

# Robotica Probabilistica

**Filtri Bayesiani**

Filteri Gaussiani

# Filtri Bayesiani Reminder

- Predizione

$$\overline{bel}(x_t) = \int p(x_t | u_t, x_{t-1}) bel(x_{t-1}) dx_{t-1}$$

- Correzione

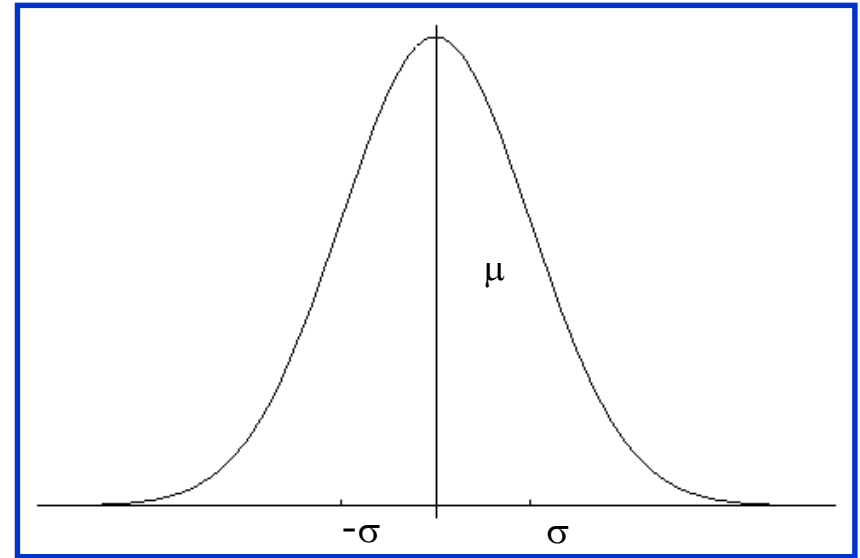
$$bel(x_t) = \eta p(z_t | x_t) \overline{bel}(x_t)$$

# Gaussiane

$$p(x) \sim N(\mu, \sigma^2):$$

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2}}$$

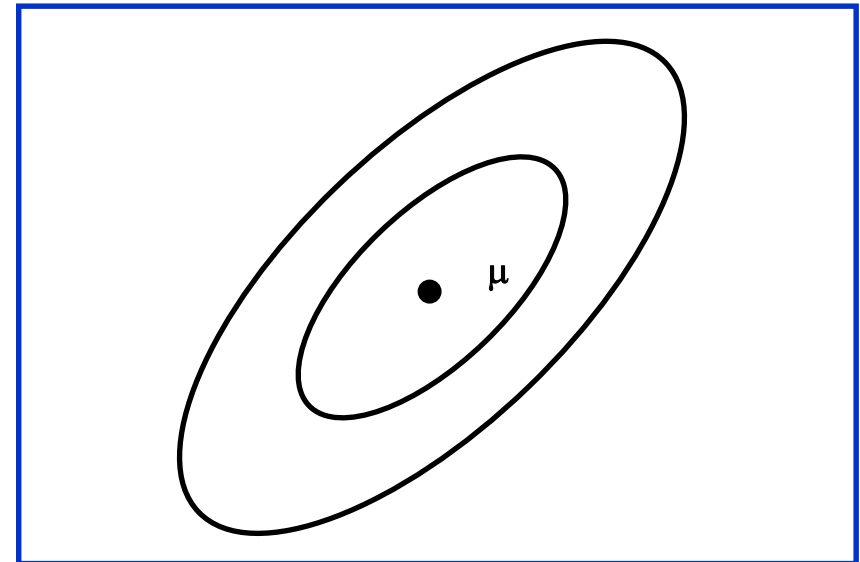
Univariata



$$p(\mathbf{x}) \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}):$$

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2} |\boldsymbol{\Sigma}|^{1/2}} e^{-\frac{1}{2} (\mathbf{x}-\boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{x}-\boldsymbol{\mu})}$$

Multivariata



# Propert  delle Gaussiane

$$\left. \begin{array}{l} X \sim N(\mu, \sigma^2) \\ Y = aX + b \end{array} \right\} \Rightarrow Y \sim N(a\mu + b, a^2\sigma^2)$$

$$\left. \begin{array}{l} X_1 \sim N(\mu_1, \sigma_1^2) \\ X_2 \sim N(\mu_2, \sigma_2^2) \end{array} \right\} \Rightarrow p(X_1) \cdot p(X_2) \sim N\left(\frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \mu_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \mu_2, \frac{1}{\sigma_1^{-2} + \sigma_2^{-2}}\right)$$

# Multivariate

$$\left. \begin{array}{l} X \sim N(\mu, \Sigma) \\ Y = AX + B \end{array} \right\} \Rightarrow Y \sim N(A\mu + B, A\Sigma A^T)$$

$$\left. \begin{array}{l} X_1 \sim N(\mu_1, \Sigma_1) \\ X_2 \sim N(\mu_2, \Sigma_2) \end{array} \right\} \Rightarrow p(X_1) \cdot p(X_2) \sim N\left(\frac{\Sigma_2}{\Sigma_1 + \Sigma_2} \mu_1 + \frac{\Sigma_1}{\Sigma_1 + \Sigma_2} \mu_2, \frac{1}{\Sigma_1^{-1} + \Sigma_2^{-1}}\right)$$

- Si rimane nel "mondo gaussiano" con trasformazioni lineari.

# Propert  delle Gaussiane

Dato:

$$x = (x_a \ x_b) \quad p(x) = N(\mu, \Sigma)$$

$$\mu = (\mu_a, \mu_b)$$

Le marginali sono Gaussiane:

$$\Sigma = \begin{pmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{pmatrix}$$

$$p(x_a) = N(\mu_a, \Sigma_{aa}),$$

$$p(x_b) = N(\mu_b, \Sigma_{bb})$$

Le condizionate sono Gaussiane:

$$p(x_a | x_b) = N(\mu_{a|b}, \Sigma_{a|b}),$$

$$p(x_b | x_a) = N(\mu_{b|a}, \Sigma_{b|a})$$

$$\mu_{a|b} = \mu_a + \Sigma_{ab} \Sigma_{bb}^{-1} (b - \mu_b)$$

$$\Sigma_{a|b} = \Sigma_{aa} - \Sigma_{ab} \Sigma_{bb}^{-1} \Sigma_{ba}$$

# Filtro di Kalman Discreto

Stima lo stato  $x$  di un processo a tempo discreto governato da una equazione alle differenze stocastica lineare

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

con una misura

$$z_t = C_t x_t + \delta_t$$

Rumore  
rappresentato come  
multivariata con  
valore medio zero

# Componenti di un Filtro di Kalman

 $A_t$ 

Matrice ( $n \times n$ ) che descrive l'evoluzione dello stato da  $t-1$  a  $t$  senza controllo o rumore.

 $B_t$ 

Matrice ( $n \times l$ ) che descrive come il controllo  $u_t$  cambia lo stato da  $t$  a  $t-1$ .

 $C_t$ 

Matrice ( $k \times n$ ) che descrive come mappare lo stato  $x_t$  in un'osservazione  $z_t$ .

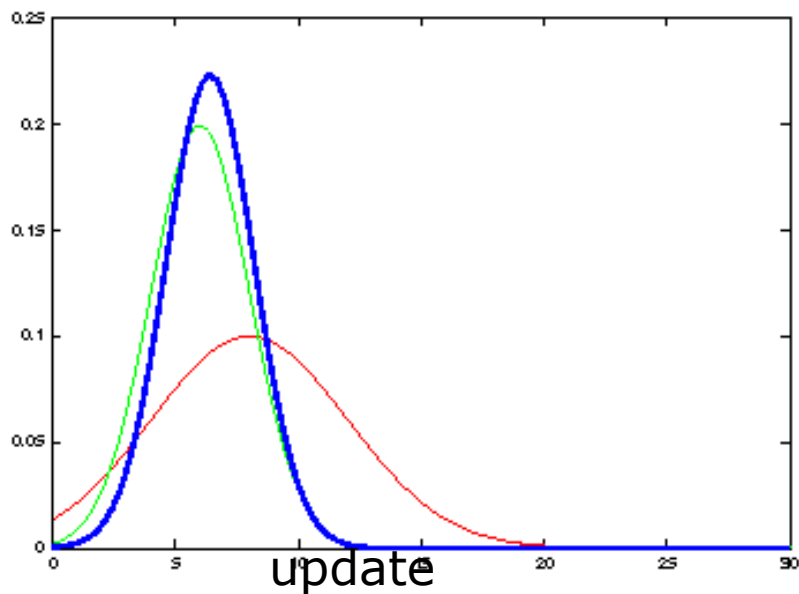
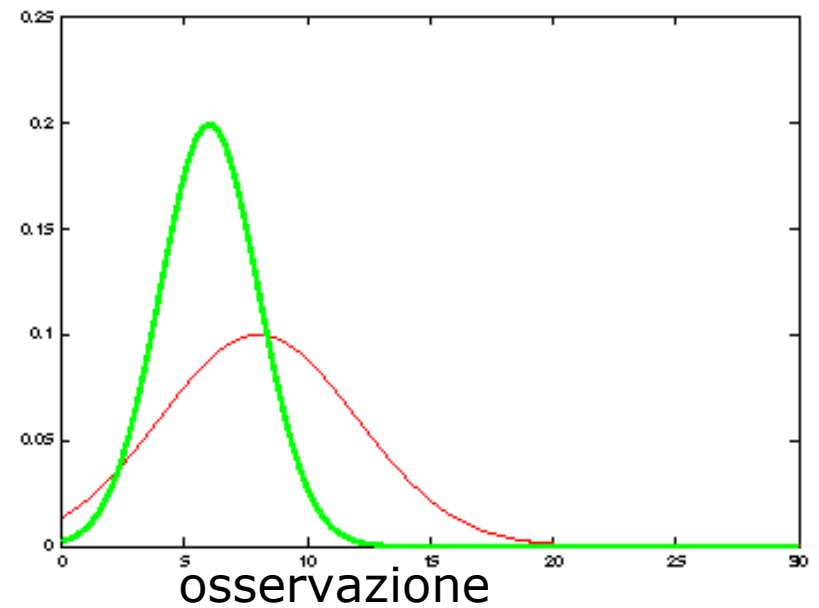
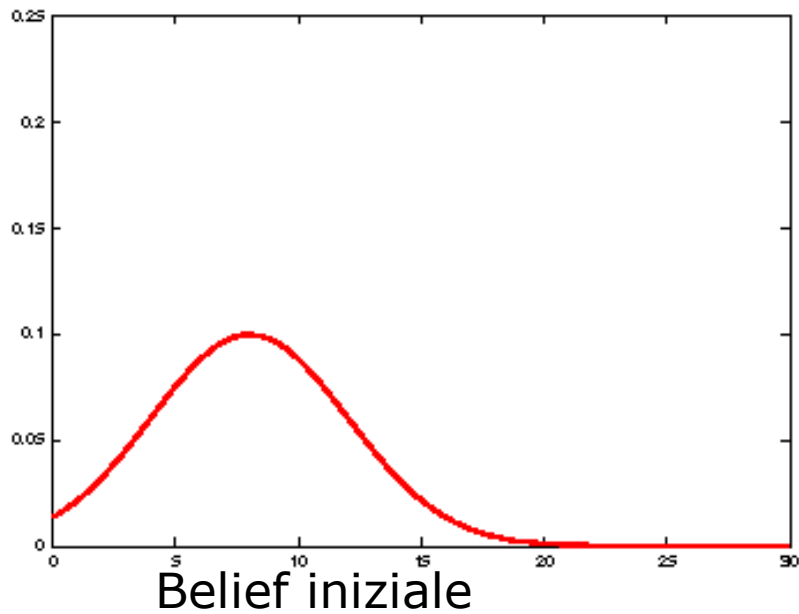
 $\varepsilon_t$ 

Variabili random che rappresentano il rumore di processo e misura. Assunte indipendenti e con distribuzione normale con covarianza  $R_t$  e  $Q_t$  rispettivamente.

 $\delta_t$



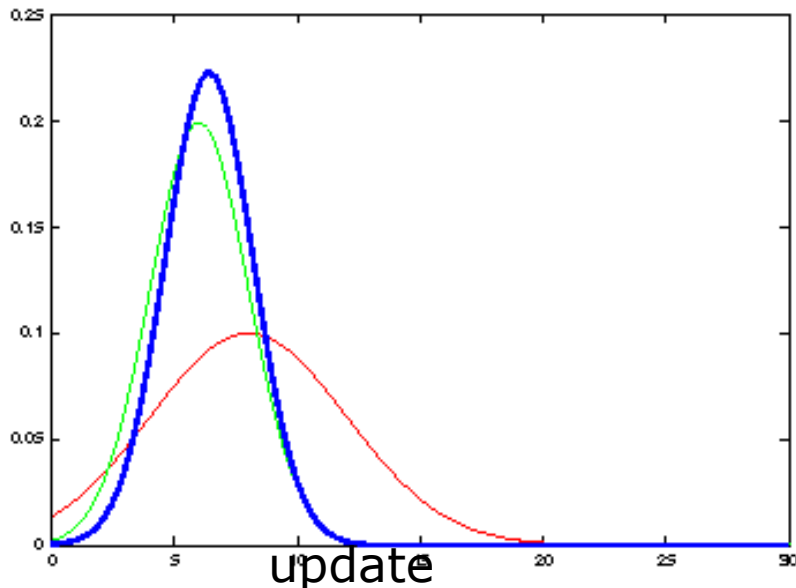
# Filtro di Kalman in 1D



# Aggiornamento Filtro di Kalman 1D

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - \bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t)\bar{\sigma}_t^2 \end{cases} \quad \text{with} \quad K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_tC_t)\bar{\Sigma}_t \end{cases} \quad \text{with} \quad K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

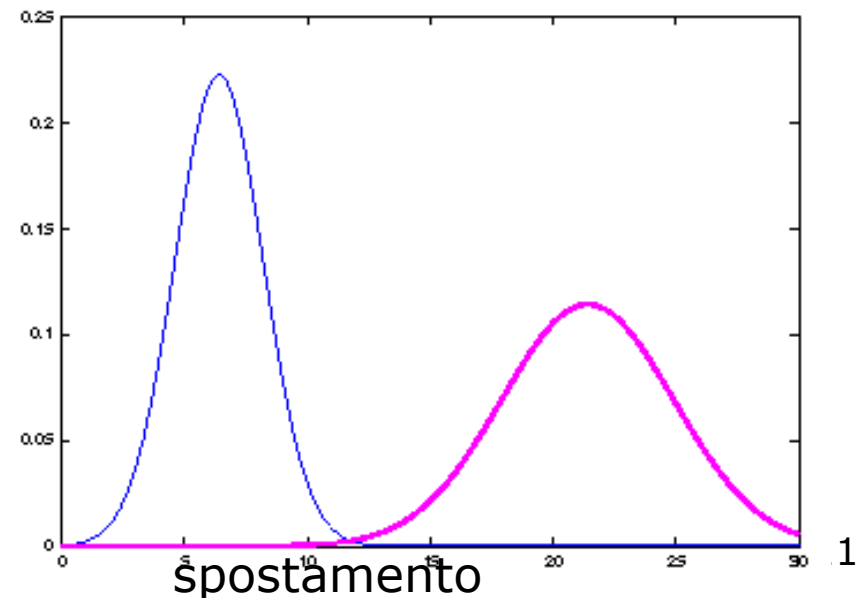
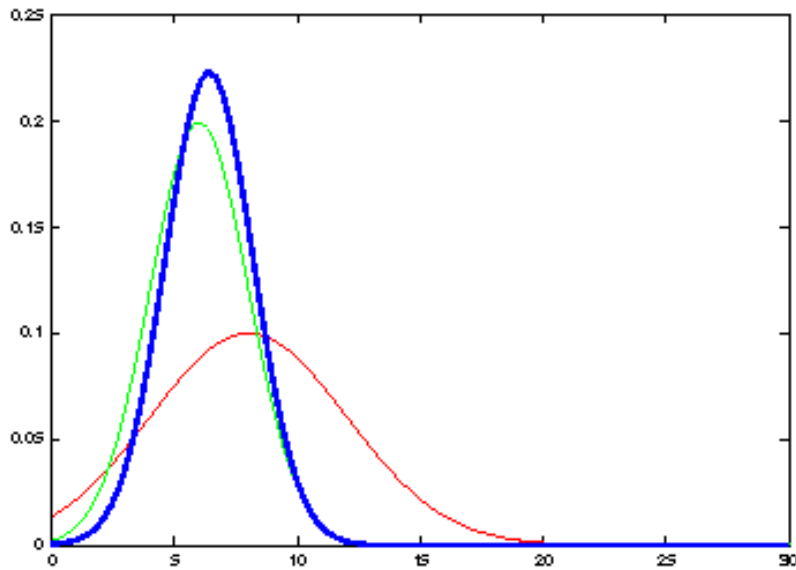


# Aggiornamento Filtro di Kalman 1D

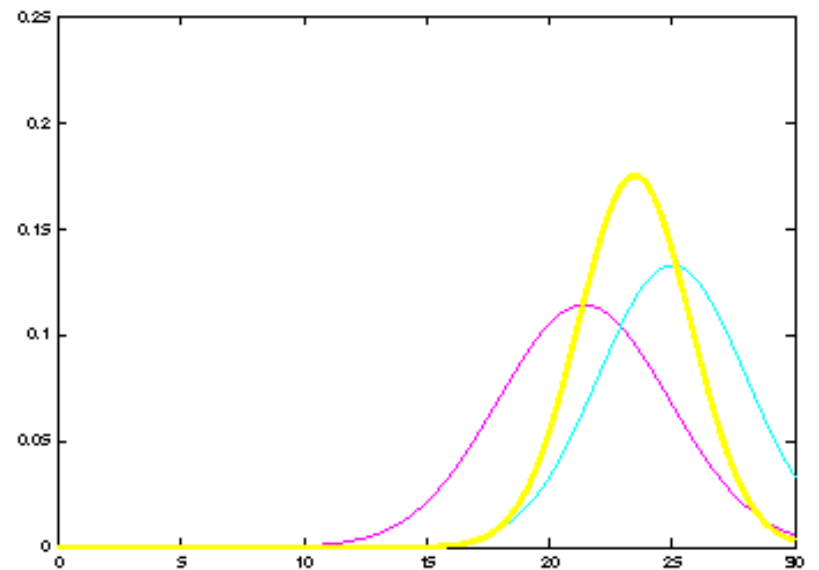
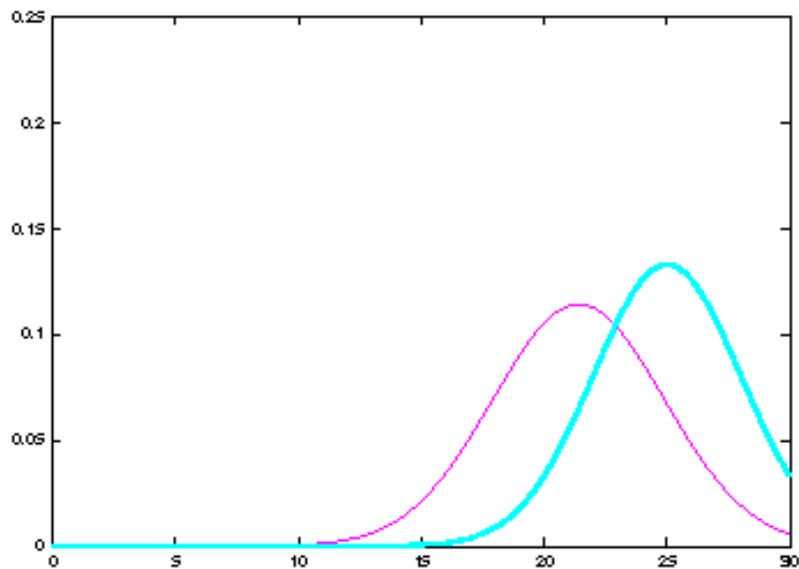
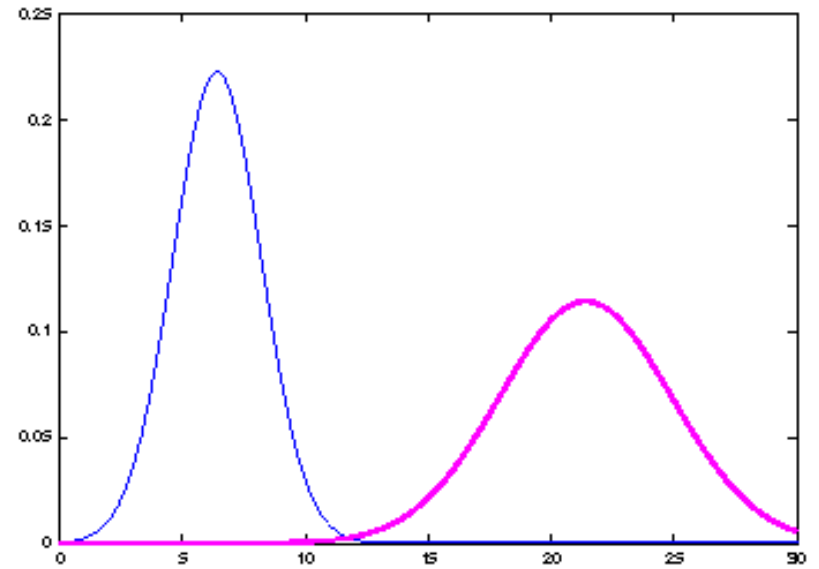
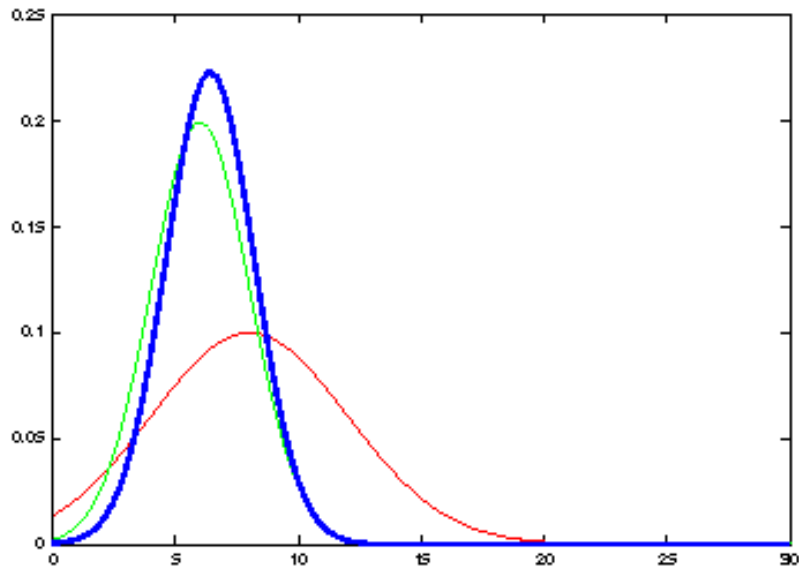
Spostamento

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_{t-1}^2 + \sigma_{act,t}^2 \end{cases}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$



# Aggiornamento Filtro Kalman



# Gaussiana Lineare: Initialization

- belief iniziale con distribuzione normale:

$$bel(x_0) = N(x_0; \mu_0, \Sigma_0)$$

# Gaussiana Lineare: Dinamica

- La dinamica è funzione lineare dello stato e del controllo più rumore additivo:

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

$$p(x_t | u_t, x_{t-1}) = N(x_t; A_t x_{t-1} + B_t u_t, R_t)$$

$$\begin{array}{ccc} \overline{bel}(x_t) = \int p(x_t | u_t, x_{t-1}) & & bel(x_{t-1}) dx_{t-1} \\ \Downarrow & & \Downarrow \\ \sim N(x_t; A_t x_{t-1} + B_t u_t, R_t) & \sim & N(x_{t-1}; \mu_{t-1}, \Sigma_{t-1}) \end{array}$$

# Gaussiana Lineare: Dinamica

$$\begin{aligned} \overline{bel}(x_t) &= \int p(x_t | u_t, x_{t-1}) \quad bel(x_{t-1}) dx_{t-1} \\ &\quad \Downarrow \qquad \qquad \qquad \Downarrow \\ &\sim N(x_t; A_t x_{t-1} + B_t u_t, R_t) \quad \sim N(x_{t-1}; \mu_{t-1}, \Sigma_{t-1}) \\ &\quad \Downarrow \\ \overline{bel}(x_t) &= \eta \int \exp\left\{-\frac{1}{2} (x_t - A_t x_{t-1} - B_t u_t)^T R_t^{-1} (x_t - A_t x_{t-1} - B_t u_t)\right\} \\ &\quad \exp\left\{-\frac{1}{2} (x_{t-1} - \mu_{t-1})^T \Sigma_{t-1}^{-1} (x_{t-1} - \mu_{t-1})\right\} dx_{t-1} \\ \overline{bel}(x_t) &= \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases} \end{aligned}$$

# Gaussiana Lineare: Osservazioni

- Le osservazioni sono funzioni lineari dello stata più rumore additivo:

$$z_t = C_t x_t + \delta_t$$

$$p(z_t | x_t) = N(z_t; C_t x_t, Q_t)$$

$$\begin{array}{ccc} \text{bel}(x_t) = \eta & p(z_t | x_t) & \overline{\text{bel}}(x_t) \\ & \Downarrow & \Downarrow \\ & \sim N(z_t; C_t x_t, Q_t) & \sim N(x_t; \overline{\mu}_t, \overline{\Sigma}_t) \end{array}$$



# Sistemi a Gaussiana Lineare: Osservazioni

$$\begin{array}{ccc}
 \text{bel}(x_t) = \eta & p(z_t | x_t) & \overline{\text{bel}}(x_t) \\
 & \Downarrow & \Downarrow \\
 & \sim N(z_t; C_t x_t, Q_t) & \sim N(x_t; \bar{\mu}_t, \bar{\Sigma}_t) \\
 & \Downarrow & 
 \end{array}$$

$$\text{bel}(x_t) = \eta \exp\left\{-\frac{1}{2}(z_t - C_t x_t)^T Q_t^{-1}(z_t - C_t x_t)\right\} \exp\left\{-\frac{1}{2}(x_t - \bar{\mu}_t)^T \bar{\Sigma}_t^{-1}(x_t - \bar{\mu}_t)\right\}$$

$$\text{bel}(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t \bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t) \bar{\Sigma}_t \end{cases} \quad \text{with} \quad K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

# Kalman Filter Algorithm

1. Algorithm **Kalman\_filter**(  $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$ ):

2. Prediction:

3.  $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$

4.  $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$

5. Correction:

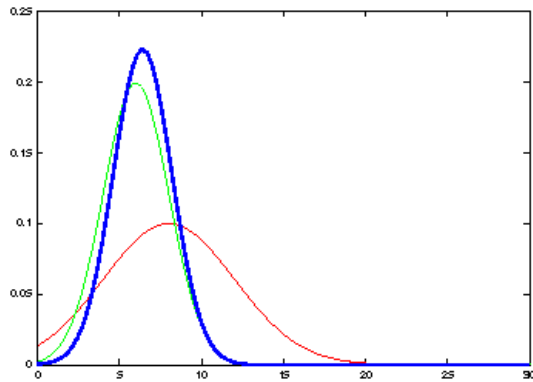
6.  $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$

7.  $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$

8.  $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$

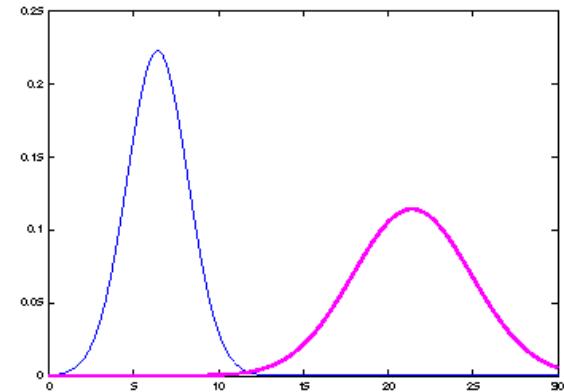
9. Return  $\mu_t, \Sigma_t$

# Il ciclo Prediction-Correction

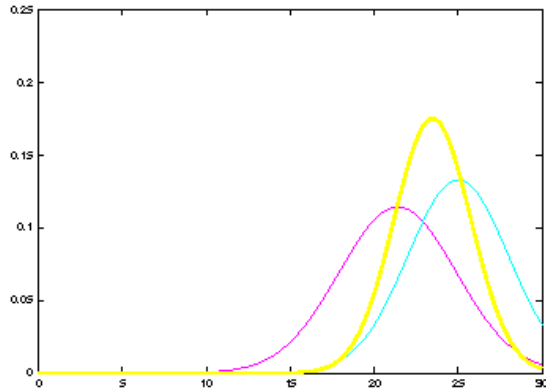


$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{act,t}^2 \end{cases}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$

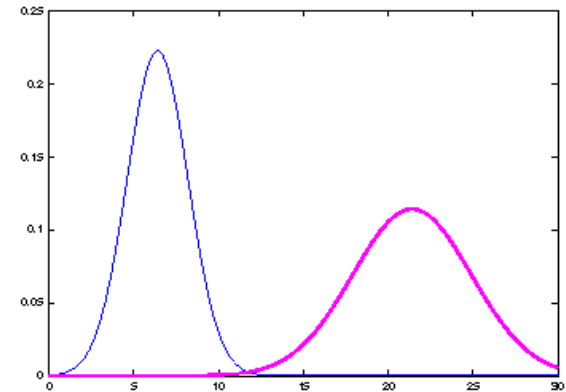


# The Prediction-Correction-Cycle



$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - \bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t)\bar{\sigma}_t^2, K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2} \end{cases}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_tC_t)\bar{\Sigma}_t, K_t = \bar{\Sigma}_tC_t^T(C_t\bar{\Sigma}_tC_t^T + Q_t)^{-1} \end{cases}$$



Correction

# Il ciclo Prediction-Correction



$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - \bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t)\bar{\sigma}_t^2 \end{cases}, K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_{t-1}^2 + \sigma_{act,t}^2 \end{cases}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t \bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t) \bar{\Sigma}_t \end{cases}, K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$



# Sommario: Filtro di Kalman

- **Efficiente**: Polinomiale nella dim delle misure  $k$  e nella dim dello stato  $n$ :  
$$O(k^{2.376} + n^2)$$
- **Ottimale per sistemi lineari Gaussiani!**
- Molti sistemi robotici sono **nonlineari!**

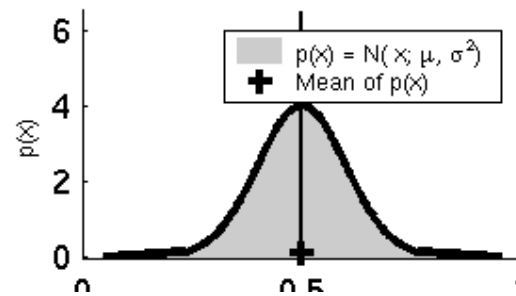
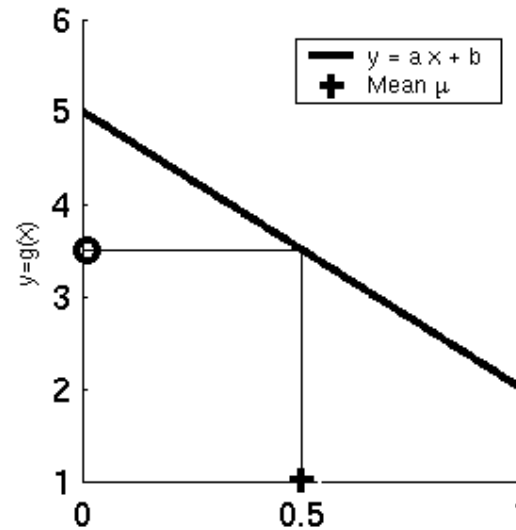
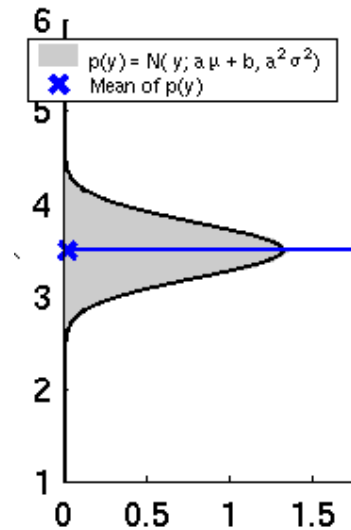
# Sistemi Dinamici Nonlineari

- Molti problemi robotici realistici richiedono funzioni nonlineari

$$x_t = g(u_t, x_{t-1})$$

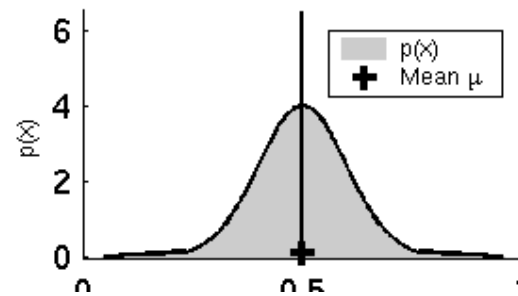
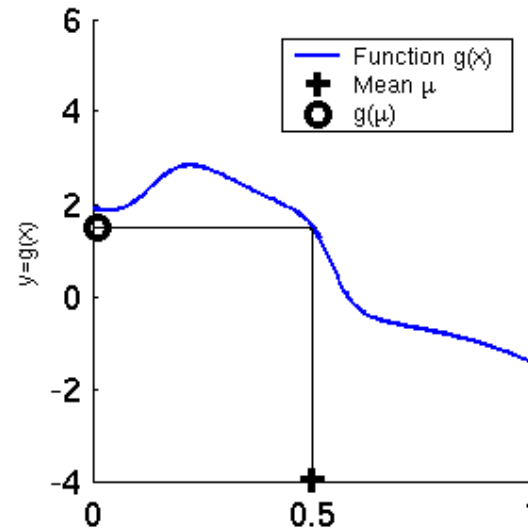
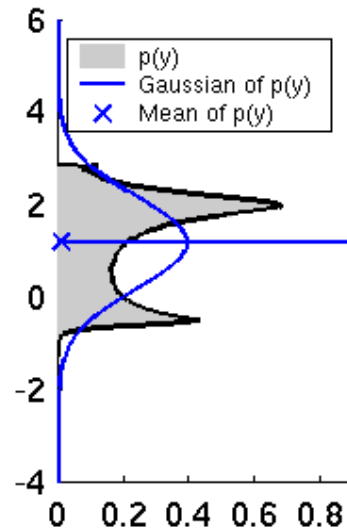
$$z_t = h(x_t)$$

# Assunzione lineare rivisitata

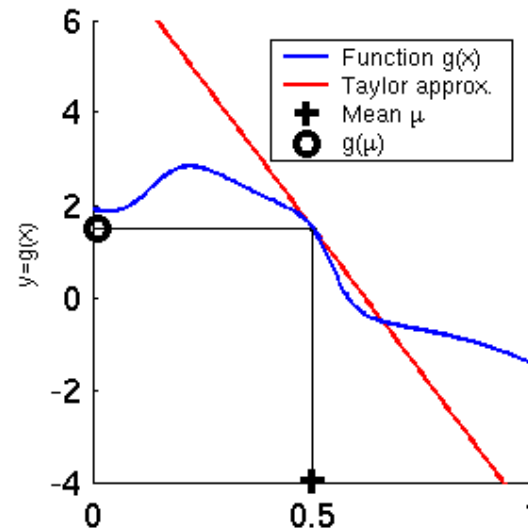
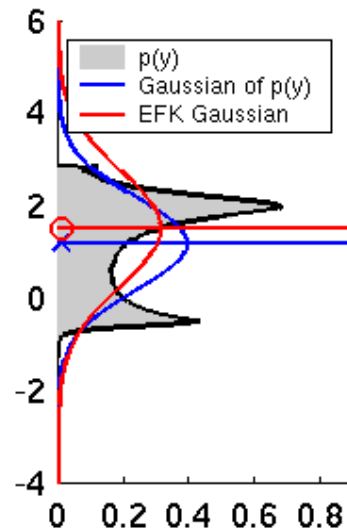




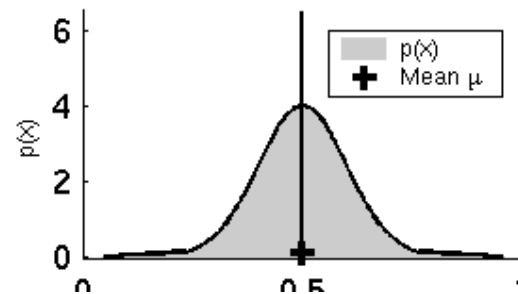
# Funzione Non-lineare



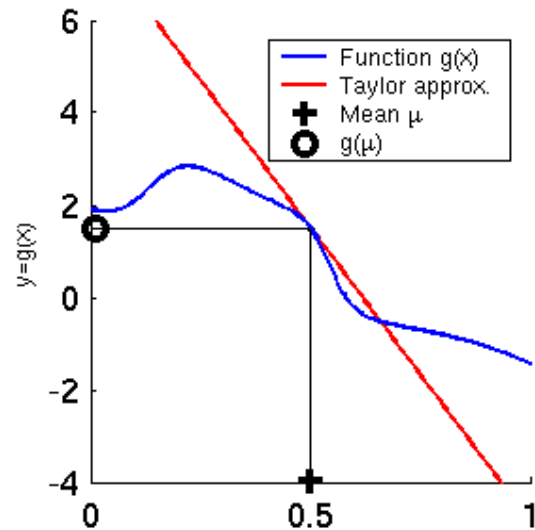
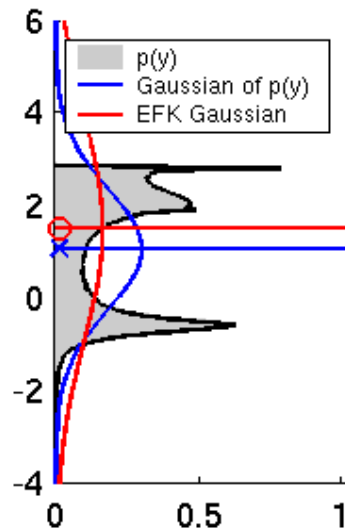
# Linearizzazione EKF (1)



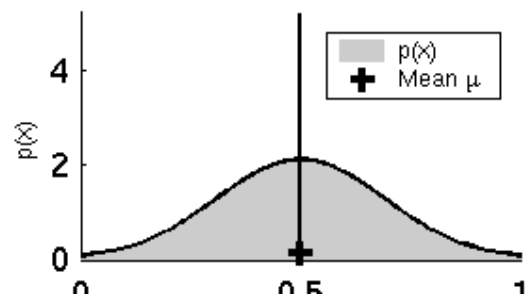
Approx lineare



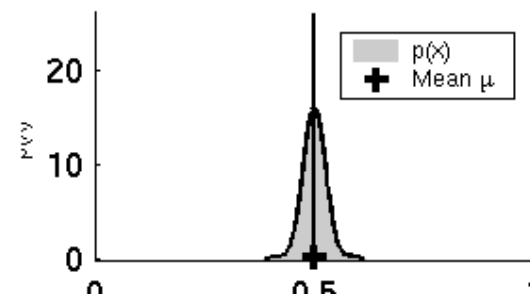
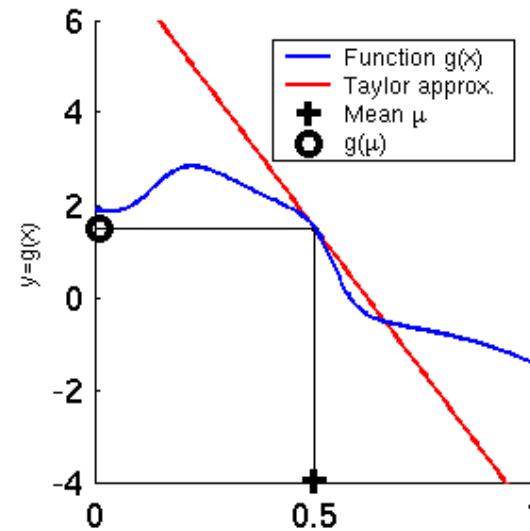
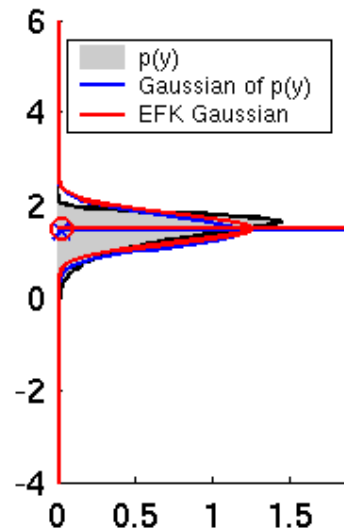
# Linearizzazione EKF (2)



Maggiore incertezza,  
maggiore approx



# Linearizzazione EKF (3)



Minore incertezza,  
minore approx

# Linearizzazione EKF: Espansione del Primo Ordine della serie di Taylor

- Predizione:

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} (x_{t-1} - \mu_{t-1})$$

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + G_t (x_{t-1} - \mu_{t-1})$$

$G_t$  corrisponde a:  
 $A_t \quad B_t$

- Correzione:

$$h(x_t) \approx h(\bar{\mu}_t) + \frac{\partial h(\bar{\mu}_t)}{\partial x_t} (x_t - \bar{\mu}_t)$$

$$h(x_t) \approx h(\bar{\mu}_t) + H_t (x_t - \bar{\mu}_t)$$

$H_t$  corrisponde a:  
 $C_t$

# Jacobiano

- Matrice  $m \times n$
- Data la funzione:

$$g(x) = \begin{pmatrix} g_1(x) \\ \dots \\ g_m(x) \end{pmatrix}$$

- Lo Jacobiano è:

$$G_x = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \dots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \dots & \frac{\partial g_m}{\partial x_n} \end{pmatrix}$$

# Algoritmo EKF

1. **Extended\_Kalman\_filter**(  $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$ ):

2. Prediction:

3.  $\bar{\mu}_t = g(u_t, \mu_{t-1})$   $\longleftarrow$   $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$   
4.  $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t$   $\longleftarrow$   $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$

5. Correction:

6.  $K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1}$   $\longleftarrow$   $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$   
7.  $\mu_t = \bar{\mu}_t + K_t (z_t - h(\bar{\mu}_t))$   $\longleftarrow$   $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$   
8.  $\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$   $\longleftarrow$   $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$

9. Return  $\mu_t, \Sigma_t$

$$H_t = \frac{\partial h(\bar{\mu}_t)}{\partial x_t} \quad G_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}}$$

# Localizzazione

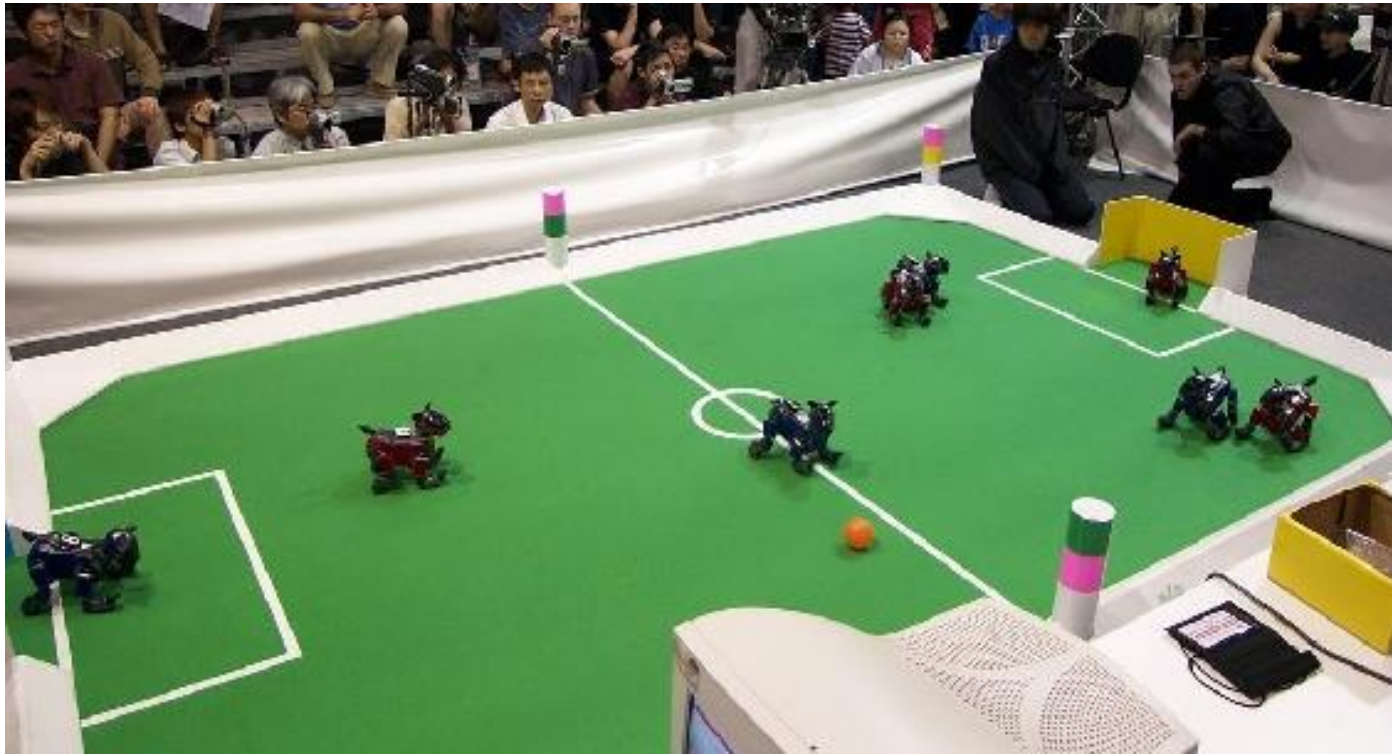
“Using sensory information to locate the robot in its environment is the most fundamental problem to providing a mobile robot with autonomous capabilities.” [Cox '91]

- **Data**
  - Mappa dell'ambiente
  - Sequenza di misure di sensori.
- **Ricerca**
  - Stima della posa del robot.
- **Problemi**
  - Position tracking
  - Global localization
  - Kidnapped robot problem (recovery)



# Localizzazione Landmark-based

Si assume una mappa definita da un insieme di landmark, l'osservazione genera un vettore  $\{z_1, \dots, z_n\}$  di misure di feature univocamente identificabili



6 marker colorati diversamente

# Localizzazione Markoviana

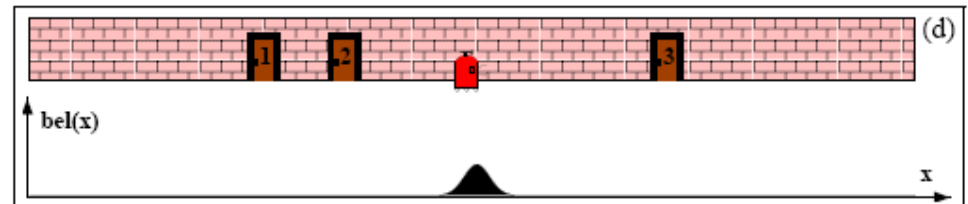
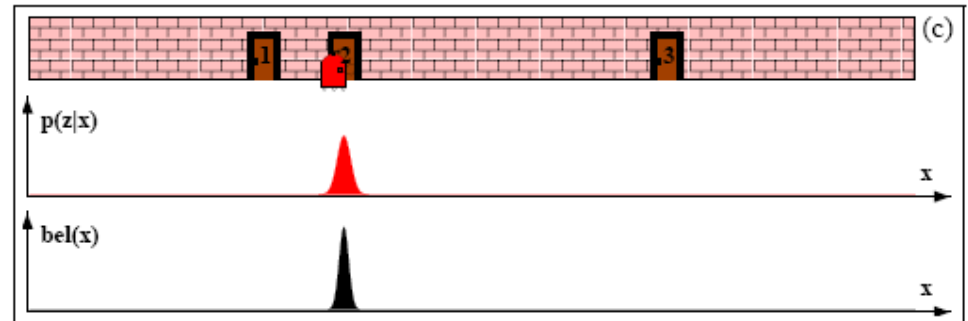
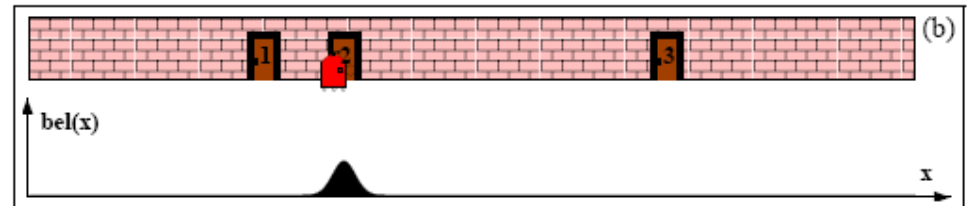
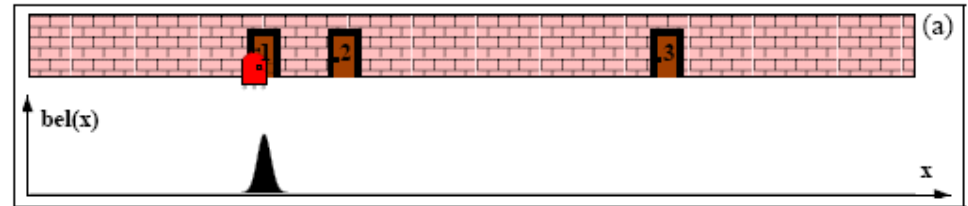
## Algoritmo generale

```
1:   Algorithm Markov_Localization( $bel(x_{t-1}), u_t, z_t, m$ ):  
2:     for all  $x_t$  do  
3:        $\overline{bel}(x_t) = \int p(x_t | u_t, x_{t-1}, m) bel(x_{t-1}) dx$   
4:        $bel(x_t) = \eta p(z_t | x_t, m) \overline{bel}(x_t)$   
5:     endfor  
6:     return  $bel(x_t)$ 
```

# Assunzione Gaussiana

La distribuzione si assume gaussiana

1. Mappa features con corrispondenze note
2. Posizione iniziale quasi nota (approx da una gaussiana)



# Caso Concreto

- Consideriamo il modello in velocità
- La mappa feature-based con corrispondenze note
- Dato belief (gaussiano) al tempo  $t-1$ 
  - valore medio e covarianza al tempo  $t-1$
- Genera stima di belief al tempo  $t$ 
  - valore medio e covarianza al tempo  $t$

# Modello in Velocità:

$$\begin{aligned} \begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} &= \begin{pmatrix} x_c + \frac{v}{\omega} \sin(\theta + \omega \Delta t) \\ y_c - \frac{v}{\omega} \cos(\theta + \omega \Delta t) \\ \theta + \omega \Delta t \end{pmatrix} \\ &= \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v}{\omega} \sin \theta + \frac{v}{\omega} \sin(\theta + \omega \Delta t) \\ \frac{v}{\omega} \cos \theta - \frac{v}{\omega} \cos(\theta + \omega \Delta t) \\ \omega \Delta t \end{pmatrix} \end{aligned}$$

Ma c'è il rumore, quindi le coordinate reali sono:

$$\begin{pmatrix} \hat{v} \\ \hat{\omega} \end{pmatrix} = \begin{pmatrix} v \\ \omega \end{pmatrix} + \begin{pmatrix} \varepsilon_{\alpha_1 |v| + \alpha_2 |\omega|} \\ \varepsilon_{\alpha_3 |v| + \alpha_4 |\omega|} \end{pmatrix}$$

Quindi:

$$\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{\hat{v}}{\hat{\omega}} \sin \theta + \frac{\hat{v}}{\hat{\omega}} \sin(\theta + \hat{\omega} \Delta t) \\ \frac{\hat{v}}{\hat{\omega}} \cos \theta - \frac{\hat{v}}{\hat{\omega}} \cos(\theta + \hat{\omega} \Delta t) \\ \hat{\omega} \Delta t \end{pmatrix}$$

# Mappa feature-based

Feature-based map: mappa come insieme di features localizzate  $\{m_1, \dots, m_n\}$

Con  $m_{i,x}$   $m_{i,y}$  si indica la locazione  $x,y$  per la feature  $i$ -esima

Legame probabilistico tra feature  $j$  nella mappa globale e feature estratta nella mappa locale

$$\begin{pmatrix} r_t^i \\ \phi_t^i \\ s_t^i \end{pmatrix} = \begin{pmatrix} \sqrt{(m_{j,x} - x)^2 + (m_{j,y} - y)^2} \\ \text{atan2}(m_{j,y} - y, m_{j,x} - x) - \theta \\ s_j \end{pmatrix} + \begin{pmatrix} \varepsilon_{\sigma_r^2} \\ \varepsilon_{\sigma_\phi^2} \\ \varepsilon_{\sigma_s^2} \end{pmatrix}$$

# Derivazione EKF

Modello di moto:

$$\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t} \sin \theta + \frac{v_t}{\omega_t} \sin(\theta + \omega_t \Delta t) \\ \frac{v_t}{\omega_t} \cos \theta - \frac{v_t}{\omega_t} \cos(\theta + \omega_t \Delta t) \\ \omega_t \Delta t + \gamma_t \Delta t \end{pmatrix} \quad u_t = \begin{pmatrix} v_t \\ \omega_t \end{pmatrix}$$

Introducendo il rumore:

$$\underbrace{\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix}}_{x_t} = \underbrace{\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t} \sin \theta + \frac{v_t}{\omega_t} \sin(\theta + \omega_t \Delta t) \\ \frac{v_t}{\omega_t} \cos \theta - \frac{v_t}{\omega_t} \cos(\theta + \omega_t \Delta t) \\ \omega_t \Delta t \end{pmatrix}}_{g(u_t, x_{t-1})} + \mathcal{N}(0, R_t)$$

Linearizzazione di Taylor:

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + G_t (x_{t-1} - \mu_{t-1})$$

Rispetto ad  $x_{t-1} = \mu_{t-1}$

Jacobiano:

$$G_t = g'(u_t, \mu_{t-1}) = \begin{pmatrix} 1 & 0 & \frac{v_t}{\omega_t} \cos \mu_{t-1, \theta} - \frac{v_t}{\omega_t} \cos(\mu_{t-1, \theta} + \omega_t \Delta t) \\ 0 & 1 & \frac{v_t}{\omega_t} \sin \mu_{t-1, \theta} - \frac{v_t}{\omega_t} \sin(\mu_{t-1, \theta} + \omega_t \Delta t) \\ 0 & 0 & 1 \end{pmatrix}$$

# 1. EKF\_localization ( $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, m$ ):

## Predizione:

3.  $G_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} = \begin{pmatrix} \frac{\partial x'}{\partial \mu_{t-1,x}} & \frac{\partial x'}{\partial \mu_{t-1,y}} & \frac{\partial x'}{\partial \mu_{t-1,\theta}} \\ \frac{\partial y'}{\partial \mu_{t-1,x}} & \frac{\partial y'}{\partial \mu_{t-1,y}} & \frac{\partial y'}{\partial \mu_{t-1,\theta}} \\ \frac{\partial \theta'}{\partial \mu_{t-1,x}} & \frac{\partial \theta'}{\partial \mu_{t-1,y}} & \frac{\partial \theta'}{\partial \mu_{t-1,\theta}} \end{pmatrix}$  Jacobiano di  $g$  rispetto alla locazione stimata

5.  $M_t = \begin{pmatrix} (\alpha_1 |v_t| + \alpha_2 |\omega_t|)^2 & 0 \\ 0 & (\alpha_3 |v_t| + \alpha_4 |\omega_t|)^2 \end{pmatrix}$  Rumore di movimento

6.  $V_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial u_t} = \begin{pmatrix} \frac{\partial x'}{\partial v_t} & \frac{\partial x'}{\partial \omega_t} \\ \frac{\partial y'}{\partial v_t} & \frac{\partial y'}{\partial \omega_t} \\ \frac{\partial \theta'}{\partial v_t} & \frac{\partial \theta'}{\partial \omega_t} \end{pmatrix}$  Jacobiano di  $g$  rispetto al controllo  $(v,w)$  per mappare il rumore di movimento

7.  $\bar{\mu}_t = g(u_t, \mu_{t-1})$  Posa predetta

8.  $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + V_t M_t V_t^T$  Covarianza predetta



# Passo di Correzione

Corrispondenza features

$$z_t^i = \begin{pmatrix} r_t^i \\ \phi_t^i \\ s_t^i \end{pmatrix} = \begin{pmatrix} \sqrt{(m_{j,x} - x)^2 + (m_{j,y} - y)^2} \\ \text{atan2}(m_{j,y} - y, m_{j,x} - x) - \theta \\ m_{j,s} \end{pmatrix} + \begin{pmatrix} \mathcal{N}(0, \sigma_r) \\ \mathcal{N}(0, \sigma_\phi) \\ \mathcal{N}(0, \sigma_s) \end{pmatrix}$$

$$z_t^i = h(x_t, j, m) + \mathcal{N}(0, Q_t) \quad h(x_t, j, m) = \begin{pmatrix} \sqrt{(m_{j,x} - x)^2 + (m_{j,y} - y)^2} \\ \text{atan2}(m_{j,y} - y, m_{j,x} - x) - \theta \\ m_{j,s} \end{pmatrix}$$

$$Q_t = \begin{pmatrix} \sigma_r & 0 & 0 \\ 0 & \sigma_\phi & 0 \\ 0 & 0 & \sigma_s \end{pmatrix} \quad \text{Covarianza dell'errore di misura addizionale}$$

segnatura

Si procede con la linearizzazione di Taylor:

$$h(x_t, j, m) \approx h(\bar{\mu}_t, j, m) + H_t^i (x_t - \bar{\mu}_t)$$

Rispetto alla posa stimata:  $\bar{\mu}_t = (\bar{\mu}_{t,x} \ \bar{\mu}_{t,y} \ \bar{\mu}_{t,\theta})^T$

H jacobiano (posa stimata):

$$H_t^i = h'(\bar{\mu}_t, j, m) = \begin{pmatrix} \frac{\partial r_t^i}{\partial \bar{\mu}_{t,x}} & \frac{\partial r_t^i}{\partial \bar{\mu}_{t,y}} & \frac{\partial r_t^i}{\partial \bar{\mu}_{t,\theta}} \\ \frac{\partial \phi_t^i}{\partial \bar{\mu}_{t,x}} & \frac{\partial \phi_t^i}{\partial \bar{\mu}_{t,y}} & \frac{\partial \phi_t^i}{\partial \bar{\mu}_{t,\theta}} \\ \frac{\partial s_t^i}{\partial \bar{\mu}_{t,x}} & \frac{\partial s_t^i}{\partial \bar{\mu}_{t,y}} & \frac{\partial s_t^i}{\partial \bar{\mu}_{t,\theta}} \end{pmatrix} \quad H_t^i = \begin{pmatrix} \frac{m_{j,x} - \bar{\mu}_{t,x}}{\sqrt{q_t}} & \frac{y_t - \bar{\mu}_{t,y}}{\sqrt{q_t}} & 0 \\ \frac{\bar{\mu}_{t,y} - y_t}{q_t} & \frac{m_{j,x} - \bar{\mu}_{t,x}}{q_t} & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Non ha effetto

with  $q_t = (m_{j,x} - \bar{\mu}_{t,x})^2 + (m_{j,y} - \bar{\mu}_{t,y})^2$ , and  $j = c_t^i$

# 1. EKF\_localization ( $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, m$ ):

## Correzione:

3.  $\hat{z}_t = \begin{pmatrix} \sqrt{(m_x - \bar{\mu}_{t,x})^2 + (m_y - \bar{\mu}_{t,y})^2} \\ \text{atan2}(m_y - \bar{\mu}_{t,y}, m_x - \bar{\mu}_{t,x}) - \bar{\mu}_{t,\theta} \end{pmatrix}$  Pred. media di misura

5.  $H_t = \frac{\partial h(\bar{\mu}_t, m)}{\partial x_t} = \begin{pmatrix} \frac{\partial r_t}{\partial \bar{\mu}_{t,x}} & \frac{\partial r_t}{\partial \bar{\mu}_{t,y}} & \frac{\partial r_t}{\partial \bar{\mu}_{t,\theta}} \\ \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,x}} & \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,y}} & \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,\theta}} \end{pmatrix}$  Jacobiano di  $h$  rispetto alla locazione

6.  $Q_t = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_r^2 \end{pmatrix}$

7.  $S_t = H_t \bar{\Sigma}_t H_t^T + Q_t$  Pred. misura di covarianza

8.  $K_t = \bar{\Sigma}_t H_t^T S_t^{-1}$  Guadagno di Kalman

9.  $\mu_t = \bar{\mu}_t + K_t (z_t - \hat{z}_t)$  Posa aggiornata

10.  $\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$  Covarianza aggiornata

1: **Algorithm EKF localization known correspondences**( $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, c_t, m$ ):

2: 
$$\bar{\mu}_t = \mu_{t-1} + \begin{pmatrix} -\frac{v_t}{\omega_t} \sin \mu_{t-1, \theta} + \frac{v_t}{\omega_t} \sin(\mu_{t-1, \theta} + \omega_t \Delta t) \\ \frac{v_t}{\omega_t} \cos \mu_{t-1, \theta} - \frac{v_t}{\omega_t} \cos(\mu_{t-1, \theta} + \omega_t \Delta t) \\ \omega_t \Delta t \end{pmatrix}$$
 Nuova posa stimata

3: 
$$G_t = \begin{pmatrix} 1 & 0 & \frac{v_t}{\omega_t} \cos \mu_{t-1, \theta} - \frac{v_t}{\omega_t} \cos(\mu_{t-1, \theta} + \omega_t \Delta t) \\ 0 & 1 & \frac{v_t}{\omega_t} \sin \mu_{t-1, \theta} - \frac{v_t}{\omega_t} \sin(\mu_{t-1, \theta} + \omega_t \Delta t) \\ 0 & 0 & 1 \end{pmatrix}$$
 Jacobiano

4: 
$$\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t$$
 Covarianza predizione:  
aggiornamento + rumore di movimento:  
$$R = V_t M_t V_t^T$$

5: 
$$Q_t = \begin{pmatrix} \sigma_r & 0 & 0 \\ 0 & \sigma_\phi & 0 \\ 0 & 0 & \sigma_s \end{pmatrix}$$
 Covarianza rumore misura addizionale

6: for all observed features  $z_t^i = (r_t^i \ \phi_t^i \ s_t^i)^T$  do

7: 
$$j = c_t^i$$
 corrispondenza

8: 
$$\delta = \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} m_{j,x} - \bar{\mu}_{t,x} \\ m_{j,y} - \bar{\mu}_{t,y} \end{pmatrix}$$
 posizione

9: 
$$q = \delta^T \delta$$

10: 
$$\hat{z}_t^i = \begin{pmatrix} \sqrt{q} \\ \text{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \\ m_{j,s} \end{pmatrix}$$
 Distanza rel. e angolo

11: 
$$H_t^i = \frac{1}{q} \begin{pmatrix} \sqrt{q} \delta_x & -\sqrt{q} \delta_y & 0 \\ \delta_y & \delta_x & -1 \\ 0 & 0 & 0 \end{pmatrix}$$
 Jacobiano del modello di misura

12: 
$$K_t^i = \bar{\Sigma}_t H_t^{i,T} (H_t^i \bar{\Sigma}_t H_t^{i,T} + Q_t)^{-1}$$
 Guadagno di Kalman per ogni osservazione

13: endfor

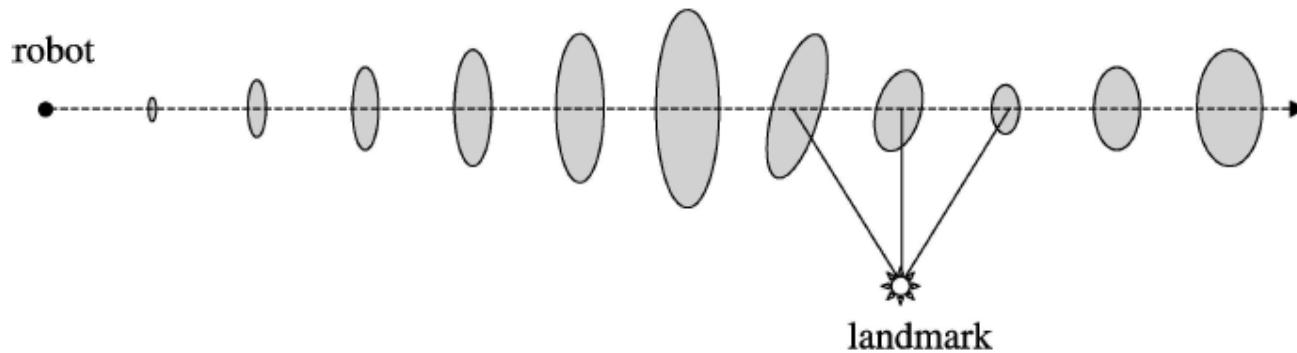
14: 
$$\mu_t = \bar{\mu}_t + \sum_i K_t^i (z_t^i - \hat{z}_t^i)$$

15: 
$$\Sigma_t = (I - \sum_i K_t^i H_t^i) \bar{\Sigma}_t$$
 Nuova Posa e covarianza

16: return  $\mu_t, \Sigma_t$

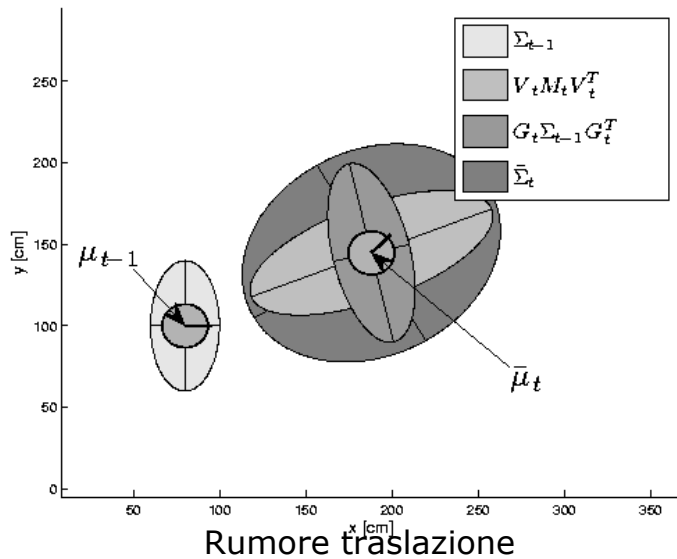
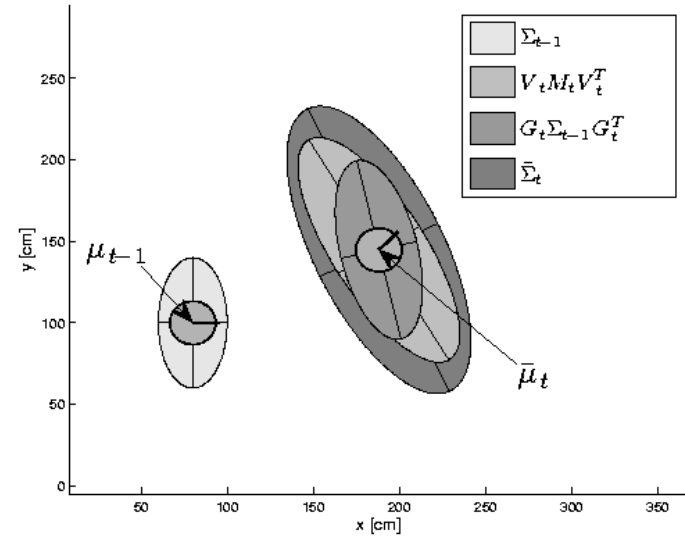
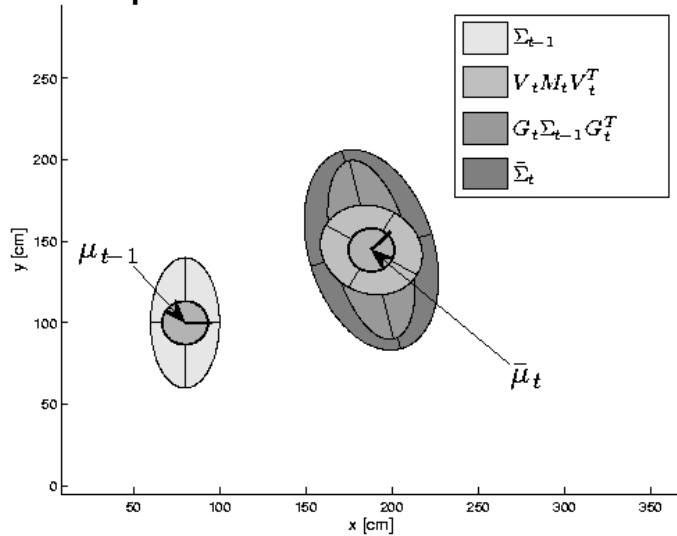
# Localizzazione EKF

Incertezza diminuisce con percezione landmark, poi aumenta di nuovo

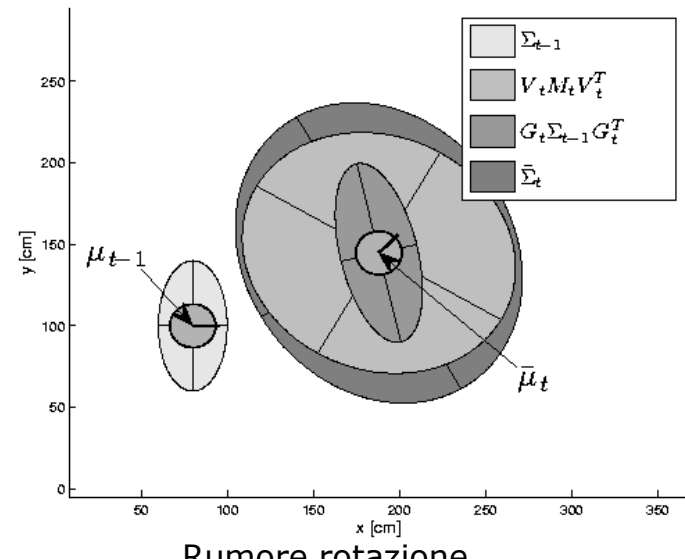


# Passo di predizione EKF

Spostamento di 90cm e 45 g

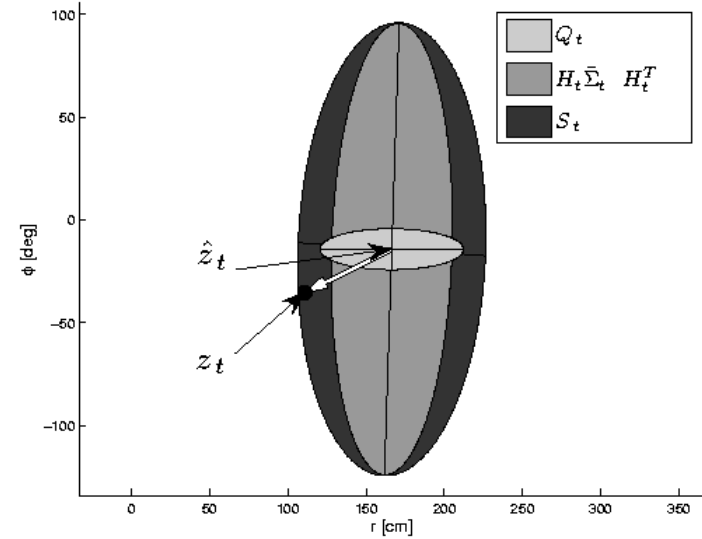
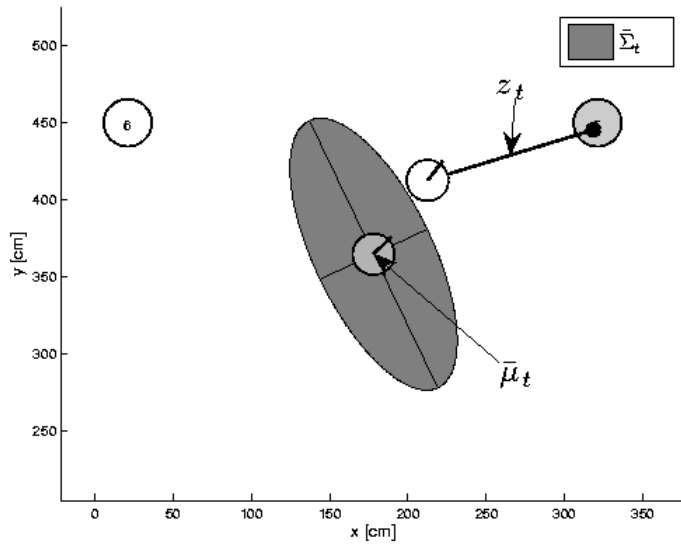


Rumore traslazione

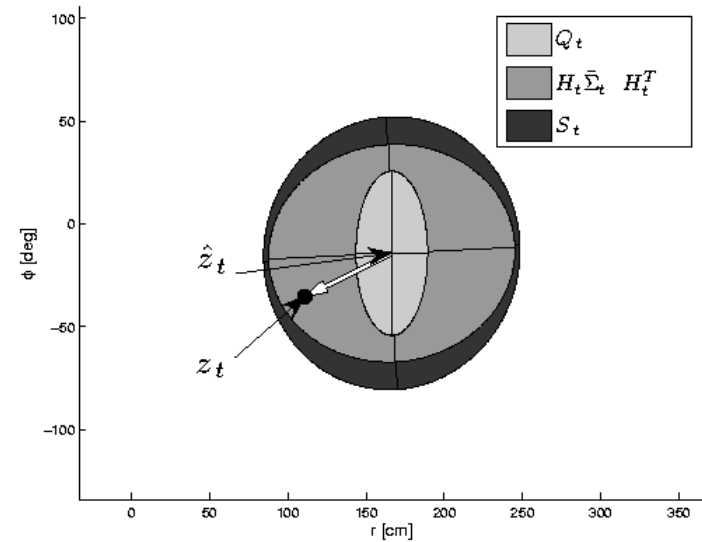
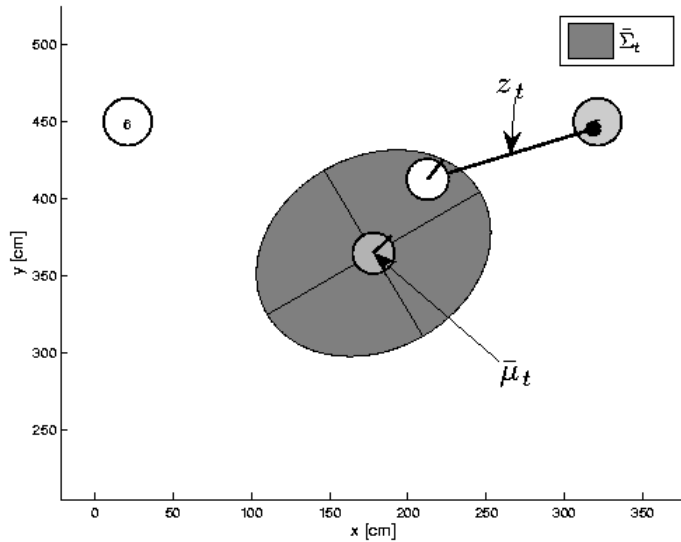


Rumore rotazione

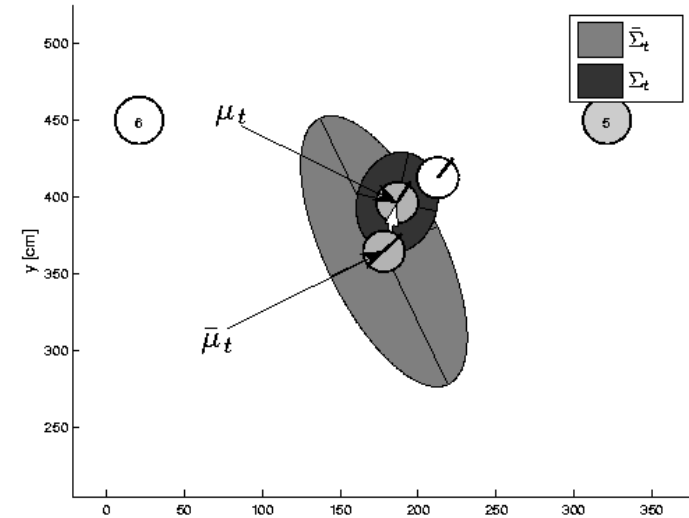
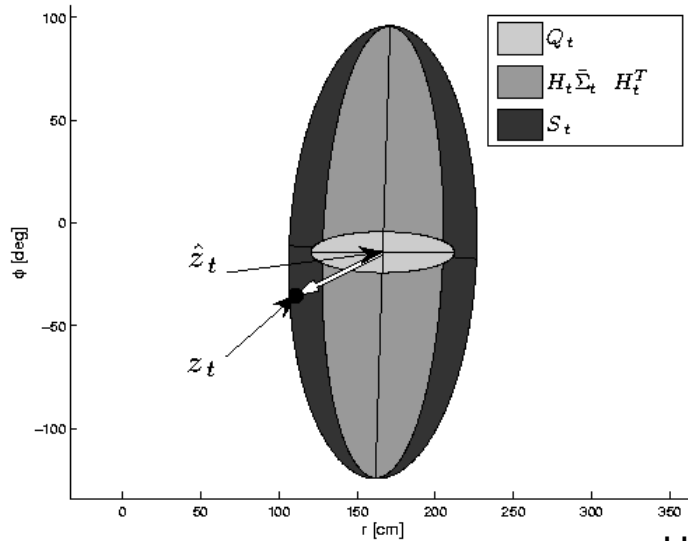
# EKF Observation Prediction Step



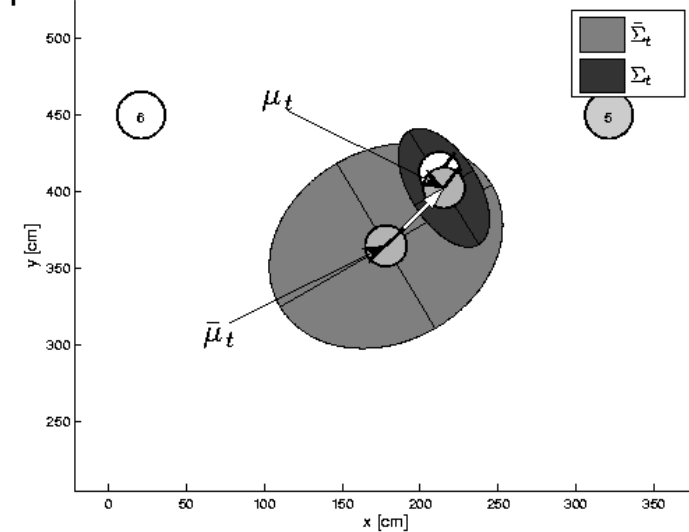
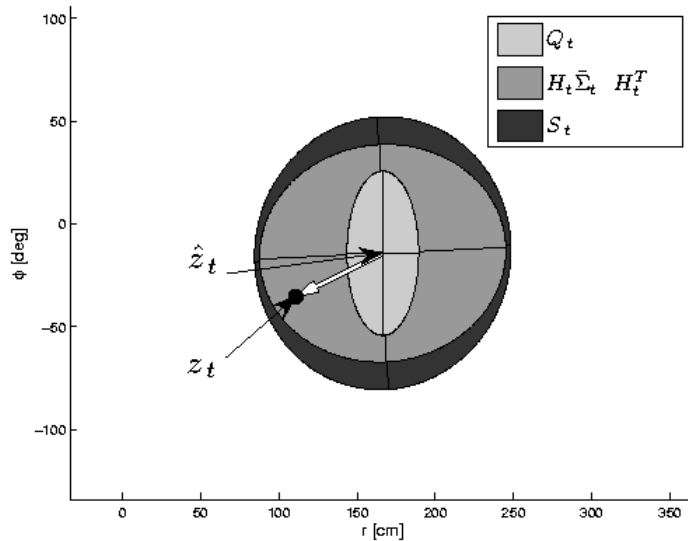
Vettore di innovazione



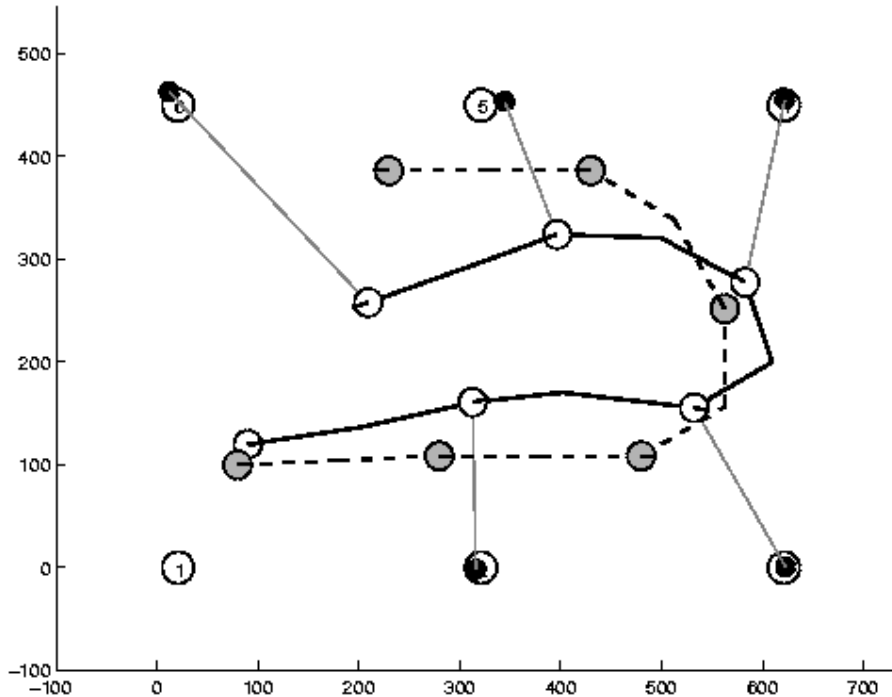
# Passo di correzione EKF



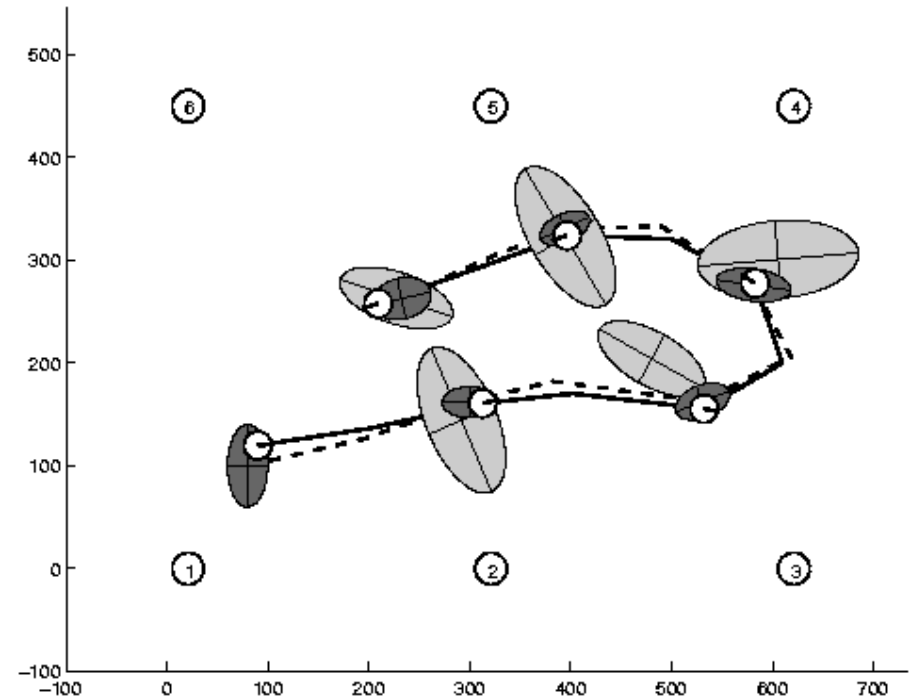
Update della stima. Vettore di innovazione e guadagno di Kalman per riposizionamento



# Estimation Sequence (1)



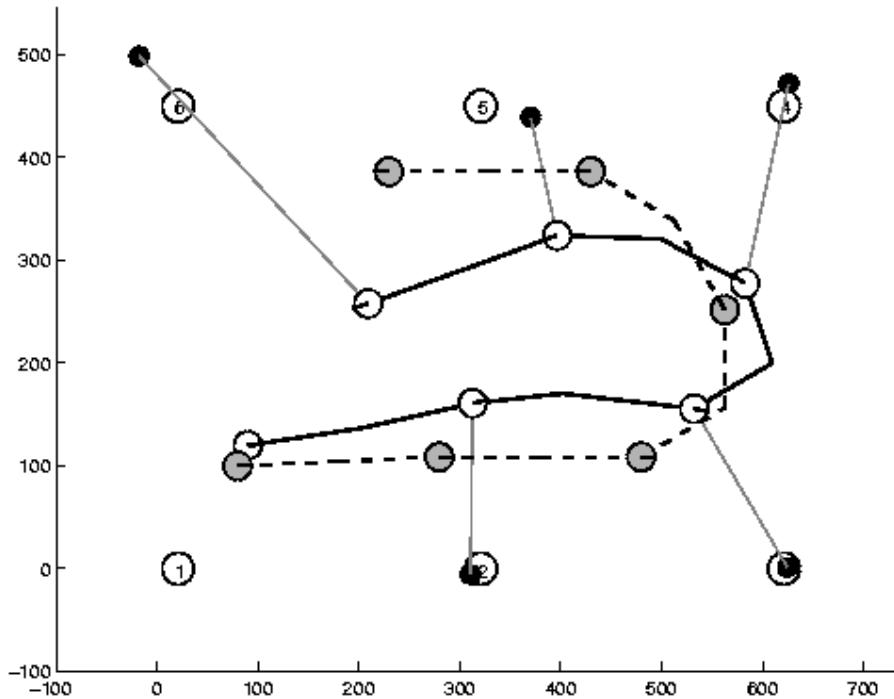
Traiettorie reali (solide),  
traiettorie controllate (a tratti)



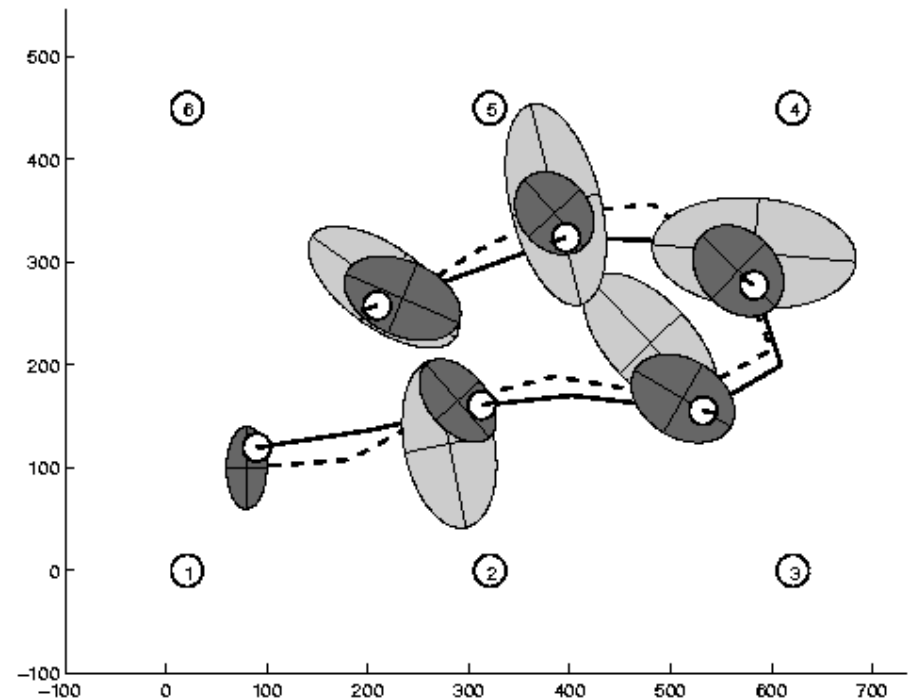
Basso errore



# Estimation Sequence (2)

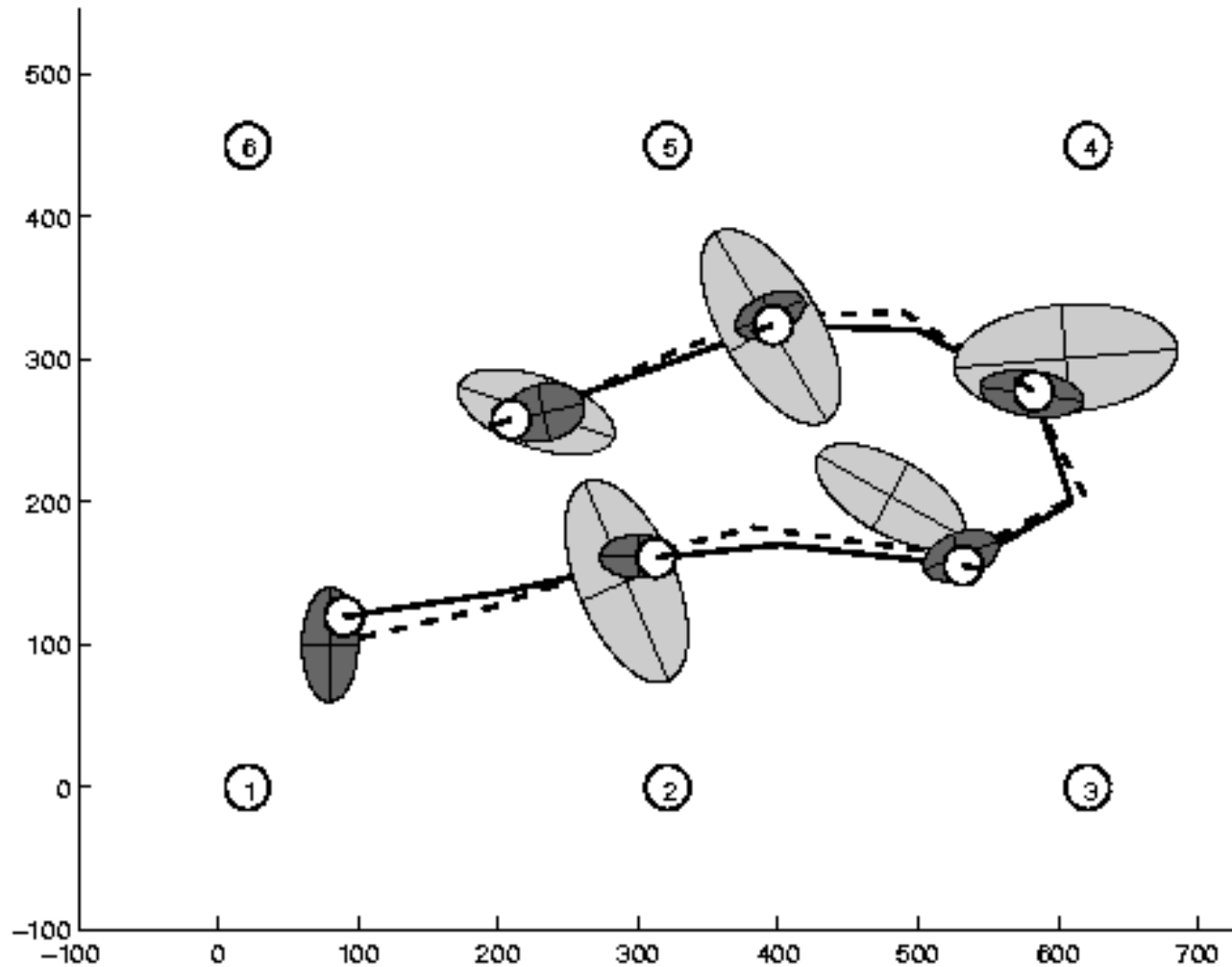


Traiettorie reali (solide),  
traiettorie controllate (a tratti)



Errore di misura

# Confronto con la GroundTruth

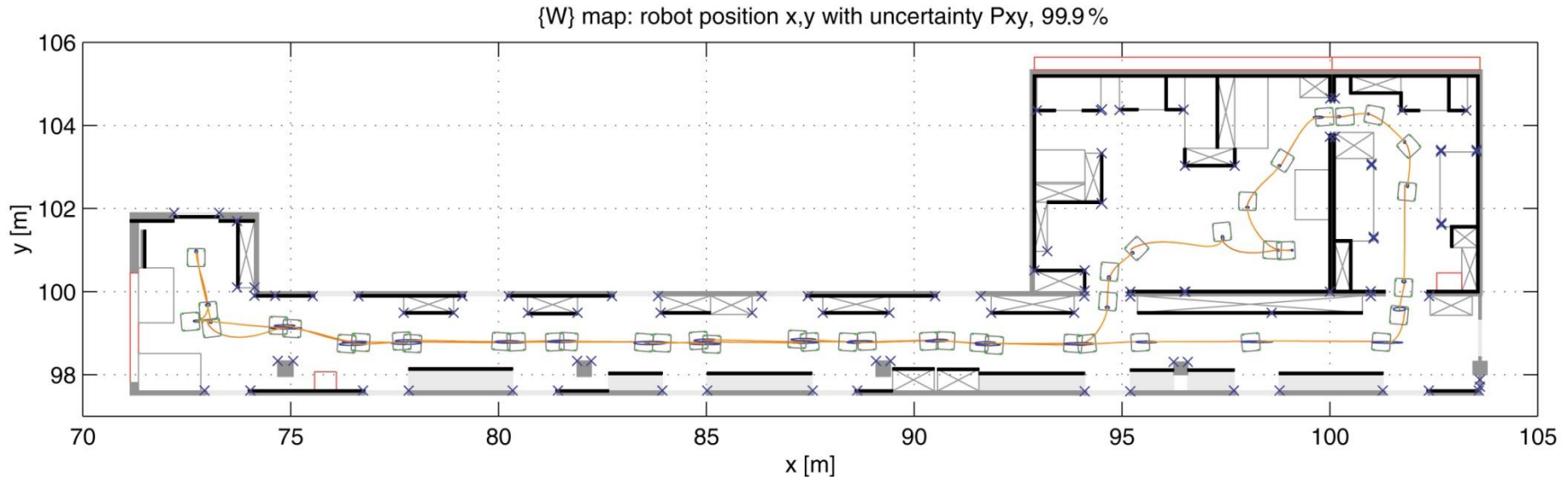


# Riassunto EKF

- **Efficiente**: Polinomiale nella dim della misura  $k$  e della dim dello stato  $n$ :  
$$O(k^{2.376} + n^2)$$
- **Non ottimale!**
- Può **divergere** se larghe nonlinearità
- Funziona bene anche quando tutte le assunzioni sono violate!

# Kalman Filter-based System

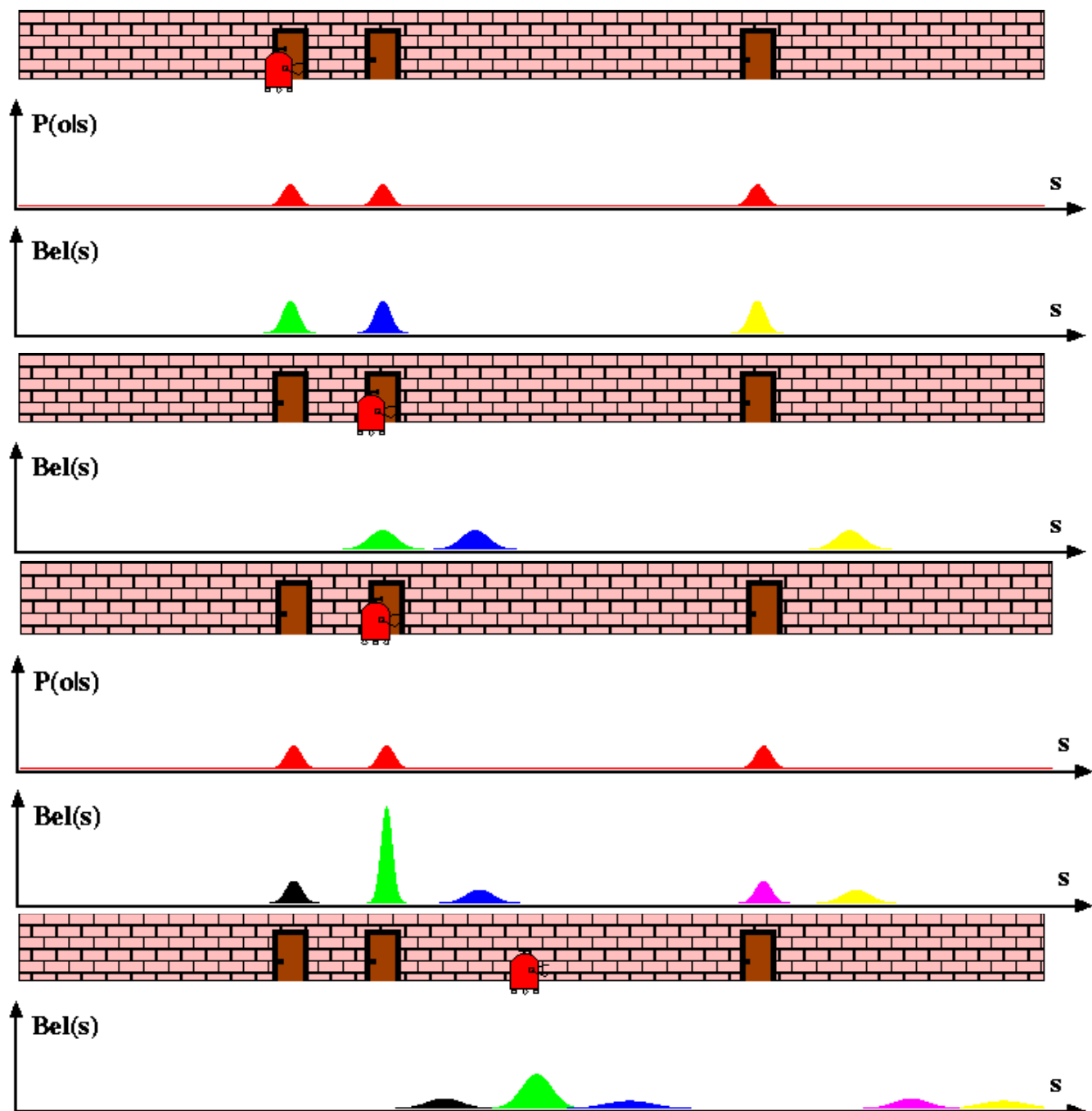
- [Arras et al. 98]:
  - Laser range-finder and vision
  - High precision (<1cm accuracy)



[Courtesy of Kai Arras]

# Multi-hypothesis Tracking

Mistura di  
Gaussiane



# Localizzazione con MHT

- Belief è rappresentato da ipotesi multiple
- Ogni ipotesi è tracciata da un filtro di Kalman
- **Problemi:**
  - **Associazione:** quale osservazione a quale ipotesi?
  - **Ipotesi:** quando aggiungere / cancellare ipotesi?
- Molta letteratura su target tracking, motion correspondence etc.

# Localizzazione MHT

Mistura di gaussiane

$$bel(x_t) = \frac{1}{\sum_l \psi_{t,l}} \sum_l \psi_{t,l} \det(2\pi\Sigma_{t,l})^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(x_t - \mu_{t,l})^T \Sigma_{t,l}^{-1} (x_t - \mu_{t,l})\right\}$$

$\psi_{t,l}$  è il peso della gaussiana

Ogni componente  $l$  si basa su un vettore di associazioni

$$bel_l(x_t) = p(x_t | z_{1:t}, u_{1:t}, c_{1:t,l})$$

Associazione ha un peso

$$\psi_{t,m} = \psi_{t,l} p(z_t | c_{1:t-1,l}, c_{t,m}, z_{1:t-1}, u_{1:t})$$

# MHT: Implemented System (1)

- Ipotesi estratte da scan LRF
- Ogni ipotesi probabilità di essere corretta:

$$H_i = \{ \hat{x}_i, \Sigma_i, P(H_i) \}$$

- Probabilità ipotesi calcolata con Bayes

$$P(H_i | s) = \frac{P(s | H_i)P(H_i)}{P(s)}$$

- Ipotesi con bassa probabilità sono cancellate.
- Nuovi candidati estratti da scan LRF.

$$C_j = \{ z_j, R_j \}$$

[Jensfelt et al. '00]



# MHT: Implemented System (2)

Robot view

Pose candidates

