

# Robotica Probabilistica

**Filtri Bayesiani**

Filteri Gaussiani

# Filtri Bayesiani Reminder

- Predizione

$$\overline{bel}(x_t) = \int p(x_t | u_t, x_{t-1}) bel(x_{t-1}) dx_{t-1}$$

- Correzione

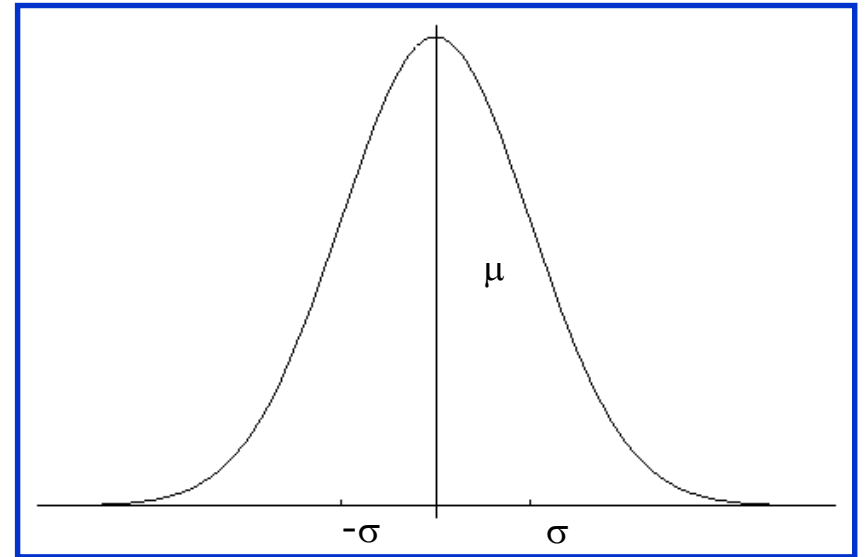
$$bel(x_t) = \eta p(z_t | x_t) \overline{bel}(x_t)$$

# Gaussiane

$$p(x) \sim N(\mu, \sigma^2):$$

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2}}$$

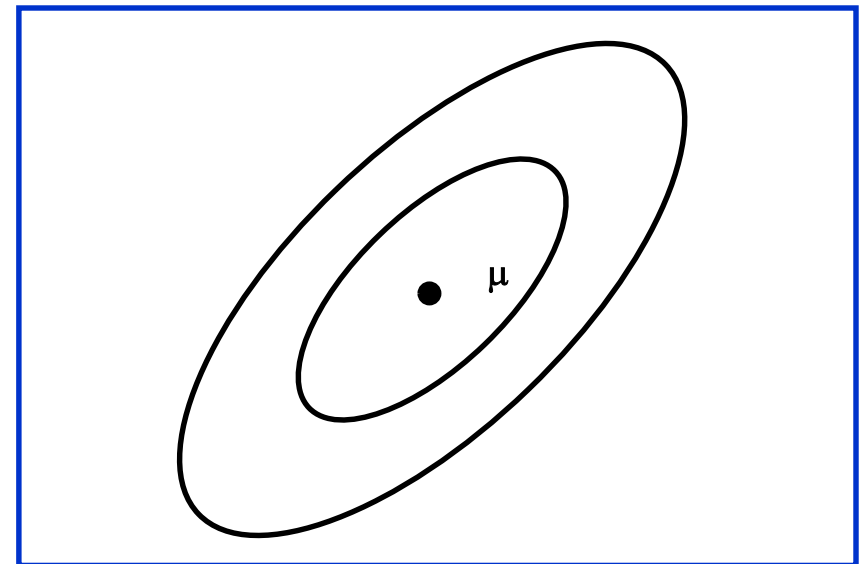
Univariata



$$p(\mathbf{x}) \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}):$$

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2} |\boldsymbol{\Sigma}|^{1/2}} e^{-\frac{1}{2} (\mathbf{x}-\boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{x}-\boldsymbol{\mu})}$$

Multivariata



# Propert  delle Gaussiane

$$\left. \begin{array}{l} X \sim N(\mu, \sigma^2) \\ Y = aX + b \end{array} \right\} \Rightarrow Y \sim N(a\mu + b, a^2\sigma^2)$$

$$\left. \begin{array}{l} X_1 \sim N(\mu_1, \sigma_1^2) \\ X_2 \sim N(\mu_2, \sigma_2^2) \end{array} \right\} \Rightarrow p(X_1) \cdot p(X_2) \sim N\left(\frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \mu_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \mu_2, \frac{1}{\sigma_1^{-2} + \sigma_2^{-2}}\right)$$

# Multivariate

$$\left. \begin{array}{l} X \sim N(\mu, \Sigma) \\ Y = AX + B \end{array} \right\} \Rightarrow Y \sim N(A\mu + B, A\Sigma A^T)$$

$$\left. \begin{array}{l} X_1 \sim N(\mu_1, \Sigma_1) \\ X_2 \sim N(\mu_2, \Sigma_2) \end{array} \right\} \Rightarrow p(X_1) \cdot p(X_2) \sim N\left(\frac{\Sigma_2}{\Sigma_1 + \Sigma_2} \mu_1 + \frac{\Sigma_1}{\Sigma_1 + \Sigma_2} \mu_2, \frac{1}{\Sigma_1^{-1} + \Sigma_2^{-1}}\right)$$

- Si rimane nel "mondo gaussiano" con trasformazioni lineari.

# Propert  delle Gaussiane

Dato:

$$x = (x_a \ x_b) \quad p(x) = N(\mu, \Sigma)$$

$$\mu = (\mu_a, \mu_b)$$

Le marginali sono Gaussiane:

$$\Sigma = \begin{pmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{pmatrix}$$

$$p(x_a) = N(\mu_a, \Sigma_{aa}),$$

$$p(x_b) = N(\mu_b, \Sigma_{bb})$$

Le condizionate sono Gaussiane:

$$p(x_a | x_b) = N(\mu_{a|b}, \Sigma_{a|b}),$$

$$p(x_b | x_a) = N(\mu_{b|a}, \Sigma_{b|a})$$

$$\mu_{a|b} = \mu_a + \Sigma_{ab} \Sigma_{bb}^{-1} (b - \mu_b)$$

$$\Sigma_{a|b} = \Sigma_{aa} - \Sigma_{ab} \Sigma_{bb}^{-1} \Sigma_{ba}$$

# Filtro di Kalman Discreto

Stima lo stato  $x$  di un processo a tempo discreto governato da una equazione alle differenze stocastica lineare

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

con una misura

$$z_t = C_t x_t + \delta_t$$

Rumore  
rappresentato come  
multivariata con  
valore medio zero

# Componenti di un Filtro di Kalman

 $A_t$ 

Matrice ( $n \times n$ ) che descrive l'evoluzione dello stato da  $t-1$  a  $t$  senza controllo o rumore.

 $B_t$ 

Matrice ( $n \times l$ ) che descrive come il controllo  $u_t$  cambia lo stato da  $t$  a  $t-1$ .

 $C_t$ 

Matrice ( $k \times n$ ) che descrive come mappare lo stato  $x_t$  in un'osservazione  $z_t$ .

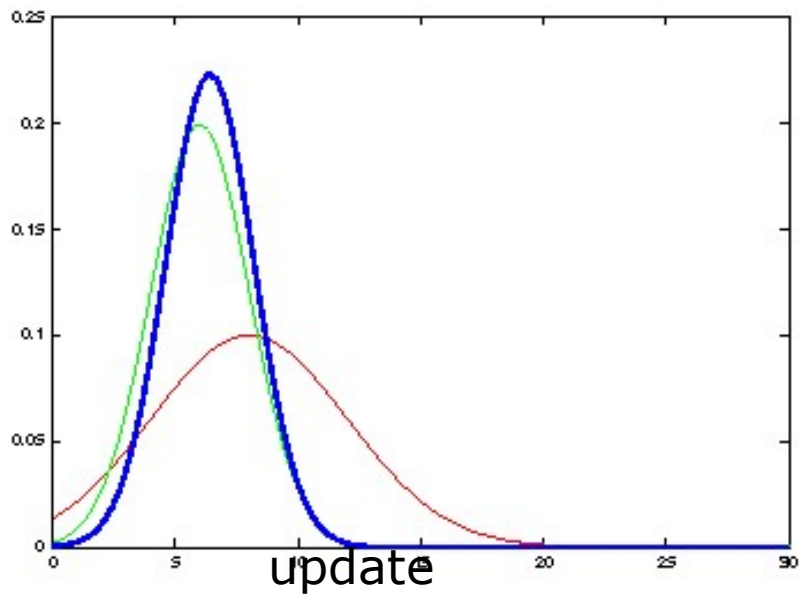
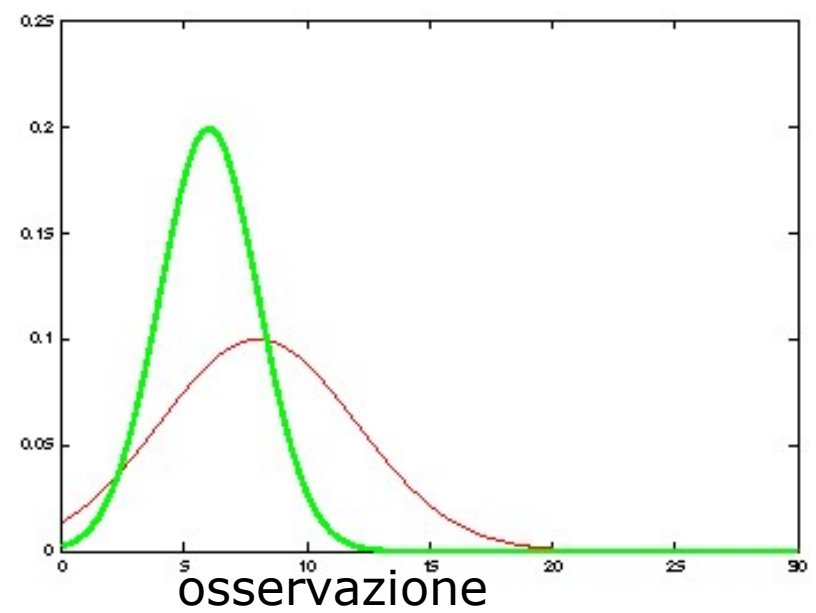
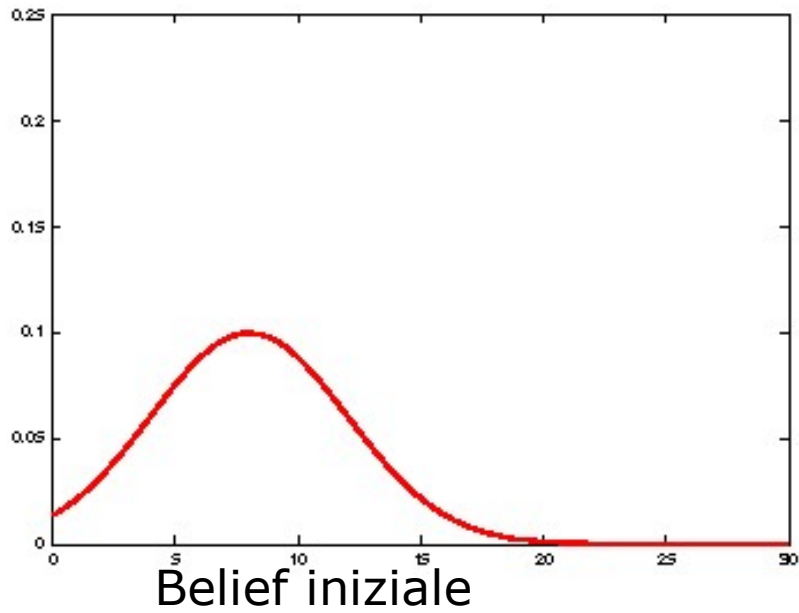
 $\varepsilon_t$ 

Variabili random che rappresentano il rumore di processo e misura. Assunte indipendenti e con distribuzione normale con covarianza  $R_t$  e  $Q_t$  rispettivamente.

 $\delta_t$



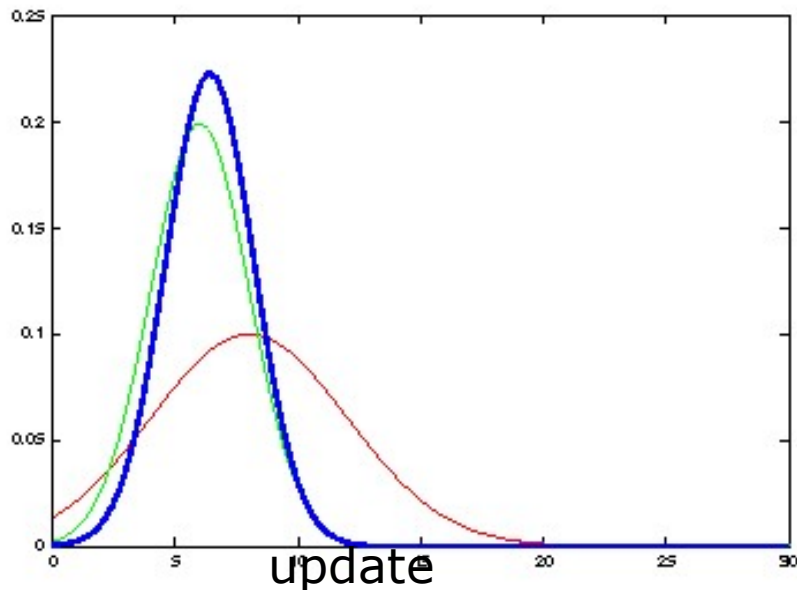
# Filtro di Kalman in 1D



# Aggiornamento Filtro di Kalman 1D

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - \bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t)\bar{\sigma}_t^2 \end{cases} \quad \text{with} \quad K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_tC_t)\bar{\Sigma}_t \end{cases} \quad \text{with} \quad K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

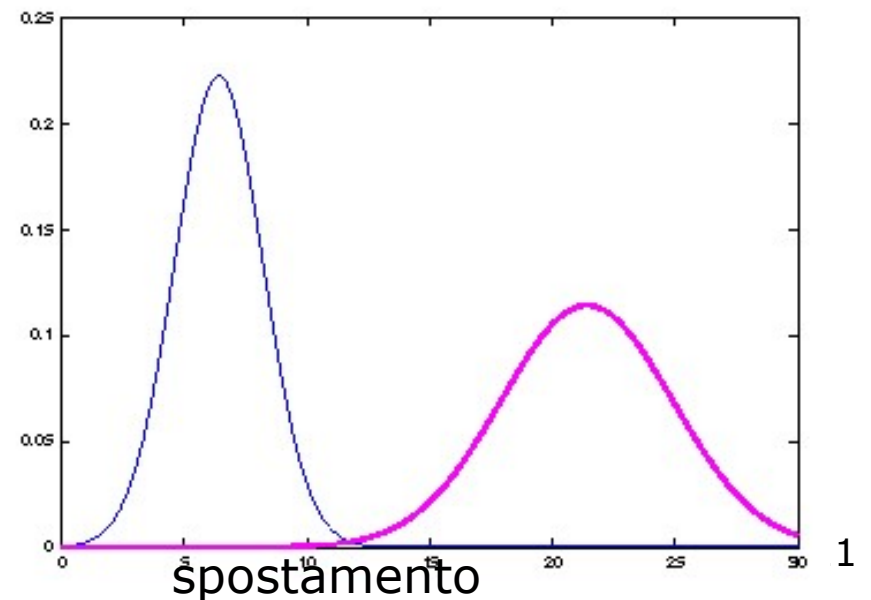
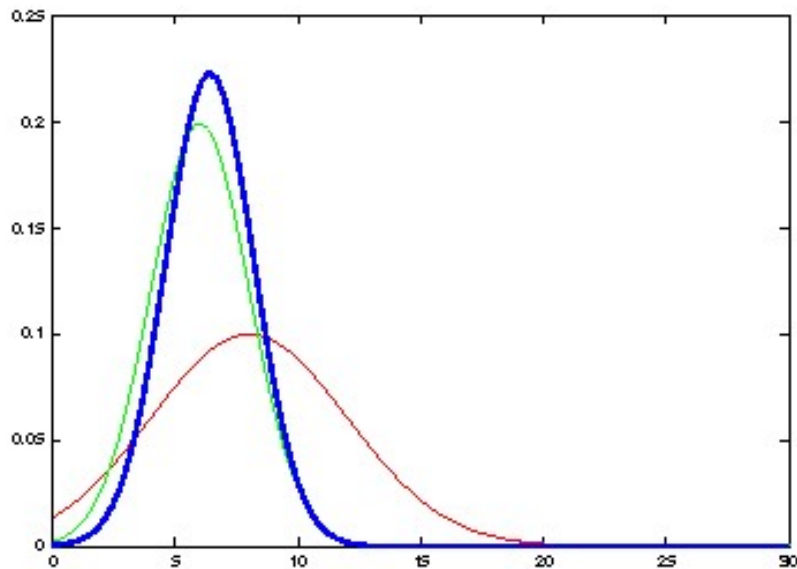


# Aggiornamento Filtro di Kalman 1D

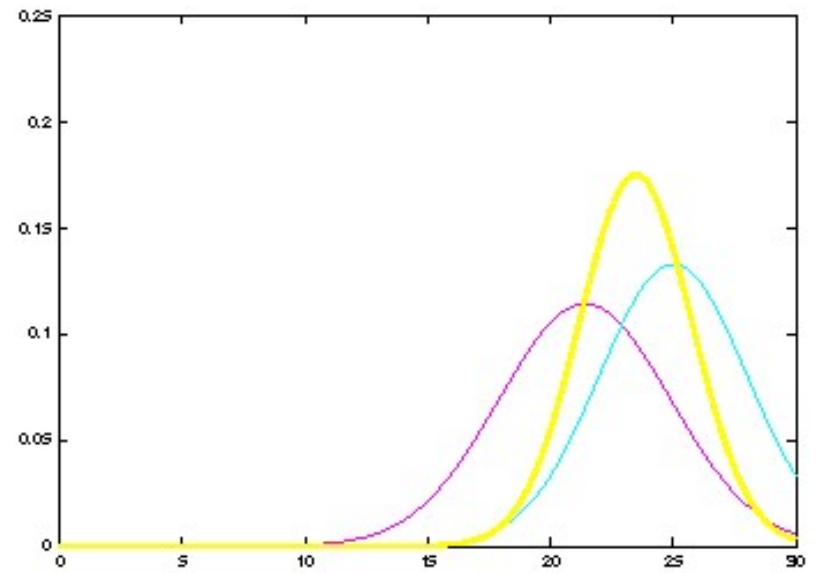
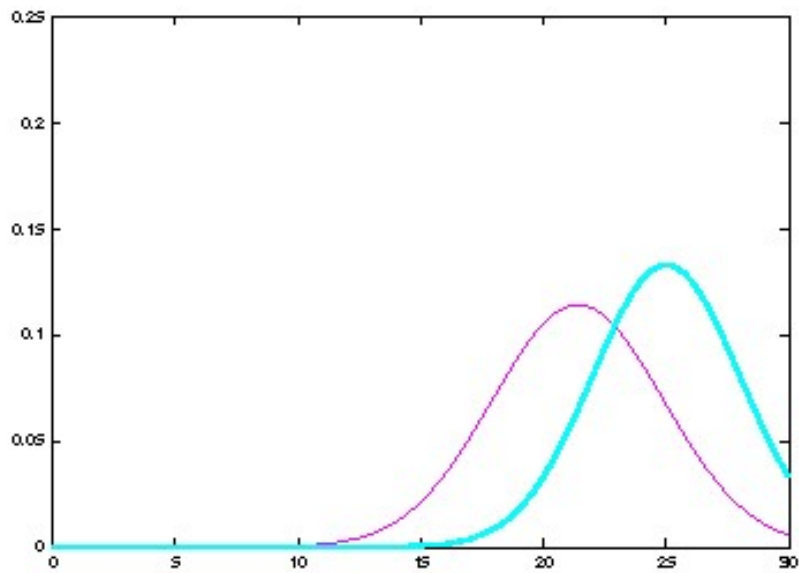
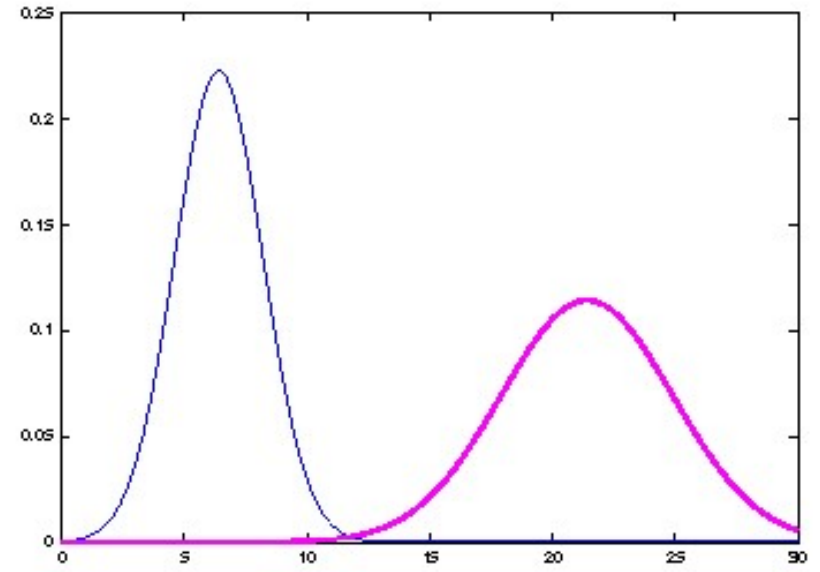
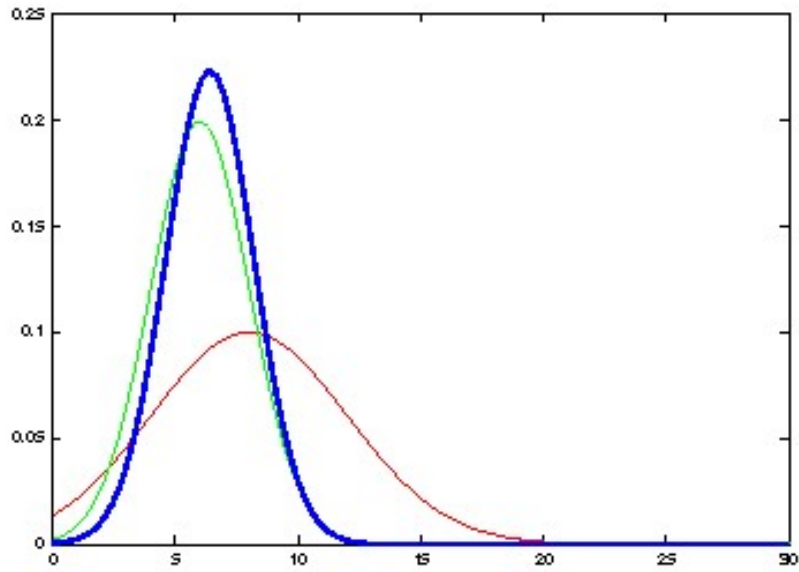
Spostamento

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_{t-1}^2 + \sigma_{act,t}^2 \end{cases}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$



# Aggiornamento Filtro Kalman



# Gaussiana Lineare: Initialization

- belief iniziale con distribuzione normale:

$$bel(x_0) = N(x_0; \mu_0, \Sigma_0)$$

# Gaussiana Lineare: Dinamica

- La dinamica è funzione lineare dello stato e del controllo più rumore additivo:

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

$$p(x_t | u_t, x_{t-1}) = N(x_t; A_t x_{t-1} + B_t u_t, R_t)$$

$$\begin{array}{ccc} \overline{bel}(x_t) = \int p(x_t | u_t, x_{t-1}) & & bel(x_{t-1}) dx_{t-1} \\ \Downarrow & & \Downarrow \\ \sim N(x_t; A_t x_{t-1} + B_t u_t, R_t) & \sim & N(x_{t-1}; \mu_{t-1}, \Sigma_{t-1}) \end{array}$$

# Gaussiana Lineare: Dinamica

$$\begin{aligned} \overline{bel}(x_t) &= \int p(x_t | u_t, x_{t-1}) \quad bel(x_{t-1}) dx_{t-1} \\ &\quad \Downarrow \qquad \qquad \qquad \Downarrow \\ &\sim N(x_t; A_t x_{t-1} + B_t u_t, R_t) \quad \sim N(x_{t-1}; \mu_{t-1}, \Sigma_{t-1}) \\ &\quad \Downarrow \\ \overline{bel}(x_t) &= \eta \int \exp\left\{-\frac{1}{2} (x_t - A_t x_{t-1} - B_t u_t)^T R_t^{-1} (x_t - A_t x_{t-1} - B_t u_t)\right\} \\ &\quad \exp\left\{-\frac{1}{2} (x_{t-1} - \mu_{t-1})^T \Sigma_{t-1}^{-1} (x_{t-1} - \mu_{t-1})\right\} dx_{t-1} \\ \overline{bel}(x_t) &= \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases} \end{aligned}$$

## Gaussiana Lineare: Osservazioni

- Le osservazioni sono funzioni lineari dello stata più rumore additivo:

$$z_t = C_t x_t + \delta_t$$

$$p(z_t | x_t) = N(z_t; C_t x_t, Q_t)$$

$$\begin{array}{ccc} \text{bel}(x_t) = \eta & p(z_t | x_t) & \overline{\text{bel}}(x_t) \\ & \Downarrow & \Downarrow \\ & \sim N(z_t; C_t x_t, Q_t) & \sim N(x_t; \overline{\mu}_t, \overline{\Sigma}_t) \end{array}$$



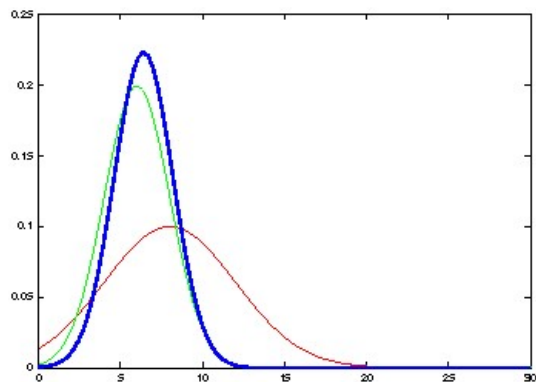
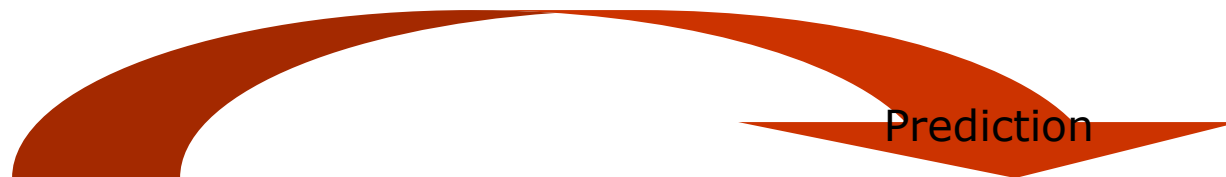
# Sistemi a Gaussiana Lineare: Osservazioni

$$\begin{aligned}
 bel(x_t) &= \eta \quad p(z_t | x_t) && \overline{bel}(x_t) \\
 &\quad \Downarrow && \Downarrow \\
 &\sim N(z_t; C_t x_t, Q_t) && \sim N(x_t; \bar{\mu}_t, \bar{\Sigma}_t) \\
 &\quad \Downarrow && \\
 bel(x_t) &= \eta \exp\left\{-\frac{1}{2}(z_t - C_t x_t)^T Q_t^{-1}(z_t - C_t x_t)\right\} \exp\left\{-\frac{1}{2}(x_t - \bar{\mu}_t)^T \bar{\Sigma}_t^{-1}(x_t - \bar{\mu}_t)\right\} \\
 \\
 bel(x_t) &= \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t \bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t) \bar{\Sigma}_t \end{cases} && \text{with } K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}
 \end{aligned}$$

# Kalman Filter Algorithm

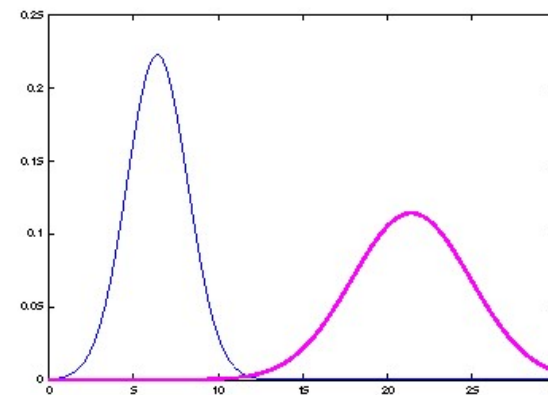
1. Algorithm **Kalman\_filter**(  $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$ ):
2. Prediction:
3.  $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$
4.  $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$
5. Correction:
6.  $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$
7.  $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$
8.  $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$
9. Return  $\mu_t, \Sigma_t$

# Il ciclo Prediction-Correction

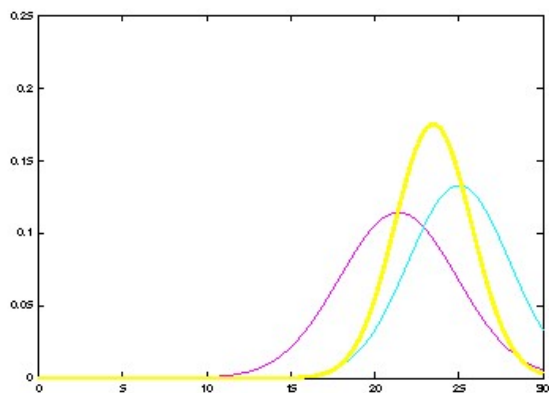


$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{act,t}^2 \end{cases}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$

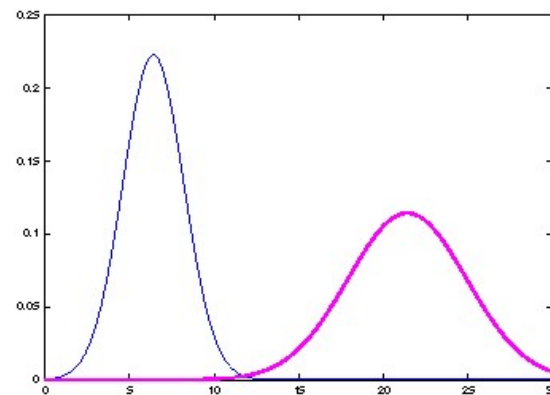


# The Prediction-Correction-Cycle



$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - \bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t)\bar{\sigma}_t^2 \end{cases}, K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_tC_t)\bar{\Sigma}_t \end{cases}, K_t = \bar{\Sigma}_tC_t^T(C_t\bar{\Sigma}_tC_t^T + Q_t)^{-1}$$



# Il ciclo Prediction-Correction



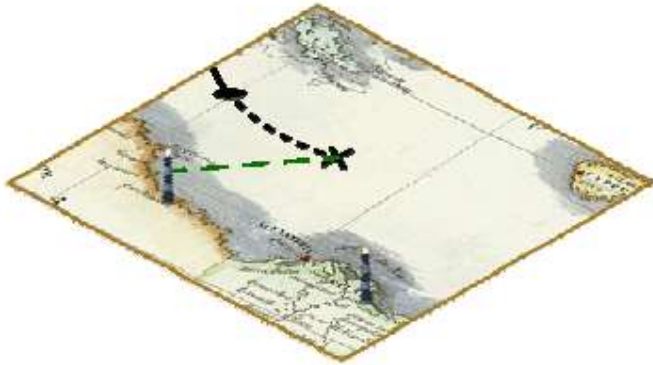
$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - \bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t)\bar{\sigma}_t^2 \end{cases}, K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_{t-1}^2 + \sigma_{act,t}^2 \end{cases}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t \bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t) \bar{\Sigma}_t \end{cases}, K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$

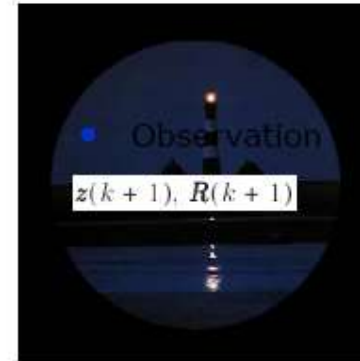
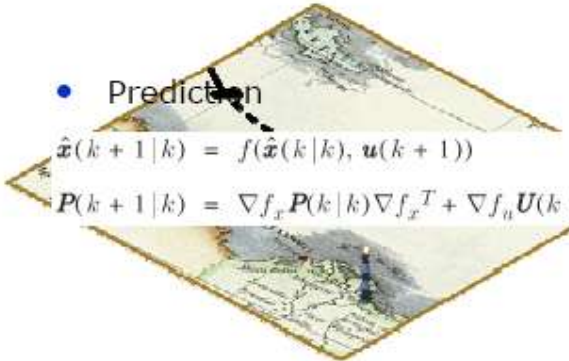




- Prediction

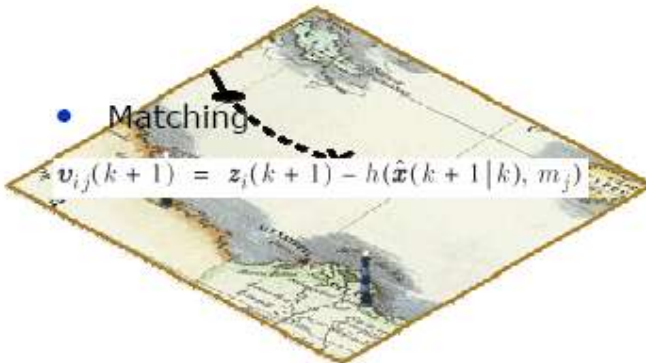
$$\hat{\mathbf{x}}(k+1|k) = f(\hat{\mathbf{x}}(k|k), \mathbf{u}(k+1))$$

$$\mathbf{P}(k+1|k) = \nabla f_x \mathbf{P}(k|k) \nabla f_x^T + \nabla f_u \mathbf{U}(k+1) \nabla f_u^T$$



- Matching

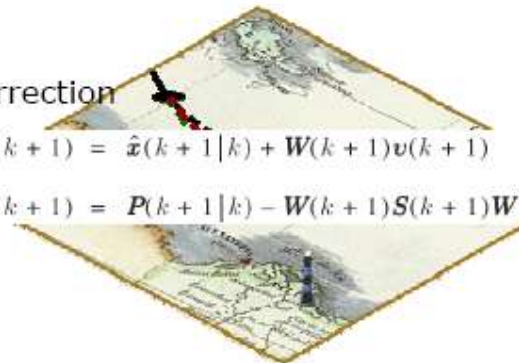
$$\mathbf{v}_{ij}(k+1) = \mathbf{z}_i(k+1) - h(\hat{\mathbf{x}}(k+1|k), m_j)$$



- Correction

$$\hat{\mathbf{x}}(k+1|k+1) = \hat{\mathbf{x}}(k+1|k) + \mathbf{W}(k+1)\mathbf{v}(k+1)$$

$$\mathbf{P}(k+1|k+1) = \mathbf{P}(k+1|k) - \mathbf{W}(k+1)\mathbf{S}(k+1)\mathbf{W}^T(k+1)$$



# Sommario: Filtro di Kalman

- **Efficiente**: Polinomiale nella dim delle misure  $k$  e nella dim dello stato  $n$ :  
$$O(k^{2.376} + n^2)$$
- **Ottimale per sistemi lineari Gaussiani!**
- Molti sistemi robotici sono **nonlineari!**