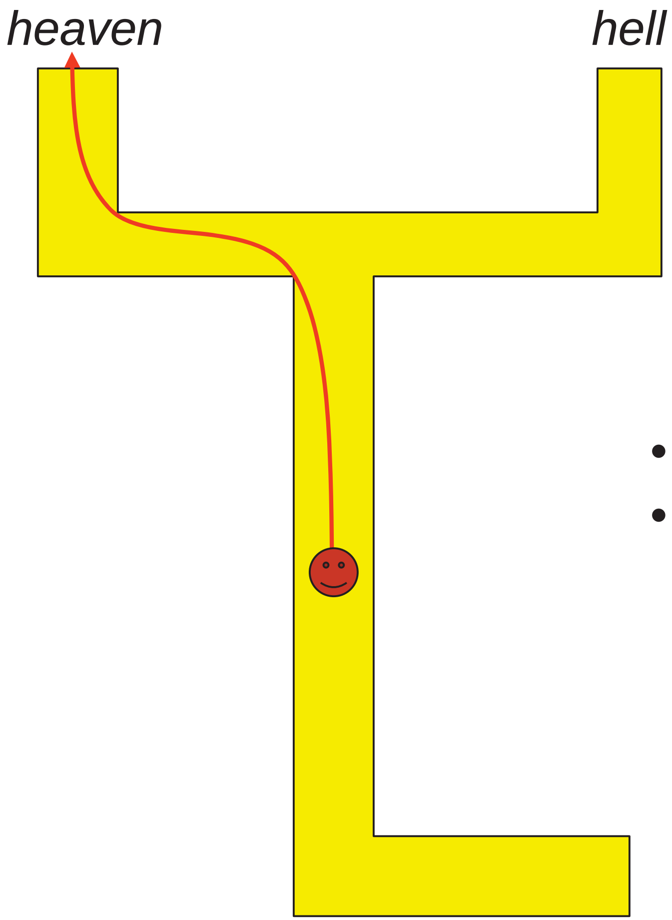
Probabilistic Robotics:

Probabilistic Planning and MDPs

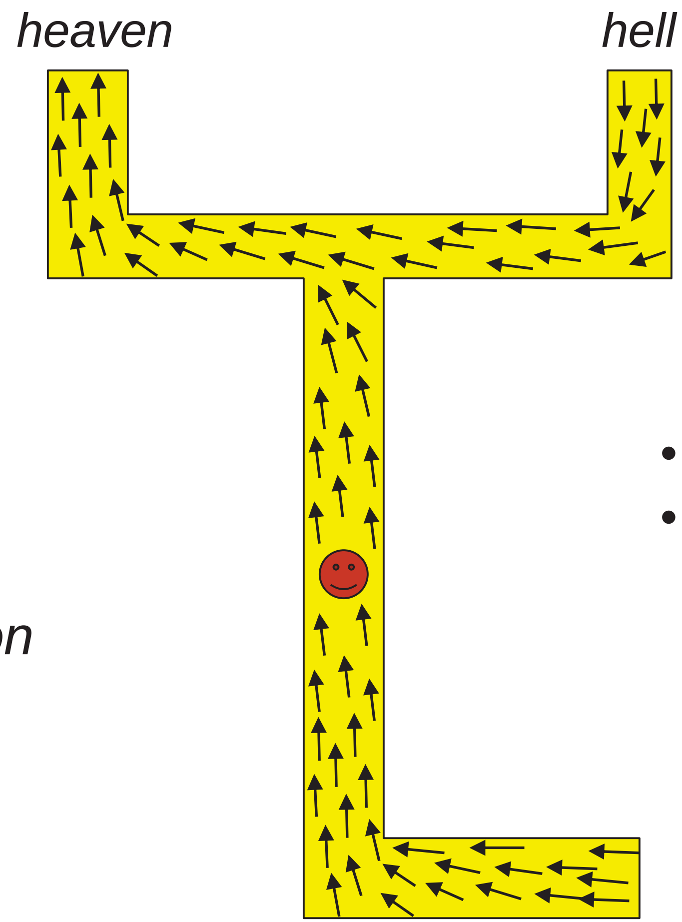
Slide credits: Wolfram Burgard, Dieter Fox, Cyrill Stachniss, Giorgio Grisetti, Maren Bennewitz, Christian Plagemann, Dirk Haehnel, Mike Montemerlo, Nick Roy, Kai Arras, Patrick Pfaff and others

Planning: Classical Situation



- *World deterministic*
- *State observable*

MDP-Style Planning

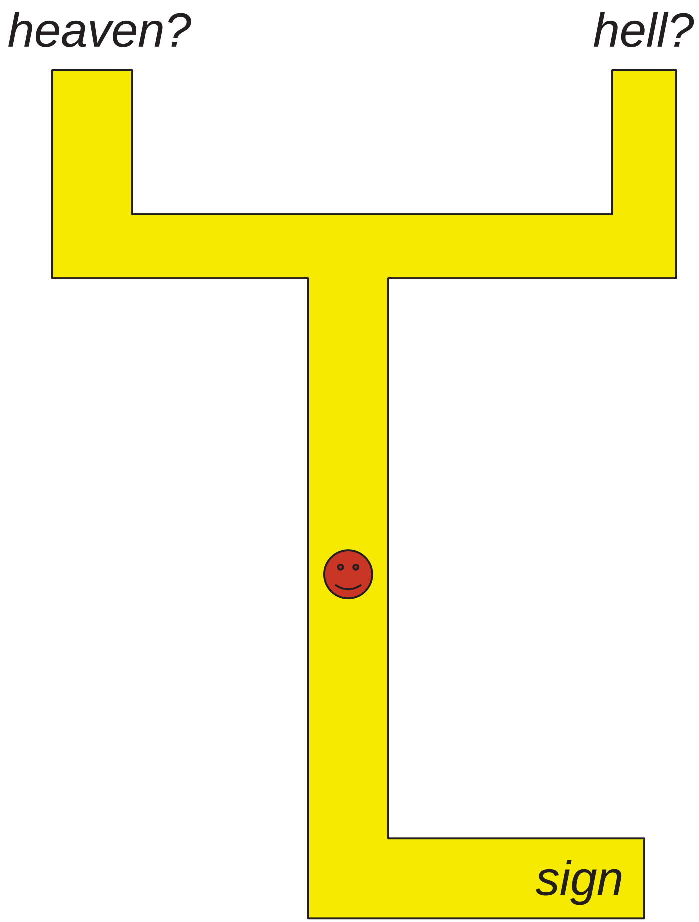


- *Policy*
- *Universal Plan*
- *Navigation function*

- *World stochastic*
- *State observable*

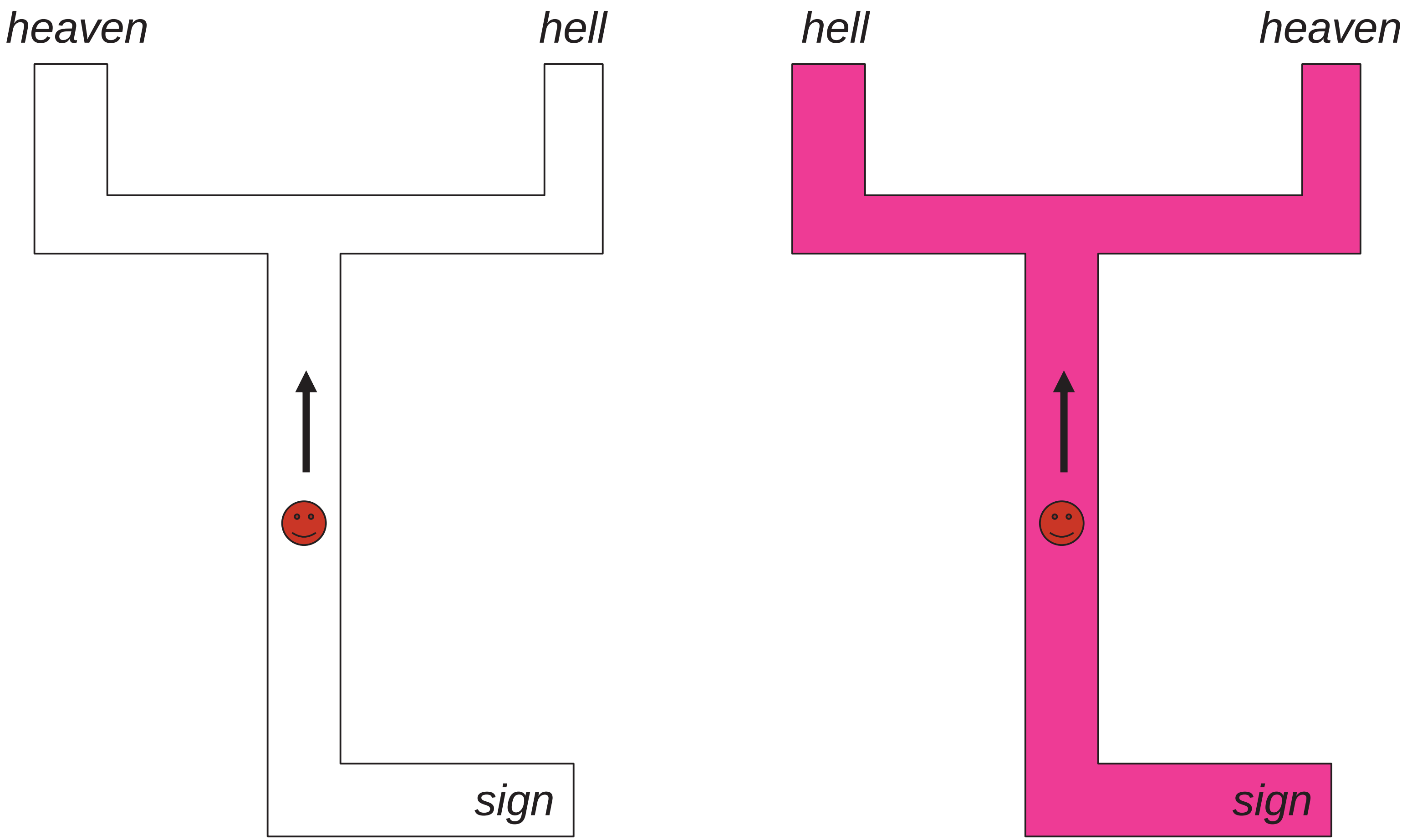
[Koditschek 87, Barto et al. 89]

Stochastic, Partially Observable

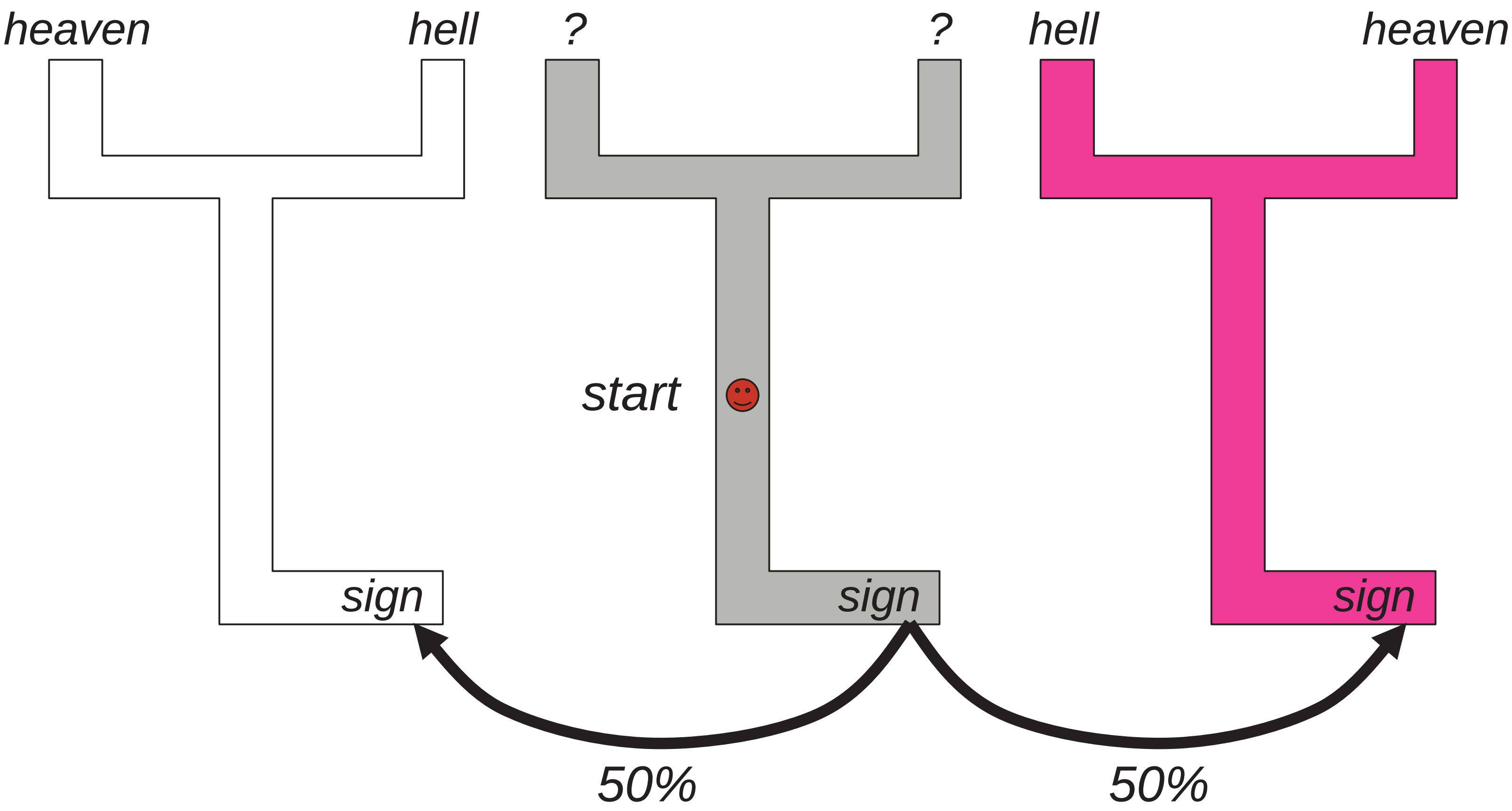


[Sondik 72] [Littman/Cassandra/Kaelbling 97]

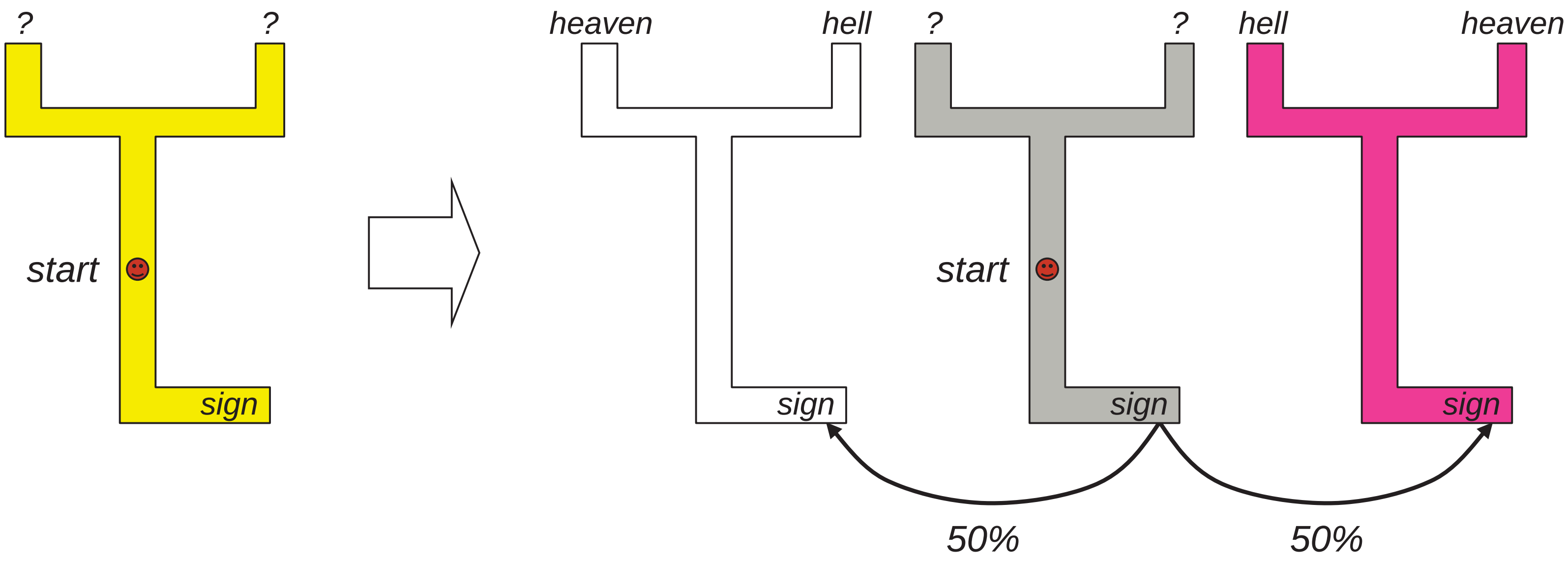
Stochastic, Partially Observable



Stochastic, Partially Observable



Stochastic, Partially Observable



A Quiz

# states	sensors	actions	size belief space?
3	perfect	deterministic	3: s_1, s_2, s_3
3	perfect	stochastic	3: s_1, s_2, s_3
3	abstract states	deterministic	2^3-1: s_1, s_2, s_3, s_{12}, s_{13}, s_{23}, s_{123}
3	stochastic	deterministic	2-dim continuous: $p(S=s_1)$, $p(S=s_2)$
3	none	stochastic	2-dim continuous: $p(S=s_1)$, $p(S=s_2)$
1-dim continuous	stochastic	deterministic	∞-dim continuous
1-dim continuous	stochastic	stochastic	∞-dim continuous
∞-dim continuous	stochastic	stochastic	aargh!

MPD Planning

- Solution for Planning problem
  - Noisy controls
  - Perfect perception
  - Generates “universal plan” (=policy)

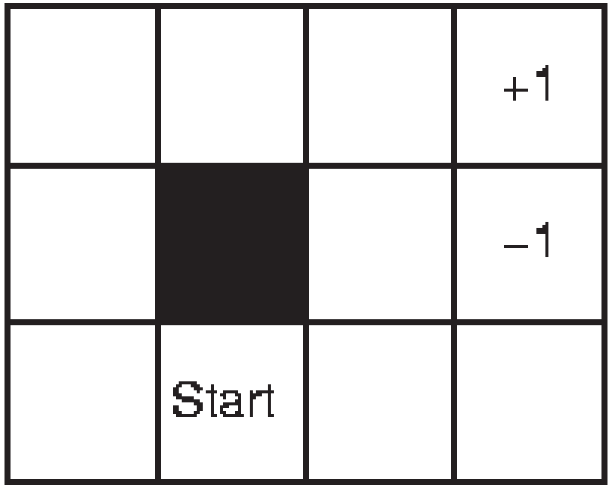
What is the problem?

- *Consider a non-deterministic robot/environment.*
- *Actions have desired outcome with a probability less then 1.*
- *What is the best action for a robot under this constraint?*

- *Example: a mobile robot does not exactly perform the desired action.*

➡ *Uncertainty about performing actions!*

Example (1)

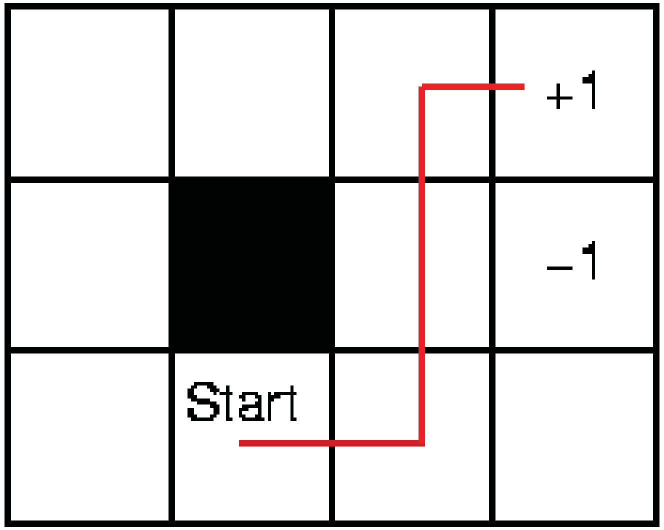


- *Bumping to wall "reflects" robot.*
- *"Reward" for free cells -0.04 (travel cost).*

- *What is the best way to reach the cell labeled with +1 without moving to –1 ?*

Example (2)

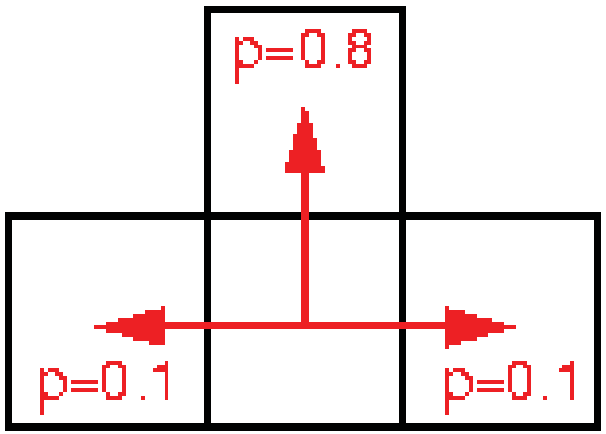
- *Deterministic Transition Model: move on the shortest path!*



Example (3)

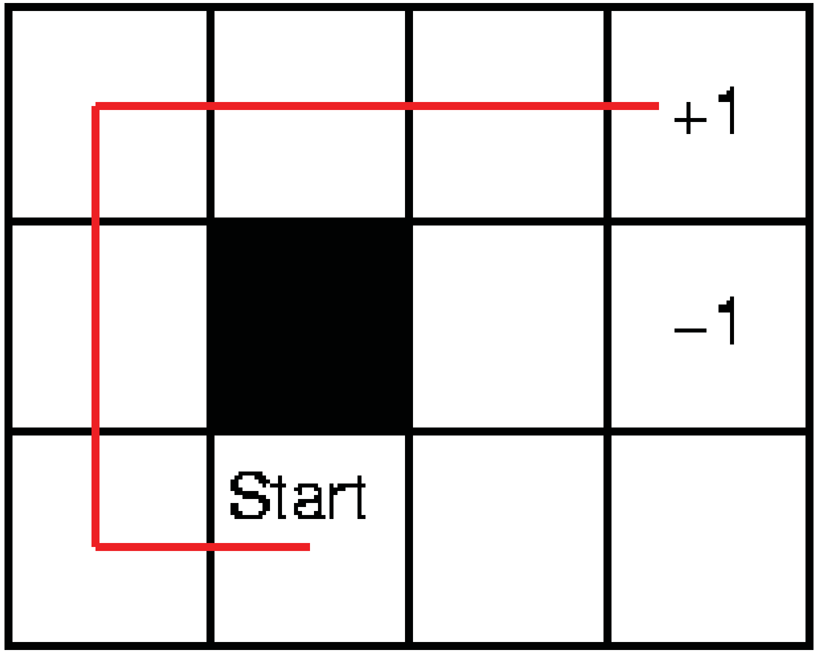
- *But now consider the non-deterministic transition model (N / E / S / W):*

(desired action)



- *What is now the best way?*

Example (4)



- *Use a longer path with lower probability to move to the cell labeled with –1.*
- *This path has the* ***highest overall utility****!*

Utility and Policy

- *Compute for every state a **utility**:*
  "What is the usage (utility) of this state for the overall task?"

- *A **Policy** is a complete mapping from states to actions ("In which state should I perform which action?").*

$$policy : States \mapsto Actions$$

Markov Decision Problem (MDP)

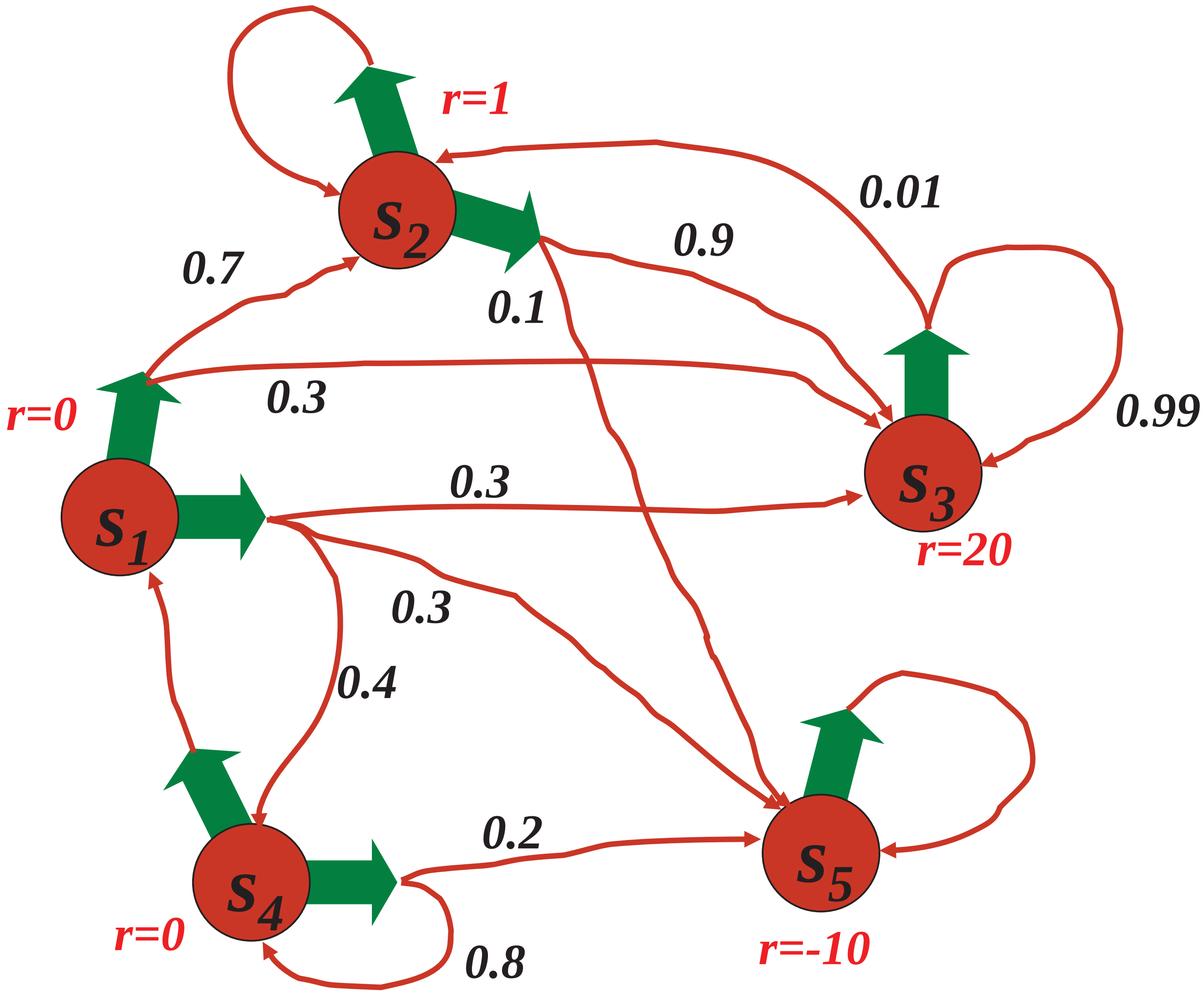
- *Compute the optimal policy in an accessible, stochastic environment with known transition model.*

Markov Property:

- *The transition probabilities depend only the current state and not on the history of predecessor states.*

⇒ *Not every decision problem is a MDP.*

Markov Decision Process (MDP)



Markov Decision Process (MDP)

- ***Given:***
- *States* x
- *Actions* u
- *Transition probabilities* $p(x'|u,x)$
- *Reward / payoff function* $r(x,u)$

- ***Wanted:***
- *Policy* $\pi(x)$ *that maximizes the future expected reward*

Rewards and Policies

- Policy (general case):

$$\pi: \quad z_{1:t-1}, u_{1:t-1} \rightarrow \ u_t$$

- Policy (fully observable case):

$$\pi: \quad x_t \rightarrow \ u_t$$

- Expected cumulative payoff:

$$R_T = E\left[\sum_{\tau=1}^{T} \gamma^{\tau} r_{t+\tau}\right]$$

  - T=1: greedy policy
  - T>1: finite horizon case, typically no discount
  - T=infty: infinite-horizon case, finite reward if discount < 1

Policies contd.

- Expected cumulative payoff of policy:

$$R_T^{\pi}(x_t) = E\left[\sum_{\tau=1}^{T} \gamma^{\tau} r_{t+\tau} \mid u_{t+\tau} = \pi\left(z_{1:t+\tau-1} u_{1:t+\tau-1}\right)\right]$$

- Optimal policy:

$$\pi^* = \underset{\pi}{\operatorname{argmax}} \quad R_T^{\pi}(x_t)$$

- 1-step optimal policy:

$$\pi_1(x) = \underset{u}{\operatorname{argmax}} \quad r(x,u)$$

- Value function of 1-step optimal policy:

$$V_1(x) = \gamma \max_u \ r(x,u)$$

2-step Policies

- Optimal policy:

$$\pi_2(x) = \operatorname*{argmax}_u \left[r(x,u) + \int V_1(x')\, p(x' \mid u, x)\, dx' \right]$$

- Value function:

$$V_2(x) = \gamma \max_u \left[r(x,u) + \int V_1(x')\, p(x' \mid u, x)\, dx' \right]$$

T-step Policies

- Optimal policy:

$$\pi_T(x) = \underset{u}{\operatorname{argmax}} \left[r(x,u) + \int V_{T-1}(x')\, p(x' \mid u, x)\, dx' \right]$$

- Value function:

$$V_T(x) = \gamma \max_u \left[r(x,u) + \int V_{T-1}(x')\, p(x' \mid u, x)\, dx' \right]$$

Infinite Horizon

- Optimal policy:

$$V_\infty(x) = \gamma \max_u \left[r(x,u) + \int V_\infty(x') \, p(x' \mid u, x) \, dx' \right]$$

- Bellman equation
- Fix point is optimal policy
- Necessary and sufficient condition

Value Iteration

- for all x do

$$\hat{V}(x) \leftarrow r_{\min}$$

- endfor

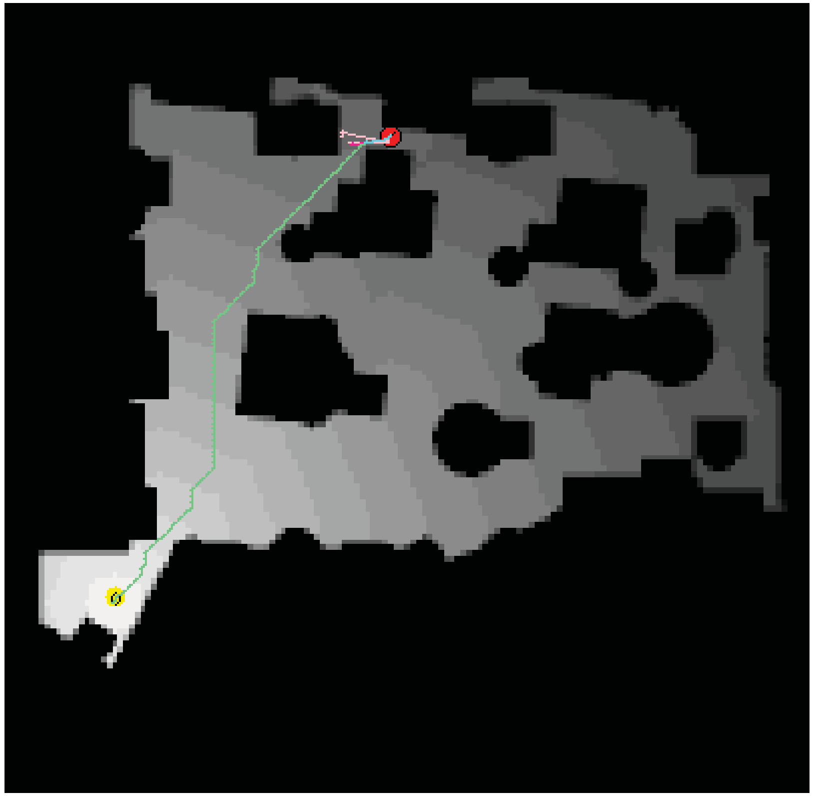
- repeat until convergence
  - for all x do

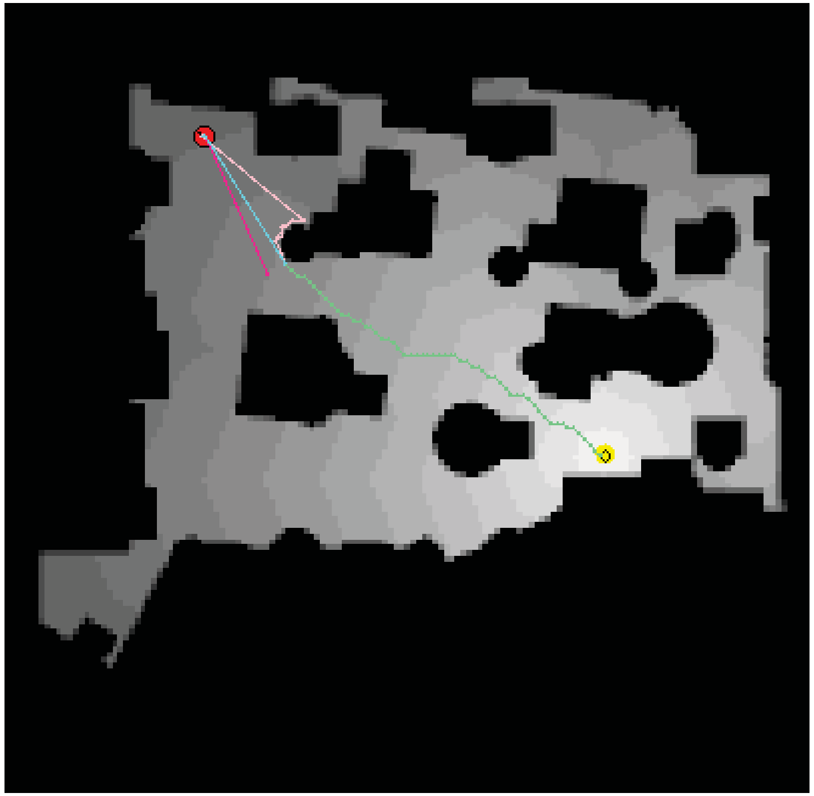
$$\hat{V}(x) \leftarrow \gamma \max_u \left[r(x,u) + \int \hat{V}(x') \, p(x' \mid u, x) dx' \right]$$

  - endfor
- endrepeat

$$\pi(x) = \operatorname*{argmax}_u \left[r(x,u) + \int \hat{V}(x') \, p(x' \mid u, x) dx' \right]$$

Value Iteration for Motion Planning





The optimal Policy

$$policy^*(i) = \operatorname*{argmax}_a \sum_j M_{ij}^a \cdot U(j)$$

$M_{ij}^a =$ *Probability of reaching state j form state i with action a.*

$U(j) =$ *Utility of state j.*

- *If we know the utility we can easily compute the optimal policy.*
- *The problem is to compute the correct utilities for all states.*

The Utility (1)

- *To compute the utility of a state we have to consider a tree of states.*
- *The utility of a state depends on the utility of all successor states.*

➡

- *Not all utility functions can be used.*
- *The utility function must have the property of separability.*
- *E.g. additive utility functions:*

$$U([s_0, s_1, \ldots s_n]) = R(s_0) + U([s_1, \ldots s_n])$$

(R = reward function)

The Utility (2)

- *The utility can be expressed similar to the policy function:*

$$U(i) = R(i) + \max_a \sum_j M_{ij}^a \cdot U(j)$$

- *The reward R(i) is the "utility" of the state itself (without considering the successors).*

Dynamic Programming

- *This Utility function is the basis for "dynamic programming".*
- *Fast solution to compute n-step decision problems.*
- *Naive solution: $O(|A|^n)$.*
- *Dynamic Programming: $O(n|A||S|)$.*
- *But what is the correct value of n?*
- *If the graph has loops:* $n \longrightarrow \infty$ *???*

Iterative Computation

Idea:

- *The Utility is computed iteratively:*

$$U_{t+1}(i) = R(i) + \max_a \sum_j M_{ij}^a \cdot U_t(j)$$

- *Optimal utility:* $U^* = \lim_{t \to \infty} U_t$
- *Abort, if change in the utility is below a threshold.*

The Value Iteration Algorithm

function VALUE-ITERATION(M, R) **returns** a utility function
  inputs: M, a transition model
    R, a reward function on states
  local variables: U, utility function, initially identical to R
    U', utility function, initially identical to R

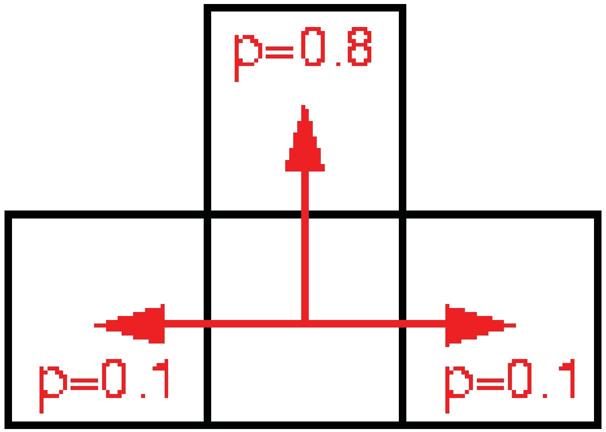
  repeat
    $U \leftarrow U'$
    for each state i **do**
      $U'[i] \leftarrow R[i] + \max_a \sum_j M_{ij}^a U[j]$
    end
  until CLOSE-ENOUGH(U, U')
  return U

Value Iteration Example

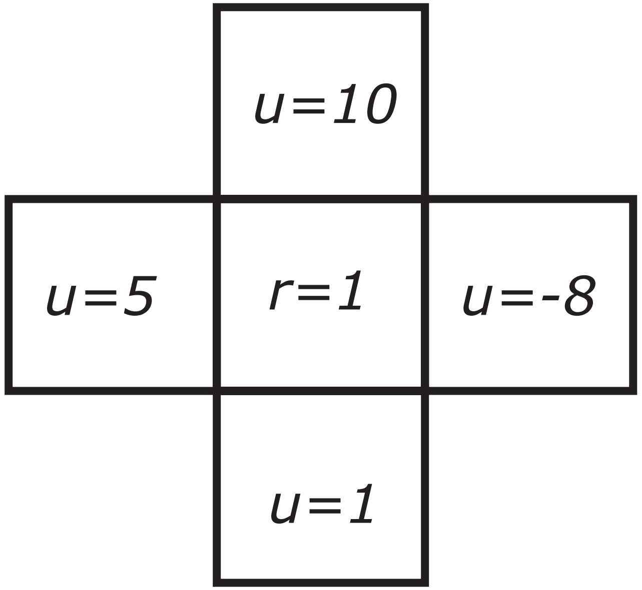
- *Calculate utility of the center cell*

$$U_{t+1}(i) = R(i) + \max_a \sum_j M^a_{ij} \cdot U_t(j)$$

(desired action=North)



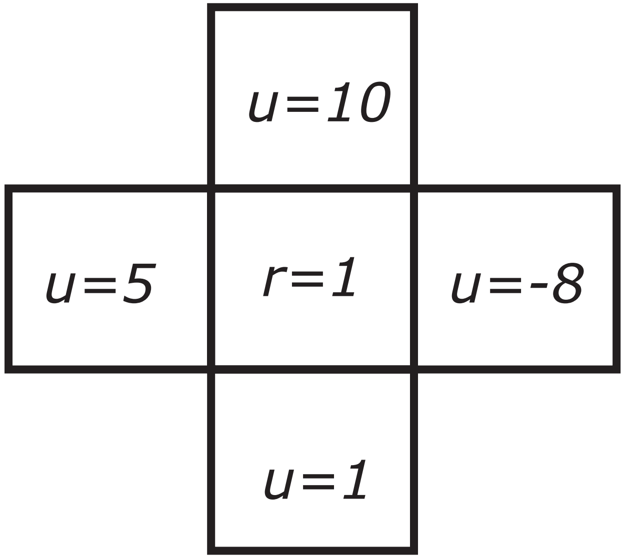
Transition Model



State Space
(u=utility, r=reward)

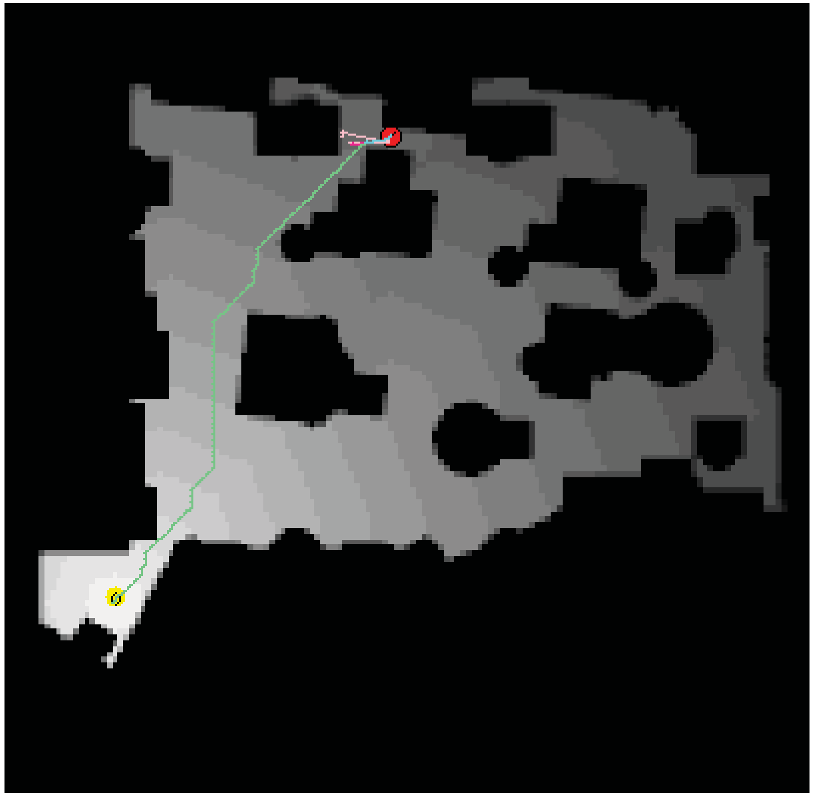
Value Iteration Example

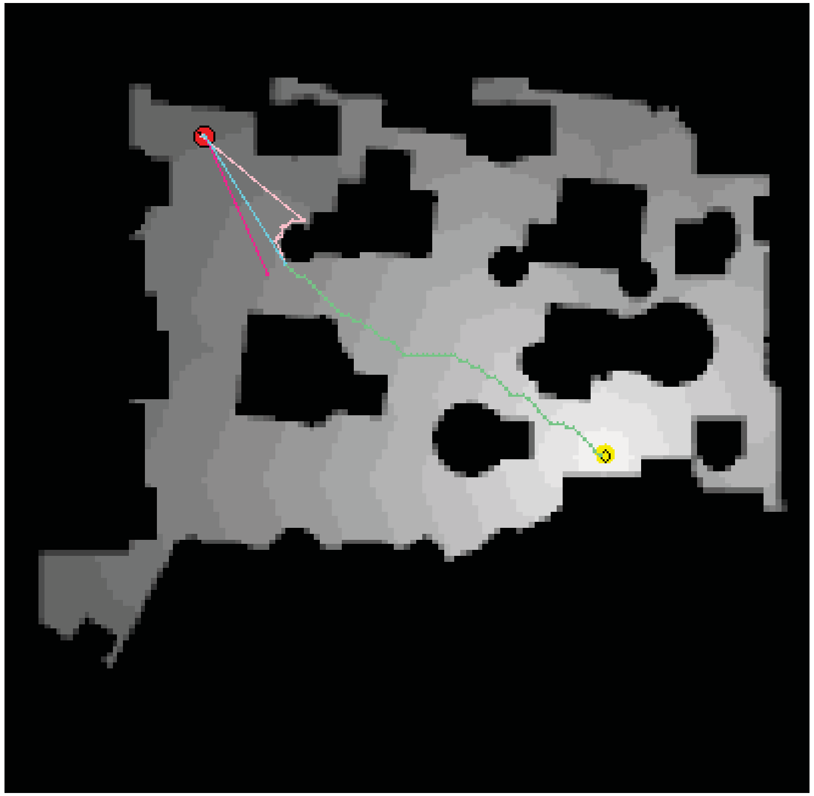
$$U_{t+1}(i) = R(i) + \max_a \sum_j M_{ij}^a \cdot U_t(j)$$



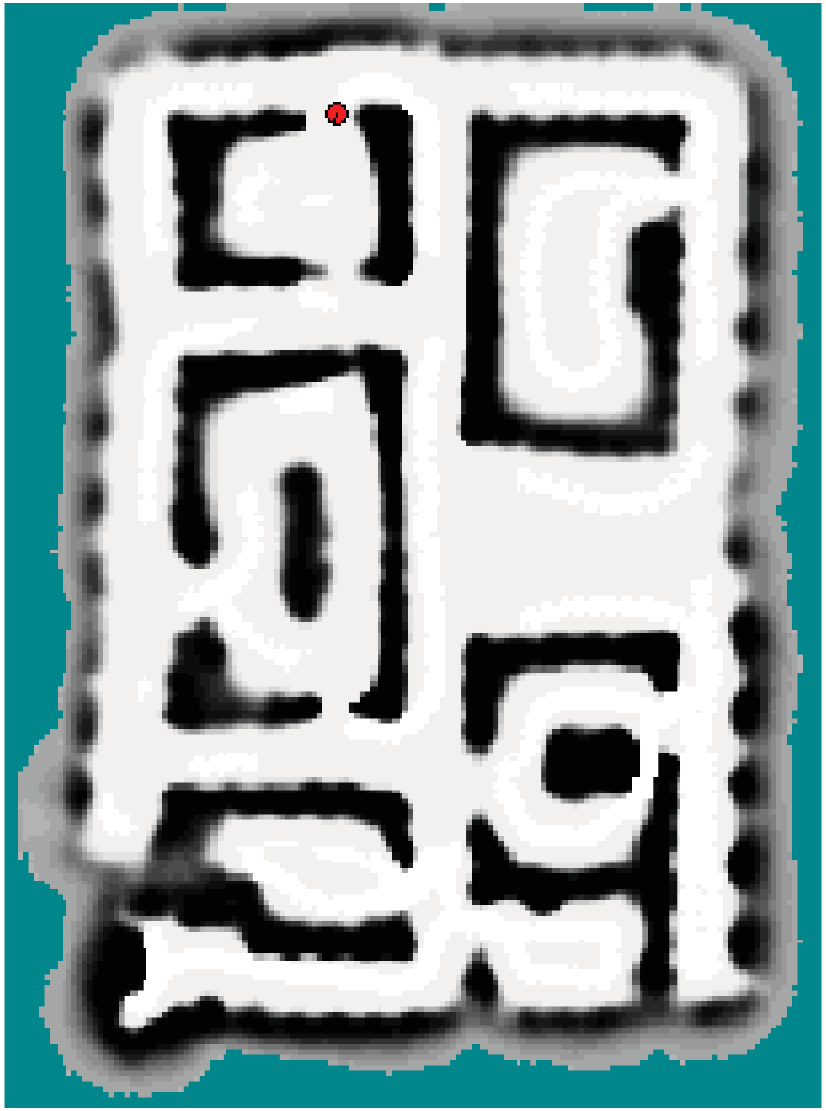
$$\begin{aligned}
&= \mathit{reward} + \max\{\\
&\quad 0.1 \cdot 1 + 0.8 \cdot 5 + 0.1 \cdot 10 \quad (\leftarrow),\\
&\quad 0.1 \cdot 5 + 0.8 \cdot 10 + 0.1 \cdot -8 \quad (\uparrow),\\
&\quad 0.1 \cdot 10 + 0.8 \cdot -8 + 0.1 \cdot 1 \quad (\rightarrow),\\
&\quad 0.1 \cdot -8 + 0.8 \cdot 1 + 0.1 \cdot 5 \quad (\downarrow)\}\\
&= 1 + \max\{5.1\,(\leftarrow), 7.7\,(\uparrow),\\
&\quad -5.3\,(\rightarrow), 0.5\,(\downarrow)\}\\
&= 1 + 7.7\\
&= 8.7
\end{aligned}$$

Value Iteration: Example

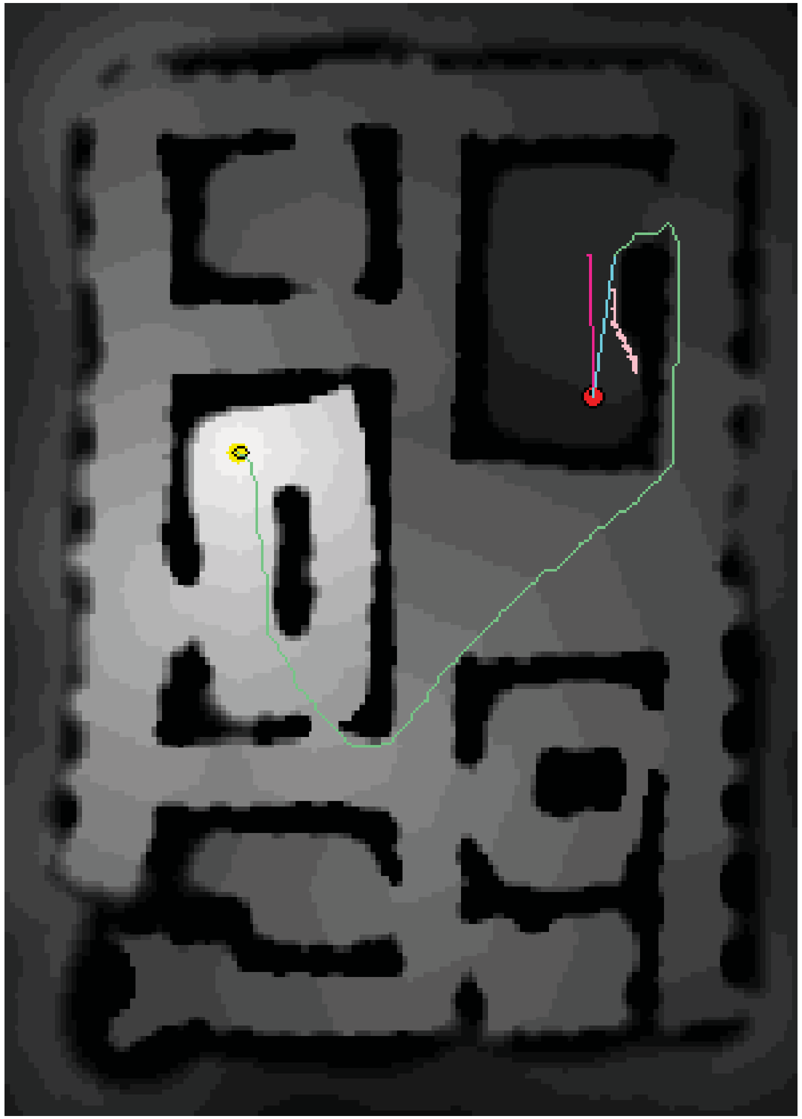




Another Example

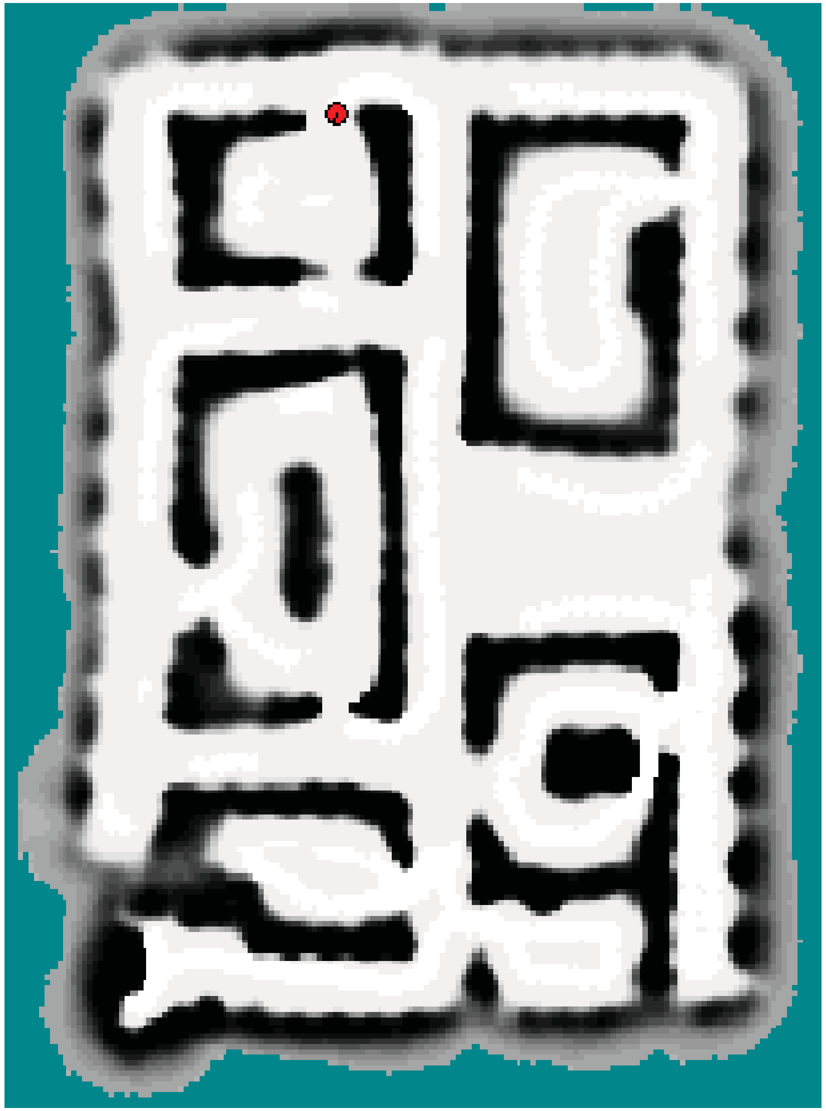


Map

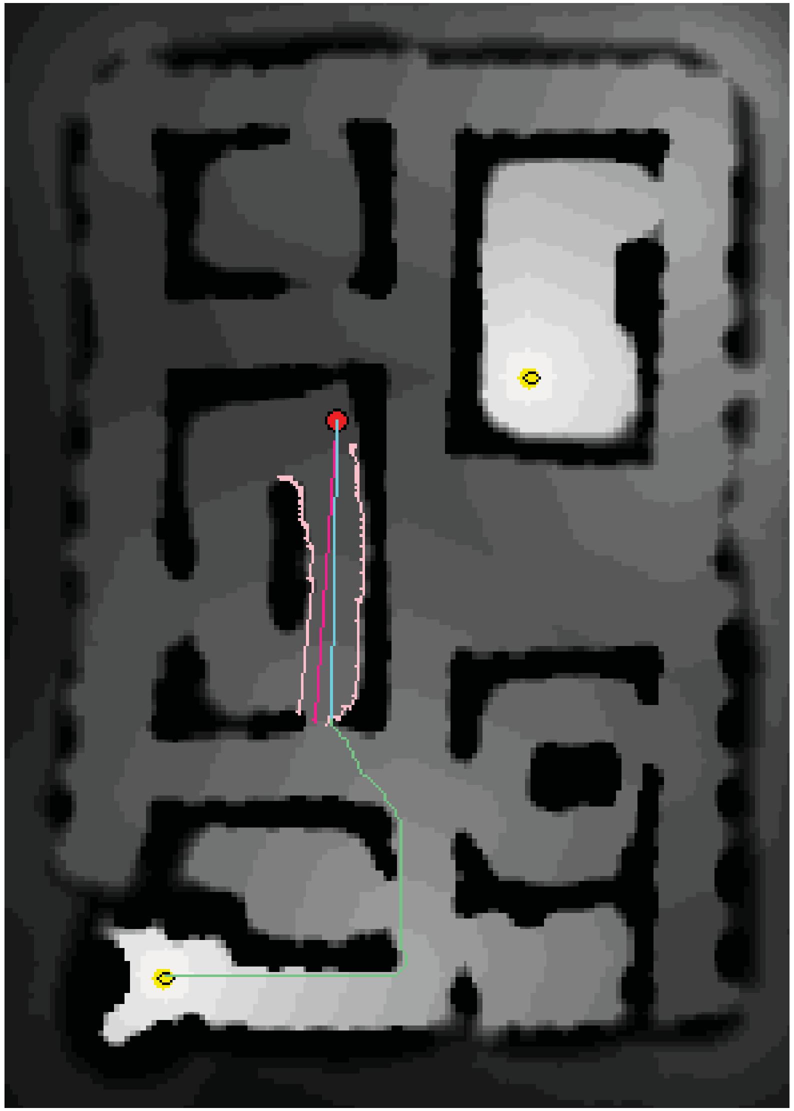


Value Function and Plan

Another Example



Map



Value Function and Plan

From Utilities to Policies

- *Computes the optimal utility function.*
- *Optimal Policy can easily be computed using the optimal utility values:*

$$policy^*(i) = \underset{a}{\operatorname{argmax}} \sum_j M_{ij}^a \cdot U^*(j)$$

- *Value Iteration is an optimal solution to the Markov Decision Problem!*

Convergence “close-enough”

- *Different possibilities to detect convergence:*
  - *RMS error – root mean square error*
  - *Policy Loss*
  - *...*

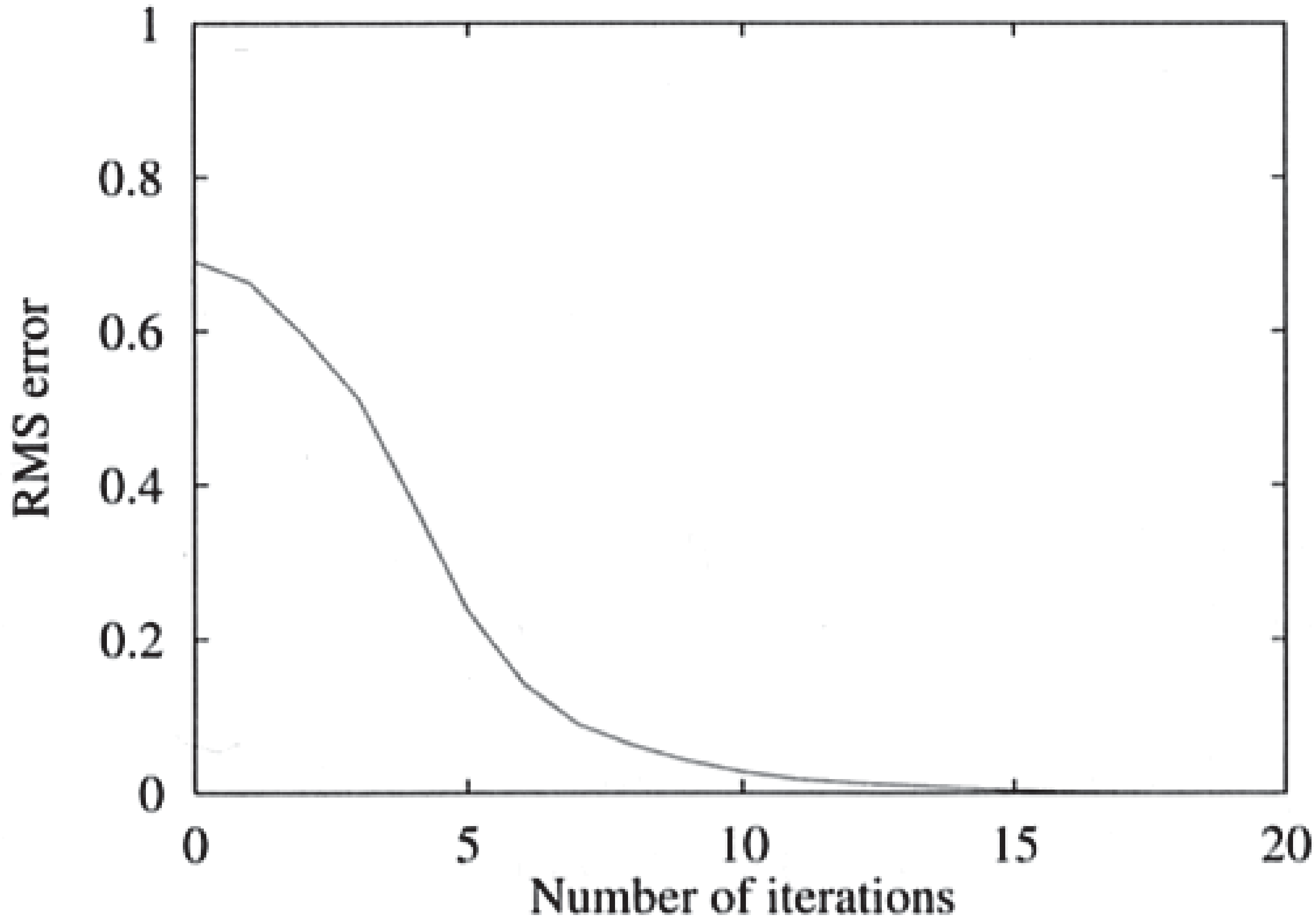
Convergence-Criteria: RMS

$$RMS = \frac{1}{|\,S\,|} \cdot \sqrt{\sum_{i=1}^{|S|} (U(i) - U'(i))^2}$$

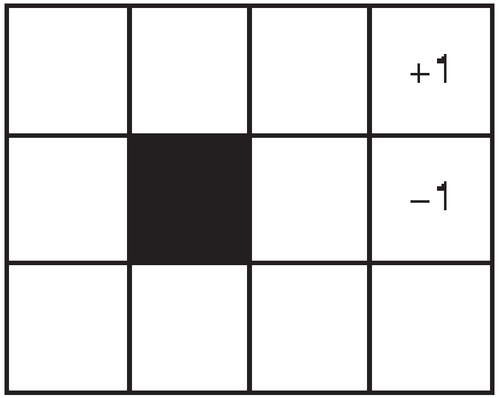
- `CLOSE-ENOUGH(U,U')` *in the algorithm can be formulated by:*

$$RMS(U, U') < \epsilon$$

Example: RMS-Convergence

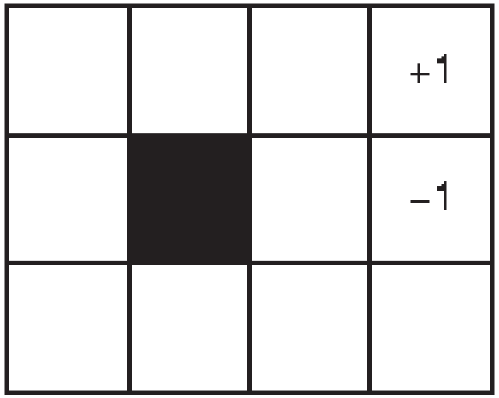


Example: Value Iteration



1. The given environment.

Example: Value Iteration



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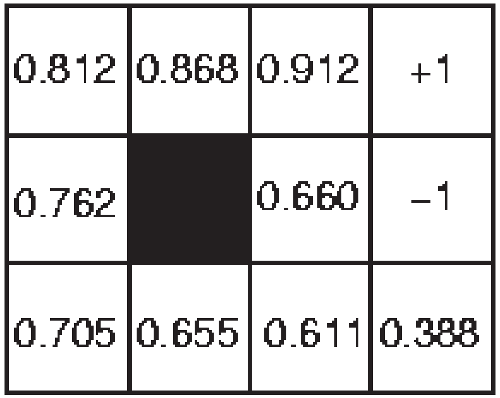
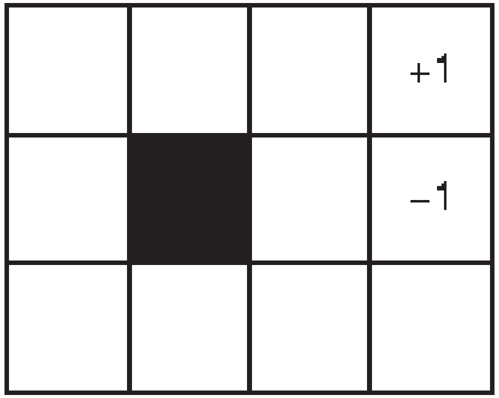


2. Calculate Utilities.

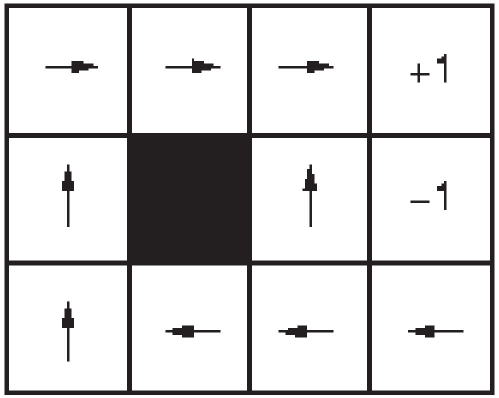
Example: Value Iteration



1. The given environment.

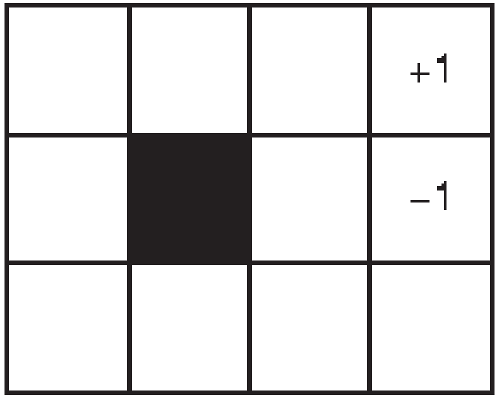
0.812	0.868	0.912	+1
0.762		0.660	−1
0.705	0.655	0.611	0.388

2. Calculate Utilities.



3. Extract optimal policy.

Example: Value Iteration



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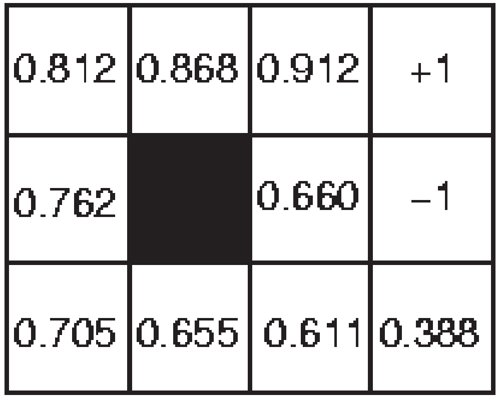

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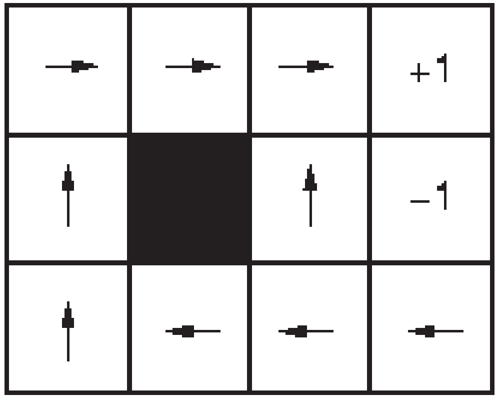

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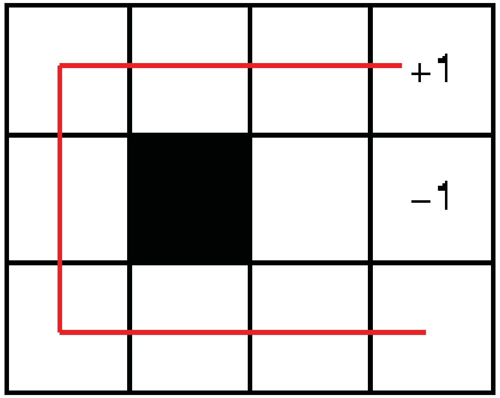
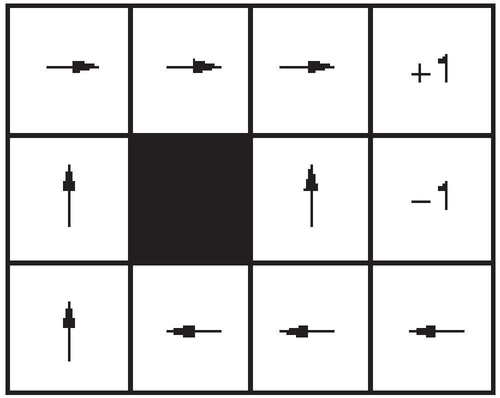


4. Execute actions.

Example: Value Iteration

0.812	0.868	0.912	+1
0.762		0.660	−1
0.705	0.655	0.611	0.388

The Utilities.



The optimal policy.

- *(3,2) has higher utility than (2,3). Why does the polity of (3,3) points to the left?*

Example: Value Iteration

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0.762		0.660	−1
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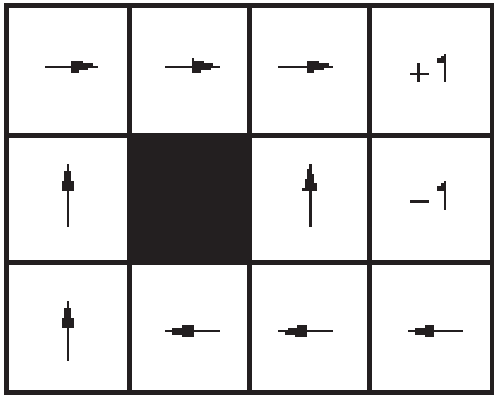
The Utilities.



The optimal policy.

- *(3,2) has higher utility than (2,3). Why does the polity of (3,3) points to the left?*
- *Because the Policy is* ***not*** *the gradient! It is:*

$$policy^*(i) = \operatorname*{argmax}_a \sum_j M_{ij}^a \cdot U(j)$$

Convergence of Policy and Utilities

- *In practice: policy converges faster than the utility values.*

- *After the relation between the utilities are correct, the policy often does not change anymore (because of the argmax).*

- *Is there an algorithm to compute the optimal policy faster?*

Policy Iteration

- ***Idea*** *for faster convergence of the policy:*
  1. *Start with one policy.*
  2. *Calculate utilities based on the current policy.*
  3. *Update policy based on policy formula.*
  4. *Repeat Step 2 and 3 until policy is stable.*

The Policy Iteration Algorithm

function POLICY-ITERATION(M, R) **returns** a policy
  inputs: M, a transition model
        R, a reward function on states
  local variables: U, a utility function, initially identical to R
                P, a policy, initially optimal with respect to U

  repeat
    $U \leftarrow$ VALUE-DETERMINATION(P, U, M, R)
    unchanged? $\leftarrow$ true
    for each state i **do**
      if $\max_a \sum_j M_{ij}^{a}\, U[j] > \sum_j M_{ij}^{P[i]}\, U[j]$ **then**
        $P[i] \leftarrow \arg\max_a \sum_j M_{ij}^{a}\, U[j]$
        unchanged? $\leftarrow$ false
    end
  until *unchanged?*
  return P

Value Determination

$$U(s_i) = R(s_i) + \sum_j P_{ij}^{\pi(s_i)} U(s_j)$$

$$U'(s_i) \leftarrow R[i] + \sum_j P_{ij}^{\pi(s_i)} U(s_j)$$