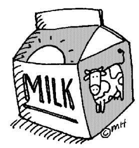
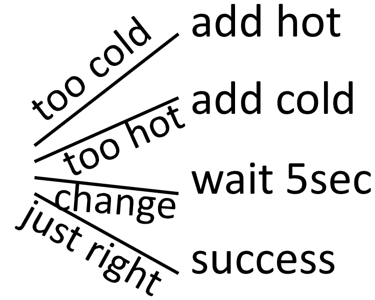
Real-world behavior is hierarchical

1. pour coffee 
2. add sugar 
3. add milk 
4. stir

1. set water temp
2. get wet
3. shampoo
4. soap
5. turn off water
6. dry off



Hierarchical Reinforcement Learning

- Exploits domain structure to facilitate learning
  - Policy constraints
  - State abstraction

- Paradigms:
  - Options
  - HAMs
  - MaxQ
  - …

Advantages of HRL

1. Faster learning
   (mitigates scaling problem)
2. Structured exploration
   (explore with sub-policies rather than primitive actions)
3. Transfer of knowledge from previous tasks
   (generalization, shaping)

Semi-Markov Decision Process

- Generalizes MDPs
- Action **a** takes **N** steps to complete in **s**
- P(**s'**,**n** | **a**, **s**), R(**s'**, **N** | **a**, **s**)
- Bellman equation:

$$V^{\pi}(s) = \sum_{s',N} P(s', N|s, \pi(s)) \left[R(s', N|s, \pi(s)) + \gamma^N V^{\pi}(s') \right].$$

$$V^{\pi}(s) = \overline{R}(s, \pi(s)) + \sum_{s',N} P(s', N|s, \pi(s)) \gamma^N V^{\pi}(s').$$

Semi-Markov Decision Process

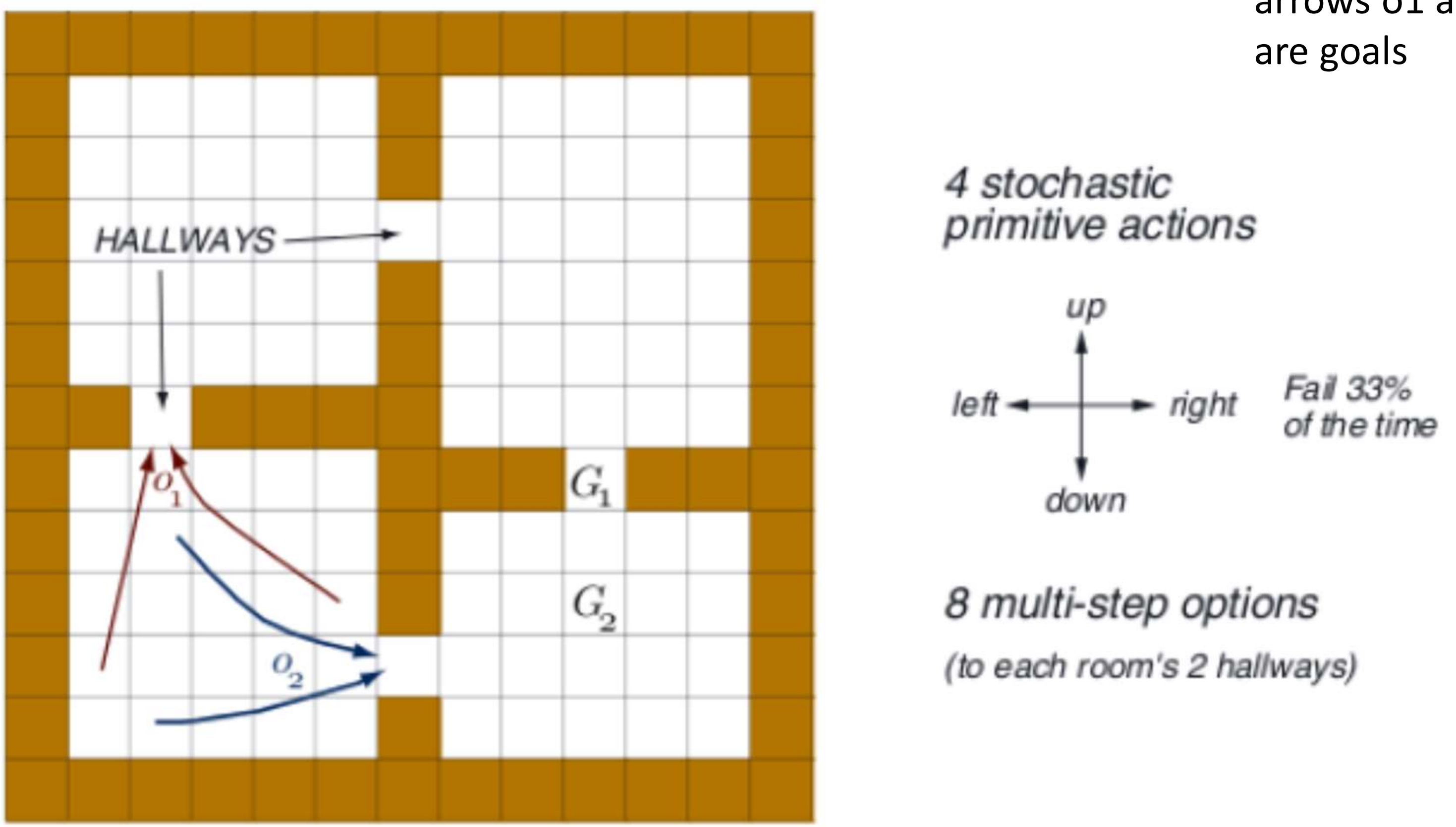
- Generalizes MDPs
- Action **a** takes **N** steps to complete in **s**
- P(**s'**,**n** | **a**, **s**), R(**s'**, **N** | **a**, **s**)
- Bellman equation:

$$V^*(s) = \max_a \left[R(s,a) + \sum_{s',\tau} \gamma^\tau p(s',\tau|s,a) V^*(s') \right]$$

$$Q^*(s,a) = R(s,a) + \sum_{s',\tau} \gamma^\tau p(s',\tau|s,a) \max_b Q^*(s',b)$$

Room Example

Gridworld environment with stochastic cell-to-cell actions and room-to-room hallway options. Two of the hallway options are suggested by the arrows o1 and o2. G1 and G2 are goals



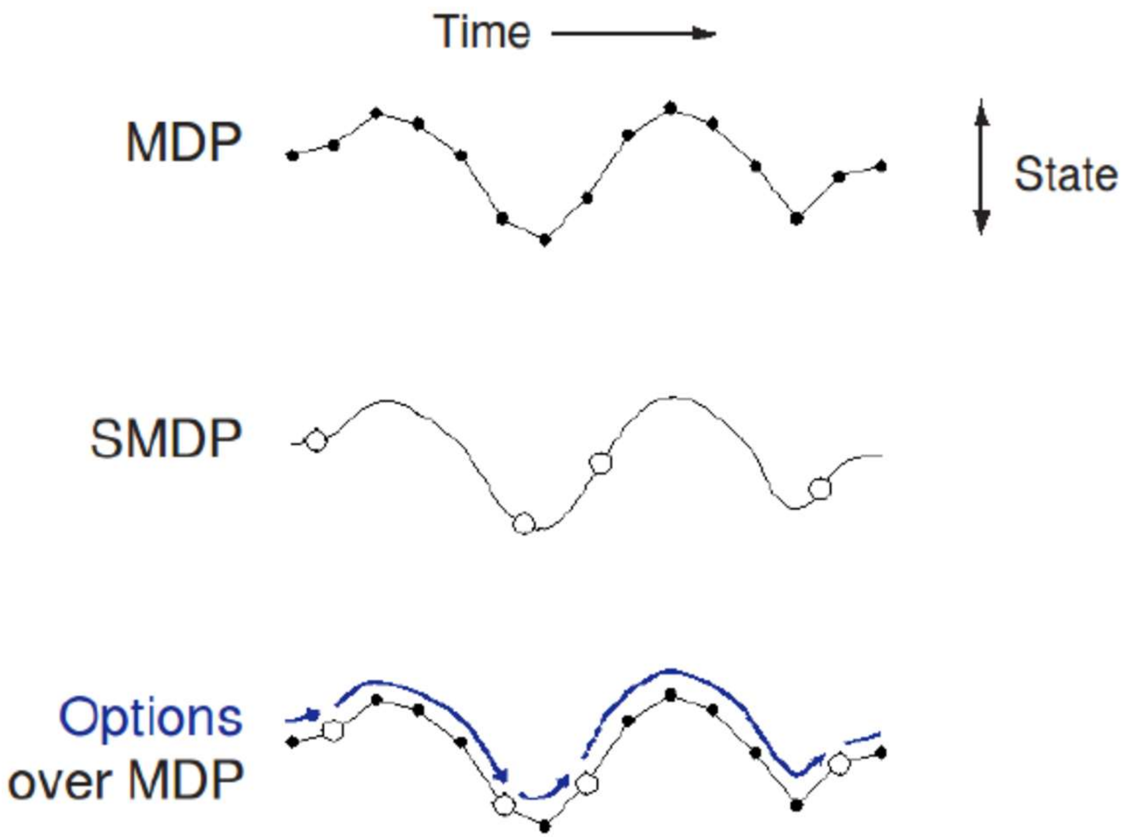
Sutton, Precup, Singh, 1999

Options

An option is a triple $o = < \mathcal{I}, \pi, \beta >$

- $\mathcal{I}$: initiation set. preconditions
- $\pi : \mathcal{S} \times \mathcal{A} \mapsto [0,1]$: option's policy behavior
- $\beta : \mathcal{S} \mapsto \left[0,1\right]$: termination condition effect

Options



Options

option's policy: π_i; global policy: μ

– reward part of option:

$$r(s,o) = \mathsf{E}\left\{r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + ... + \gamma^k r_{t+k} | o, s_t = s\right\}$$

– prediction-state part:

$$p(s'|s,o) = \sum_{k=1}^{\infty} p(s',k|s,o)\gamma^k$$

policy over options $\mu : \mathcal{S} \times \mathcal{O} \to [0,1]$

$$\begin{aligned} V^\mu(s) &= \mathsf{E}\left\{r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} .... | \mu, s_t = s\right\} \\ &= \mathsf{E}\left\{r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + ... + \gamma^{k-1} r_{t+k} + \gamma^k V^\mu(s_{t+k}) | \mu, s_t = s\right\} \\ &= \mathsf{E}\left[r(s,o) + \sum_{s_{t+k}} p(s_{t+k}|s,o) V^\mu(s') | \mu, s_t = s\right] \end{aligned}$$

Options

option's policy: π_i; global policy: μ

– reward part of option:

$$r(s,o) = \mathsf{E}\left\{ r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + ... + \gamma^k r_{t+k} | o, s_t = s \right\}$$

– prediction-state part:

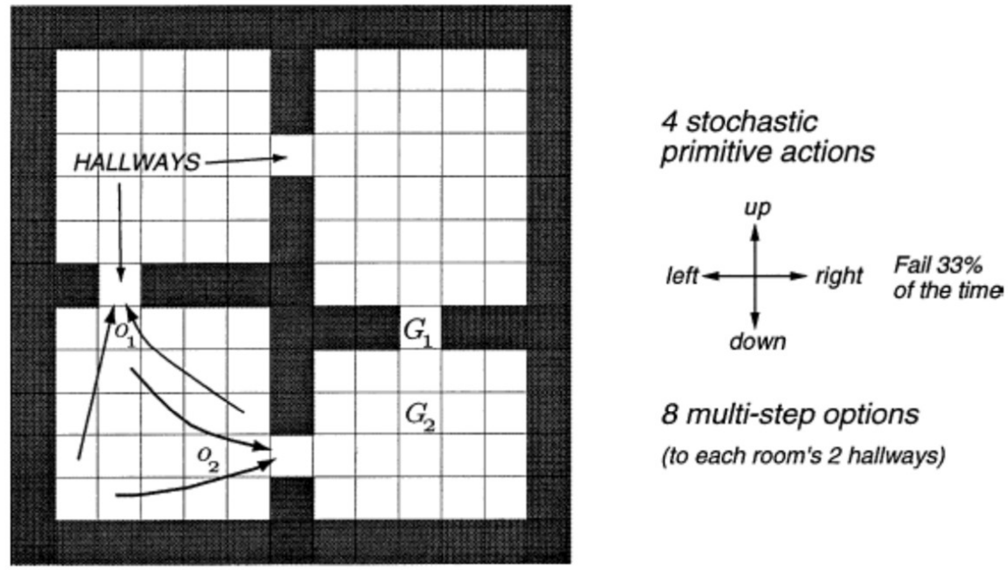
$$p(s'|s,o) = \sum_{k=1}^{\infty} p(s',k|s,o)\gamma^k$$

policy over options $\mu : \mathcal{S} \times \mathcal{O} \to [0,1]$

$$\begin{aligned}
Q^{\mu}(s,o) &= \mathsf{E}\left\{ r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} .... | o\mu, s_t = s \right\} \\
&= \mathsf{E}\left\{ r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + ... + \gamma^{k-1} r_{t+k} + \gamma^k V^{\mu}(s_{t+k}) | \mu, s_t = s \right\} \\
&= \mathsf{E}\left\{ r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + ... + \gamma^{k-1} r_{t+k} \right. \\
&\left. + \max_{o'} \mu(s_{t+k}, o') Q^{\mu}(s_{t+k}, o') | o\mu, s_t = s \right\} \\
&= \mathsf{E}\left\{ r(s,o) + \sum p(s_{t+k}|s,o) \max_{o'} \mu(s_{t+k}, o') Q^{\mu}(s_{t+k}, o') \right\}
\end{aligned}$$

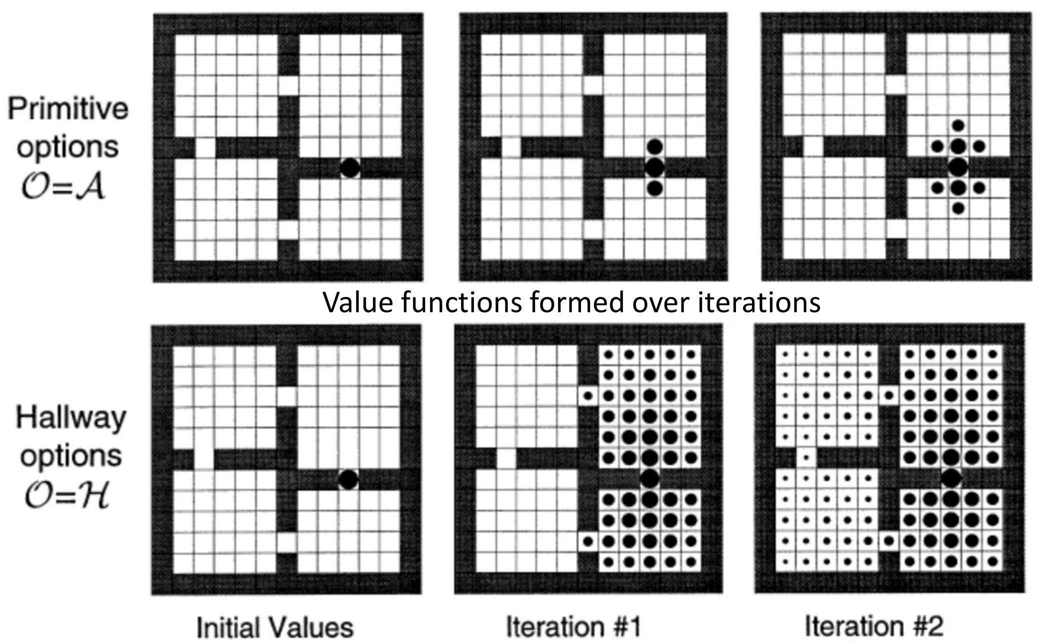
Options

Gridworld environment with stochastic cell-to-cell actions and room-to-room hallway options. Two of the hallway options are suggested by the arrows o1 and o2. G1 and G2 are goals

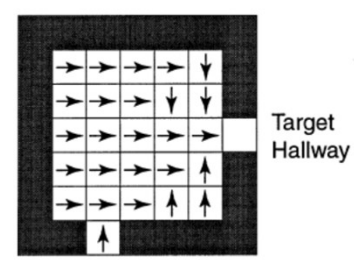


hallway options enables planning to proceed room-by-room rather than cell-by-cell.

The area of the disk is proportional to the estimated value of the state.

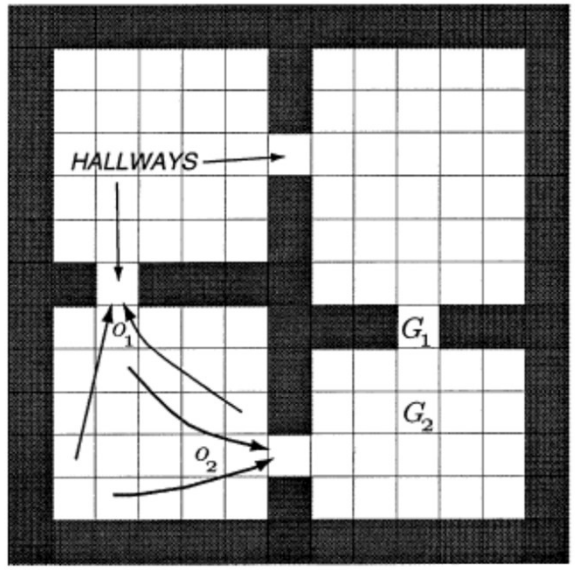


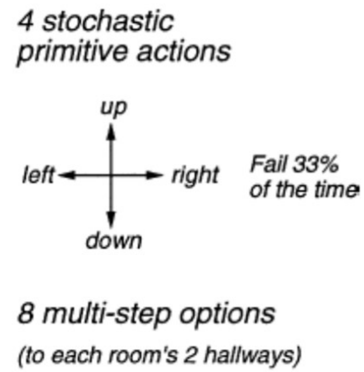
hallway options take the agent from anywhere within the room to one of the two hallway cells leading out of the room



Options

Gridworld environment with stochastic cell-to-cell actions and room-to-room hallway options. Two of the hallway options are suggested by the arrows o1 and o2. G1 and G2 are goals

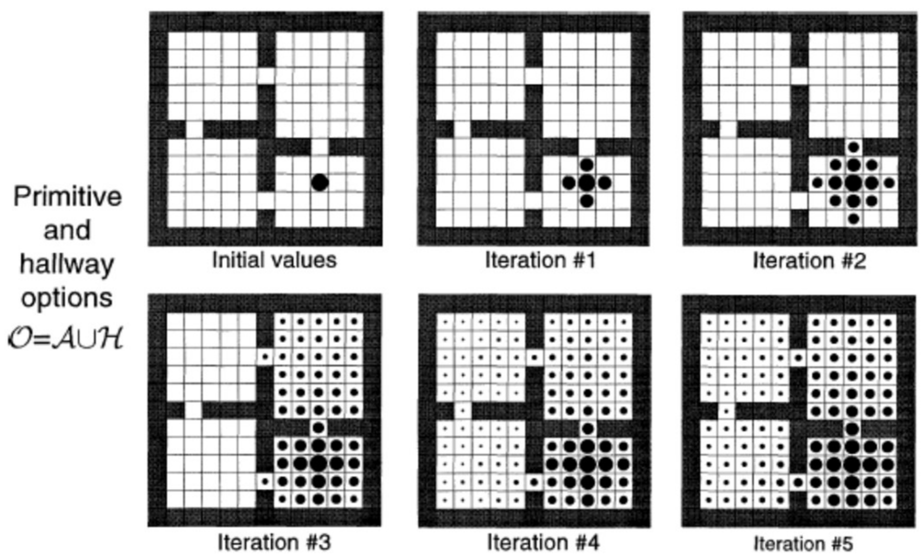




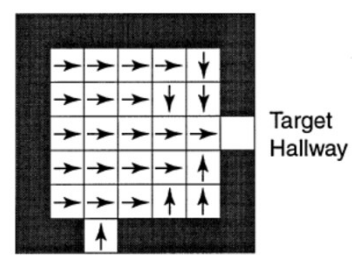
Second goal

hallway options enables planning to proceed room-by-room rather than cell-by-cell.

The area of the disk is proportional to the estimated value of the state.



hallway options take the agent from anywhere within the room to one of the two hallway cells leading out of the room

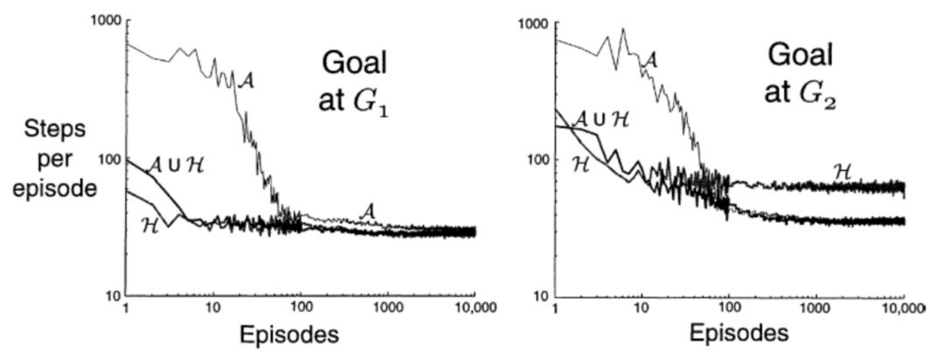


Options

SMDP Q-learning: given the set of defined options.
– execute the current selected option (e.g use epsilon greedy $Q(s, o)$) to termination.
– compute $r(s_t, o)$, then update $Q(s_t, o)$ as Q-learning/SARSA.

$$Q(s,o) \leftarrow Q(s,o) + \alpha\Big[r + \gamma^k \max_{o' \in \mathcal{O}_{s'}} Q(s',o') \quad Q(s,o)\Big]$$

If primitive actions are included as options, then optimal with options is like optimal without options



$\alpha = \frac{1}{8}$ except with H, G_1 ($\alpha = \frac{1}{16}$) and $A \cup H, G_2$ ($\alpha = \frac{1}{4}$)

Options

SMDP Q-learning: given the set of defined options.

– execute the current selected option (e.g use epsilon greedy $Q(s, o)$) to termination.

– compute $r(s_t, o)$, then update $Q(s_t, o)$ as Q-learning/SARSA.

Intra-option Q-learning: partially defined options

– after each primitive action, update all the options (off-policy learning).

– converge to correct values, "under same assumptions as 1-step Q-learning" (Sutton)

$$Q_{k+1}(s_t, o) = (1 - \alpha_k) Q_k(s_t, o) + \alpha_k \left[r_{t+1} + \gamma U_k(s_t, o) \right]$$

Markov Options with deterministic policies

where

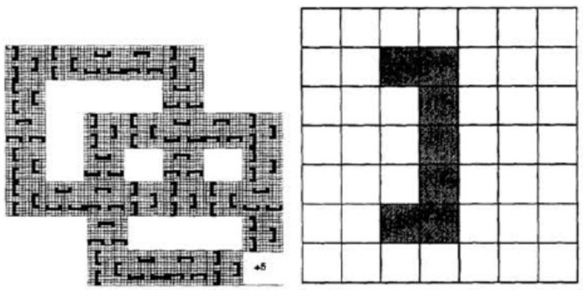
$$U_k(s, o) = (1 - \beta(s)) Q_k(s, o) + \beta(s) \max_{o' \in \mathcal{O}} Q_k(s, o')$$

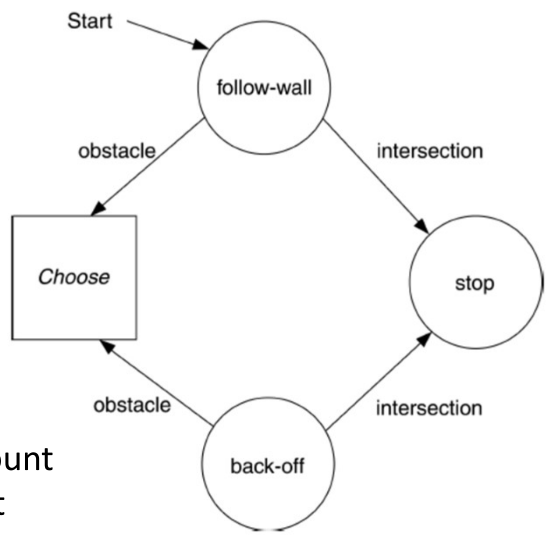
See [Sutton, Precup, Singh 1999]

Hierarchies of Abstract Machines

Partially specified Programs [Parr Russell 97]

- MDP State and Machine State (*hierarchical abstract machines HAMs*)





$$r_c \leftarrow r_c + \beta_c r$$
$$\beta_c \leftarrow \beta \beta_c$$

Accumulated reward and discount since the previous choice point

For each choice point:

$$Q([s_c, m_c], c) \leftarrow Q([s_c, m_c], c) + \alpha[r_c + \beta_c * V([t, n]) - Q([s_c, m_c], c)]$$
$$r_c \leftarrow 0$$
$$\beta_c \leftarrow 1$$

Theorem 1 For any MDP M and HAM H, let $\mathcal{C}$ be the set of choice points in $H \circ M$. Then there exists an MDP, which we will call *reduce*$(H \circ M)$, with states $\mathcal{C}$ such that the optimal policy for *reduce*$(H \circ M)$ corresponds to the optimal policy for M that is consistent with H.

Theorem 2 For any MDP M and any HAM H, HAMQ-learning will converge to the optimal action choice for every choice point in *reduce*$(H \circ M)$ with probability 1.

Task Hierarchy

MAXQ Task hierarchy [Dietterich 2000]

- Directed acyclic graph of subtasks
- Hierarchy of SMDS to be simultaneously learned
- Leaves are the primitive MDP actions

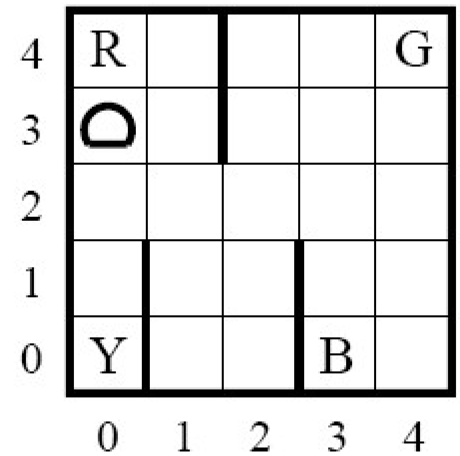
Traditionally, task structure is provided as prior knowledge to the learning agent

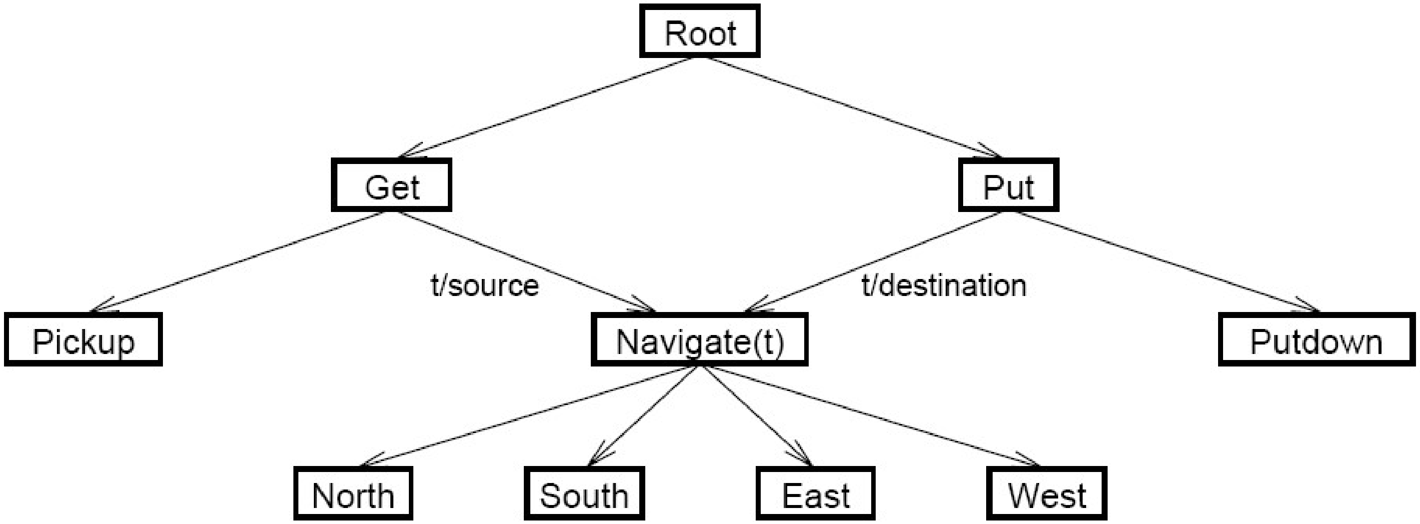
Each task associated with termination, Actions, and pseudo reward function: $\langle T_i, A_i, R_i \rangle$

Hierarchical policy is a set of policies, one for each subtask

Taxi Domain

- Motivational Example
- Reward: -1 actions, -10 illegal, 20 mission.
- 500 states
- Task Graph:





HSMQ Alg. (Task Decomposition)

function HSMQ(state s, subtask p) **returns** float
    Let $TotalReward = 0$
    while p is not terminated **do**
        Choose action $a = \pi_x(s)$ according to exploration policy π_x
        Execute a.
            if a is primitive, Observe one-step reward r
            else $r := HSMQ(s, a)$, which invokes subroutine a and
                returns the total reward received while a executed.
        $TotalReward := TotalReward + r$
        Observe resulting state s'
        Update $Q(p, s, a) := (1 - \alpha)Q(p, s, a) + \alpha \left[r + \max_{a'} Q(p, s', a')\right]$
    end // **while**
    return $TotalReward$
end

This algorithm converges to a recursively optimal policy for the original MDP provided that it is GLIE and the learning rates α suitably decreases

MAXQ Alg. (Value Fun. Decomposition)

- Compactness in the representation of the hierarchical value function (decomposition)
- Re-write Q(p, s, a) as

$$Q(p, s, a) = V(a, s) + C(p, s, a)$$

$$V(p, s) = \max_a [V(a, s) + C(p, s, a)]$$

where *V(a, s)* is the expected total reward while executing action a, and *C(p, s, a)* is the expected reward of completing parent task *p* after *a* has returned

Hierarchical Structure

- MDP decomposed in task M0, … , Mn

Theorem 1 *Given a task graph over tasks $M_0, \ldots, M_n$ and a hierarchical policy π, each subtask M_i defines a semi-Markov decision process with states S_i, actions A_i, probability transition function $P_i^\pi(s', N|s,a)$, and* expected *reward function $\overline{R}(s,a) = V^\pi(a,s)$, where $V^\pi(a,s)$ is the projected value function for child task M_a in state s. If a is a primitive action, $V^\pi(a,s)$ is defined as the expected immediate reward of executing a in s: $V^\pi(a,s) = \sum_{s'} P(s'|s,a)R(s'|s,a)$.*

- Q for the subtask i

$$Q^\pi(i,s,a) = V^\pi(a,s) + \sum_{s',N} P_i^\pi(s',N|s,a)\gamma^N Q^\pi(i,s',\pi(s')),$$

$$Q^\pi(i,s,a) = V^\pi(a,s) + C^\pi(i,s,a).$$

Value Decomposition

Definition 6 *The* completion function, $C^\pi(i, s, a)$, *is the expected discounted cumulative reward of* completing *subtask* M_i *after invoking the subroutine for subtask* M_a *in state* s. *The reward is discounted back to the point in time where* a *begins execution.*

$$C^\pi(i, s, a) = \sum_{s', N} P_i^\pi(s', N|s, a)\gamma^N Q^\pi(i, s', \pi(s')) \tag{9}$$

With this definition, we can express the Q function recursively as

$$Q^\pi(i, s, a) = V^\pi(a, s) + C^\pi(i, s, a). \tag{10}$$

Finally, we can re-express the definition for $V^\pi(i, s)$ as

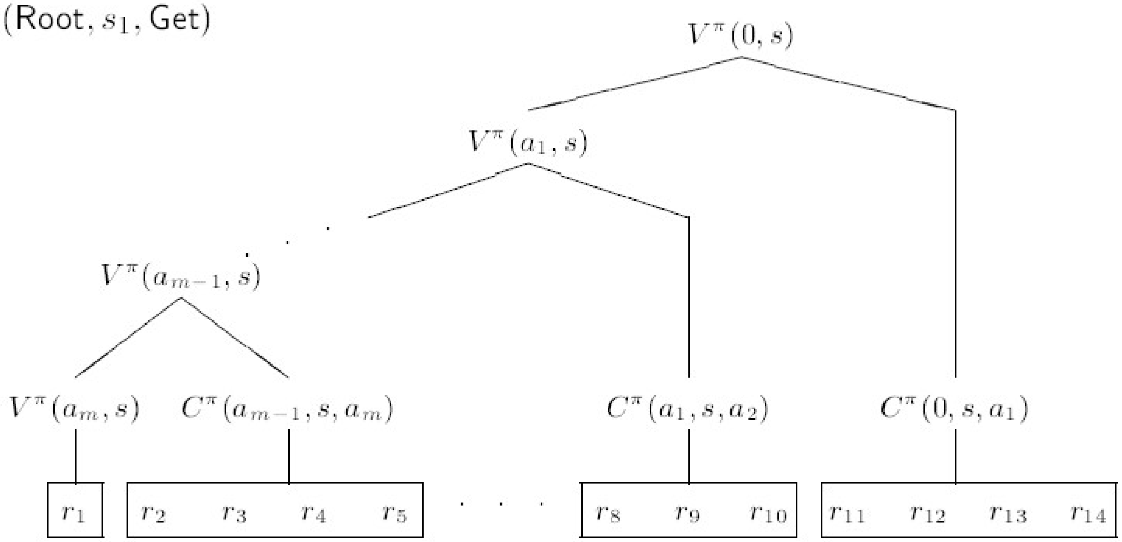
$$V^\pi(i, s) = \begin{cases} Q^\pi(i, s, \pi_i(s)) & \text{if } i \text{ is composite} \\ \sum_{s'} P(s'|s, i)R(s'|s, i) & \text{if } i \text{ is primitive} \end{cases} \tag{11}$$

Value Decomposition

- The value function can be decomposed as follows

$$V^\pi(0,s) = V^\pi(a_m,s) + C^\pi(a_{m-1},s,a_m) + \ldots + C^\pi(a_1,s,a_2) + C^\pi(0,s,a_1)$$

$$\begin{aligned} V^\pi(\text{Root},s_1) &= V^\pi(\text{North},s_1) + C^\pi(\text{Navigate}(R),s_1,\text{North}) + \\ &\quad C^\pi(\text{Get},s_1,\text{Navigate}(R)) + C^\pi(\text{Root},s_1,\text{Get}) \\ &= -1 + 0 + -1 + 12 \\ &= 10 \end{aligned}$$



MAXQ Alg.

- An example

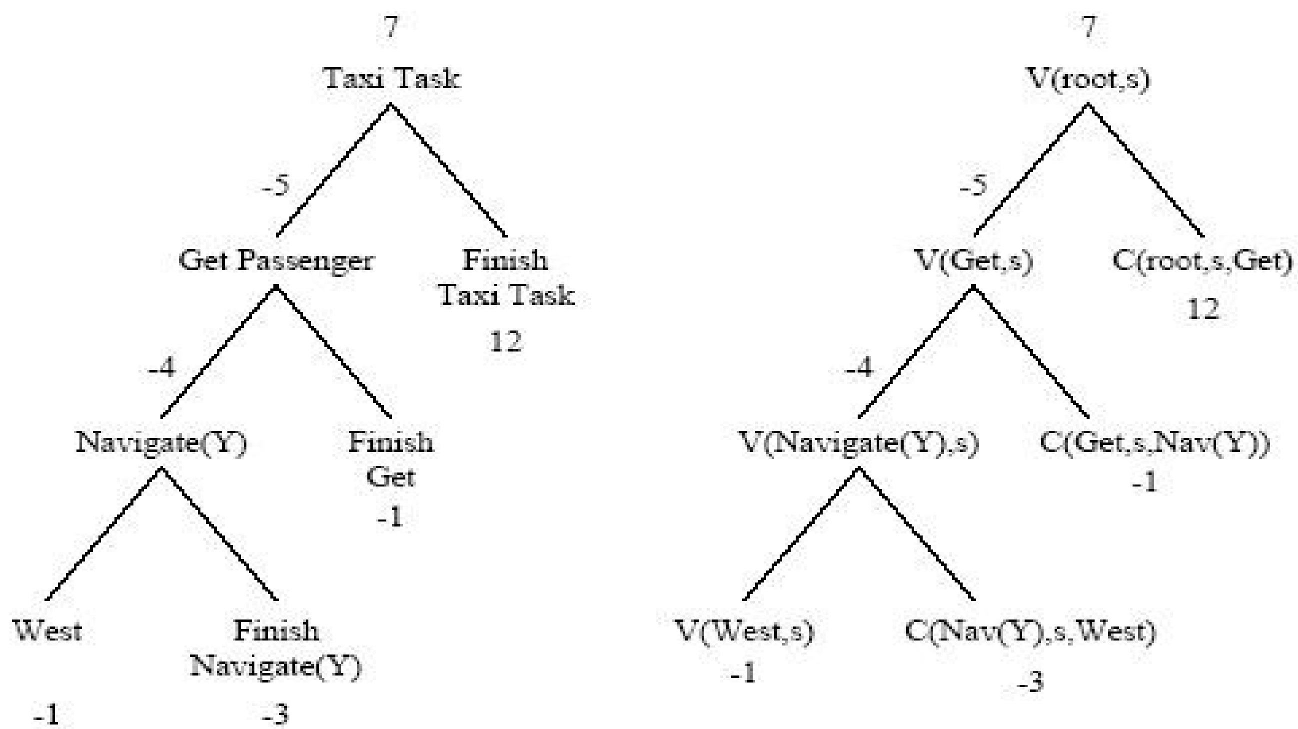


Fig. 5. An example of the MAXQ value function decomposition for the state in which the taxi is at location (2,2), the passenger is at (0,0), and wishes to get to (3,0). The left tree gives English descriptions, and the right tree uses formal notation.

MAXQ Alg.

$$
\begin{aligned}
V(\text{root}, s) = V(\text{west}, s) &+ C(\text{navigate}(Y), s, \text{west}) \\
&+ C(\text{get}, s, \text{navigate}(Y)) \\
&+ C(\text{root}, s, \text{get}).
\end{aligned}
$$

Passenger at Y

	0	1	2	3	4
4	10	9	8	7	6
3	11	10	9	8	7
2	12	11	10	9	8
1	13	10	9	8	7
0	14	9	8	7	6

Passenger In Taxi

	0	1	2	3	4
4	G	18	13	12	11
3	18	17	14	13	12
2	17	16	15	14	13
1	16	15	14	13	12
0	15	14	13	12	11

Fig. 4. Value function for the case where the passenger is at (0,0) (location Y) and wishes to get to (0,4) (location R).

MAXQQ Alg.

function MAXQQ(state s, subtask p) **returns** float
Let $TotalReward = 0$
while p is not terminated **do**
Choose action $a = \pi_x(s)$ according to exploration policy π_x
Execute a.
if a is primitive, Observe one-step reward r
else $r := MAXQQ(s, a)$, which invokes subroutine a and returns the total reward received while a executed.
$TotalReward := TotalReward + r$
Observe resulting state s'
if a is a primitive
$V(a, s) := (1 - \alpha)V(a, s) + \alpha r$
else a is a subroutine
$C(p, a, s) := (1 - \alpha)C(p, s, a) + \alpha \max_{a'} [V(a', s') + C(p, s', a')]$
end // **while**
return $TotalReward$
end

This algorithm converges to a recursively optimal policy for the original MDP provided that it is GLIE and the learning rates α suitably decreases

Optimality in HRL

Hierarchically optimal vs. recursively optimal

- Hierarchical optimality: The learnt policy is the best policy consistent with the given hierarchy. Task's policy depends not only on its children's policies, but also on its context.

- Recursive optimality: The policy for a parent task is optimal given the learnt policies of its children. (Context-free task's policy).