

Robotica Probabilistica

Filtri Bayesiani

Filteri Gaussiani

Filtri Bayesiani Reminder

- Predizione

$$\overline{bel}(x_t) = \int p(x_t | u_t, x_{t-1}) bel(x_{t-1}) dx_{t-1}$$

- Correzione

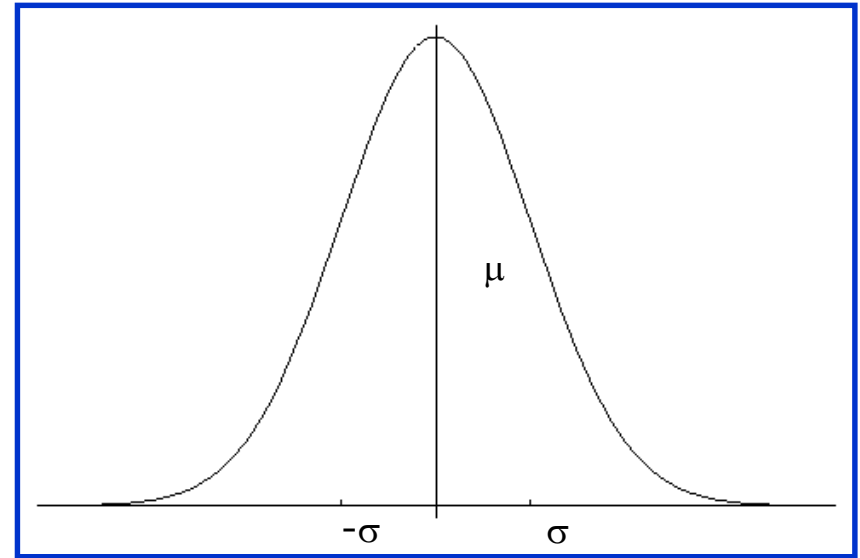
$$bel(x_t) = \eta p(z_t | x_t) \overline{bel}(x_t)$$

Gaussiane

$p(x) \sim N(\mu, \sigma^2)$:

$$p(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2}}$$

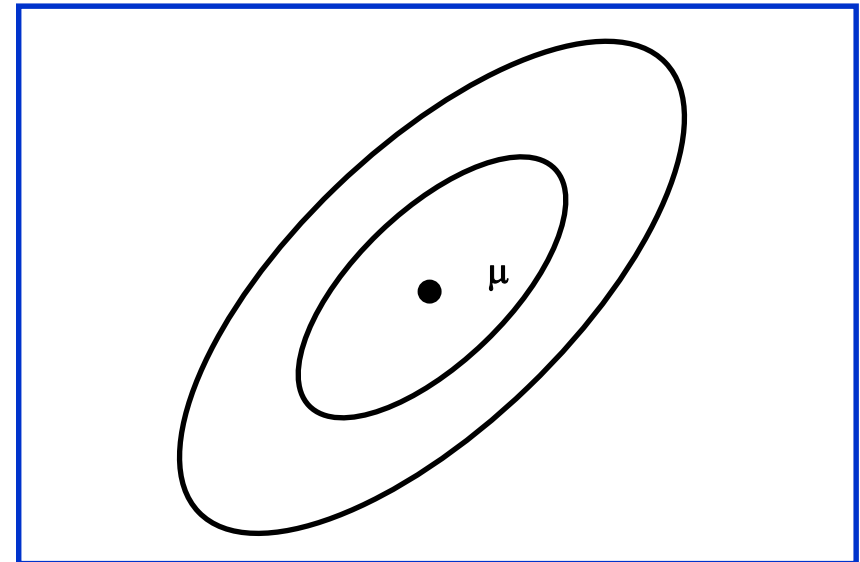
Univariata



$p(\mathbf{x}) \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$:

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2} |\boldsymbol{\Sigma}|^{1/2}} e^{-\frac{1}{2} (\mathbf{x}-\boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{x}-\boldsymbol{\mu})}$$

Multivariata



Ellissoide di equidensità definito dagli autovettori ed autovalori della matrice di covarianza

Propert  delle Gaussiane

$$\left. \begin{array}{l} X \sim N(\mu, \sigma^2) \\ Y = aX + b \end{array} \right\} \Rightarrow Y \sim N(a\mu + b, a^2\sigma^2)$$

$$\left. \begin{array}{l} X_1 \sim N(\mu_1, \sigma_1^2) \\ X_2 \sim N(\mu_2, \sigma_2^2) \end{array} \right\} \Rightarrow p(X_1) \cdot p(X_2) \sim N\left(\frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \mu_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \mu_2, \frac{1}{\sigma_1^{-2} + \sigma_2^{-2}}\right)$$

Multivariate

$$\left. \begin{array}{l} X \sim N(\mu, \Sigma) \\ Y = AX + B \end{array} \right\} \Rightarrow Y \sim N(A\mu + B, A\Sigma A^T)$$

$$\left. \begin{array}{l} X_1 \sim N(\mu_1, \Sigma_1) \\ X_2 \sim N(\mu_2, \Sigma_2) \end{array} \right\} \Rightarrow p(X_1) \cdot p(X_2) \sim N\left(\frac{\Sigma_2}{\Sigma_1 + \Sigma_2} \mu_1 + \frac{\Sigma_1}{\Sigma_1 + \Sigma_2} \mu_2, \frac{1}{\Sigma_1^{-1} + \Sigma_2^{-1}}\right)$$

- Si rimane nel "mondo gaussiano" con trasformazioni lineari.

Filtro di Kalman Discreto

Stima lo stato x di un processo a tempo discreto governato da una equazione alle differenze stocastica lineare

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

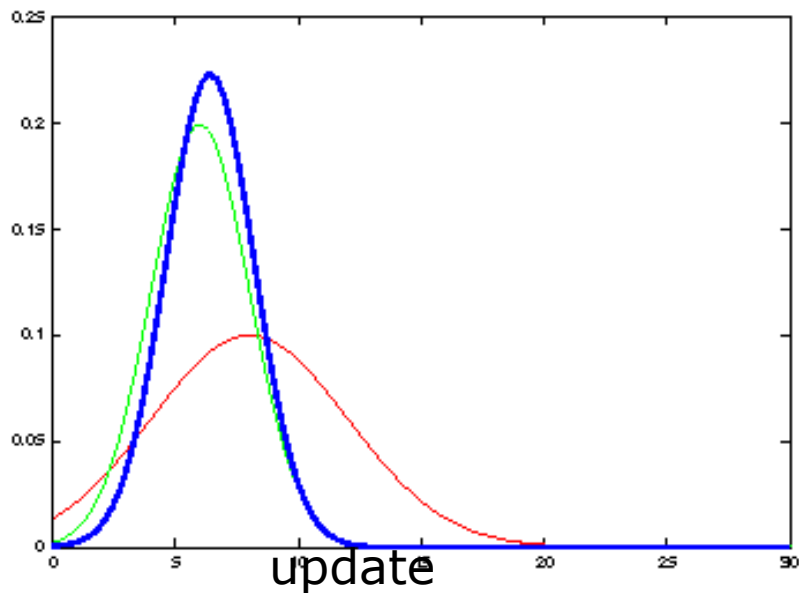
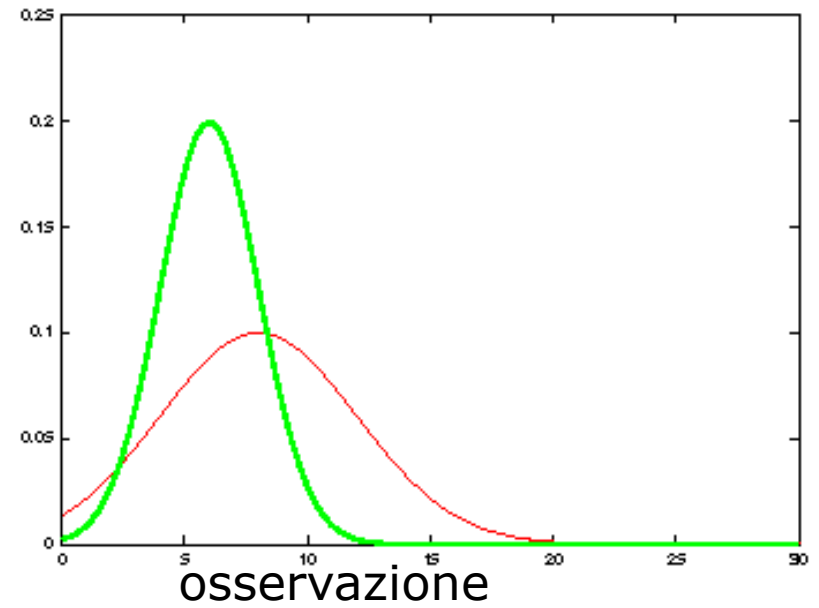
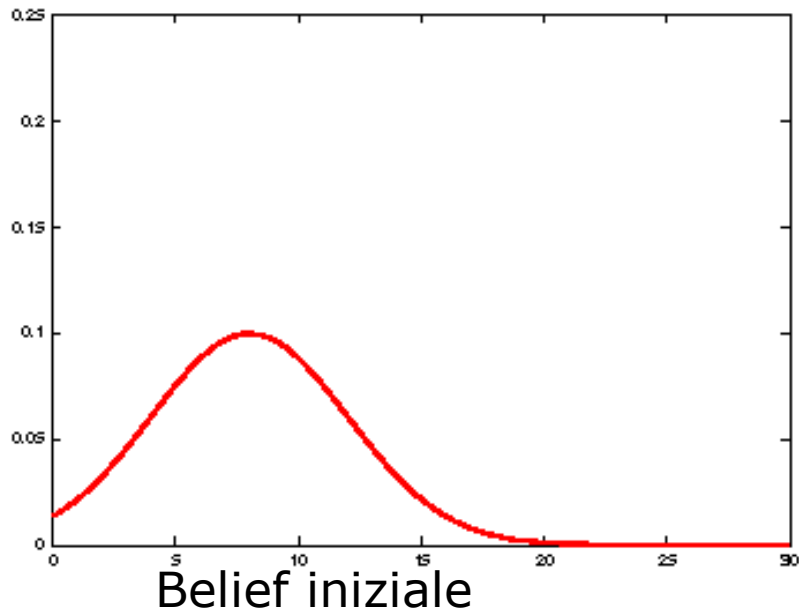
con una misura

$$z_t = C_t x_t + \delta_t$$

Componenti di un Filtro di Kalman

- A_t Matrice ($n \times n$) che descrive l'evoluzione dello stato da t a $t-1$ senza controllo o rumore.
- B_t Matrice ($n \times l$) che descrive come il controllo u_t cambia lo stato da t a $t-1$.
- C_t Matrice ($k \times n$) che descrive come mappare lo stato x_t in un'osservazione z_t .
- ε_t Variabili random che rappresentano il rumore di processo e misura. Assunte indipendenti e con distribuzione normale con covarianza R_t e Q_t rispettivamente.
- δ_t

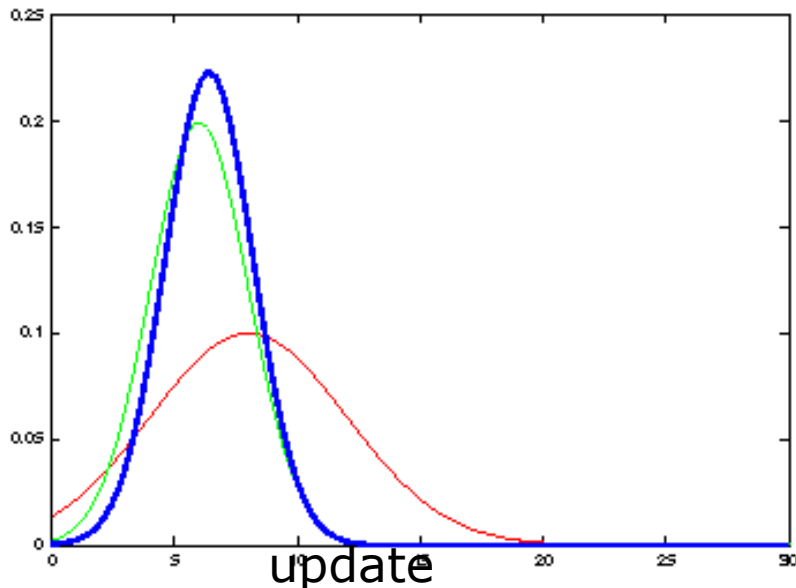
Filtro di Kalman in 1D



Aggiornamento Filtro di Kalman 1D

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - \bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t)\bar{\sigma}_t^2 \end{cases} \quad \text{with} \quad K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_tC_t)\bar{\Sigma}_t \end{cases} \quad \text{with} \quad K_t = \bar{\Sigma}_tC_t^T(C_t\bar{\Sigma}_tC_t^T + Q_t)^{-1}$$

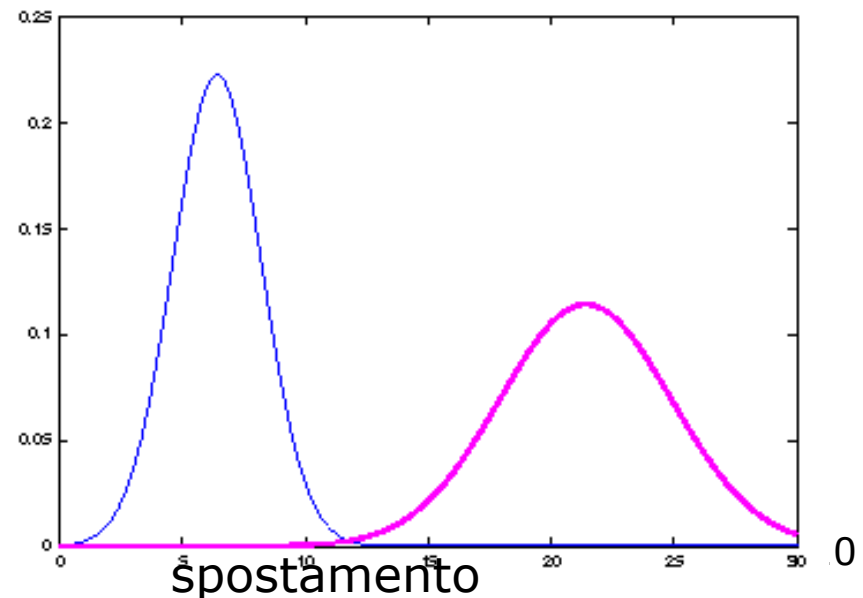
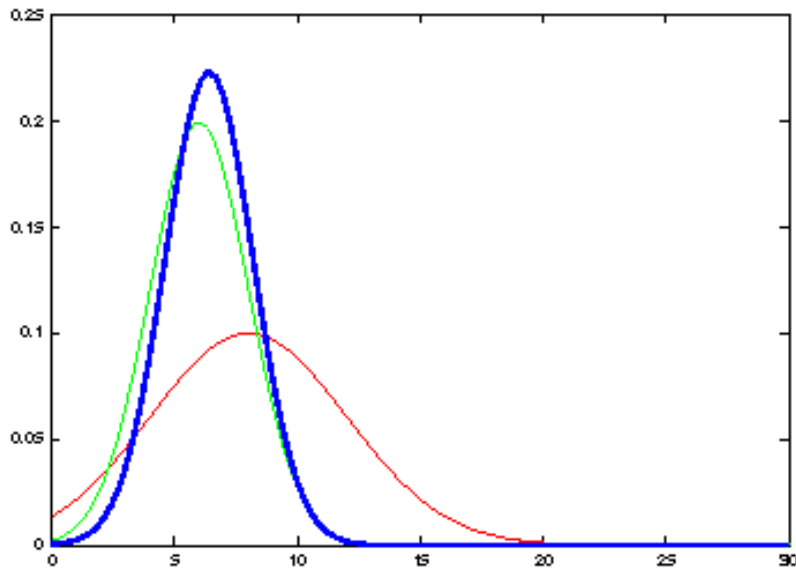


Aggiornamento Filtro di Kalman 1D

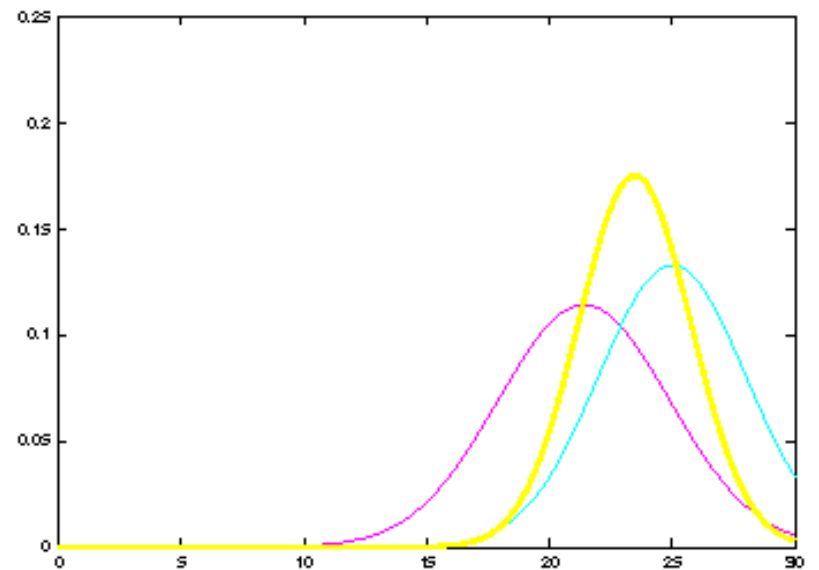
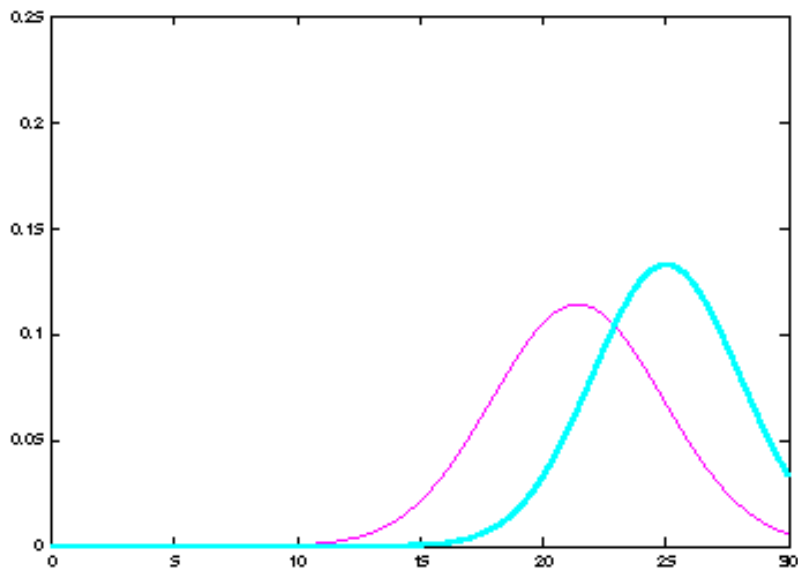
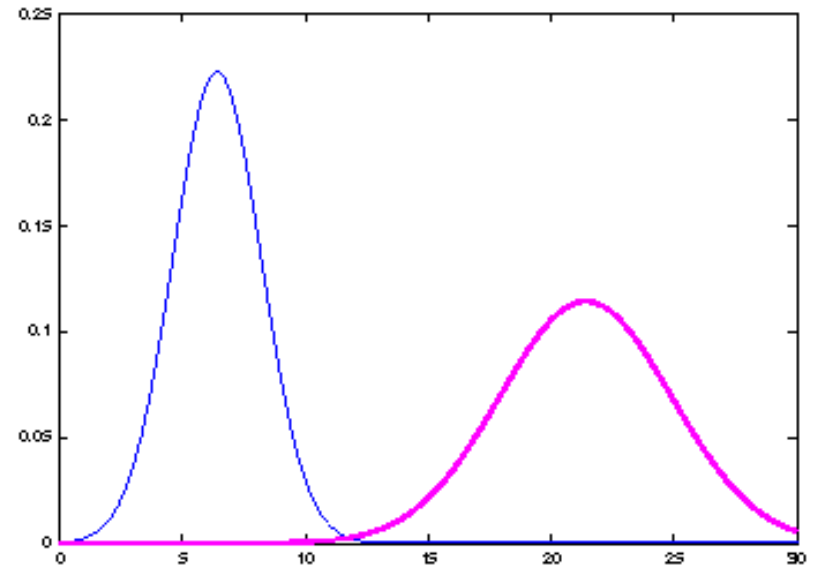
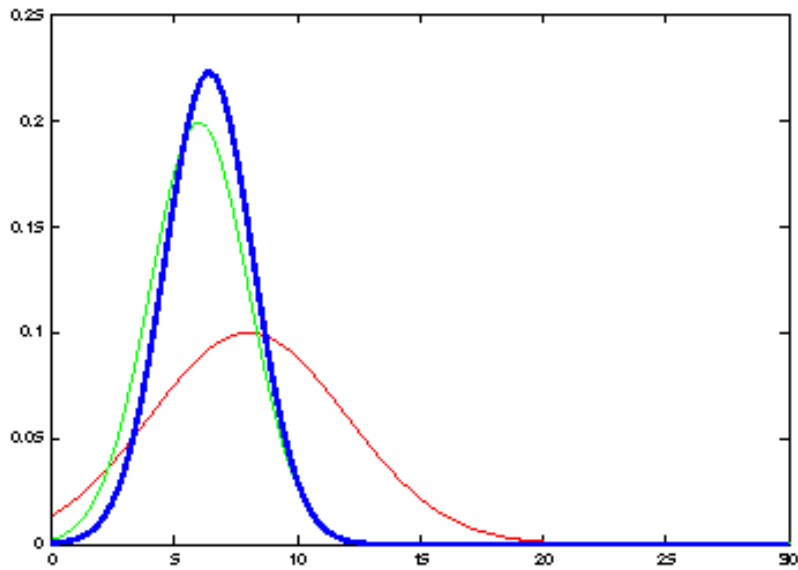
Spostamento

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_{t-1}^2 + \sigma_{act,t}^2 \end{cases}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$



Aggiornamento Filtro Kalman



Gaussiana Lineare: Initialization

- belief iniziale con distribuzione normale:

$$bel(x_0) = N(x_0; \mu_0, \Sigma_0)$$

Gaussiana Lineare: Dinamica

- La dinamica è funzione lineare dello stato e del controllo più rumore additivo:

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

$$p(x_t | u_t, x_{t-1}) = N(x_t; A_t x_{t-1} + B_t u_t, R_t)$$

$$\begin{array}{ccc} \overline{bel}(x_t) = \int p(x_t | u_t, x_{t-1}) & & bel(x_{t-1}) dx_{t-1} \\ \Downarrow & & \Downarrow \\ \sim N(x_t; A_t x_{t-1} + B_t u_t, R_t) & \sim & N(x_{t-1}; \mu_{t-1}, \Sigma_{t-1}) \end{array}$$

Gaussiana Lineare: Dinamica

$$\begin{aligned} \overline{bel}(x_t) &= \int p(x_t | u_t, x_{t-1}) \quad bel(x_{t-1}) dx_{t-1} \\ &\quad \Downarrow \qquad \qquad \qquad \Downarrow \\ &\sim N(x_t; A_t x_{t-1} + B_t u_t, R_t) \quad \sim N(x_{t-1}; \mu_{t-1}, \Sigma_{t-1}) \\ &\quad \Downarrow \\ \overline{bel}(x_t) &= \eta \int \exp\left\{-\frac{1}{2} (x_t - A_t x_{t-1} - B_t u_t)^T R_t^{-1} (x_t - A_t x_{t-1} - B_t u_t)\right\} \\ &\quad \exp\left\{-\frac{1}{2} (x_{t-1} - \mu_{t-1})^T \Sigma_{t-1}^{-1} (x_{t-1} - \mu_{t-1})\right\} dx_{t-1} \\ \overline{bel}(x_t) &= \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases} \end{aligned}$$

Gaussiana Lineare: Osservazioni

- Le osservazioni sono funzioni lineari dello stato più rumore additivo:

$$z_t = C_t x_t + \delta_t$$

$$p(z_t | x_t) = N(z_t; C_t x_t, Q_t)$$

$$\begin{array}{ccc} \text{bel}(x_t) = \eta & p(z_t | x_t) & \overline{\text{bel}}(x_t) \\ & \Downarrow & \Downarrow \\ & \sim N(z_t; C_t x_t, Q_t) & \sim N(x_t; \overline{\mu}_t, \overline{\Sigma}_t) \end{array}$$

Sistemi a Gaussiana Lineare: Osservazioni

$$\begin{aligned} \text{bel}(x_t) &= \eta \quad p(z_t | x_t) & \overline{\text{bel}}(x_t) \\ & \quad \Downarrow & \quad \Downarrow \\ & \sim N(z_t; C_t x_t, Q_t) & \sim N(x_t; \bar{\mu}_t, \bar{\Sigma}_t) \\ & \quad \Downarrow & \\ \text{bel}(x_t) &= \eta \exp\left\{-\frac{1}{2}(z_t - C_t x_t)^T Q_t^{-1}(z_t - C_t x_t)\right\} \exp\left\{-\frac{1}{2}(x_t - \bar{\mu}_t)^T \bar{\Sigma}_t^{-1}(x_t - \bar{\mu}_t)\right\} \\ \text{bel}(x_t) &= \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t \bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t) \bar{\Sigma}_t \end{cases} & \text{with } K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1} \end{aligned}$$

Kalman Filter Algorithm

1. Algorithm **Kalman_filter**($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$):

2. Prediction:

3. $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$

4. $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$

5. Correction:

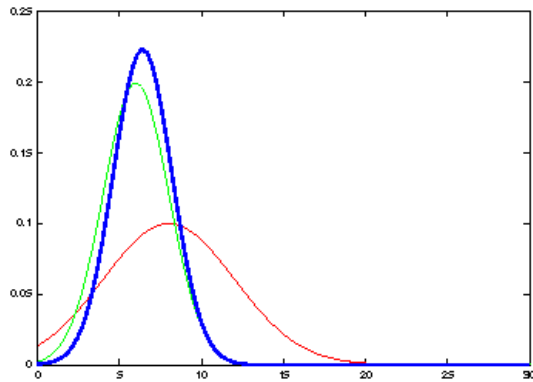
6. $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$

7. $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$

8. $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$

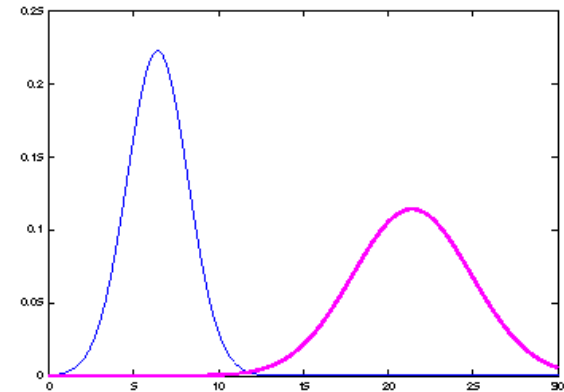
9. Return μ_t, Σ_t

Il ciclo Prediction-Correction

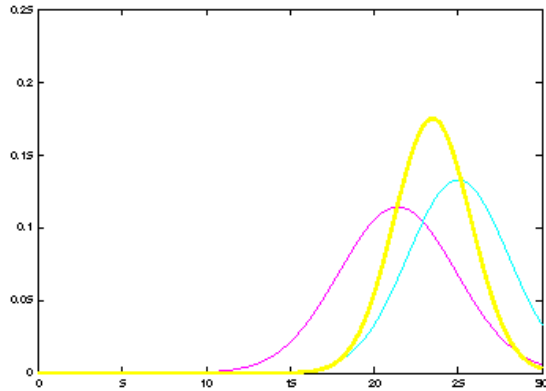


$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{act,t}^2 \end{cases}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$

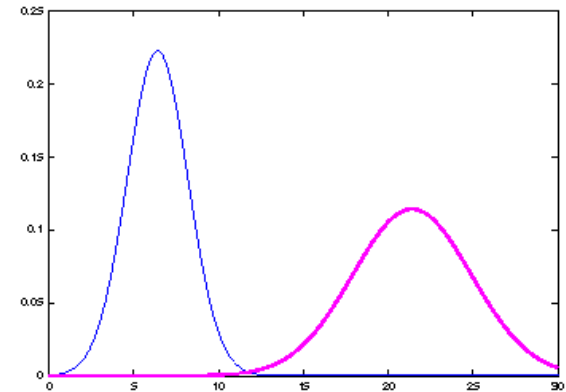


The Prediction-Correction-Cycle



$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - \bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t)\bar{\sigma}_t^2, K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2} \end{cases}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_tC_t)\bar{\Sigma}_t, K_t = \bar{\Sigma}_tC_t^T(C_t\bar{\Sigma}_tC_t^T + Q_t)^{-1} \end{cases}$$



Correction

Il ciclo Prediction-Correction



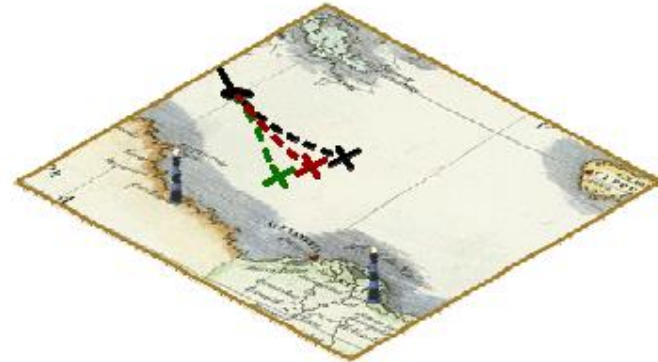
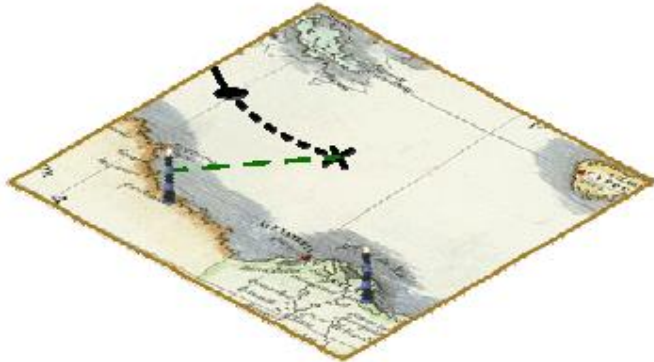
$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - \bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t)\bar{\sigma}_t^2 \end{cases}, K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_{t-1}^2 + \sigma_{act,t}^2 \end{cases}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t \bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t) \bar{\Sigma}_t \end{cases}, K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$





Sommario: Filtro di Kalman

- **Efficiente**: Polinomiale nella dim delle misure k e nella dim dello stato n :
$$O(k^{2.376} + n^2)$$
- **Ottimale per sistemi lineari Gaussiani!**
- Molti sistemi robotici sono **nonlineari!**

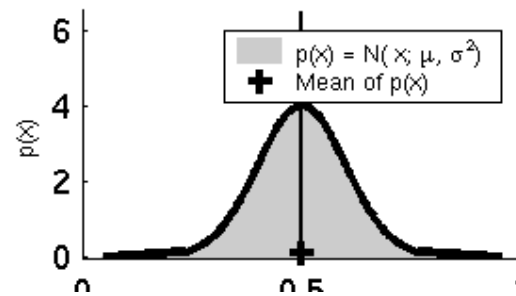
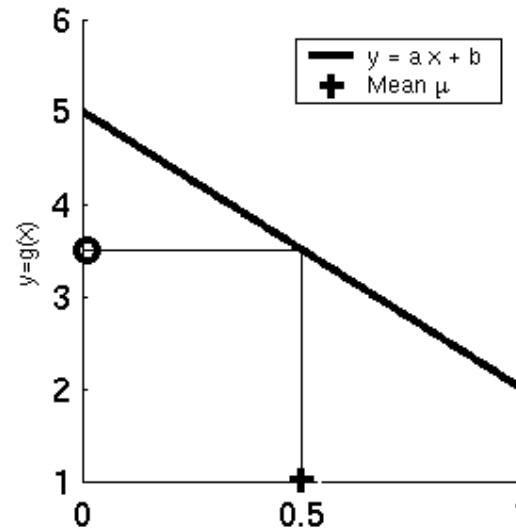
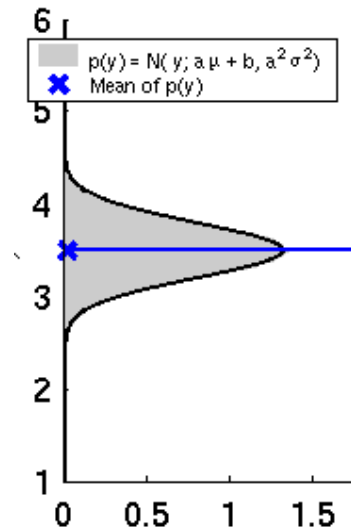
Sistemi Dinamici Nonlineari

- Molti problemi robotici realistici richiedono funzioni nonlineari

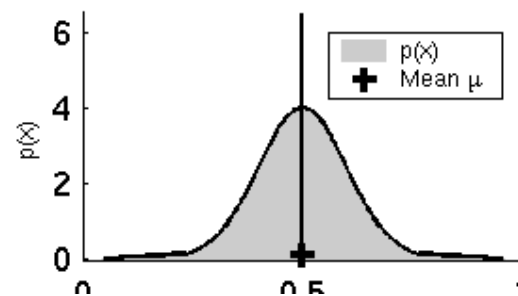
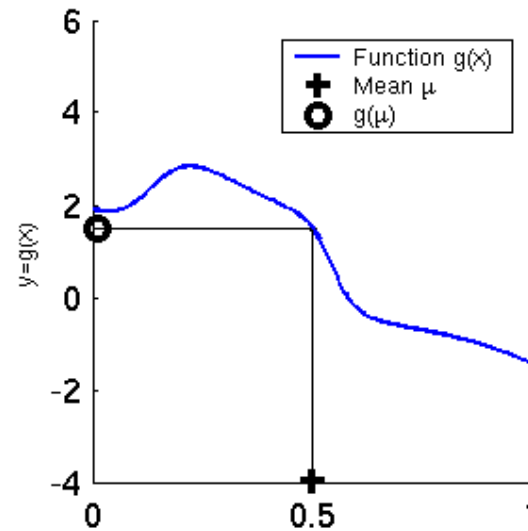
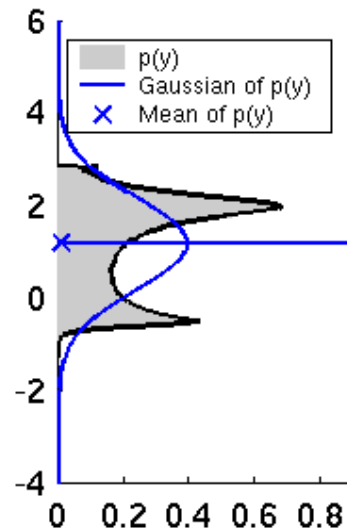
$$x_t = g(u_t, x_{t-1})$$

$$z_t = h(x_t)$$

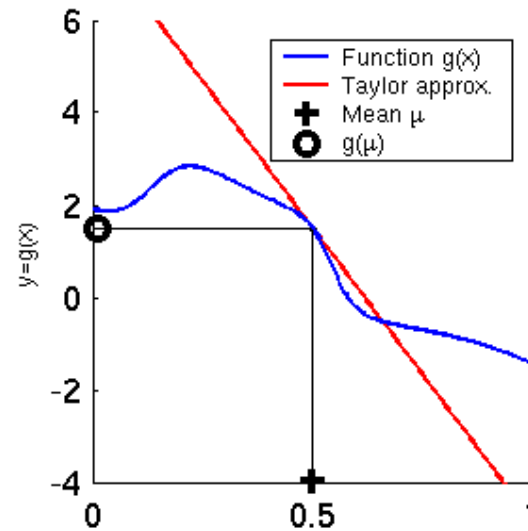
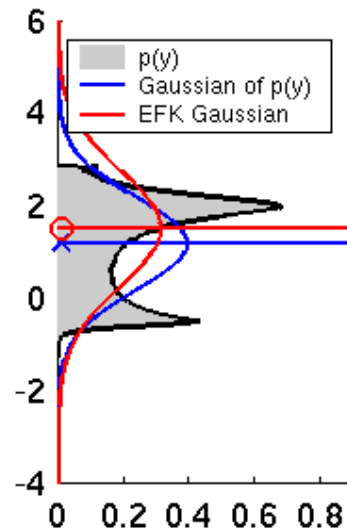
Assunzione lineare rivisitata



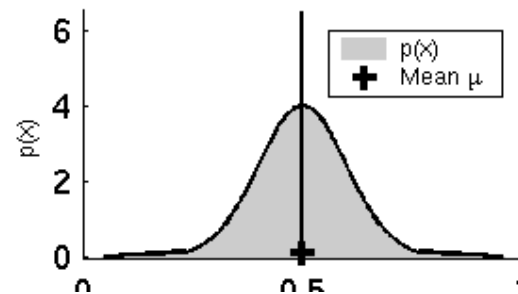
Funzione Non-lineare



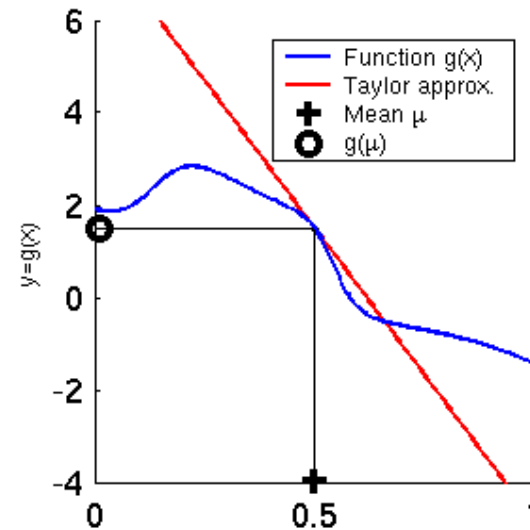
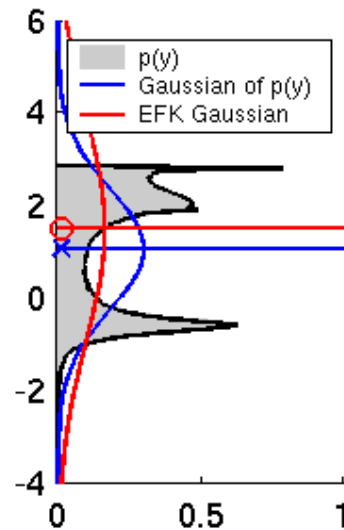
Linearizzazione EKF (1)



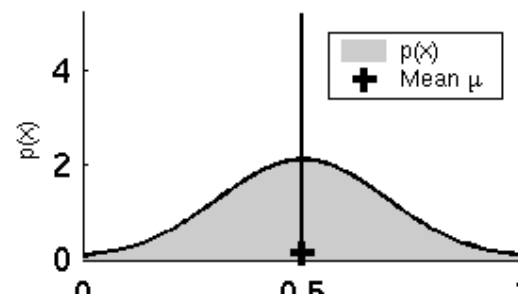
Approx lineare



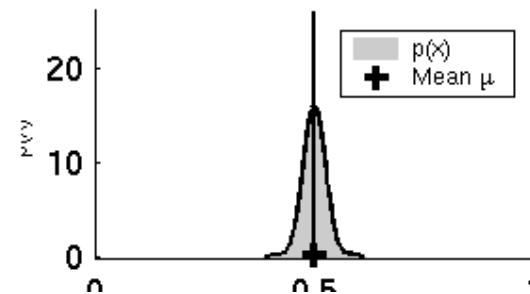
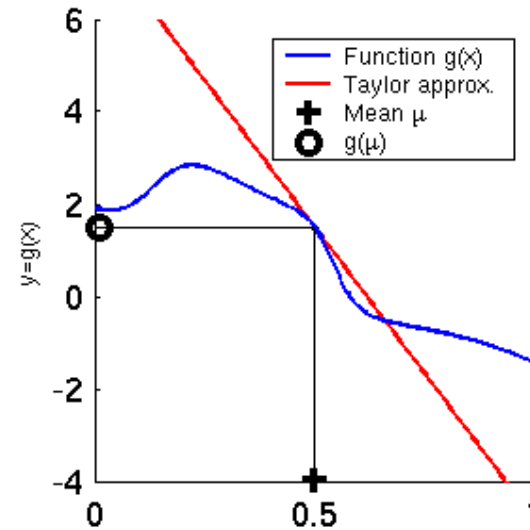
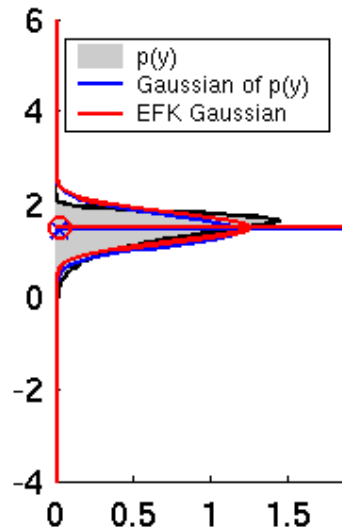
Linearizzazione EKF (2)



Maggiore incertezza,
maggiore approx



Linearizzazione EKF (3)



Minore incertezza,
minore approx

Linearizzazione EKF: Espansione del Primo Ordine della serie di Taylor

- Predizione:

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} (x_{t-1} - \mu_{t-1})$$

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + G_t (x_{t-1} - \mu_{t-1})$$

G_t corrisponde a:
 $A_t \quad B_t$

- Correzione:

$$h(x_t) \approx h(\bar{\mu}_t) + \frac{\partial h(\bar{\mu}_t)}{\partial x_t} (x_t - \bar{\mu}_t)$$

$$h(x_t) \approx h(\bar{\mu}_t) + H_t (x_t - \bar{\mu}_t)$$

H_t corrisponde a:
 C_t

Linearizzazione EKF: Espansione del Primo Ordine della serie di Taylor

- Linearization points: best estimate and 0 (assuming zero-mean noise)
- Prediction (linearize around μ_{t-1} and 0):

Jacobians

$$g(u_t, x_{t-1}, \varepsilon_t) \approx g(u_t, \mu_{t-1}, 0) + \frac{\partial g(u_t, \mu_{t-1}, 0)}{\partial x_{t-1}} (x_{t-1} - \mu_{t-1}) + \frac{\partial g(u_t, \mu_{t-1}, \varepsilon_t)}{\partial \varepsilon_t} \varepsilon_t$$

$$g(u_t, x_{t-1}, \varepsilon_t) \approx g(u_t, \mu_{t-1}) + G_t (x_{t-1} - \mu_{t-1}) + W_t \varepsilon_t$$

- Correction (linearize around $\bar{\mu}_t$ and 0):

Jacobians

$$h(x_t, \delta_t) \approx h(\bar{\mu}_t, 0) + \frac{\partial h(\bar{\mu}_t, 0)}{\partial x_t} (x_t - \bar{\mu}_t) + \frac{\partial h(\bar{\mu}_t, \delta_t)}{\partial \delta_t} \delta_t$$

$$h(x_t, \delta_t) \approx h(\bar{\mu}_t, 0) + H_t (x_t - \bar{\mu}_t) + V_t \delta_t$$

Algoritmo EKF

1. **Extended_Kalman_filter**($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$):

2. Prediction:

3. $\bar{\mu}_t = g(u_t, \mu_{t-1})$ \longleftarrow $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$
4. $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t$ \longleftarrow $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$

5. Correction:

6. $K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1}$ \longleftarrow $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$
7. $\mu_t = \bar{\mu}_t + K_t (z_t - h(\bar{\mu}_t))$ \longleftarrow $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$
8. $\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$ \longleftarrow $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$

9. Return μ_t, Σ_t

$$H_t = \frac{\partial h(\bar{\mu}_t)}{\partial x_t} \quad G_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}}$$

Localizzazione

“Using sensory information to locate the robot in its environment is the most fundamental problem to providing a mobile robot with autonomous capabilities.” [Cox '91]

- **Data**
 - Mappa dell'ambiente
 - Sequenza di misure di sensori.
- **Ricerca**
 - Stima della posa del robot.
- **Problemi**
 - Position tracking
 - Global localization
 - Kidnapped robot problem (recovery)

Localizzazione Landmark-based

Si assume una mappa definita da un insieme di landmark, l'osservazione genera un vettore $\{z_1, \dots, z_n\}$ di misure di feature univocamente identificabili



6 marker colorati diversamente

Localizzazione Markoviana

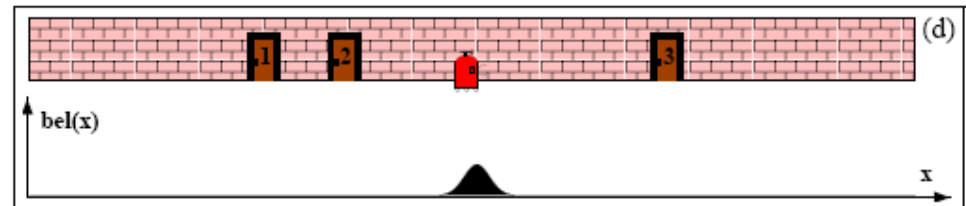
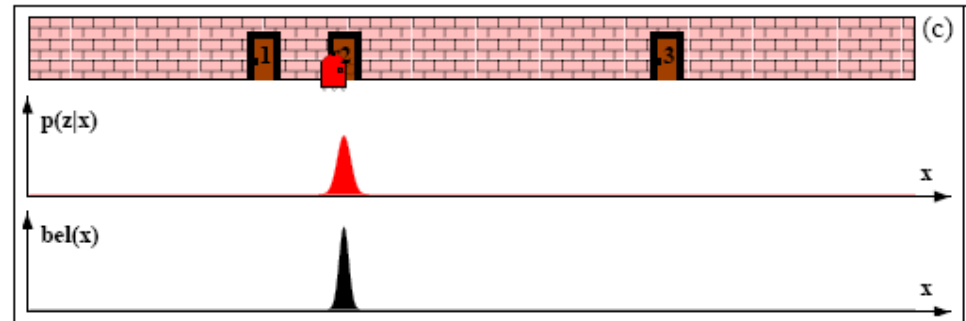
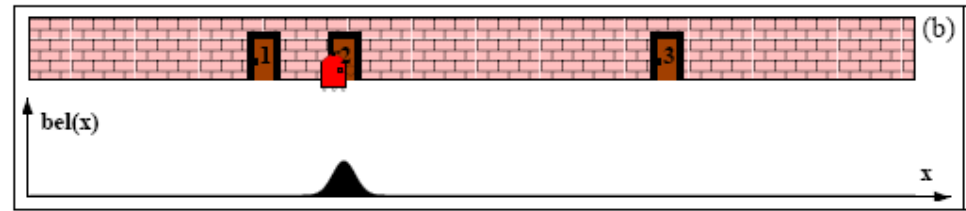
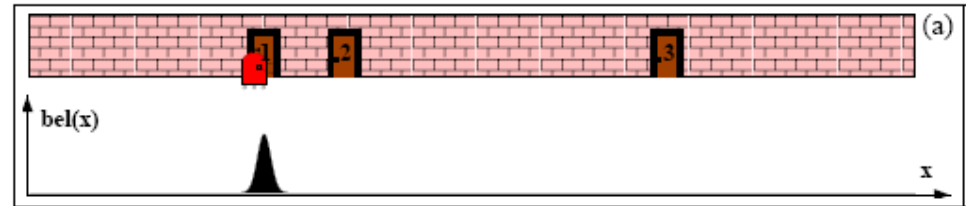
Algoritmo generale

```
1:   Algorithm Markov_Localization( $bel(x_{t-1}), u_t, z_t, m$ ):  
2:     for all  $x_t$  do  
3:        $\overline{bel}(x_t) = \int p(x_t | u_t, x_{t-1}, m) bel(x_{t-1}) dx$   
4:        $bel(x_t) = \eta p(z_t | x_t, m) \overline{bel}(x_t)$   
5:     endfor  
6:     return  $bel(x_t)$ 
```

Assunzione Gaussiana

La distribuzione si assume gaussiana

1. Mappa features con corrispondenze note
2. Posizione iniziale quasi nota (approx da una gaussiana)



Caso Concreto

- Consideriamo il modello in velocità
- La mappa feature-based con corrispondenze note
- Dato belief (gaussiano) al tempo $t-1$
 - valore medio e covarianza al tempo $t-1$
- Genera stima di belief al tempo t
 - valore medio e covarianza al tempo t

Modello in Velocità:

$$\begin{aligned} \begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} &= \begin{pmatrix} x_c + \frac{v}{\omega} \sin(\theta + \omega \Delta t) \\ y_c - \frac{v}{\omega} \cos(\theta + \omega \Delta t) \\ \theta + \omega \Delta t \end{pmatrix} \\ &= \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v}{\omega} \sin \theta + \frac{v}{\omega} \sin(\theta + \omega \Delta t) \\ \frac{v}{\omega} \cos \theta - \frac{v}{\omega} \cos(\theta + \omega \Delta t) \\ \omega \Delta t \end{pmatrix} \end{aligned}$$

Ma c'è il rumore, quindi le coordinate reali sono:

$$\begin{pmatrix} \hat{v} \\ \hat{\omega} \end{pmatrix} = \begin{pmatrix} v \\ \omega \end{pmatrix} + \begin{pmatrix} \varepsilon_{\alpha_1 |v| + \alpha_2 |\omega|} \\ \varepsilon_{\alpha_3 |v| + \alpha_4 |\omega|} \end{pmatrix}$$

Quindi:

$$\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{\hat{v}}{\hat{\omega}} \sin \theta + \frac{\hat{v}}{\hat{\omega}} \sin(\theta + \hat{\omega} \Delta t) \\ \frac{\hat{v}}{\hat{\omega}} \cos \theta - \frac{\hat{v}}{\hat{\omega}} \cos(\theta + \hat{\omega} \Delta t) \\ \hat{\omega} \Delta t \end{pmatrix}$$

Mappa feature-based

Feature-based map: mappa come insieme di features localizzate $\{m_1, \dots, m_n\}$

Con $m_{i,x}$ $m_{i,y}$ si indica la locazione x,y per la feature i -esima

Legame probabilistico tra feature j nella mappa globale e feature estratta nella mappa locale

$$\begin{pmatrix} r_t^i \\ \phi_t^i \\ s_t^i \end{pmatrix} = \begin{pmatrix} \sqrt{(m_{j,x} - x)^2 + (m_{j,y} - y)^2} \\ \text{atan2}(m_{j,y} - y, m_{j,x} - x) - \theta \\ s_j \end{pmatrix} + \begin{pmatrix} \varepsilon_{\sigma_r^2} \\ \varepsilon_{\sigma_\phi^2} \\ \varepsilon_{\sigma_s^2} \end{pmatrix}$$

Derivazione EKF

Modello di moto:

$$\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t} \sin \theta + \frac{v_t}{\omega_t} \sin(\theta + \omega_t \Delta t) \\ \frac{v_t}{\omega_t} \cos \theta - \frac{v_t}{\omega_t} \cos(\theta + \omega_t \Delta t) \\ \omega_t \Delta t + \gamma_t \Delta t \end{pmatrix} \quad u_t = \begin{pmatrix} v_t \\ \omega_t \end{pmatrix}$$

Introducendo il rumore:

$$\underbrace{\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix}}_{x_t} = \underbrace{\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t} \sin \theta + \frac{v_t}{\omega_t} \sin(\theta + \omega_t \Delta t) \\ \frac{v_t}{\omega_t} \cos \theta - \frac{v_t}{\omega_t} \cos(\theta + \omega_t \Delta t) \\ \omega_t \Delta t \end{pmatrix}}_{g(u_t, x_{t-1})} + \mathcal{N}(0, R_t)$$

Linearizzazione di Taylor:

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + G_t (x_{t-1} - \mu_{t-1})$$

Rispetto ad $x_{t-1} = \mu_{t-1}$

Jacobiano:

$$G_t = g'(u_t, \mu_{t-1}) = \begin{pmatrix} 1 & 0 & \frac{v_t}{\omega_t} \cos \mu_{t-1, \theta} - \frac{v_t}{\omega_t} \cos(\mu_{t-1, \theta} + \omega_t \Delta t) \\ 0 & 1 & \frac{v_t}{\omega_t} \sin \mu_{t-1, \theta} - \frac{v_t}{\omega_t} \sin(\mu_{t-1, \theta} + \omega_t \Delta t) \\ 0 & 0 & 1 \end{pmatrix}$$

1. EKF_localization ($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, m$):

Predizione:

3. $G_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} = \begin{pmatrix} \frac{\partial x'}{\partial \mu_{t-1,x}} & \frac{\partial x'}{\partial \mu_{t-1,y}} & \frac{\partial x'}{\partial \mu_{t-1,\theta}} \\ \frac{\partial y'}{\partial \mu_{t-1,x}} & \frac{\partial y'}{\partial \mu_{t-1,y}} & \frac{\partial y'}{\partial \mu_{t-1,\theta}} \\ \frac{\partial \theta'}{\partial \mu_{t-1,x}} & \frac{\partial \theta'}{\partial \mu_{t-1,y}} & \frac{\partial \theta'}{\partial \mu_{t-1,\theta}} \end{pmatrix}$ Jacobiano di g rispetto alla locazione stimata

5. $M_t = \begin{pmatrix} (\alpha_1 |v_t| + \alpha_2 |\omega_t|)^2 & 0 \\ 0 & (\alpha_3 |v_t| + \alpha_4 |\omega_t|)^2 \end{pmatrix}$ Rumore di movimento

6. $V_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial u_t} = \begin{pmatrix} \frac{\partial x'}{\partial v_t} & \frac{\partial x'}{\partial \omega_t} \\ \frac{\partial y'}{\partial v_t} & \frac{\partial y'}{\partial \omega_t} \\ \frac{\partial \theta'}{\partial v_t} & \frac{\partial \theta'}{\partial \omega_t} \end{pmatrix}$ Jacobiano di g rispetto al controllo (v,w) per mappare il rumore di movimento

7. $\bar{\mu}_t = g(u_t, \mu_{t-1})$ Posa predetta

8. $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + V_t M_t V_t^T$ Covarianza predetta

Passo di Correzione

Corrispondenza features

$$z_t^i = \begin{pmatrix} r_t^i \\ \phi_t^i \\ s_t^i \end{pmatrix} = \begin{pmatrix} \sqrt{(m_{j,x} - x)^2 + (m_{j,y} - y)^2} \\ \text{atan2}(m_{j,y} - y, m_{j,x} - x) - \theta \\ m_{j,s} \end{pmatrix} + \begin{pmatrix} \mathcal{N}(0, \sigma_r) \\ \mathcal{N}(0, \sigma_\phi) \\ \mathcal{N}(0, \sigma_s) \end{pmatrix}$$

$$z_t^i = h(x_t, j, m) + \mathcal{N}(0, Q_t) \quad h(x_t, j, m) = \begin{pmatrix} \sqrt{(m_{j,x} - x)^2 + (m_{j,y} - y)^2} \\ \text{atan2}(m_{j,y} - y, m_{j,x} - x) - \theta \\ m_{j,s} \end{pmatrix}$$

$$Q_t = \begin{pmatrix} \sigma_r & 0 & 0 \\ 0 & \sigma_\phi & 0 \\ 0 & 0 & \sigma_s \end{pmatrix} \quad \text{Covarianza dell'errore di misura addizionale}$$

segnatura

Si procede con la linearizzazione di Taylor:

$$h(x_t, j, m) \approx h(\bar{\mu}_t, j, m) + H_t^i (x_t - \bar{\mu}_t)$$

Rispetto alla posa stimata: $\bar{\mu}_t = (\bar{\mu}_{t,x} \ \bar{\mu}_{t,y} \ \bar{\mu}_{t,\theta})^T$

H jacobiano (posa stimata):

$$H_t^i = h'(\bar{\mu}_t, j, m) = \begin{pmatrix} \frac{\partial r_t^i}{\partial \bar{\mu}_{t,x}} & \frac{\partial r_t^i}{\partial \bar{\mu}_{t,y}} & \frac{\partial r_t^i}{\partial \bar{\mu}_{t,\theta}} \\ \frac{\partial \phi_t^i}{\partial \bar{\mu}_{t,x}} & \frac{\partial \phi_t^i}{\partial \bar{\mu}_{t,y}} & \frac{\partial \phi_t^i}{\partial \bar{\mu}_{t,\theta}} \\ \frac{\partial s_t^i}{\partial \bar{\mu}_{t,x}} & \frac{\partial s_t^i}{\partial \bar{\mu}_{t,y}} & \frac{\partial s_t^i}{\partial \bar{\mu}_{t,\theta}} \end{pmatrix} \quad H_t^i = \begin{pmatrix} \frac{m_{j,x} - \bar{\mu}_{t,x}}{\sqrt{q_t}} & \frac{y_t - \bar{\mu}_{t,y}}{\sqrt{q_t}} & 0 \\ \frac{\bar{\mu}_{t,y} - y_t}{q_t} & \frac{m_{j,x} - \bar{\mu}_{t,x}}{q_t} & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Non ha effetto

with $q_t = (m_{j,x} - \bar{\mu}_{t,x})^2 + (m_{j,y} - \bar{\mu}_{t,y})^2$, and $j = c_t^i$

1. EKF_localization ($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, m$):

Correzione:

3. $\hat{z}_t = \begin{pmatrix} \sqrt{(m_x - \bar{\mu}_{t,x})^2 + (m_y - \bar{\mu}_{t,y})^2} \\ \text{atan2}(m_y - \bar{\mu}_{t,y}, m_x - \bar{\mu}_{t,x}) - \bar{\mu}_{t,\theta} \end{pmatrix}$ Pred. media di misura

5. $H_t = \frac{\partial h(\bar{\mu}_t, m)}{\partial x_t} = \begin{pmatrix} \frac{\partial r_t}{\partial \bar{\mu}_{t,x}} & \frac{\partial r_t}{\partial \bar{\mu}_{t,y}} & \frac{\partial r_t}{\partial \bar{\mu}_{t,\theta}} \\ \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,x}} & \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,y}} & \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,\theta}} \end{pmatrix}$ Jacobiano di h rispetto alla locazione

6. $Q_t = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_r^2 \end{pmatrix}$

7. $S_t = H_t \bar{\Sigma}_t H_t^T + Q_t$ Pred. misura di covarianza

8. $K_t = \bar{\Sigma}_t H_t^T S_t^{-1}$ Guadagno di Kalman

9. $\mu_t = \bar{\mu}_t + K_t (z_t - \hat{z}_t)$ Posa aggiornata

10. $\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$ Covarianza aggiornata

1: **Algorithm EKF localization known correspondences**($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, c_t, m$):

2:
$$\bar{\mu}_t = \mu_{t-1} + \begin{pmatrix} -\frac{v_t}{\omega_t} \sin \mu_{t-1, \theta} + \frac{v_t}{\omega_t} \sin(\mu_{t-1, \theta} + \omega_t \Delta t) \\ \frac{v_t}{\omega_t} \cos \mu_{t-1, \theta} - \frac{v_t}{\omega_t} \cos(\mu_{t-1, \theta} + \omega_t \Delta t) \\ \omega_t \Delta t \end{pmatrix}$$
 Nuova posa stimata

3:
$$G_t = \begin{pmatrix} 1 & 0 & \frac{v_t}{\omega_t} \cos \mu_{t-1, \theta} - \frac{v_t}{\omega_t} \cos(\mu_{t-1, \theta} + \omega_t \Delta t) \\ 0 & 1 & \frac{v_t}{\omega_t} \sin \mu_{t-1, \theta} - \frac{v_t}{\omega_t} \sin(\mu_{t-1, \theta} + \omega_t \Delta t) \\ 0 & 0 & 1 \end{pmatrix}$$
 Jacobiano

4:
$$\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t$$
 Covarianza predizione:
aggiornamento + rumore di movimento:
$$R = V_t M_t V_t^T$$

5:
$$Q_t = \begin{pmatrix} \sigma_r & 0 & 0 \\ 0 & \sigma_\phi & 0 \\ 0 & 0 & \sigma_s \end{pmatrix}$$
 Covarianza rumore misura addizionale

6: for all observed features $z_t^i = (r_t^i \ \phi_t^i \ s_t^i)^T$ do

7:
$$j = c_t^i$$
 corrispondenza

8:
$$\delta = \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} m_{j,x} - \bar{\mu}_{t,x} \\ m_{j,y} - \bar{\mu}_{t,y} \end{pmatrix}$$
 posizione

9:
$$q = \delta^T \delta$$

10:
$$\hat{z}_t^i = \begin{pmatrix} \sqrt{q} \\ \text{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \\ m_{j,s} \end{pmatrix}$$
 Distanza rel. e angolo

11:
$$H_t^i = \frac{1}{q} \begin{pmatrix} \sqrt{q} \delta_x & -\sqrt{q} \delta_y & 0 \\ \delta_y & \delta_x & -1 \\ 0 & 0 & 0 \end{pmatrix}$$
 Jacobiano del modello di misura

12:
$$K_t^i = \bar{\Sigma}_t H_t^{i,T} (H_t^i \bar{\Sigma}_t H_t^{i,T} + Q_t)^{-1}$$
 Guadagno di Kalman per ogni osservazione

13: endfor

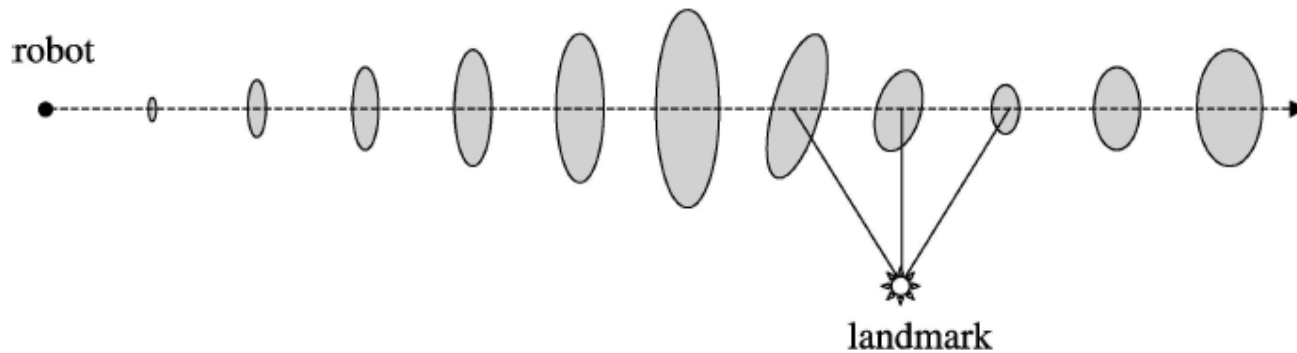
14:
$$\mu_t = \bar{\mu}_t + \sum_i K_t^i (z_t^i - \hat{z}_t^i)$$

15:
$$\Sigma_t = (I - \sum_i K_t^i H_t^i) \bar{\Sigma}_t$$
 Nuova Posa e covarianza

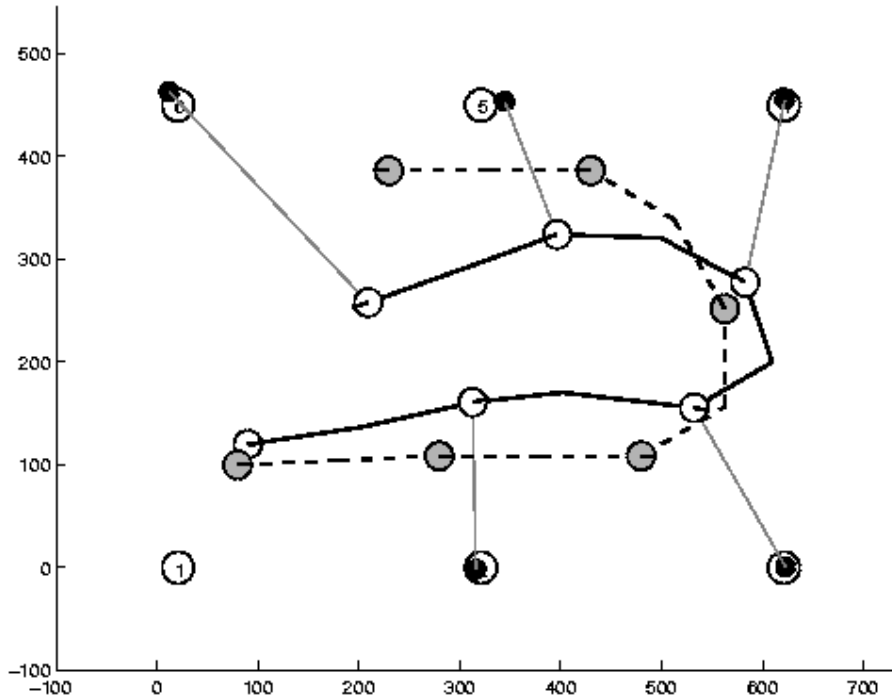
16: return μ_t, Σ_t

Localizzazione EKF

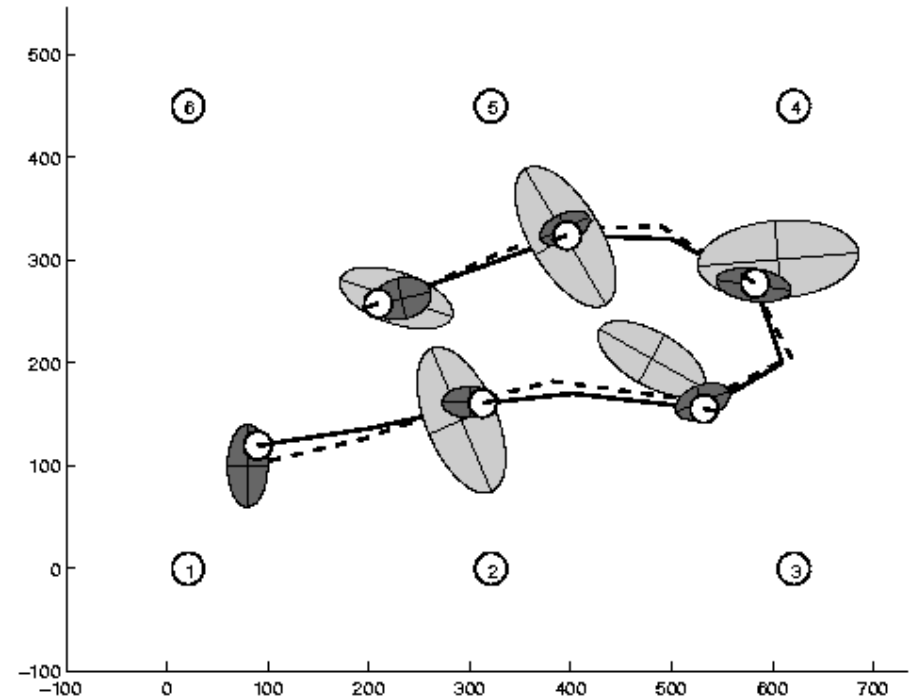
Incertezza diminuisce con percezione landmark, poi aumenta di nuovo



Estimation Sequence (1)

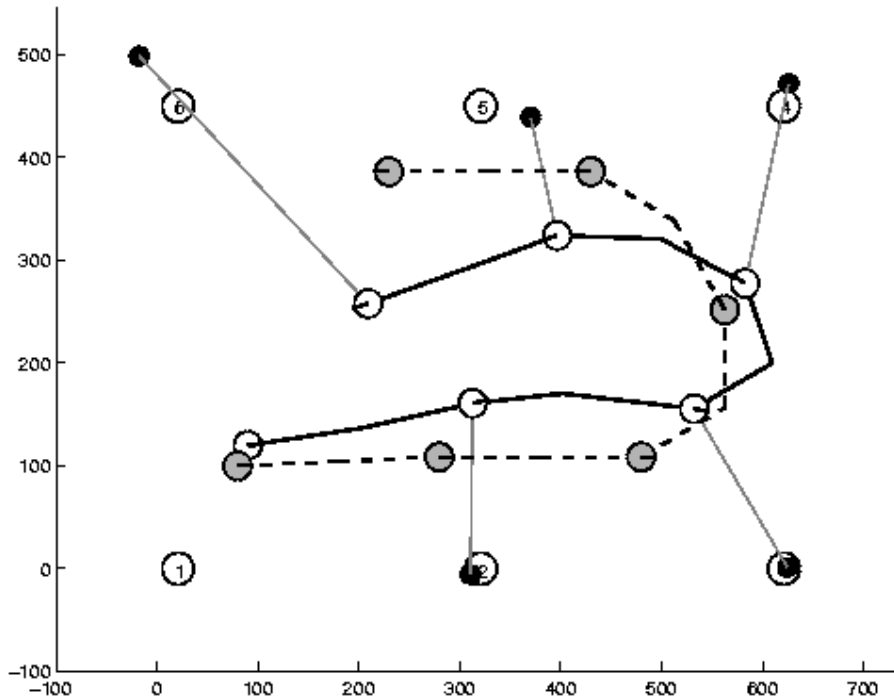


Traiettorie reali (solide),
traiettorie controllate (a tratti)

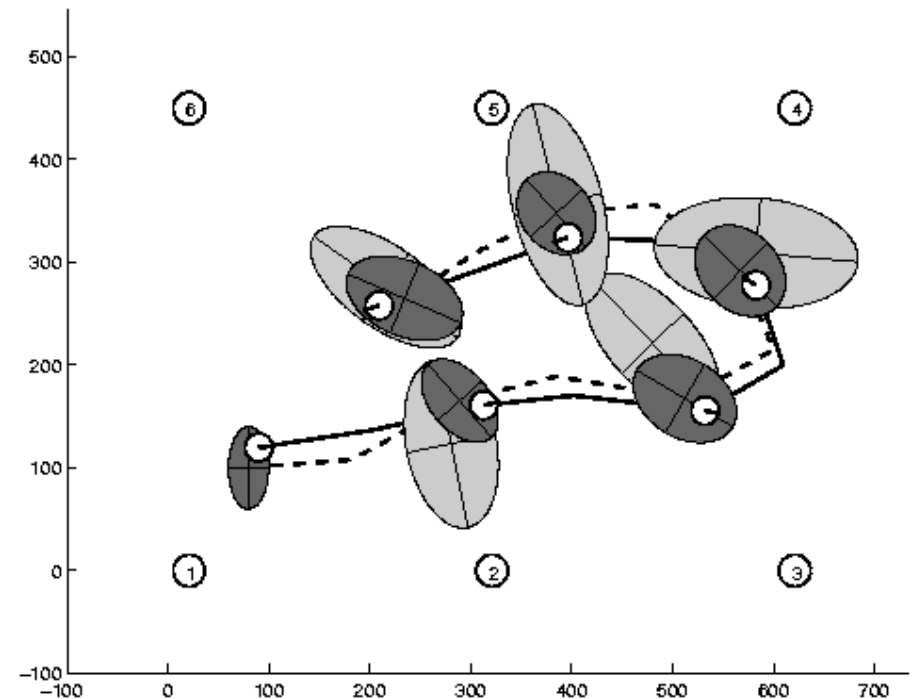


Basso errore

Estimation Sequence (2)

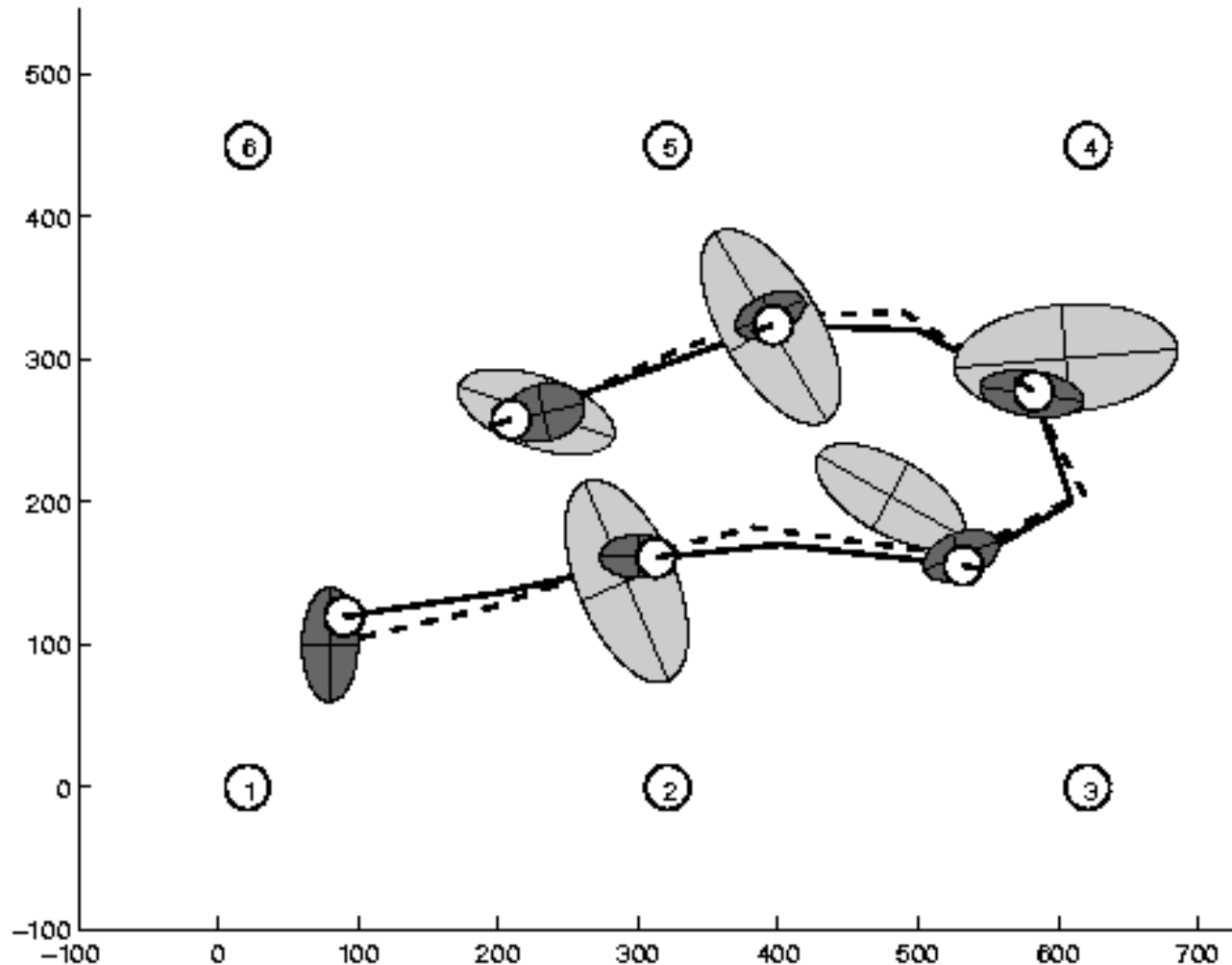


Traiettorie reali (solide),
traiettorie controllate (a tratti)



Errore di misura

Confronto con la GroundTruth

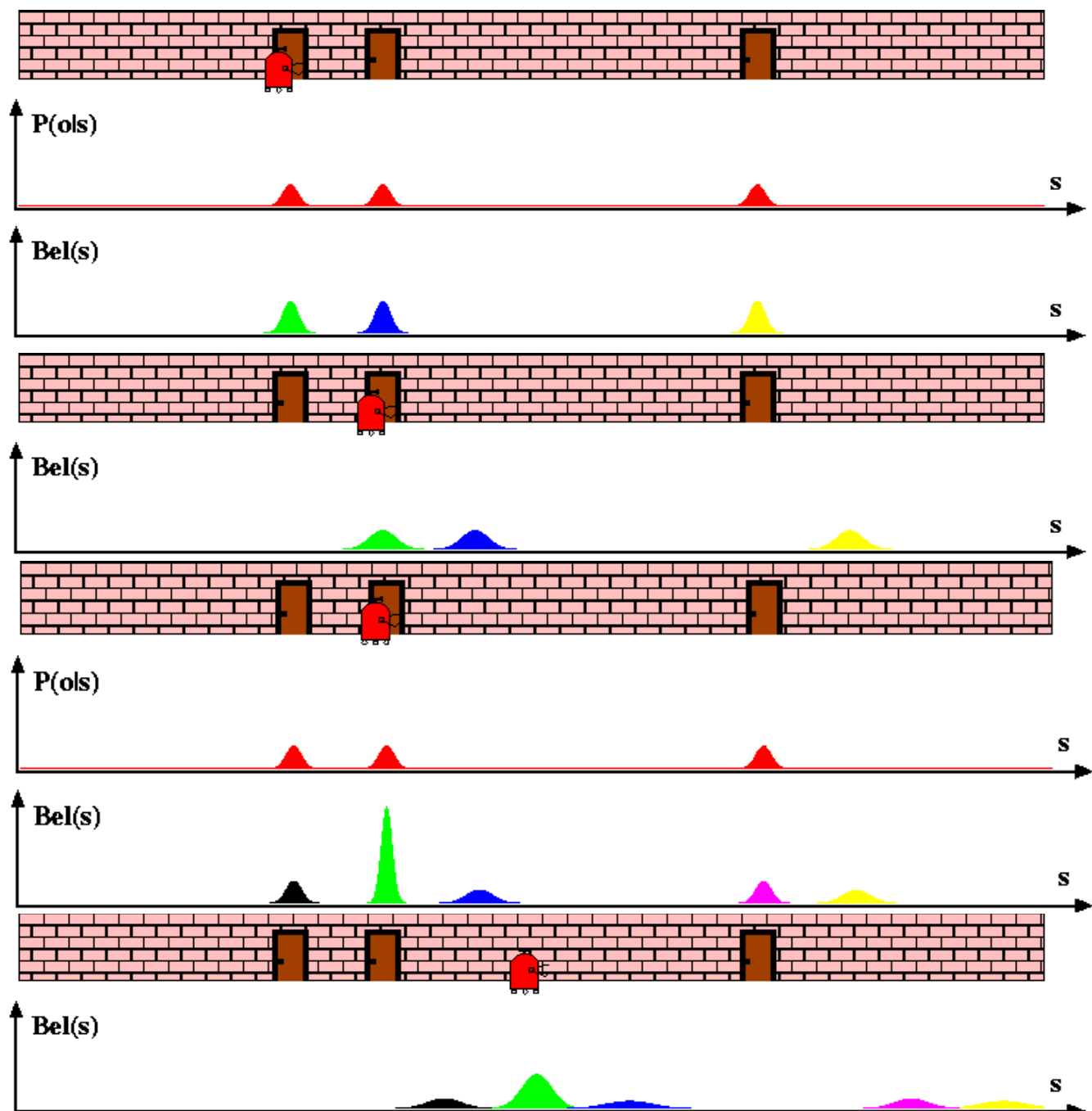


Riassunto EKF

- **Efficiente**: Polinomiale nella dim della misura k e della dim dello stato n :
$$O(k^{2.376} + n^2)$$
- **Non ottimale!**
- Può **divergere** se larghe nonlinearità
- Funziona bene anche quando tutte le assunzioni sono violate!

Multi-hypothesis Tracking

Mistura di Gaussiane



Localizzazione con MHT

- Belief è rappresentato da ipotesi multiple
- Ogni ipotesi è tracciata da un filtro di Kalman
- **Problemi:**
 - **Associazione:** quale osservazione a quale ipotesi?
 - **Ipotesi:** quando aggiungere / cancellare ipotesi?
- Molta letteratura su target tracking, motion correspondence etc.

Localizzazione MHT

Mistura di gaussiane

$$bel(x_t) = \frac{1}{\sum_l \psi_{t,l}} \sum_l \psi_{t,l} \det(2\pi\Sigma_{t,l})^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(x_t - \mu_{t,l})^T \Sigma_{t,l}^{-1} (x_t - \mu_{t,l})\right\}$$

$\psi_{t,l}$ è il peso della gaussiana

Ogni componente l si basa su un vettore di associazioni

$$bel_l(x_t) = p(x_t | z_{1:t}, u_{1:t}, c_{1:t,l})$$

Associazione ha un peso

$$\psi_{t,m} = \psi_{t,l} p(z_t | c_{1:t-1,l}, c_{t,m}, z_{1:t-1}, u_{1:t})$$

MHT: Implemented System (1)

- Ipotesi estratte da scan LRF
- Ogni ipotesi probabilità di essere corretta:

$$H_i = \{ \hat{x}_i, \Sigma_i, P(H_i) \}$$

- Probabilità ipotesi calcolata con Bayes

$$P(H_i | s) = \frac{P(s | H_i)P(H_i)}{P(s)}$$

- Ipotesi con bassa probabilità sono cancellate.
- Nuovi candidati estratti da scan LRF.

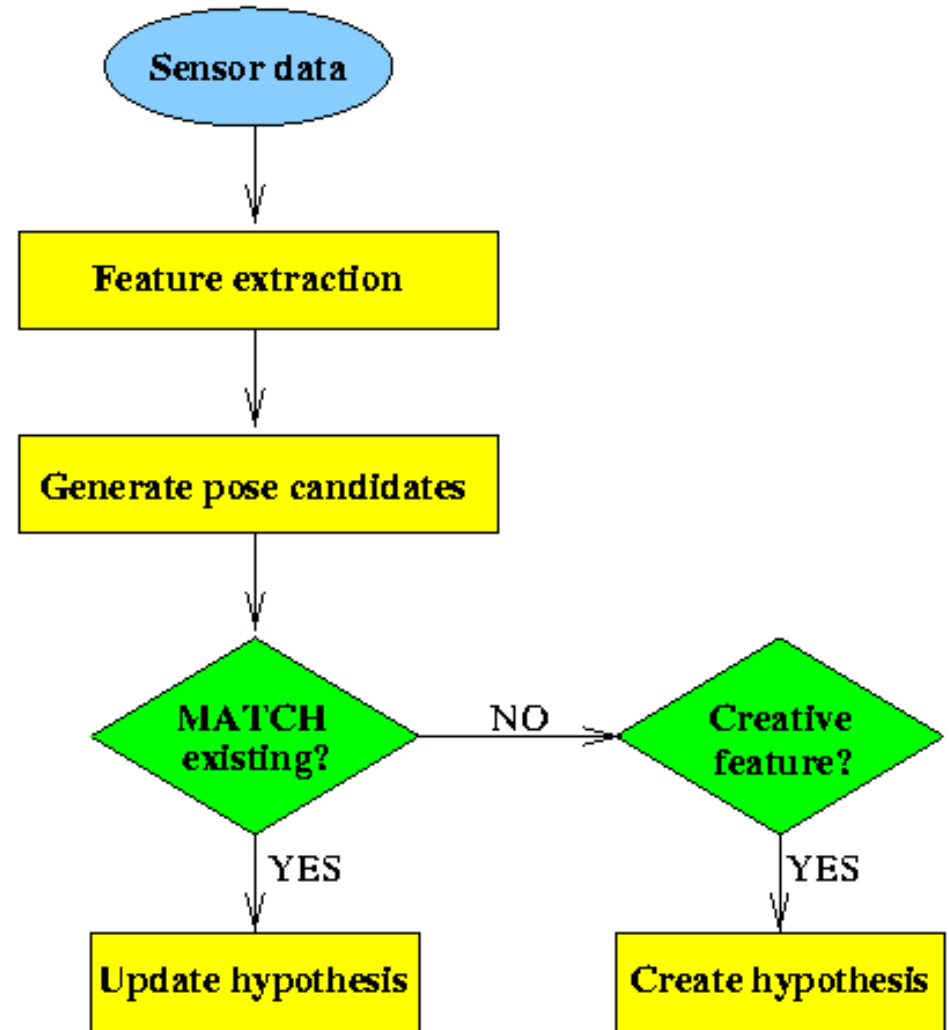
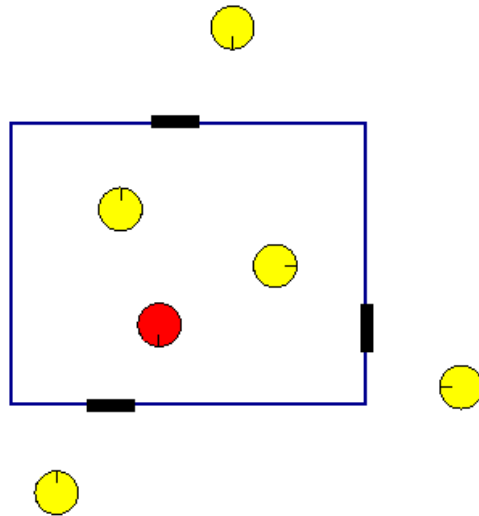
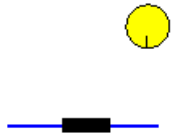
$$C_j = \{ z_j, R_j \}$$

[Jensfelt et al. '00]

MHT: Implemented System (2)

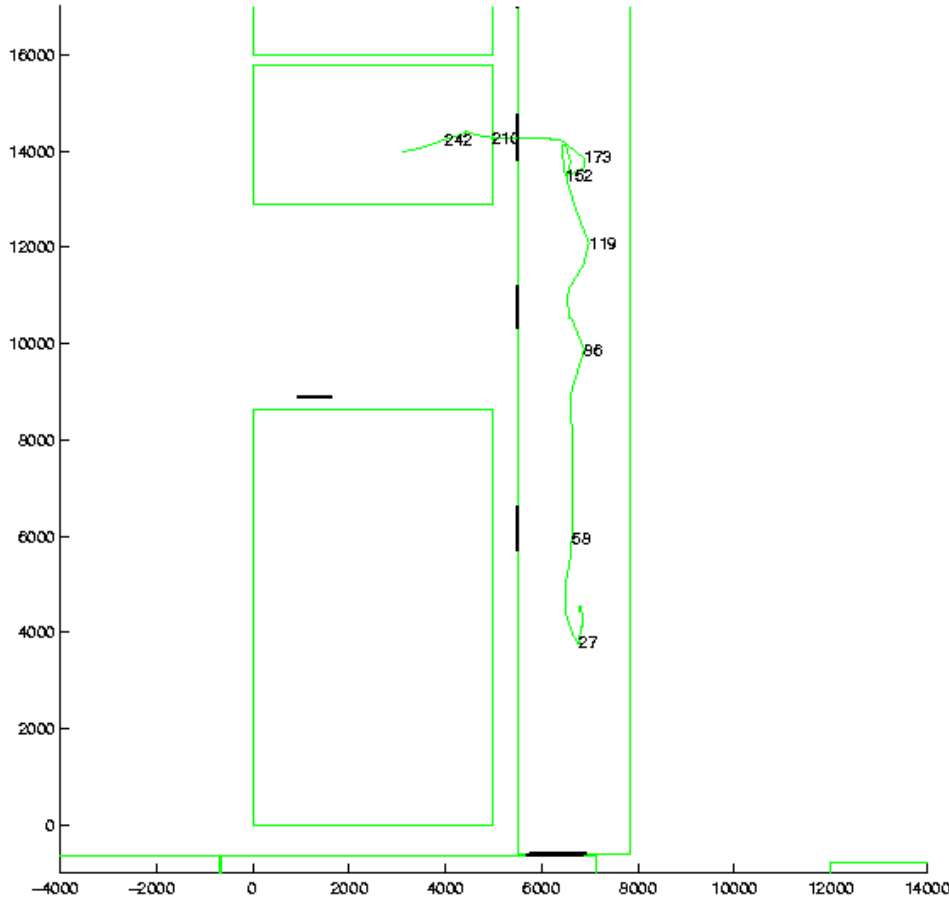
Robot view

Pose candidates

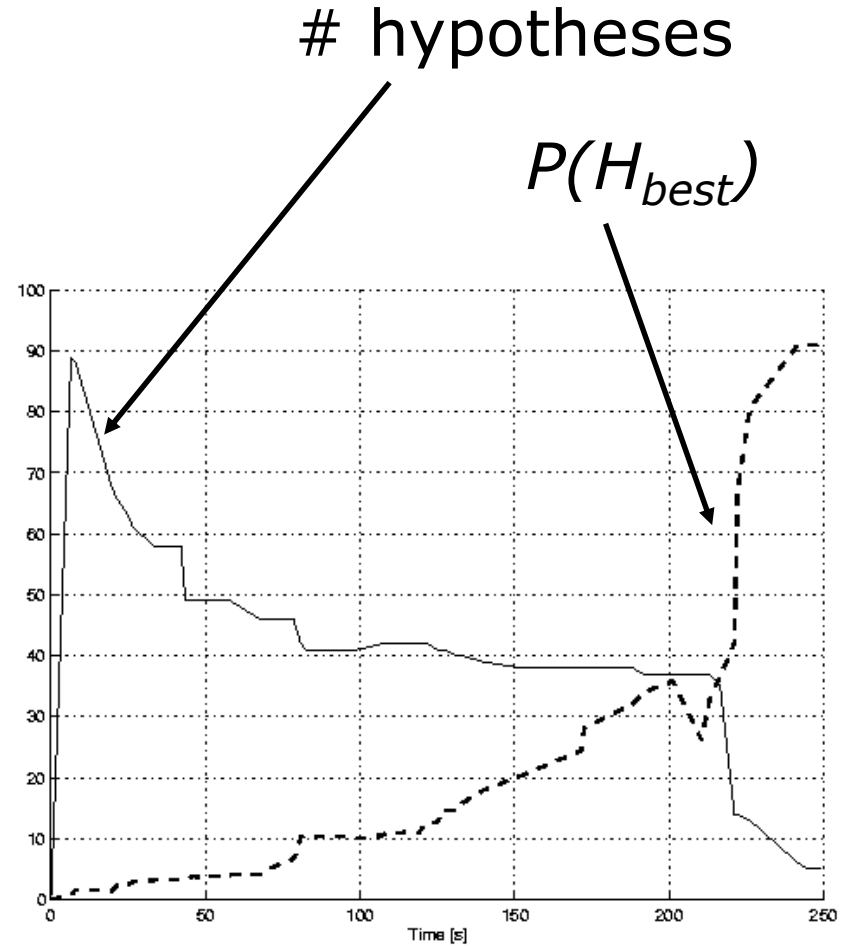


MHT: Implemented System (3)

Example run



Map and trajectory



#hypotheses vs. time