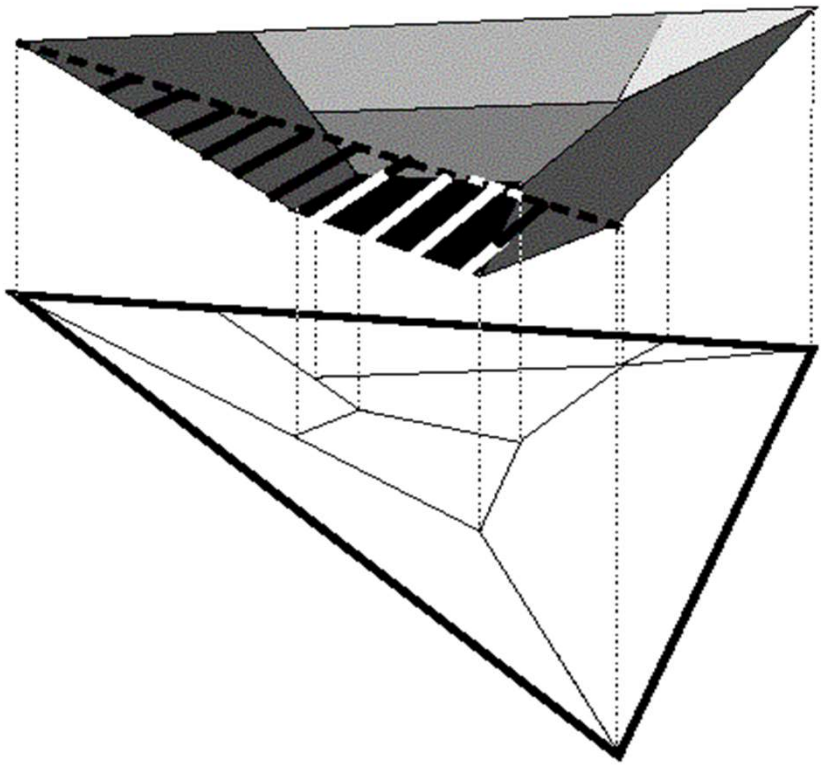












- $|S| = 3$
- Hyper-planes
- Finite number of regions over the simplex

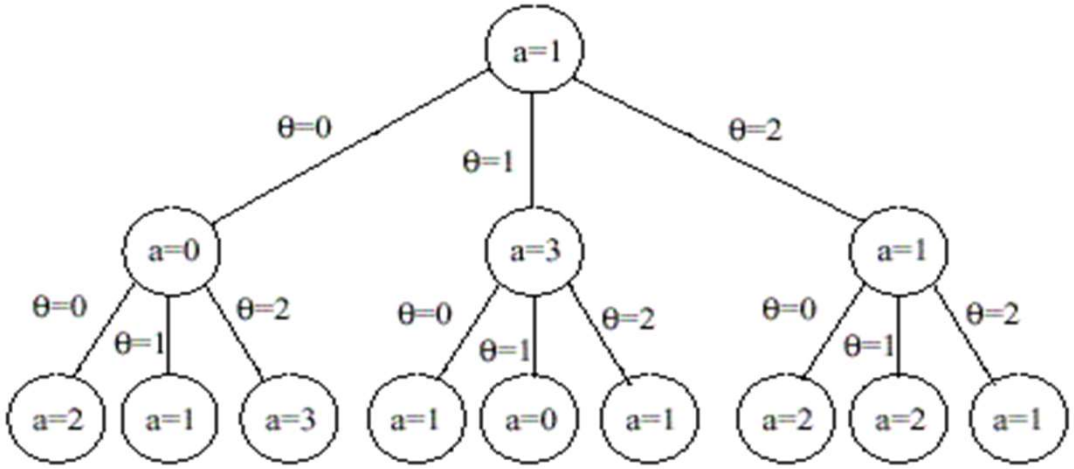
- *Sample value function for* $|S| = 3$

- Repeat the process for value functions of 3-horizon,…, and k-horizon POMDP

$$V_t^*(b) = \max_{a \in A} [\sum_i b_i q_i^a + \sum_{i,j,z} b_i p_{ij}^a r_{jz}^a V_{t-1}^*[T(b \mid a, z)]]$$

Alternate Value function interpretation

- A decision tree
  - Nodes represent an action decision
  - Branches represent observation made
- Too many trees to be generated!



Algorithm POMDP(T):

$\Upsilon = (0, \ldots, 0)$
for $\tau = 1$ *to* T *do*
  $\Upsilon' = \emptyset$
  for all $(u'; v_1^k, \ldots, v_N^k)$ *in* Υ *do*
    for all control actions u *do*
      for all measurements z *do*
        for $j = 1$ *to* N *do*

$$v_{j,u,z}^k = \sum_{i=1}^{N} v_i^k \; p(z \mid x_i) \; p(x_i \mid u, x_j)$$

        endfor
      endfor
    endfor
  endfor
  for all control actions u *do*
    for all $k(1), \ldots, k(M) = (1, \ldots, 1)$ *to* $(|\Upsilon|, \ldots, |\Upsilon|)$ *do*
      for $i = 1$ *to* N *do*

$$v_i' = \gamma \left[r(x_i, u) + \sum_z v_{u,z,i}^{k(z)} \right]$$

      endfor
      add $(u; v_1', \ldots, v_N')$ *to* Υ'
    endfor
  endfor
  optional: prune Υ'
  $\Upsilon = \Upsilon'$
endfor
return Υ

Why Pruning is Essential

- Each **update introduces additional linear components** to V.
- Each **measurement squares the number of linear components**.
- Thus, an un-pruned value function for T=20 includes more than $10^{547,864}$ linear functions.
- At T=30 we have $10^{561,012,337}$ linear functions.
- The pruned value functions at T=20, in comparison, contains only 12 linear components.
- The combinatorial explosion of linear components in the value function are the major reason why **POMDPs are impractical for most applications**.

POMDP Summary

- POMDPs compute the optimal action in partially observable, stochastic domains.
- For finite horizon problems, the resulting value functions are piecewise linear and convex.
- In each iteration the number of linear constraints grows exponentially.

POMDP Approximations

- QMDPs
- Point-based value iteration
- AMDPs

QMDPs

- QMDPs only consider state uncertainty in the first step

- After that, the world becomes fully observable.

"One drawback is that these policies will not take actions to gain information" [Littman Cassandra Kaelbling 1995]

1: **Algorithm QMDP**($b = (p_1, \ldots, p_N)$)**:**

2: $\hat{V} = \mathbf{MDP\_discrete\_value\_iteration}()$

3: *for all control actions* u *do*

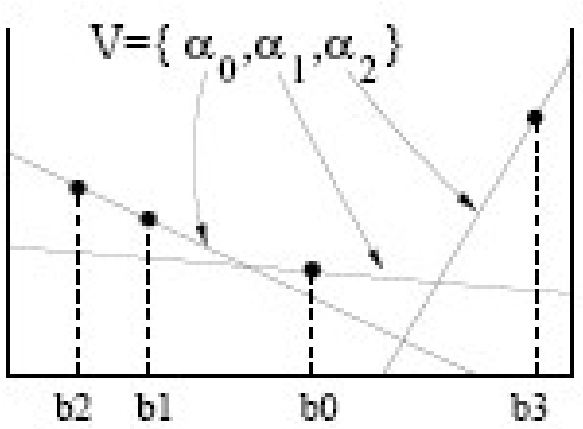
4: $$Q(x_i, u) = r(x_i, u) + \sum_{j=1}^{N} \hat{V}(x_j)\, p(x_j \mid u, x_i)$$

5: *endfor*

6: *return* $$\arg\max_u \sum_{i=1}^{N} p_i\, Q(x_i, u)$$

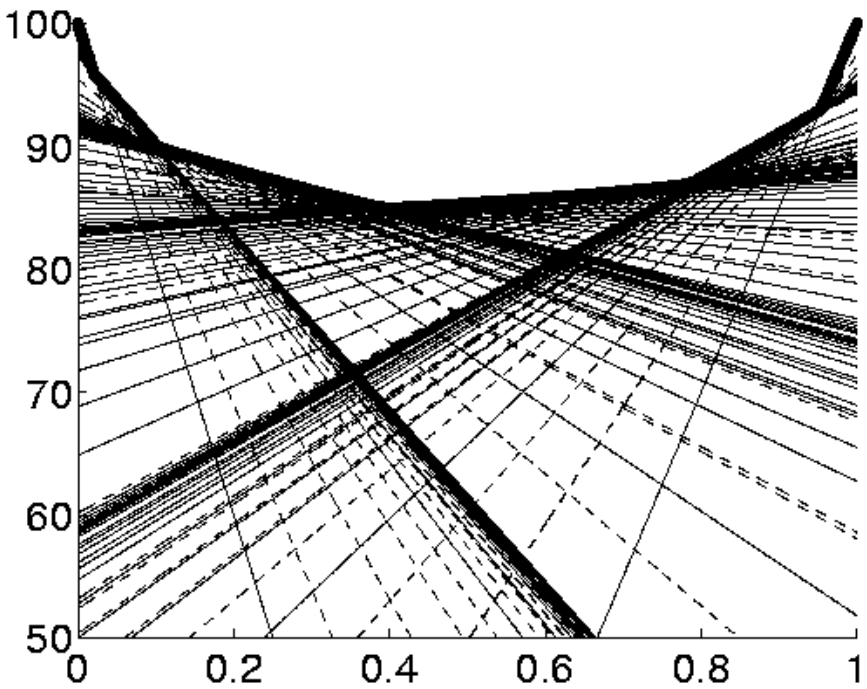
Point-based Value Iteration

- Maintains a set of example beliefs
  - Only considers constraints that maximize value function for at least one of the examples
  - Occasionally add new belief points
  - Can do point updates in polytime, no pruning

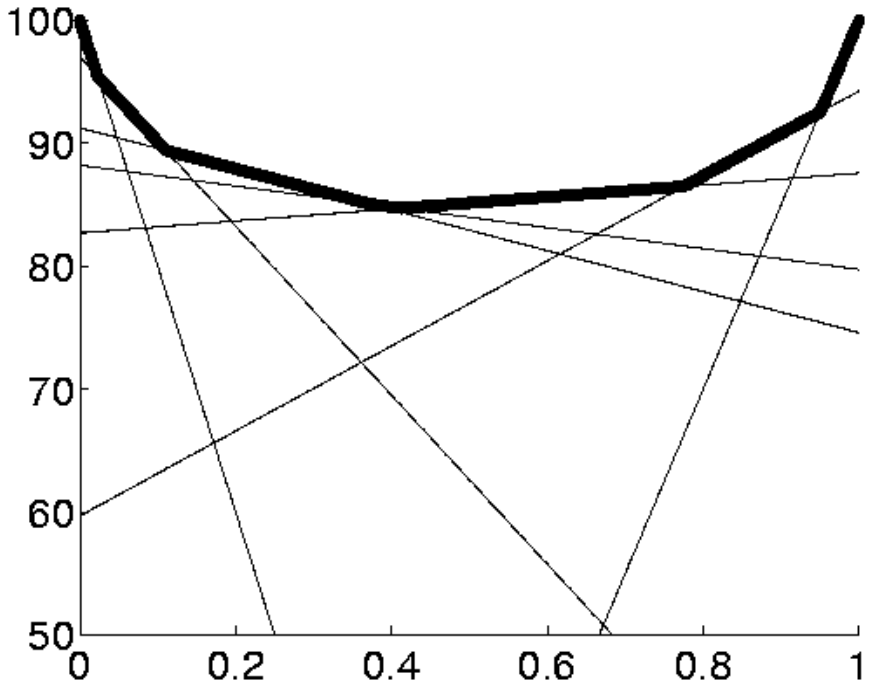


Point-based Value Iteration

Value functions for T=30



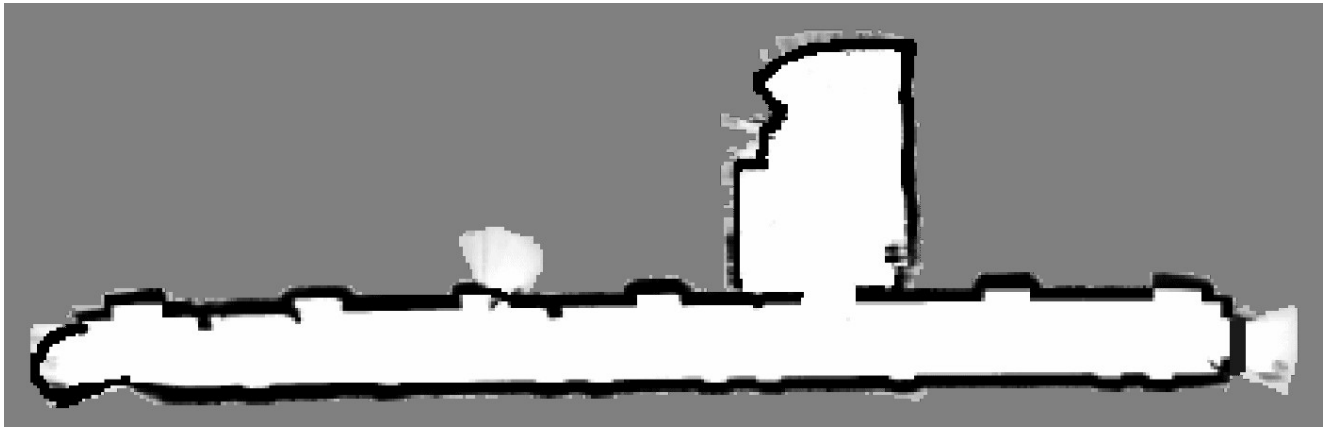
Exact value function



PBVI

Example Application

The goal is to search for and tag a moving opponent [Rosencrantz et al., 2003]



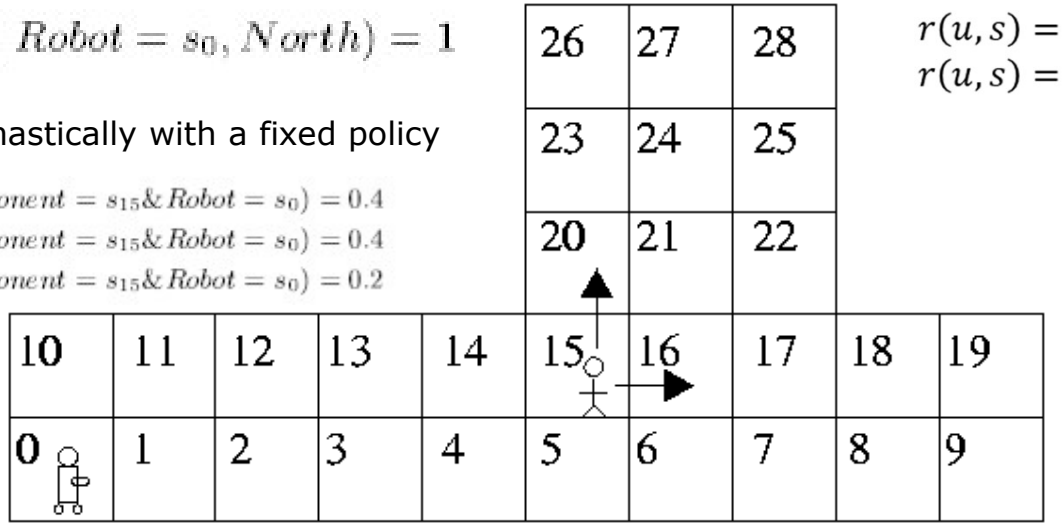
Robot deterministic and fully observable

$$Pr(Robot = s_{10} \mid Robot = s_0, North) = 1$$

Opponent moves stochastically with a fixed policy

$$Pr(Opponent = s_{16} \mid Opponent = s_{15} \& Robot = s_0) = 0.4$$
$$Pr(Opponent = s_{20} \mid Opponent = s_{15} \& Robot = s_0) = 0.4$$
$$Pr(Opponent = s_{15} \mid Opponent = s_{15} \& Robot = s_0) = 0.2$$

Opponent observable only on the same cell



$r(u,s) = -1,$
$r(u,s) = 10, r(u,s) = -10$

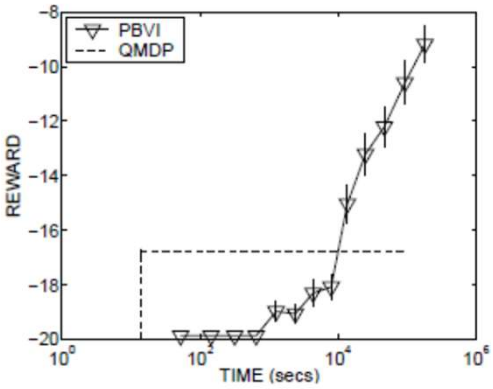
870 states

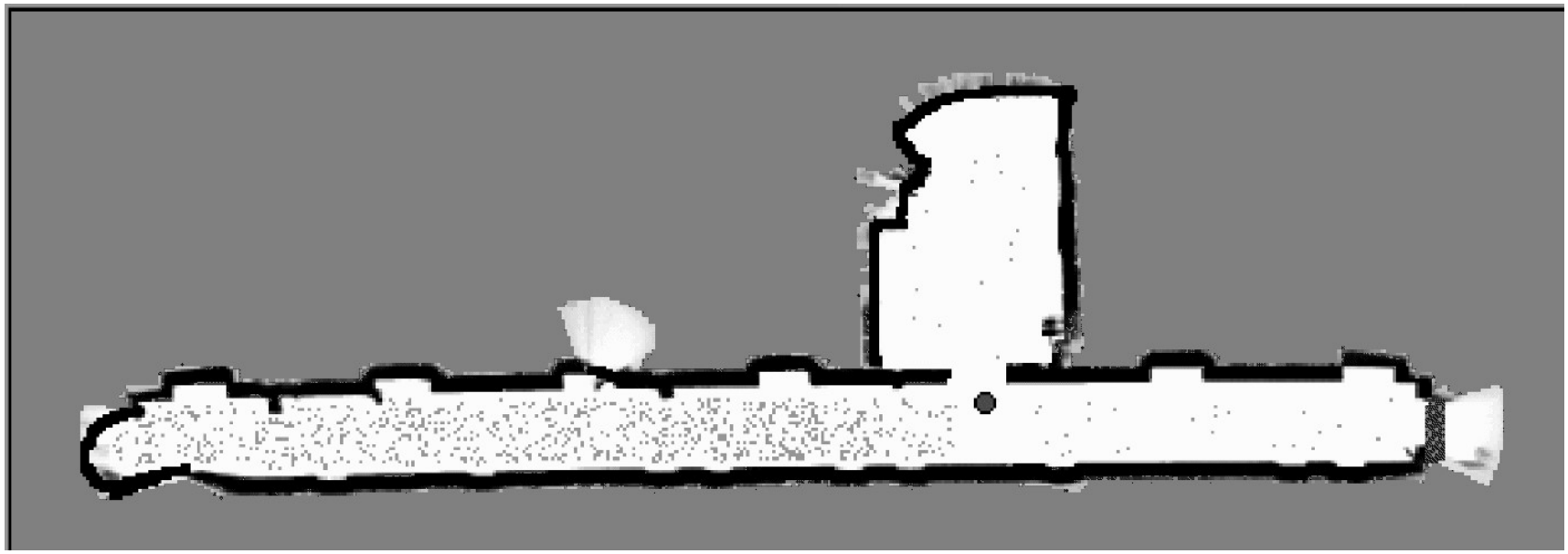
cross-product of Robot and Opponent states

Example Application

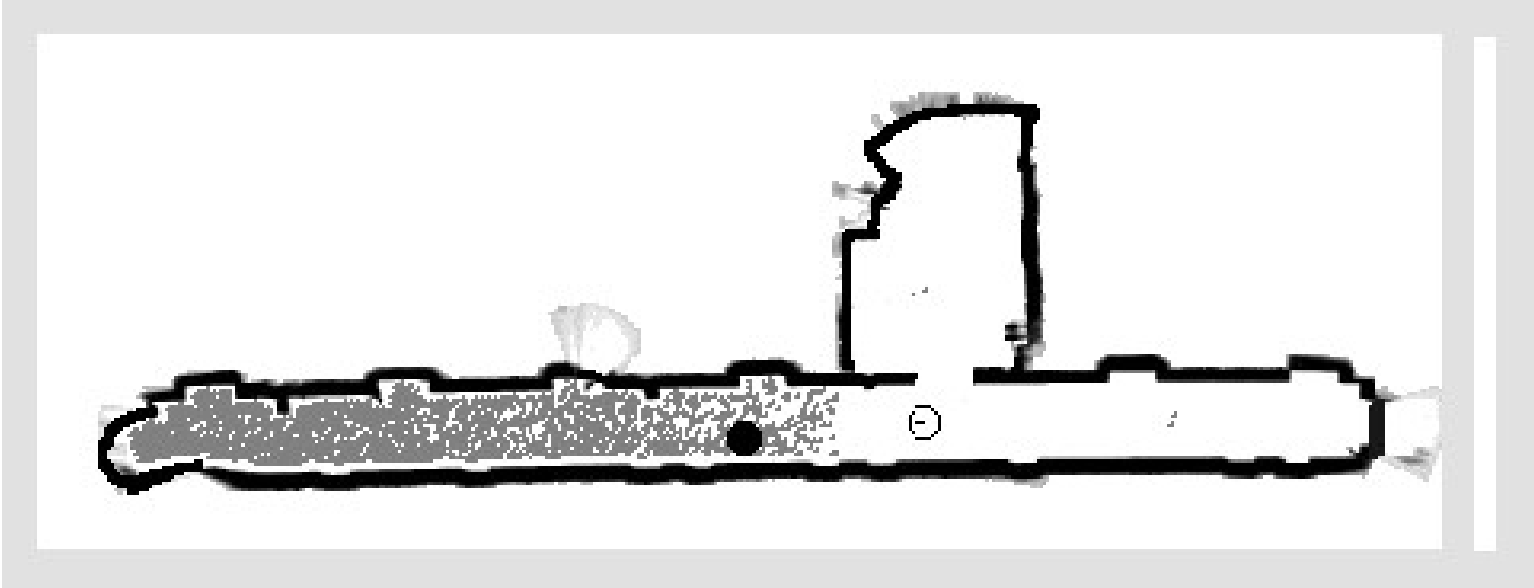
"PBVI performs best when its belief set is uniformly dense in the set B of reachable beliefs. So, we initialize the set B to contain the initial belief b0 and expand B by greedily choosing new reachable beliefs that improve the worst-case density as rapidly as possible" [Pineau, Gordon, Thrun 2003]

"PBVI tries to generate one new belief from each previous belief; so, B at most doubles in size on each expansion" [Pineau, Gordon, Thrun 2003]





Dimensionality Reduction on Beliefs



Augmented MDPs

- Augmentation adds uncertainty component to state space, e.g.,

$$\bar{b} = \begin{pmatrix} \arg\max_x b(x) \\ H_b(x) \end{pmatrix}, \qquad H_b(x) = -\int b(x) \log b(x) dx$$

- Planning is performed by MDP in augmented state space
- Transition, observation and payoff models must be learned

Due fasi: learning (2-19) e value iteration (20-27)

n campioni per ogni b- e u

b(x) è gaussiana simmetrica

Aggiornamento basato su frequenza

Value Iteration

Algorithm AMDP_value_iteration( ):

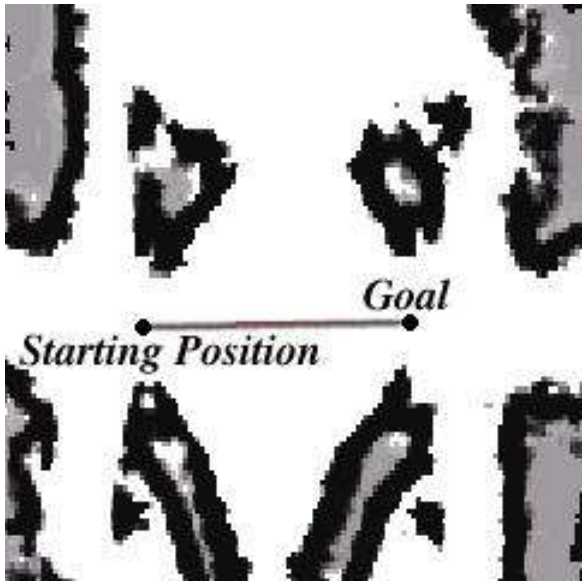
for all $\bar{b}$ *do* // *learn model*
  for all u *do*
    for all $\bar{b}$ *do* // *initialize model*
      $\hat{\mathcal{P}}(\bar{b}, u, \bar{b}') = 0$
    endfor
    $\hat{\mathcal{R}}(\bar{b}, u) = 0$
    repeat n *times* // *learn model*
      generate b *with* $f(b) = \bar{b}$
      sample $x \sim b(x)$ // *belief sampling*
      sample $x' \sim p(x' \mid u, x)$ // *motion model*
      sample $z \sim p(z \mid x')$ // *measurement model*
      calculate $b' = B(b, u, z)$ // *Bayes filter*
      calculate $\bar{b}' = f(b')$ // *belief state statistic*
      $\hat{\mathcal{P}}(\bar{b}, u, \bar{b}') = \hat{\mathcal{P}}(\bar{b}, u, \bar{b}') + \frac{1}{n}$ // *learn transitions prob's*
      $\hat{\mathcal{R}}(\bar{b}, u) = \hat{\mathcal{R}}(\bar{b}, u) + \frac{r(u,s)}{n}$ // *learn payoff model*
    endrepeat
  endfor
endfor
for all $\bar{b}$ // *initialize value function*
  $\hat{V}(\bar{b}) = r_{\min}$
endfor
repeat until convergence // *value iteration*
  for all $\bar{b}$ *do*
    $$\hat{V}(\bar{b}) = \gamma \max_u \left[\hat{\mathcal{R}}(u, \bar{b}) + \sum_{\bar{b}'} \hat{V}(\bar{b}') \, \hat{\mathcal{P}}(\bar{b}, u, \bar{b}') \right]$$
endfor
return $\hat{V}, \hat{\mathcal{P}}, \hat{\mathcal{R}}$ // *return value fct & model*

Algorithm policy_AMDP($\hat{V}$, $\hat{\mathcal{P}}$, $\hat{\mathcal{R}}$, b):

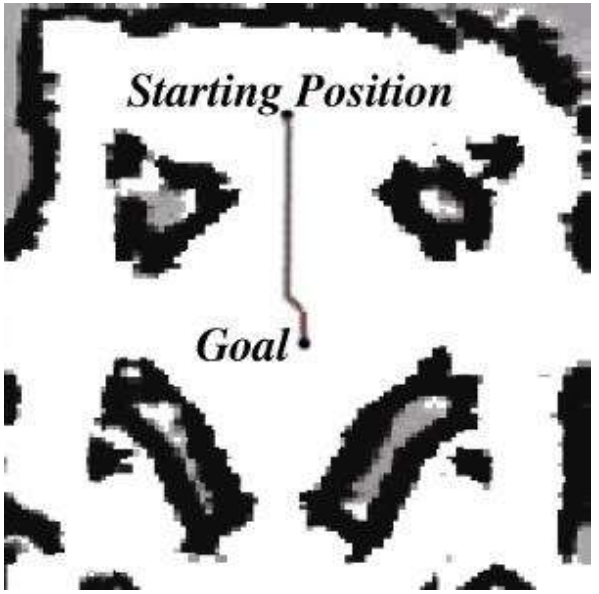
$\bar{b} = f(b)$

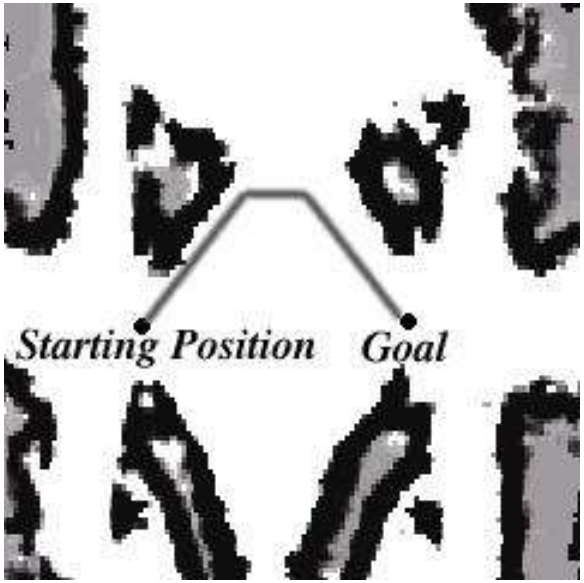
return $\arg\max_u \left[\hat{\mathcal{R}}(u, \bar{b}) + \sum_{\bar{b}'} \hat{V}(\bar{b}') \, \hat{\mathcal{P}}(\bar{b}, u, \bar{b}') \right]$

Coastal Navigation

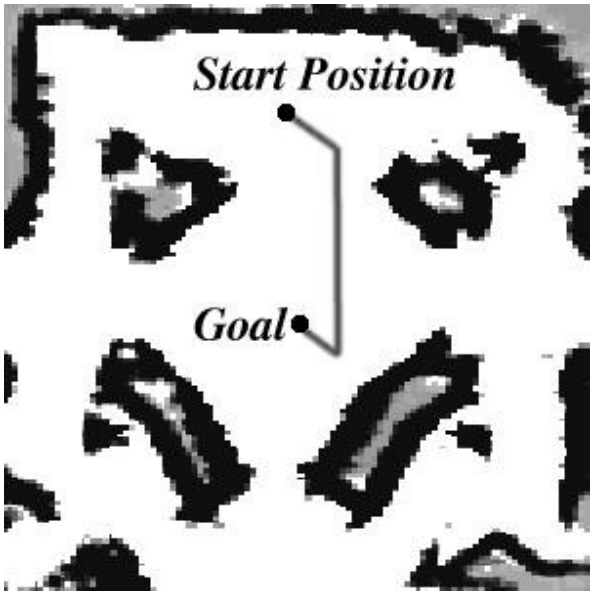


No measure uncertainty

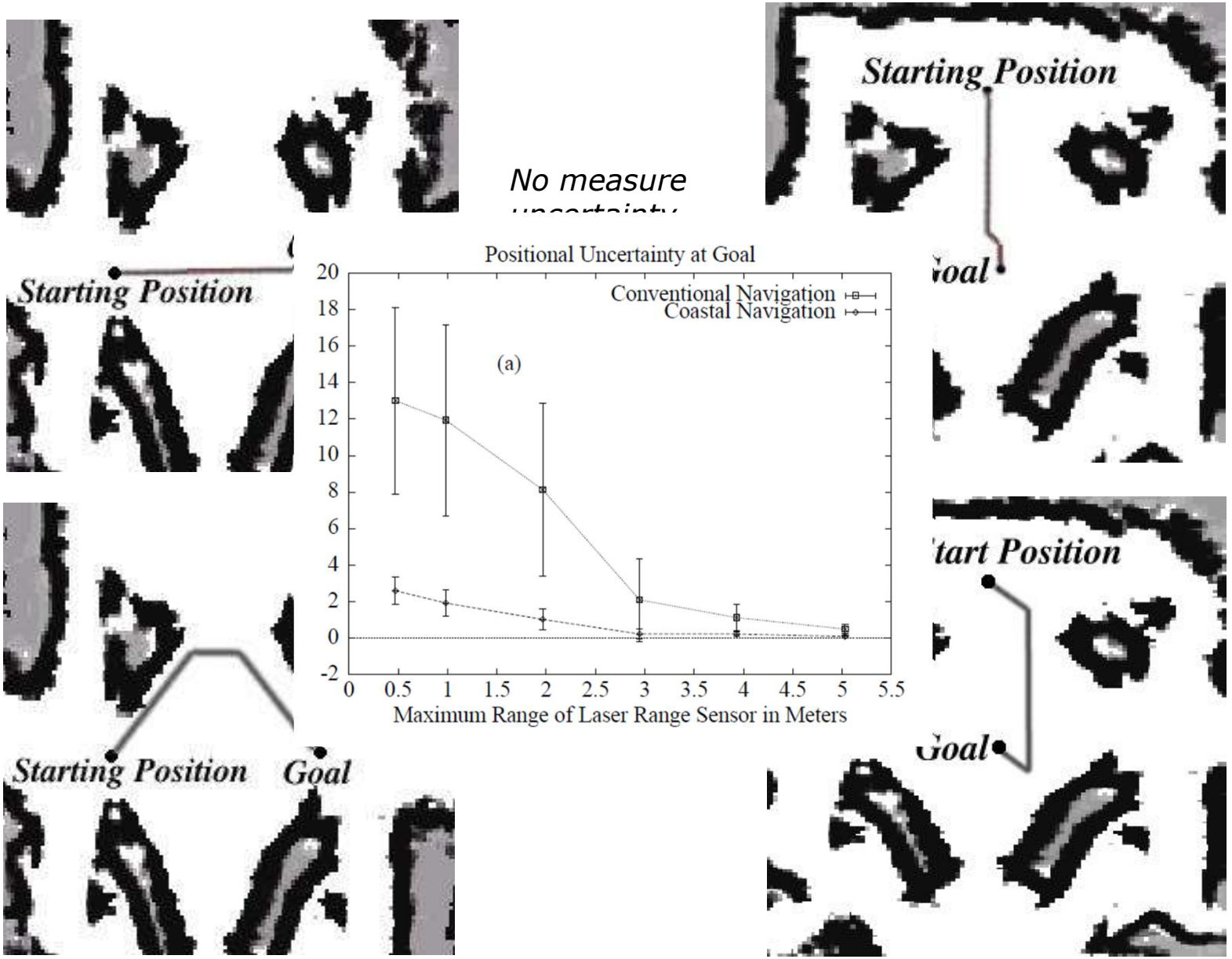




Uncertainty measure

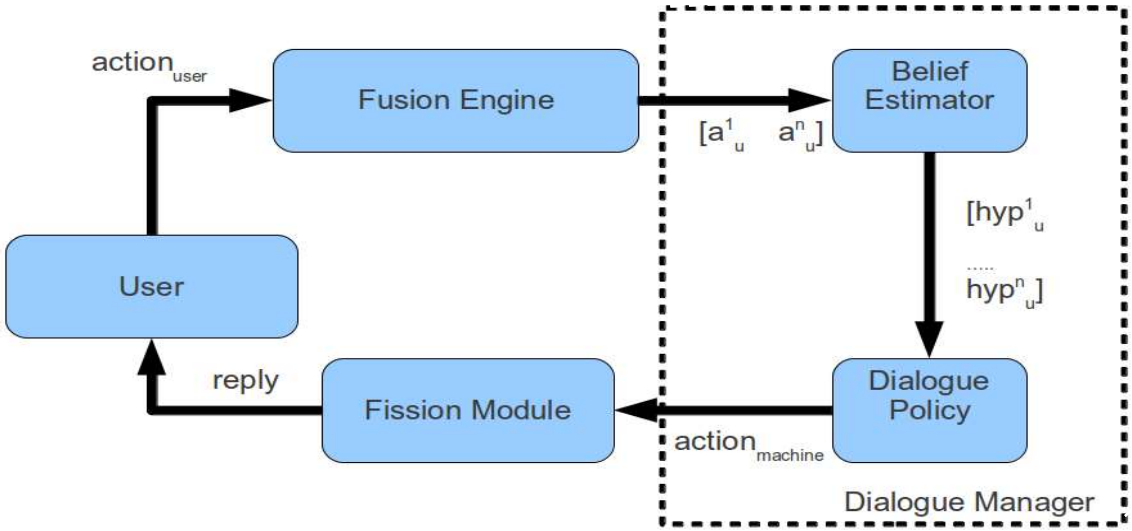


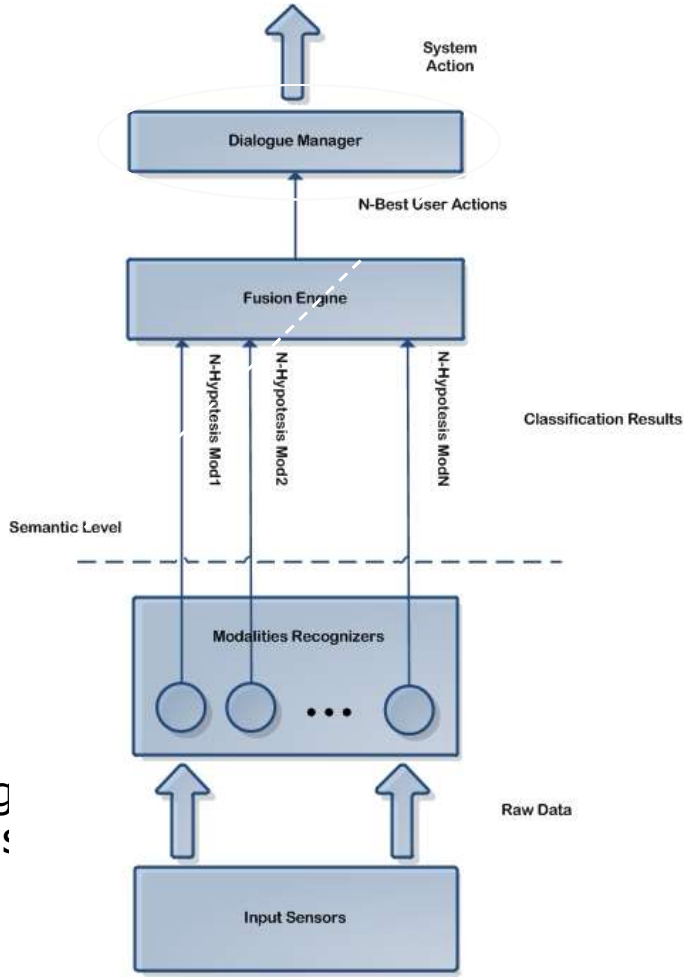
Coastal Navigation



Multimodal Communication

- Dialogue manager



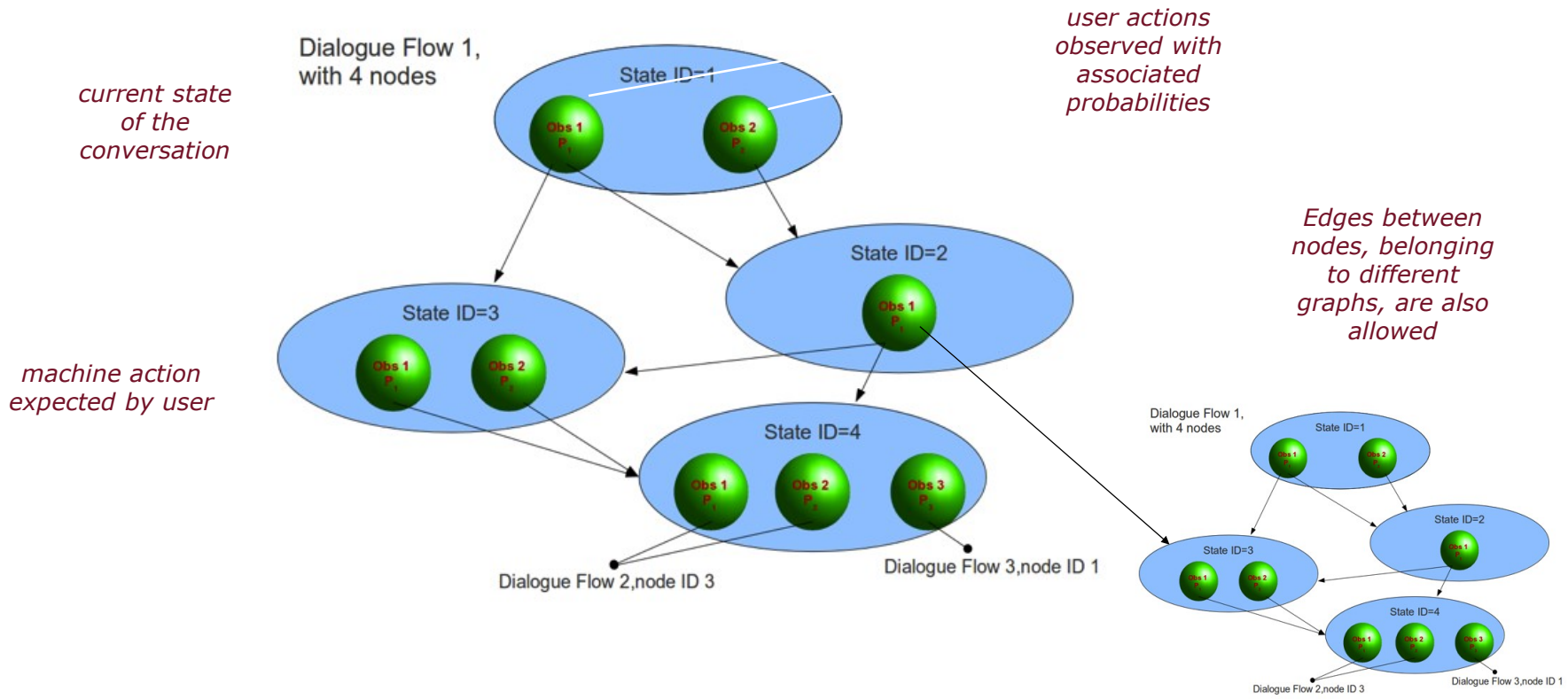


- Dialogue state estimation according to the interaction
- User intentions recognition from context and disambig multiple hypotheses arising due to noisy or ambiguous
- Dialogue coordination and action execution

A Dialogue System for Multimodal Human-Robot Interaction , L. Lucignano, F. Cutugno, S. Rossi, A. Finzi, In Proc. ACM International Conference on Multimodal Interaction - ICMI 2013

Multimodal Communication

- Dialogue manager
  - The system is provided with a set of interaction models named "dialogue flows", which describe how the dialogue can develop

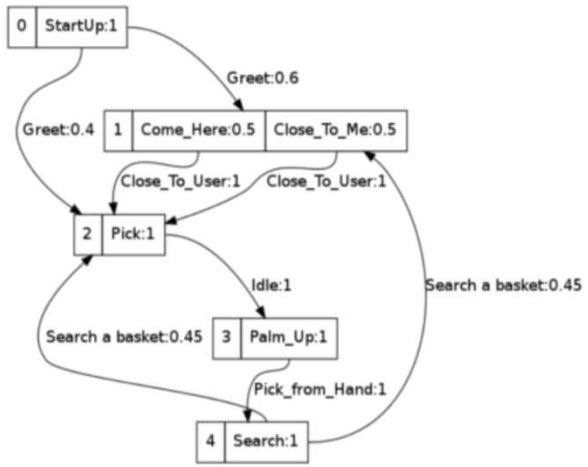


Multimodal Communication

- Dialogue manager
  - The system is provided with a set of interaction models named “dialogue flows”, which describe how the dialogue can develop

XML description of a dialogue flow

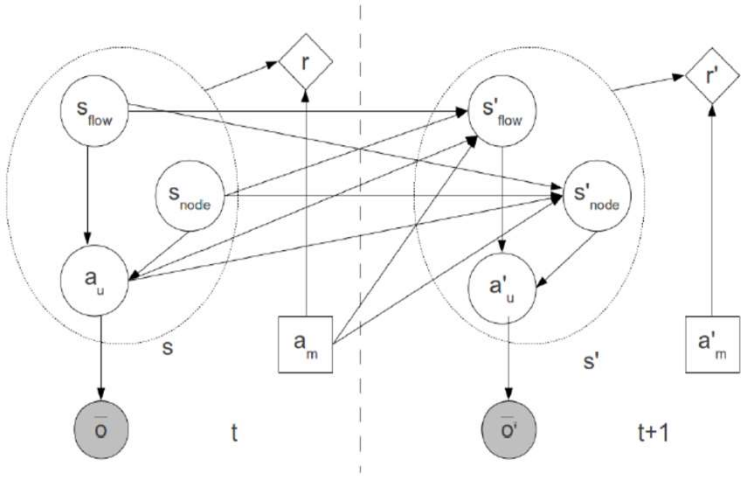
```
<?xml version="1.0" encoding="UTF-8"?>
<Dialog label="Scenario1">
    <Node ID="0" label="Start" startingP="1">
                        <Observ name="StartUp" P="1">
                            <MachineAction name="Greet">
                                <Link_To ID="1">0.6</Link_To>
                                <Link_To ID="2">0.4</Link_To>
                            </MachineAction>
                        </Observ>
    </Node>
    <Node ID="1" label="Robot is far, user wants it closer" >
                        <Observ name="Come_Here" P="0.5">
                            <MachineAction name="Close_To_User">
                                <Link_To ID="2">1</Link_To>
                          </MachineAction>
                        </Observ>
                        <Observ name="Close_To_Me" P="0.5">
                            <MachineAction name="Close_To_User">
                                <Link_To ID="2">1</Link_To>
                          </MachineAction>
                        </Observ>

    </Node>
     <Node ID="2" label="User ask to pick" >
                        <Observ name="Pick" P="1">
                            <MachineAction name="Idle">
                                <Link_To ID="3">1</Link_To>
                          </MachineAction>
                        </Observ>
    </Node>
    <Node ID="3" label="User's palm is up with a ball " >
                        <Observ name="Palm_Up" P="1">
                            <MachineAction name="Pick_from_Hand">
                                <Link_To ID="4">1</Link_To>
                          </MachineAction>
                        </Observ>
    </Node>
    <Node ID="4" label="Search a basket to place the ball" >
                        <Observ name="Search" P="1">
                            <MachineAction name="Search a basket">
                                <Link_To ID="1">0.45</Link_To>
                                <Link_To ID="2">0.45</Link_To>
                            </MachineAction>
                        </Observ>
```



Multimodal Communication

- Dialogue manager
  - The Dialogue is represented by a Partially Observable Markov Decision Problem [Young10, Jurafsky00] extended to the multimodal case [Lucignano et al. 2013]
  - POMDP state is a tuple

$$\left(\underbrace{s_{flow}}_{\text{dialogue flow ID}} , \underbrace{s_{node}}_{\text{flow state ID}} , \underbrace{a_u}_{\text{last user's action}} \right)$$



  - POMDP soved using approximation methods:
    - **Point Based Value Iteration** [Pineau et al. 2003], that approximates the value function only at a finite set of belief points
    - **Augmented MDP**, that performs the optimization in a summary space rather than in the original space [Roy et al. 2000]

Monte Carlo POMDPs

- POMDPs with continuous state and action spaces
- Represent beliefs by samples
- Estimate value function on sample sets
- Simulate control and observation transitions between beliefs

1: **Algorithm MC-POMDP(b_0, V):**
2: *repeat until convergence*
3: *sample* $x \sim b(x)$ // *initialization*
4: *initialize* $\mathcal{X}$ *with* M *samples of* $b(x)$
5: *repeat until episode over*
6: *for all control actions* u *do* // *update value function*
7: $Q(u) = 0$
8: *repeat* n *times*
9: *select random* $x \in \mathcal{X}$
10: *sample* $x' \sim p(x' \mid u, x)$
11: *sample* $z \sim p(z \mid x')$
12: $\mathcal{X}' = \textbf{Particle\_filter}(\mathcal{X}, u, z)$
13: $Q(u) = Q(u) + \frac{1}{n}\,\gamma\,[r(x,u) + V(\mathcal{X}')]$
14: *endrepeat*
15: *endfor*
16: $V(\mathcal{X}) = \max_u Q(u)$ // *update value function*
17: $u^* = \operatorname{argmax}_u Q(u)$ // *select greedy action*
18: *sample* $x' \sim p(x' \mid u, x)$ // *simulate state transition*
19: *sample* $z \sim p(z \mid x')$
20: $\mathcal{X}' = \textbf{Particle\_filter}(\mathcal{X}, u, z)$ // *compute new belief*
21: *set* $x = x'$; $\mathcal{X} = \mathcal{X}'$ // *update state and belief*
22: *endrepeat*
23: *endrepeat*
24: *return* V