

Pseudopalindrome closure operators in free monoids

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Abstract

We consider involutory antimorphisms ϑ of a free monoid A^* and their fixed points, called ϑ -palindromes or pseudopalindromes. A ϑ -palindrome reduces to a usual palindrome when ϑ is the reversal operator. For any word $w \in A^*$ the right (resp. left) ϑ -palindrome closure of w is the shortest ϑ -palindrome having w as a prefix (resp. suffix). We prove some results relating ϑ -palindrome closure operators with periodicity and conjugacy, and derive some interesting closure properties for the languages of finite Sturmian and episturmian words. In particular, a finite word w is Sturmian if and only if both its palindromic closures are so. Moreover, in such a case, both the palindromic closures of w share the same minimal period of w . A new characterization of finite Sturmian words follows, in terms of periodicity and special factors of their palindromic closures. Some weaker results can be extended to the episturmian case. By using the right ϑ -palindrome closure, we extend the construction of standard episturmian words via directive words. In this way one obtains a family of infinite words, called ϑ -standard words, which are morphic images of episturmian words, as well as a wider family of infinite words including the Thue-Morse word on two symbols.

1 Introduction

For a word w over an arbitrary alphabet A , the right (resp. left) palindrome closure $w^{(+)}$ (resp. $w^{(-)}$) is the shortest palindrome having w as a prefix (resp. suffix). The palindromic closure operators were introduced in [5] by the first author. In that work it was shown how such operators can be used to construct the standard Sturmian words; starting from the empty word, one successively adds a letter from $\{a, b\}$ and applies the palindromic closure. Iterating the process, on the base of a given infinite word over $\{a, b\}$ called directive word, one yields a sequence of palindromes, each being a prefix (and suffix) of the next one. In this way, one obtains as a limit an infinite standard Sturmian word (or an infinite power of a finite standard word). Conversely, every standard Sturmian

word can be generated by this procedure. Droubay, Justin, and Pirillo [11] later extended this construction to arbitrary alphabets, thus introducing the family of standard episturmian words.

In this paper, we start from a more general point of view. We call ϑ -*palindromes* the fixed points of an involutory antimorphism of the free monoid A^* , and define accordingly the right (resp. left) ϑ -palindromic closure of a word $w \in A^*$ as the shortest ϑ -palindrome having w as a prefix (resp. suffix).

In Section 2 we recall some basic definitions and results concerning combinatorics on words and involutory antimorphisms. Moreover, we consider ϑ -symmetric words, i.e., words which are the product of two ϑ -palindromes. Some lemmas relating ϑ -symmetric words, periodicity, and conjugacy are proved. In particular, one has that the fractional root of a ϑ -palindrome is ϑ -symmetric.

In Section 3, we discuss some general properties of the ϑ -palindromic closure operators. It is shown that the right and left ϑ -palindromic closures of a word w have the same minimal period, which is in general different from the minimal period of w . The main result of the section is Theorem 3.4, which states that a nonempty word w has the same minimal period of its ϑ -palindromic closures if and only if its fractional root z_w is ϑ -symmetric.

In Section 4, we introduce the notion of elementary ϑ -palindrome action, which consists in appending a letter to a word and then taking the right ϑ -palindrome closure. Such actions can be naturally extended from letters to a finite or infinite word w by an iterative composition of the elementary ϑ -palindrome actions corresponding to the successive letters of w . If w is an infinite word, then, starting from the empty word, one generates an infinite word called ϑ -standard. If ϑ is the reversal operator, one obtains a standard episturmian word.

In Sections 5 and 6, we consider Sturmian and episturmian words respectively. In Section 5 we prove that both closures $w^{(+)}$ and $w^{(-)}$ of a finite Sturmian word w are Sturmian themselves, and share the same minimal period of w since the fractional root of w is symmetric, i.e., the product of two palindromes. Moreover, there exists a standard Sturmian word s such that $w^{(+)}$ and $w^{(-)}$ are both factors of s . From the preceding results, a new characterization of finite Sturmian words can be given in terms of the minimal period and of the right special factors of its right palindrome closure (cf. Theorem 5.11).

Some of the previous results can be extended to episturmian words; however, in this more general setting only weaker results can be proved. In Section 6, we show that if w is a factor of some standard episturmian word s , then one of the two words $w^{(+)}$ and $w^{(-)}$ is a factor of s too. However, in general, the minimal period of $w^{(+)}$ and $w^{(-)}$ is different from that of w , since the fractional root of w can be non-symmetric.

In Section 7, we consider ϑ -standard words. The main result is that any ϑ -standard word is a morphic image, by an injective morphism (depending on ϑ), of the standard episturmian word having the same directive word.

In Section 8, a generalization of the method of construction of ϑ -standard words is introduced, by assuming that ϑ can vary among all involutory antimorphisms of A^* at each step of the iterating process, which is directed by a bi-sequence of letters and operators. In this way, one gets a wider family of infinite words, including the Thue-Morse word on two symbols.

2 Preliminaries

Let A be a nonempty finite set, or *alphabet*, and A^* (resp. A^+) the *free monoid* (resp. *semigroup*) generated by A . The elements of A are called *letters* and those of A^* *words*. The identity element of A^* is called *empty word* and denoted by ε . A nonempty word w can be written uniquely as a sequence of letters $w = a_1 a_2 \cdots a_n$, with $a_i \in A$, $i = 1, \dots, n$. The integer n is called the *length* of w and is denoted by $|w|$. The length of ε is conventionally 0.

Let $w \in A^*$. A word v is a *factor* of w if there exist words r and s such that $w = rvs$. A factor v of w is *proper* if $v \neq w$. If $w = vs$ for some s (resp. $w = rv$ for some word r), then v is called a *prefix* (resp. a *suffix*) of w . A factor v of w is *median* if $w = rvs$ and $|r| = |s|$. A factor u of w is called a *border* of w if it is both a prefix and a suffix of w .

We shall denote respectively by $\text{Fact}(w)$, $\text{Pref}(w)$, and $\text{Suff}(w)$ the sets of all factors, prefixes, and suffixes of the word w .

A positive integer p is a *period* of $w = a_1 \cdots a_n$ if whenever $1 \leq i, j \leq |w|$ one has that

$$i \equiv j \pmod{p} \implies a_i = a_j .$$

As is well known [15], a word w has a period $p \leq |w|$ if and only if it has a border of length $|w| - p$. We denote by π_w the minimal period of w ; we shall set $\pi_\varepsilon = 1$. If w is nonempty, the *fractional root* z_w of w is its prefix of length $|z_w| = \pi_w$. A word w is called *unbordered* if $\pi_w = |w|$.

We can write any nonempty word w as

$$w = z_w^k z'$$

where z_w is the fractional root of w , $k \geq 1$, and z' is a proper prefix of z_w . The fractional root z_w of w is *primitive*, i.e., $z_w \neq v^h$ with $h > 1$ and $v \in A^*$. Moreover, by using the classic periodicity theorem of Fine and Wilf (cf. [15]) one easily obtains that for any $w \in A^+$ and $k > 1$,

$$\pi_w = \pi_{z_w^k} . \tag{1}$$

Two words $u, v \in A^*$ are *conjugate* if there exist $\lambda, \mu \in A^*$ such that $u = \lambda\mu$ and $v = \mu\lambda$. As is well known (cf. [15]), conjugacy is an equivalence relation in A^* . If u and v are conjugate, we shall write $u \sim v$. The following lemma, whose easy proof is in [8], relates the periodicity of a word and conjugacy classes of its factors.

Lemma 2.1. *A word w has a period $p \leq |w|$ if and only if all its factors of length p are in the same conjugacy class.*

An infinite word (from left-to-right) x over the alphabet A is a mapping $x : \mathbb{N}_+ \longrightarrow A$ where \mathbb{N}_+ is the set of positive integers. We can represent x as

$$x = x_1 x_2 \cdots x_n \cdots ,$$

where for any $i > 0$, $x_i = x(i) \in A$. A (finite) *factor* of x is either the empty word or any sequence $x_i \cdots x_j$ with $i \leq j$, i.e., any block of consecutive letters of x . If $i = 1$, then u is a *prefix* of x . We shall denote by $\text{Fact}(x)$ and $\text{Pref}(x)$ the sets of finite factors and prefixes of x respectively. The set of all infinite words over A is denoted by A^ω . Moreover, we set $A^\infty = A^* \cup A^\omega$.

Let $w \in A^\infty$. A factor u of w is a *right special* factor of w if there exist two letters $a, b \in A$, $a \neq b$, such that ua and ub are factors of w . We shall denote by R_w the minimal integer k (if it exists) such that w has no right special factor of length k . One sets $R_\varepsilon = 0$. If $w \in A^*$, then $0 \leq R_w < |w|$.

The following noteworthy inequality (cf. [6]) relates the minimal period π_w of a finite word w and R_w :

$$\pi_w \geq R_w + 1 . \quad (2)$$

2.1 Involutionary antimorphisms of a free monoid

If $w = a_1 \cdots a_n \in A^*$, $a_i \in A$, $i = 1, \dots, n$, the *mirror image*, or *reversal*, of w is the word

$$\tilde{w} = a_n \cdots a_1 .$$

One sets $\tilde{\varepsilon} = \varepsilon$.

We recall that a map $\varphi : A^* \rightarrow A^*$ is an (*endo-*)*morphism* of A^* if for all $u, v \in A^*$ one has:

$$\varphi(uv) = \varphi(u)\varphi(v) .$$

A map $\varphi : A^* \rightarrow A^*$ is called an *antimorphism* of A^* if for all $u, v \in A^*$ one has

$$\varphi(uv) = \varphi(v)\varphi(u) .$$

If φ is bijective, then the morphism (resp. antimorphism) is called *automorphism* (resp. *antiautomorphism*). The morphism or antimorphism φ is *involutionary* if $\varphi^2 = \text{id}$. We shall often use, for simplicity, the exponential notation w^φ for $\varphi(w)$.

Let $R : A^* \rightarrow A^*$ be the map defined by

$$w^R = \tilde{w}$$

for any $w \in A^*$. The map R , called *reversal* operator, is clearly an involutionary antiautomorphism of A^* .

Let τ be an involutionary permutation of the alphabet A . It can be extended to a unique automorphism τ of the free monoid A^* . The map $\vartheta = \tau \circ R = R \circ \tau$ is the *unique* involutionary antimorphism of A^* extending the permutation τ . One has, for $w = a_1 \cdots a_n$, $a_i \in A$, $i = 1, \dots, n$,

$$w^\vartheta = a_n^\tau \cdots a_1^\tau .$$

Any involutionary antimorphism of A^* can be constructed in this way; for example, the reversal R is obtained by extending the identity map of A .

If $A = \{a, b\}$, then there exist only two involutionary antimorphisms, namely, the reversal R and the antimorphism $e = E \circ R$, called *exchange antimorphism*, extending the exchange map E defined on A as $E(a) = b$ and $E(b) = a$.

If the alphabet A has cardinality n , then the number of all involutionary antimorphisms of A^* equals the number of the involutionary permutations over n elements. As is well known, this number is given by

$$n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{2^k (n-2k)! k!}$$

(sequence A000085 in [16]).

Let ϑ be an involutory antimorphism of A^* . A word $w \in A^*$ is called ϑ -palindrome if it is a fixpoint of ϑ , i.e., $w = w^\vartheta$. The set of all ϑ -palindromes of A^* is denoted by $PAL_\vartheta(A)$ or simply PAL_ϑ when there is no ambiguity.

An R -palindrome is usually called *palindrome* and PAL_R is denoted by PAL . In less precise terms, a word which is a ϑ -palindrome with respect to a given but unspecified involutory antimorphism ϑ , is also called *pseudopalindrome*.

Examples 2.1. Let $A = \{a, b\}$, e be the exchange antimorphism, and $w = abaabb$. One has $w^e = aabbab$. The word $abbaab$ is an e -palindrome.

Let $A = \{a, b, c\}$ and τ be the involutory permutation defined as $\tau(a) = b$, $\tau(b) = a$, and $\tau(c) = c$. Setting $\vartheta = \tau \circ R$, the word $abcacbcab$ is a ϑ -palindrome.

A word is called ϑ -symmetric if it is the product of two ϑ -palindromes. An R -symmetric word is simply called *symmetric*. In particular, any ϑ -palindrome is ϑ -symmetric.

Some combinatorial properties of symmetric words were studied in [4], and more recently in [3], where the term symmetric was used. One easily verifies that all words on the alphabet $\{a, b\}$ of length ≤ 5 are symmetric. The word $w = abaabb$ is not symmetric but it is e -symmetric, because it is the product of the two words ab and $aabb$ which are e -palindromes.

In the following, we shall assume that ϑ is a fixed involutory antimorphism of A^* . To simplify the notation, for any $w \in A^*$, we shall denote by \bar{w} the word w^ϑ , so that for all $u, v \in A^*$ one has

$$|\bar{u}| = |u|, \quad \overline{uv} = \bar{v}\bar{u}, \quad \text{and} \quad \overline{(\bar{u})} = u .$$

Lemma 2.2. *A word w is a conjugate of \bar{w} if and only if it is ϑ -symmetric.*

Proof. If $w = \alpha\beta$ with $\alpha, \beta \in PAL_\vartheta$, then $\bar{w} = \beta\alpha$, so that $w \sim \bar{w}$. Conversely, suppose that w and \bar{w} are conjugate. One can write $w = \lambda\mu$ and $\bar{w} = \mu\lambda$ for some $\lambda, \mu \in A^*$. Thus $w = \bar{\lambda}\bar{\mu} = \lambda\mu$. Since $|\lambda| = |\bar{\lambda}|$, one obtains $\lambda = \bar{\lambda}$ and $\mu = \bar{\mu}$. \square

Lemma 2.3. *A ϑ -palindrome $w \in A^+$ has a period $p \leq |w|$ if and only if it has a ϑ -palindromic prefix (suffix) of length $|w| - p$.*

Proof. If w has a period $p \leq |w|$, then it has a border v of length $|w| - p$, so that we can write $w = \lambda v = v\mu$ for some words λ and μ . Since w is a ϑ -palindrome, one has

$$w = v\mu = \bar{v}\bar{\lambda} .$$

Therefore, $v = \bar{v}$. Conversely, if the ϑ -palindrome w has the ϑ -palindromic prefix v , one has

$$w = v\mu = \bar{\mu}v ,$$

so that v is a border of w and $|w| - |v|$ is a period of w . \square

Lemma 2.4. *Let $w \in A^+$ and z_w be its fractional root. The word $z_{\bar{w}}$ is a conjugate of \bar{z}_w .*

Proof. Let w be a nonempty word. Since ϑ acts on the alphabet as a permutation, one derives that p is a period of w if and only if it is a period of \bar{w} .

Therefore one has $\pi_w = \pi_{\bar{w}}$. We can write $w = z_w^k z'$ with $k \geq 1$ and z' a proper prefix of z_w , and

$$\bar{w} = \bar{z}' \bar{z}_w^k = z_{\bar{w}}^h z''$$

with $h \geq 1$ and z'' a proper prefix of $z_{\bar{w}}$. Since $|w| = |\bar{w}|$ and $|\bar{z}_w| = |z_{\bar{w}}| = \pi_{\bar{w}}$, one has $h = k$ and, by Lemma 2.1, $\bar{z}_w \sim z_{\bar{w}}$. \square

Corollary 2.5. *Let $w \in A^+$ be a ϑ -palindrome having a period $p \leq |w|$. Any factor u of w of length p is ϑ -symmetric. In particular, z_w is ϑ -symmetric.*

Proof. Since $w = \bar{w}$ and $|u| = p$, by Lemma 2.1 one has $u \sim \bar{u}$. Hence, by Lemma 2.2 one obtains $u \in PAL_{\vartheta}^2$. As $|z_w| = \pi_w$, one derives $z_w \in PAL_{\vartheta}^2$. \square

3 Pseudopalindrome closure operators

Let ϑ be an involutory antimorphism of A^* . We define in A^* two closure operators associating to each word w respectively the shortest ϑ -palindrome having w as a prefix, and the shortest ϑ -palindrome having w as a suffix. We prove that the minimal periods of these two ϑ -palindrome closures of w are equal. Moreover, if w is nonempty, their fractional roots are conjugate. The main result of the section is that the minimal period of the ϑ -palindrome closures of a nonempty word w is equal to the minimal period of w if and only if the fractional root of w is ϑ -symmetric.

Lemma 3.1. *For any word $w \in A^*$, there exists a unique shortest ϑ -palindrome having w as a prefix (resp. suffix).*

Proof. Let us observe that certainly there exists a ϑ -palindrome having w as a prefix, namely, $w\bar{w}$. Now suppose that $w\lambda_1$ and $w\lambda_2$ ($\lambda_1, \lambda_2 \in A^*$) are two ϑ -palindromes having w as a prefix, of shortest length $k = |w\lambda_1| = |w\lambda_2|$. One has $0 \leq |\lambda_1| = |\lambda_2| = \ell \leq |w|$. Hence, if u is the prefix of w of length ℓ , one derives that $\lambda_1 = \lambda_2 = \bar{u}$.

In a similar way, one proves that there exists a unique shortest ϑ -palindrome having w as a suffix. \square

For any word $w \in A^*$, we denote by $w^{\oplus\vartheta}$ (resp. $w^{\ominus\vartheta}$) the shortest ϑ -palindrome in A^* having w as a prefix (resp. suffix). We call $w^{\oplus\vartheta}$ (resp. $w^{\ominus\vartheta}$) the *right* (resp. *left*) ϑ -palindrome closure of w . To simplify the notation, we shall write w^{\oplus} and w^{\ominus} for $w^{\oplus\vartheta}$ and $w^{\ominus\vartheta}$ respectively, when no confusion arises.

When ϑ is the reversal operator R , w^{\oplus} and w^{\ominus} are respectively the shortest palindrome having w as a prefix and the shortest palindrome having w as a suffix. As usual, they will be denoted by $w^{(+)}$ and $w^{(-)}$ (cf. [5]).

For any word w , we denote by $P_{\vartheta}(w)$ (resp. $Q_{\vartheta}(w)$) the longest ϑ -palindromic prefix (resp. ϑ -palindromic suffix) of w . When there is no ambiguity, we shall simply write P and Q instead of $P_{\vartheta}(w)$ and $Q_{\vartheta}(w)$, respectively.

Proposition 3.2. *If w is a word and $w = sQ = Pt$, then $w^{\oplus} = sQ\bar{s}$ and $w^{\ominus} = \bar{t}Pt$.*

Proof. Let $w = sQ$ and $w^{\oplus} = sQ\lambda$ with $\lambda \in A^*$. Since w^{\oplus} is a ϑ -palindrome, one has

$$w^{\oplus} = sQ\lambda = \bar{\lambda}Q\bar{s} .$$

If $|s| \geq |\bar{\lambda}|$, then $s = \bar{\lambda}\delta$, $\delta \in A^*$. Since $w^\oplus = \bar{\lambda}\delta Q\lambda$, it follows that δQ is a ϑ -palindrome. One can write $w = sQ = \bar{\lambda}\delta Q$, so that $\delta = \varepsilon$ since by hypothesis Q is the ϑ -palindromic suffix of w of maximal length. Hence $\lambda = \bar{s}$. In a similar way, one proves that $w^\ominus = \bar{t}Pt$. \square

As a consequence of the definition, one derives that

$$(\bar{w})^\oplus = w^\ominus . \quad (3)$$

Example 3.1. Let $w = abaabb$. One has $P_R(w) = aba$, $Q_R(w) = bb$, $w^{(+)} = abaabbaaba$, and $w^{(-)} = bbaabaabb$. If $\vartheta = e = E \circ R$, one has $P_e(w) = ab$, $Q_e(w) = aabb$, $w^\oplus = abaabbab$, and $w^\ominus = aabbabaabb$.

Proposition 3.3. *Let $w \in A^+$ and $w = sQ = Pt$. One has that $z_{w^\oplus} \sim z_{w^\ominus}$ so that $\pi_{w^\oplus} = \pi_{w^\ominus}$. If $w \notin PAL_\vartheta$, then*

$$z_{w^\oplus} = \bar{s}\bar{t} \text{ and } z_{w^\ominus} = \bar{t}s .$$

Proof. If w is a ϑ -palindrome, then the result is trivial. Let us then suppose $w = sQ = Pt \notin PAL_\vartheta$, so that $s, t \in A^+$. By Proposition 3.2, one has:

$$w^\oplus = sQ\bar{s} = Pt\bar{s} = \bar{s}\bar{t}P , \quad (4)$$

$$w^\ominus = \bar{t}Pt = \bar{t}sQ = Q\bar{s}\bar{t} . \quad (5)$$

Since P and Q are proper ϑ -palindromic prefixes and suffixes of w^\oplus and w^\ominus respectively, by Lemma 2.3 one has that $p = |\bar{s}\bar{t}| = |\bar{t}s| > 0$ is a period of w^\oplus and of w^\ominus .

Let us now prove that P is the longest proper ϑ -palindromic prefix (suffix) of w^\oplus . By contradiction, suppose that T is a ϑ -palindromic prefix of w^\oplus of length greater than $|P|$. If $|T| \leq |Pt|$, then T would be a ϑ -palindromic prefix of w longer than P , which is absurd. If $|Pt| < |T| < |w^\oplus|$, then one would contradict the fact that w^\oplus is the shortest ϑ -palindrome having w as a prefix. Therefore, by Lemma 2.3, $p = \pi_{w^\oplus}$. Since by (4), $\bar{s}\bar{t}$ is a prefix of w^\oplus and $|\bar{s}\bar{t}| = \pi_{w^\oplus}$, one has $z_{w^\oplus} = \bar{s}\bar{t}$. In a similar way, one shows that Q is the longest proper ϑ -palindromic prefix (suffix) of w^\ominus , so that $p = \pi_{w^\ominus}$ and $z_{w^\ominus} = \bar{t}s$. From this it follows $z_{w^\oplus} \sim z_{w^\ominus}$. \square

Example 3.2. Let $w = z_w = abaabb$ (see Example 3.1). If $\vartheta = R$, then $s = abaa$, $t = abb$, $z_{w^{(+)}} = abaabba = \bar{s}\bar{t}$, and $z_{w^{(-)}} = bbaabaa = \bar{t}s$, so that $z_{w^{(+)}} \sim z_{w^{(-)}}$.

If $\vartheta = e = E \circ R$, one has $s = ab = \bar{s}$, $t = aabb = \bar{t}$, $z_{w^\oplus} = abaabb = \bar{s}\bar{t}$, and $z_{w^\ominus} = aabbab = \bar{t}s$, so that $z_{w^\oplus} \sim z_{w^\ominus}$.

Theorem 3.4. *Let $w \in A^+$. One has $\pi_w = \pi_{w^\oplus}$ if and only if z_w is ϑ -symmetric.*

Proof. We first prove the “if” part. Suppose $z_w = \alpha\beta$ with $\alpha, \beta \in PAL_\vartheta$, so that

$$w = (\alpha\beta)^n z'$$

where $n \geq 1$ and $z' \in \text{Pref}(\alpha\beta)$. Moreover, let $w = Pt = sQ$ as before, so that by Proposition 3.3 one has $z_{w^\oplus} = \bar{s}\bar{t}$. Since $\pi_{w^\oplus} \geq \pi_w$, it suffices to show that $|\bar{s}\bar{t}| \leq \pi_w$. We distinguish two cases, depending on the length of z' .

The first possibility is $z' \in \text{Pref}(\alpha)$. Let $u \in A^*$ be such that $\alpha = z'u = \bar{u}z'$. Then the word $\bar{z}'\beta(\alpha\beta)^{n-1}z'$ is a ϑ -palindromic suffix of w , and therefore a suffix of Q . This implies $|s| \leq |u|$, because $w = sQ = \bar{u}z'\beta(\alpha\beta)^{n-1}z'$. In a similar way, since $(\alpha\beta)^{n-1}\alpha$ is a ϑ -palindromic prefix of w (and then of P), one has $|t| \leq |\beta z'|$ because $w = Pt = (\alpha\beta)^{n-1}\alpha\beta z'$. In conclusion, one gets $|s\bar{t}| \leq |u| + |\beta z'| = |\alpha\beta| = \pi_w$, as desired.

The second case occurs when z' is not a prefix of α , so that $z' = \alpha z''$ with $z'' \in \text{Pref}(\beta)$. Let v be the word such that $\beta = z''v = \bar{v}z''$. Then $\bar{z}''(\alpha\beta)^{n-1}\alpha z''$ is a ϑ -palindromic suffix of w , so that one derives $|s| \leq |\alpha v|$ following the above arguments. Moreover, since $(\alpha\beta)^n\alpha \in \text{PAL}_\vartheta \cap \text{Pref}(w)$, one obtains $|t| \leq |z''|$, which implies $|s\bar{t}| \leq |\alpha v| + |z''| = |\alpha\beta| = \pi_w$.

Let us now prove the “only if” part. If $\pi_w = \pi_{w^\oplus}$, then $z_w = z_{w^\oplus}$. Moreover, since w^\oplus is a ϑ -palindrome beginning with z_w , it has the suffix \bar{z}_w . As $|z_w| = |\bar{z}_w| = \pi_w$, one has by Lemma 2.1 that $z_w \sim \bar{z}_w$. By Lemma 2.2 it follows $z_w \in \text{PAL}_\vartheta^2$. \square

Corollary 3.5. *Let $L_\vartheta = \{w \in A^* \mid z_w \in \text{PAL}_\vartheta^2\}$. If $w \in L_\vartheta$, then $w^\oplus, w^\ominus \in L_\vartheta$ and $\pi_{w^\oplus} = \pi_{w^\ominus} = \pi_w$.*

Proof. Let $w \in L_\vartheta$. By Corollary 2.5, $z_{w^\oplus}, z_{w^\ominus} \in \text{PAL}_\vartheta^2$ so that $w^\oplus, w^\ominus \in L_\vartheta$. By Theorem 3.4 and Proposition 3.3, $\pi_w = \pi_{w^\oplus} = \pi_{w^\ominus}$. \square

Corollary 3.6. *Let $w \in A^+$. If z_w is ϑ -symmetric, then $z_w^2 A^* \cap w^\oplus A^* \neq \emptyset$.*

Proof. Since $z_w \in \text{PAL}_\vartheta^2$, by the previous theorem one has $\pi_w = \pi_{w^\oplus}$ so that $z_w = z_{w^\oplus}$. One can write

$$w^\oplus = z_{w^\oplus}^k z' = z_w^k z' ,$$

with $k \geq 1$ and $z' \in \text{Pref}(z_w)$. If $k > 1$, one has that z_w^2 is a prefix of w^\oplus . If $k = 1$, then $w^\oplus = z_w z'$, so that $w^\oplus \in \text{Pref}(z_w^2)$. \square

Let us remark that the converse of the statement of the preceding corollary is not true in general, as shown in the last example reported below.

Examples 3.3. Let $w = abaabb$ (see Examples 3.1 and 3.2). One has that $w = z_w \notin \text{PAL}^2$, so that $\pi_{w^{(+)}} = \pi_{w^{(-)}} = 7 \neq 6 = \pi_w$. For $\vartheta = e$, since $z_w \in \text{PAL}_e^2$, one has $\pi_{w^\oplus} = \pi_{w^\ominus} = \pi_w = 6$.

Let $w = aabaa$. One has that $z_w = aab \notin \text{PAL}_e^2$. One has $w^\oplus = aabaabbabb$ and $\pi_{w^\oplus} = 10 \neq 3 = \pi_w$.

Let $w = abccbab$. One has $z_w = abccb \in \text{PAL}^2$ and $\pi_w = 5$. Thus $w^{(+)} = abccbabcba$ and $w^{(-)} = babccbab$. Moreover, $z_w = z_{w^{(+)}} \sim z_{w^{(-)}} = babcc$. In this case z_w^2 is a prefix of $w^{(+)}$.

Let $w = babaab$. One has $z_w = baba \in \text{PAL}^2$, $w^{(+)} = babaabab$, $w^{(-)} = baababab$, so that $\pi_{w^{(+)}} = \pi_{w^{(-)}} = \pi_w = 5$. In this case $w^{(+)}$ is a prefix of z_w^2 .

Let $w = (aba^2b^2)^2a$, whose fractional root is $z_w = aba^2b^2 \notin \text{PAL}^2$. One has that z_w^2 is a prefix of w , and then of $w^{(+)}$.

The following two lemmas will be useful in the sequel.

Lemma 3.7. *If a word $u \in A^*$ and a letter $x \in A$ are ϑ -palindromes, then the word ux has a fractional root z_{ux} which is ϑ -symmetric.*

Proof. If ux is unbordered, then $z_{ux} = ux \in PAL_{\vartheta}^2$, so that z_{ux} is ϑ -symmetric.

If $|z_{ux}| < |ux|$, then z_{ux} is a prefix of u and $\pi_{ux} = |z_{ux}| \leq |u|$ is a period of u . By Corollary 2.5, it follows that z_{ux} is ϑ -symmetric. \square

Let us observe that the preceding result is not in general true if the letter x is not a ϑ -palindrome. For instance, let $A = \{a, b\}$ and ϑ equal to the exchange antimorphism e . The word $w = aabb$ is an e -palindrome, but the word $wb = aabbb = z_{wb}$ is not e -symmetric.

Lemma 3.8. *Let $u \in A^*$ and $w = (ux)^{\oplus}$, where $x \in A$. If p is any prefix of w of length $|p| > |u|$, then $p^{\oplus} = w$.*

Proof. The word w is a ϑ -palindrome having p as a prefix, so that $|p^{\oplus}| \leq |w|$. Moreover, p has the prefix ux , so that

$$|(ux)^{\oplus}| \leq |p^{\oplus}| \leq |w| .$$

Therefore, $|p^{\oplus}| = |w|$. Since w is a ϑ -palindrome of minimal length having p as a prefix, it follows by Lemma 3.1 that $w = p^{\oplus}$. \square

4 Iterated pseudopalindrome closures

Let ϑ be a fixed involutory antimorphism of A^* and \oplus the right ϑ -palindrome closure operator. For any letter $a \in A$ we denote by D_a^{ϑ} , or simply D_a , the map $D_a : A^* \rightarrow PAL_{\vartheta}$ defined as: for all $v \in A^*$

$$D_a(v) = (va)^{\oplus}.$$

We call the operators D_a , $a \in A$, the *elementary ϑ -palindrome (right) actions* of the letters of A on A^* . One can extend inductively the definition of the operators D_a from the letters of the alphabet A to the words of A^* by setting $D_{\varepsilon} = \text{id}$ and for any $a \in A$ and $w \in A^*$,

$$D_{wa} = D_a \circ D_w.$$

Hence, if $w = a_1 a_2 \cdots a_n$, $a_i \in A$, $i = 1, \dots, n$, one has:

$$D_w = D_{a_n} \circ D_{a_{n-1}} \circ \cdots \circ D_{a_1}.$$

Thus the action of the operator D_w on the words of A^* is obtained by successive elementary ϑ -palindrome actions with an iterated process which is directed by the word w . Since for any $w, u \in A^*$ and $a \in A$ the word $D_w(u)$ is a prefix of $D_{wa}(u)$, we can define for an infinite word x an operator $D_x : A^* \rightarrow A^{\omega}$ by setting, for any $u \in A^*$:

$$D_x(u) = \lim_{n \rightarrow \infty} D_{w_n}(u) , \tag{6}$$

where $\{w_n\} = \text{Pref}(x) \cap A^n$ for $n \geq 0$.

The words u and x are called, respectively, the *seed* and the *directive word* of $D_x(w)$. In the following, we shall consider mainly the case when the seed u is equal to the empty word. Therefore, we set for any $w \in A^{\omega}$

$$\psi_{\vartheta}(w) = D_w(\varepsilon) . \tag{7}$$

From this definition one has $\psi_\vartheta(\varepsilon) = \varepsilon$ and, for any $w \in A^*$ and $a \in A$,

$$\psi_\vartheta(wa) = (\psi_\vartheta(w)a)^\oplus .$$

For any $w, v \in A^*$, one has:

$$\psi_\vartheta(wv) \in \psi_\vartheta(w)A^* \cap A^*\psi_\vartheta(w).$$

If $x = x_1x_2 \cdots x_n \cdots \in A^\omega$, from (6) and (7) it follows

$$\psi_\vartheta(x) = \lim_{n \rightarrow \infty} \psi_\vartheta(x_1 \cdots x_n) .$$

The infinite word $\psi_\vartheta(x)$ will be called the ϑ -*standard (infinite) word* directed by x . The directive word of a ϑ -standard word t will be also denoted by $\Delta(t)$. A ϑ -standard word will be called, without specifying the antimorphism ϑ , a *pseudostandard word*.

Examples 4.1. Let $A = \{a, b\}$ and $x = (ab)^\omega$. If $\vartheta = R$, one obtains

$$\psi_R(a) = a, \quad \psi_R(ab) = aba, \quad \psi_R(aba) = abaaba, \dots$$

In this case, $\psi_R((ab)^\omega) = abaababaabaab \cdots$ is the famous *Fibonacci word* f .

If $\vartheta = e$, one has

$$\psi_e(a) = ab, \quad \psi_e(ab) = abbaab, \quad \psi_e(aba) = abbaabbaab, \dots$$

In this case, $\psi_e((ab)^\omega) = \mu(f)$, where μ is the Thue-Morse morphism (cf. [15]) defined as $\mu(a) = ab$, $\mu(b) = ba$.

Proposition 4.1. *Let $s = \psi_\vartheta(x)$ be a ϑ -standard word. The following hold:*

1. w is a prefix of s if and only if w^\oplus is a prefix of s ,
2. the set of all palindromic prefixes of s is given by $\psi_\vartheta(\text{Pref}(x))$,
3. s is closed under ϑ , i.e., if $w \in \text{Fact}(s)$, then $\bar{w} \in \text{Fact}(s)$.

Proof. If w^\oplus is a prefix of s , then trivially w is a prefix of s . Conversely, suppose that w is a prefix of s and that $\Delta(s) = x = x_1x_2 \cdots x_n \cdots$ with $x_i \in A$, $i > 0$. Let us set $u_1 = \varepsilon$ and for $n > 1$, $u_{n+1} = \psi_\vartheta(x_1 \cdots x_n)$, so that $u_{n+1} = (u_nx_n)^\oplus$. If $w = \varepsilon$, then trivially $w^\oplus = \varepsilon \in \text{Pref}(s)$. If $w \neq \varepsilon$, we consider the least n such that $|u_n| < |w| \leq |u_{n+1}|$. By Lemma 3.8 one has $w^\oplus = u_{n+1} \in \text{Pref}(s)$. This proves point 1.

By the definition of ϑ -standard words, all the words in the set $\psi_\vartheta(\text{Pref}(x))$ are ϑ -palindromic prefixes of s . Conversely, if w is a ϑ -palindromic prefix of s , then by following the same argument used for point 1, one has that there exists an integer n such that $w = w^\oplus = u_n \in \psi_\vartheta(\text{Pref}(x))$. This proves point 2.

Let w be a factor of s . Since there are infinitely many ϑ -palindromic prefixes of s , there exists a ϑ -palindromic prefix u having w as a factor. Therefore, also \bar{w} is a factor of u and of s . This concludes the proof. \square

Proposition 4.2. *Let t be a ϑ -standard word. If w is a factor of t , then either w^\oplus or w^\ominus are factors of t .*

Proof. We suppose that $w \notin PAL_{\vartheta}$, otherwise the result is trivial. By Proposition 4.1, $\text{Fact}(t)$ is closed under ϑ , so that also \bar{w} is a factor of t . Let p be a prefix of t such that $p = \lambda u$, where u is either w or \bar{w} and λ is of minimal length. If $\lambda = \varepsilon$, then u is a prefix of t , so that by the preceding proposition, u^{\oplus} is a factor of t . Suppose $\lambda \neq \varepsilon$ and let Q be the longest ϑ -palindromic suffix of p . One can write $p = \lambda u = sQ$ with $s \in A^*$. We now show that Q is the longest ϑ -palindromic suffix of u . Indeed, otherwise one would have $Q = \mu u = \bar{u}\bar{\mu}$ with $\mu \in A^+$, so that

$$p = \lambda u = s\mu u = s\bar{u}\bar{\mu} .$$

Since $\lambda = s\mu$ and $|\mu| > 0$, one has $|s| < |\lambda|$ and this contradicts the minimality of $|\lambda|$. Hence we can write $p = \lambda u = \lambda s'Q$, where $u = s'Q$ and Q is the longest ϑ -palindromic suffix of u . Thus $p^{\oplus} = \lambda s'Q s' \bar{\lambda} = \lambda u^{\oplus} \bar{\lambda}$. Since p^{\oplus} is a ϑ -palindromic prefix of t by the preceding proposition, it follows that $u^{\oplus} \in \text{Fact}(t)$.

We have proved that in all cases, u^{\oplus} is a factor of t . Therefore, if $u = w$, one has $w^{\oplus} \in \text{Fact}(t)$; if $u = \bar{w}$, by (3) one has $w^{\ominus} \in \text{Fact}(t)$. \square

An R -standard word has been called in [11] *standard episturmian word*. In the case of a binary alphabet A , a standard episturmian word which is directed by an infinite word over A with infinitely many occurrences of both letters, is a *standard* or *characteristic Sturmian word*.

Some remarkable combinatorial properties of ϑ -palindromic prefixes of ϑ -standard words, which are related to a suitable extension of the Fine and Wilf theorem, were recently studied in [1, and references therein].

In the next two sections, we consider Sturmian and episturmian words, and give some combinatorial results which are mainly concerned with palindrome closures of their factors. In Section 7, we consider again pseudostandard words, proving that any pseudostandard word is a morphic image of a standard episturmian word having the same directive word.

5 Sturmian words

An infinite word is *Sturmian* if for any $n \geq 0$ it has $n + 1$ distinct factors of length n . Therefore from the definition one has that a Sturmian word is on a binary alphabet, that in the following will be denoted by $\mathcal{A} = \{a, b\}$.

An equivalent definition of Sturmian words can be given in terms of special factors. More precisely, an infinite word over a binary alphabet is Sturmian if it has exactly one right special factor of each length.

A Sturmian word s is called *standard* if it can be defined as follows. For any sequence $d_0, d_1, \dots, d_n, \dots$ of integers such that $d_0 \geq 0$ and $d_i > 0$ for $i > 0$, one defines, inductively, the sequence of words $(s_n)_{n \geq 0}$ where

$$s_0 = b, s_1 = a, \text{ and } s_{n+1} = s_n^{d_n-1} s_{n-1}, \text{ for } n \geq 1 .$$

The sequence $(s_n)_{n \geq 0}$ converges to a limit s which is, as one can prove (cf. [2]), an infinite Sturmian word, called *standard*. The sequence $(d_n)_{n \geq 0}$ is called the *directive sequence* for s . If $d_i = 1$ for all $i \geq 0$, one obtains the Fibonacci word f . We shall denote by *Stand* the set of all the words s_n , $n \geq 0$ of any standard sequence $(s_n)_{n \geq 0}$. Any word of *Stand* is called *finite standard (Sturmian) word*.

It was proved in [5] that a standard Sturmian word s is an R -standard infinite word whose directive word is over \mathcal{A} and has infinitely many occurrences of both letters. Moreover, one has

$$\Delta(s) = a^{d_0} b^{d_1} a^{d_2} b^{d_3} \dots$$

where $(d_n)_{n \geq 0}$ is the directive sequence for s .

Let St be the set of *finite Sturmian words*, i.e., factors of infinite Sturmian words over the alphabet \mathcal{A} . We recall that for any infinite Sturmian word there exists an infinite standard Sturmian word having the same set of factors (cf. [2]). Therefore one easily derives that St is the set of factors of $Stand$.

We recall the following characterization of $Stand$ given in [10]:

$$Stand = \mathcal{A} \cup (PAL^2 \cap PAL\{ab, ba\}) , \quad (8)$$

i.e., a word $w \in \mathcal{A}^*$ is standard if and only if it is a letter or it satisfies the following equation:

$$w = \alpha\beta = \gamma xy ,$$

with $\alpha, \beta, \gamma \in PAL$ and $\{x, y\} = \mathcal{A}$. Hence, a standard word is symmetric, whereas the converse is not true in general.

The palindromic prefixes of all standard words are called *central words* (cf. [2]). The set of all central words is usually denoted by PER . The following important characterization of central words holds:

Proposition 5.1. *A word w is central over \mathcal{A} if and only if w is a power of a letter of \mathcal{A} or it satisfies the equation*

$$w = w_1 a w_2 = w_2 b a w_1$$

for some words w_1 and w_2 . Moreover, in this latter case, w_1 and w_2 are central words over \mathcal{A} , $p = |w_1| + 2$ and $q = |w_2| + 2$ are coprime periods of w , and $\min\{p, q\}$ is the minimal period of w .

From the preceding proposition, one derives that a word w is central if and only if it has two coprime periods p, q such that $|w| = p + q - 2$. Moreover, by (8) one easily derives (cf. [10]) that

$$Stand = \mathcal{A} \cup PER\{ab, ba\} . \quad (9)$$

Lemma 5.2. *If $s \in Stand$, then $s \sim \bar{s}$.*

Proof. The result is trivial if $s \in \mathcal{A}$. If s is not a letter, then by (8), $s \in PAL^2$ and the result follows from Lemma 2.2. \square

Corollary 5.3. *If a word w is a conjugate of a standard word, then $w \sim \tilde{w}$.*

Proof. Let s be a standard word such that $w \sim s$. One has $\tilde{w} \sim \tilde{s}$. Since by the preceding lemma $s \sim \tilde{s}$, the result follows. \square

We recall the following characterization of finite Sturmian words given in [8]:

Theorem 5.4. *A nonempty word w is a finite Sturmian word if and only if its fractional root is a conjugate of a standard word.*

Corollary 5.5. *Let w be a nonempty word and z_w be its fractional root. Then w is a finite Sturmian word if and only if so is z_w^2 .*

Proof. This is a straightforward consequence of the preceding theorem, since by (1) the fractional roots of w and of z_w^2 are both equal to z_w . \square

Theorem 5.6. *If $w \in St$, then $w^{(+)}, w^{(-)} \in St$ and $\pi_w = \pi_{w^{(+)}} = \pi_{w^{(-)}}$.*

Proof. Let w be a finite Sturmian word. The result is trivial when $w = \varepsilon$; let us then suppose $w \in A^+$. By Theorem 5.4, $z_w \sim s$ for some $s \in Stand$. By Corollary 5.3, $z_w \sim \tilde{z}_w$ and by Lemma 2.2, $z_w \in PAL^2$. By Theorem 3.4 and Proposition 3.3, one has $\pi_{w^{(+)}} = \pi_{w^{(-)}} = \pi_w$. This implies that $z_{w^{(+)}} = z_w \sim s$, so that by Theorem 5.4 it follows $w^{(+)} \in St$. Since by Proposition 3.3, $z_{w^{(-)}} \sim z_{w^{(+)}} = z_w \sim s$, by applying again Theorem 5.4, one obtains $w^{(-)} \in St$. \square

From the previous results and from Corollary 3.6 one derives the following:

Corollary 5.7. *Let w be a nonempty Sturmian word. One has*

$$z_w^2 \mathcal{A}^* \cap w^{(+)} \mathcal{A}^* \neq \emptyset .$$

The following proposition shows that the left and right palindromic closures of a finite Sturmian word are factors of a suitable infinite standard Sturmian word.

Proposition 5.8. *Let $w \in St$. There exists an infinite standard Sturmian word s such that $w^{(+)}, w^{(-)} \in \text{Fact}(s)$.*

Proof. If $w \in St$, then by (1) one has that for any $k > 1$, the fractional root of z_w^k is z_w . By Theorem 5.4, z_w is a conjugate of a standard word; therefore, by using again Theorem 5.4, one has $z_w^k \in St$. By Proposition 3.3 and Theorem 5.6, one has $z_w = z_{w^{(+)}} \sim z_{w^{(-)}}$. This implies $z_{w^{(-)}} \in \text{Fact}(z_w^2)$. Therefore, there exists an integer $m > 1$ such that $w^{(+)}, w^{(-)} \in \text{Fact}(z_w^m)$. Since $z_w^m \in St$, there exists an infinite standard Sturmian word s such that $z_w^m \in \text{Fact}(s)$. Hence, $w^{(+)}, w^{(-)} \in \text{Fact}(s)$. \square

Let us observe that in general, if s is an infinite Sturmian word, then $w^{(+)} \in \text{Fact}(s)$ does not imply $w^{(-)} \in \text{Fact}(s)$. For instance, in the case of the Fibonacci word f , one has that $(abab)^{(+)} = ababa$ is a factor of f , whereas $(abab)^{-} = babab$ cannot be a factor of f . Indeed otherwise, since $aabaa \in \text{Fact}(f)$, the “balance” condition for Sturmian words (cf. [2]) would not be satisfied.

Proposition 5.9. *Let w be a nonempty word. The following conditions are equivalent:*

1. w is a prefix of a standard Sturmian word,
2. $w^{(+)}$ is central,
3. the fractional root z_w is a standard word.

Proof. 1. \Leftrightarrow 2. This is a consequence of Proposition 4.1. Indeed, w is a prefix of a standard Sturmian word if and only if $w^{(+)}$ is a prefix of a standard Sturmian word, and this occurs if and only if $w^{(+)}$ is a central word.

2. \Rightarrow 3. Trivial if $\pi_w = \pi_{w^{(+)}} = 1$. Then assume by Proposition 5.1 that $w^{(+)} = w_1xyw_2$, with $\{x, y\} = \{a, b\}$ and $|w_1| < |w_2|$, so that by (9) one has $z_{w^{(+)}} = w_1xy \in \text{Stand}$. Since $z_w = z_{w^{(+)}}$, the result follows.

3. \Rightarrow 2. Since $z_w \in \text{PAL}^2$, by Theorem 3.4 one has $z_{w^{(+)}} = z_w$. The word z_w is standard, so that, as a consequence of the construction via directive sequences, one derives that for any $k \geq 1$, $z_w^k \in \text{Pref}(s)$, where s is an infinite standard Sturmian word. Now

$$w^{(+)} = z_{w^{(+)}}^k z' = z_w^k z' \in \text{Pref}(z_w^{k+1})$$

for some $z' \in \text{Pref}(z_w)$. Hence $w^{(+)}$ is a palindromic prefix of a standard word, so that $w^{(+)} \in \text{PER}$. \square

From Theorem 5.6 a new characterization of finite Sturmian words can be given. We need the following lemma that summarizes some results proved in [7]:

Lemma 5.10. *Let $w \in A^*$. If $\pi_w = R_w + 1$, then w is Sturmian. Conversely, if w is a Sturmian palindrome, then $\pi_w = R_w + 1$.*

Theorem 5.11. *A word w is Sturmian if and only if*

$$\pi_{w^{(+)}} = R_{w^{(+)}} + 1 .$$

Proof. By Theorem 5.6, w is Sturmian if and only if $w^{(+)}$ is Sturmian. By the previous lemma, the result follows. \square

In a perfectly symmetric way, one derives that a word w is Sturmian if and only if $\pi_{w^{(-)}} = R_{w^{(-)}} + 1$.

We observe that if $w \in \text{St}$, then from the preceding proposition and Theorem 5.6 one derives $\pi_w = R_{w^{(+)}} + 1$. However, this condition does not assure in general that w is Sturmian, as shown by the following example: let $w = \text{abaabb} \notin \text{St}$; one has $\pi_w = 6$, $\pi_{w^{(+)}} = 7$, and $R_{w^{(+)}} = 5$.

A characterization of finite Sturmian words similar to Theorem 5.11 is given by Theorem 5.13, which was proved in [8]. Here we shall give a different proof, which is based on Theorem 5.11 and on the following lemma.

Lemma 5.12. *If $w \in A^+$ and $\pi_w = R_w + 1$, then $R_w = R_{z_w^2}$.*

Proof. By (1) and (2) one has that for any $k > 1$

$$|z_w| = \pi_w = \pi_{z_w^k} \geq R_{z_w^k} + 1 .$$

Since $\text{Fact}(z_w^2) \subseteq \text{Fact}(z_w^k)$ and $\pi_w = R_w + 1$, one has that for all $k > 1$

$$|z_w| - 1 \geq R_w \geq R_{z_w^k} \geq R_{z_w^2} .$$

As any factor of z_w^k of length at most $|z_w| - 1$ is also a factor of z_w^2 , it follows that $R_{z_w^k} = R_{z_w^2}$ for any $k > 1$. By the definition of fractional root, there exists an integer $h \geq 1$ such that $w \in \text{Fact}(z_w^h)$, so that $R_w \leq R_{z_w^h} = R_{z_w^2}$. From this $R_w = R_{z_w^2}$. \square

Theorem 5.13. *A nonempty word w is Sturmian if and only if*

$$\pi_w = R_{z_w^2} + 1 .$$

Proof. Let w be a nonempty Sturmian word. By Theorem 5.11, $\pi_{w^{(+)}} = R_{w^{(+)}} + 1$. From Theorem 5.6, $\pi_{w^{(+)}} = \pi_w$ and $z_{w^{(+)}} = z_w$. By the preceding lemma, one derives $R_{w^{(+)}} = R_{z_w^2}$ and $\pi_w = R_{z_w^2} + 1$.

Conversely, if $\pi_w = R_{z_w^2} + 1$, then $\pi_{z_w^2} = R_{z_w^2} + 1$, so that by Lemma 5.10, $z_w^2 \in St$. By Corollary 5.5, the result follows. \square

6 Episturmian words

Episturmian words are a natural generalization of infinite Sturmian words in the case of alphabets with more than two letters. They have been introduced in [11] and their theory has been developed in several papers (see for instance [12, 14]).

An infinite word $t \in A^\omega$ is a *standard episturmian* word if it is an R -standard word over A . An infinite word $s \in A^\omega$ is called *episturmian* if there exists a standard episturmian word $t \in A^\omega$ such that $\text{Fact}(s) = \text{Fact}(t)$. It was proved in [11] that an infinite word s is episturmian if and only if s has at most one right special factor of each length and $\text{Fact}(s)$ is closed under reversal.

Of course, any (standard) Sturmian word is a (standard) episturmian word over a two-letter alphabet.

Proposition 6.1. *Let w be a nonempty prefix of a standard episturmian word. The fractional root z_w is symmetric, so that $\pi_w = \pi_{w^{(+)}}$.*

Proof. Let u be the longest palindromic prefix of s whose length is less than $|w|$. We can write $w = ux\xi$ with $\xi \in A^*$, so that by Lemma 3.8 one obtains $w^{(+)} = (ux)^{(+)}$. One has

$$\pi_{ux} \leq \pi_w \leq \pi_{w^{(+)}} = \pi_{(ux)^{(+)}} . \quad (10)$$

Since u is a palindrome, by Lemma 3.7 one has that z_{ux} is symmetric, so that by Theorem 3.4, $\pi_{ux} = \pi_{(ux)^{(\pm)}}$. By (10), $\pi_w = \pi_{w^{(+)}}$, that is equivalent to $z_w \in PAL^2$ by Theorem 3.4. \square

Example 6.1. Let t be the standard episturmian word, called *Tribonacci word*, $t = \psi_R((abc)^\omega)$,

$$t = abacabaabacababac \dots .$$

The fractional roots of the nonempty prefixes of t are the symmetric words

$$a, ab, abac, abacaba, abacabaabacab, \dots$$

Let us observe that in the case of a ϑ -standard word s , the fractional root of a prefix of s is not in general ϑ -symmetric. For instance, consider in the case of $A = \{a, b\}$ and $\vartheta = e$, any e -standard word s having a directive word beginning with a^2b . The word s has the prefix $ababbaabab = (ababb)^\oplus$. Let $w = ababb$. One has $z_w = w \notin PAL_e^2$. In fact, one has $\pi_w = 5$ and $\pi_{w^\oplus} = 6$.

The finite factors of (standard) episturmian words are also called *finite episturmian words*. Differently from the Sturmian case, the fractional root of a finite episturmian word can be non-symmetric, as shown in the following example.

Example 6.2. The word $v = aabaacaabaacaaba$ is a prefix of a standard episturmian word. The word $w = z_w = baac$ is a non-symmetric factor of v . However, $w^{(+)} = baacaab$ and $w^{(-)} = caabaac$ are factors of v .

Let us observe that Corollary 5.5 cannot be extended to the case of episturmian finite words, since there exist finite episturmian words w such that z_w^2 is not a finite episturmian word, as shown by the following:

Example 6.3. The word $w = bac = z_w$ is a finite episturmian word. However, $z_w^2 = (bac)^2$ is not factor of any episturmian word. Indeed, as shown in [11], the number of all palindromic factors in a finite episturmian word u has to be equal to $|u| + 1$. The number of palindromic factors of $(bac)^2$ is 4, and $|z_w^2| + 1 = 7$.

From Proposition 4.2, one has that if w is a finite episturmian word, then either $w^{(+)}$ or $w^{(-)}$ are episturmian. One can ask the question whether, similarly to the Sturmian case (cf. Theorem 5.6), both $w^{(+)}$ and $w^{(-)}$ are episturmian. A positive answer to this problem has been very recently given by Zamboni. A proof can be found in [9].

7 Pseudostandard words

Let ϑ be an involutory antimorphism of A^* . We recall (cf. Section 4) that the map ψ_ϑ defined by (7) satisfies, for any $x \in A^\omega$,

$$\psi_\vartheta(x) = \lim_{n \rightarrow \infty} \psi_\vartheta(w_n) \ ,$$

where $\{w_n\} = \text{Pref}(x) \cap A^n$ for any $n \geq 0$. The word $\psi_\vartheta(x)$ is ϑ -standard, $\psi_\vartheta(A^\omega)$ is the set of all ϑ -standard infinite words, and $\psi_\vartheta(A^*)$ is the set of their ϑ -palindromic prefixes.

As we have seen in Section 2.1, the reversal operator R is the basic involutory antimorphism of A^* , because any other is obtained by composing R with an involutory permutation. Therefore, it is natural to ask whether any pseudostandard word can be obtained, by a suitable morphism, from a standard episturmian word. As we shall see later, the answer to this problem is positive (cf. Theorem 7.1). To this end, we introduce the endomorphism μ_ϑ of A^* by setting $\mu_\vartheta(a) = a^\oplus$ for each $a \in A$. Thus for every letter a one has:

$$\mu_\vartheta(a) = \begin{cases} a & \text{if } a = \bar{a} \\ a\bar{a} & \text{if } a \neq \bar{a} \end{cases} \ .$$

We observe that μ_ϑ is injective, since $\mu_\vartheta(A)$ is a prefix code.

Example 7.1. If $\vartheta = R$, then $\mu_R = \text{id}$. If $\vartheta = e$ is the ‘‘exchange’’ antimorphism of $\{a, b\}^*$, then $\mu_e(a) = ab$ and $\mu_e(b) = ba$, i.e., μ_e is the Thue-Morse morphism.

The main result of this section is the following:

Theorem 7.1. *For any $w \in A^\omega$, one has*

$$\psi_\vartheta(w) = \mu_\vartheta(\psi_R(w)) \ . \tag{11}$$

By this theorem, any ϑ -standard word is a morphic image (by μ_ϑ) of the standard episturmian word having the same directive word. Moreover, the set of palindromic prefixes of ϑ -standard words over A is a morphic image of the palindromic prefixes of standard episturmian words. In particular, the Thue-Morse morphism sends standard Sturmian words to words constructible via iterated e -palindromic closure: $\mu(\psi_R(x)) = \psi_e(x)$. For instance, $\psi_e((ab)^\omega) = \mu(f)$ where f is the Fibonacci word.

To prove Theorem 7.1, we need some lemmas and propositions concerning the morphism μ_ϑ and the antimorphism ϑ .

In the following, we shall drop the subscript ϑ from μ_ϑ when the context is clear. One easily verifies that for any $a \in A$, one has

$$\overline{a^\oplus} = a^\oplus \quad \text{and} \quad (\bar{a})^\oplus = \widetilde{a^\oplus} . \quad (12)$$

Lemma 7.2. *For all $w \in A^*$, $\mu(\tilde{w}) = \overline{\mu(w)}$.*

Proof. Let $w = a_1 \cdots a_n$, with $a_i \in A$, $i = 1, \dots, n$. From (12),

$$\mu(\tilde{w}) = a_n^\oplus \cdots a_1^\oplus = \overline{a_n^\oplus} \cdots \overline{a_1^\oplus} = \overline{a_1^\oplus \cdots a_n^\oplus} = \overline{\mu(w)} . \quad \square$$

Corollary 7.3. *The morphism μ sends palindromes into ϑ -palindromes and vice-versa. Formally, for any $w \in A^*$,*

$$w \in PAL \iff \mu(w) \in PAL_\vartheta , \quad (13)$$

$$w \in PAL_\vartheta \iff \mu(w) \in PAL . \quad (14)$$

Proof. From the previous lemma, since μ is injective one immediately obtains

$$w = \tilde{w} \iff \mu(w) = \mu(\tilde{w}) = \overline{\mu(w)} ,$$

proving (13).

Let $w = a_1 \cdots a_n$, $a_i \in A$, $i = 1, \dots, n$. By (12), $w = \bar{w}$ is equivalent to:

$$\mu(w) = \mu(\bar{w}) = \mu(\bar{a}_n \cdots \bar{a}_1) = (\bar{a}_n)^\oplus \cdots (\bar{a}_1)^\oplus = \widetilde{a_n^\oplus} \cdots \widetilde{a_1^\oplus} = \widetilde{\mu(w)}$$

as desired. \square

Let $\| \cdot \|_\vartheta : A^* \rightarrow \mathbb{Z}_2$ be the morphism of A^* in the additive group \mathbb{Z}_2 of the integers mod 2, defined by the rule: for all $a \in A$,

$$\|a\|_\vartheta = \begin{cases} 0 & \text{if } a = \bar{a} \\ 1 & \text{if } a \neq \bar{a} \end{cases} .$$

In other terms, for any $w \in A^*$, $\|w\|_\vartheta$ counts, modulo 2, the occurrences of letters in w which are not ϑ -palindromes. Note that one has obviously $\|w\|_\vartheta = \|\bar{w}\|_\vartheta$ for any word w . Let us observe that if $\vartheta = R$, then $\|w\| = 0$ for all $w \in A^*$; if $\vartheta = e$, then $\|w\|_e = (|w| \bmod 2)$ for all $w \in \{a, b\}^*$. In the following, we shall denote $\| \cdot \|_\vartheta$ simply by $\| \cdot \|$ when there is no ambiguity.

Lemma 7.4. *If $w \in \mu(A^*) \cup PAL_\vartheta$, then $\|w\| = 0$.*

Proof. It is clear from the definition that $\|\mu(u)\| = 0$ for all $u \in A^*$. Indeed, any letter which is not a ϑ -palindrome is sent by μ in two non- ϑ -palindromic letters. Let $w = a_1 a_2 \cdots a_n \in PAL_\vartheta$, $a_i \in A$, $i = 1, \dots, n$. Since $a_i = \bar{a}_{n+1-i}$ for $1 \leq i \leq n$, it follows that:

- if $n = |w|$ is even, then $w = v\bar{v}$,
- if n is odd, then $w = v\bar{c}\bar{v}$,

where $v = a_1 \cdots a_{\lfloor n/2 \rfloor}$ and $c \in A \cap PAL_\vartheta$. In both cases,

$$\|w\| = \|v\| + \|\bar{v}\| = 2\|v\| = 0 . \quad \square$$

Proposition 7.5. *Let $w \in A^*$. Then $PAL_\vartheta \cap \text{Suff}(\mu(w)) = \mu(PAL \cap \text{Suff}(w))$.*

Proof. The “ \supseteq ” inclusion is a consequence of (13). Now we prove the inverse inclusion. Let s be a suffix of $\mu(w)$ which is not in $\mu(\text{Suff}(w))$. If $w = a_1 \cdots a_n$, with $a_i \in A$ for $1 \leq i \leq n$, then $\mu(w) = a_1^\oplus \cdots a_n^\oplus$, so that s has to be of the form $s = \bar{a}_i \mu(u)$ for some $i \in \{1, \dots, n\}$ (such that $a_i \neq \bar{a}_i$) and $u \in \text{Suff}(w)$. Hence, by Lemma 7.4, $\|s\| = \|a_i\| + \|\mu(u)\| = 1$, and therefore $s \notin PAL_\vartheta$, again by Lemma 7.4. \square

Theorem 7.6. *For all $w \in A^*$, one has*

$$(\mu(w))^\oplus = \mu(w^{(+)}) , \quad (15)$$

$$(\mu(w))^\ominus = \mu(w^{(-)}) . \quad (16)$$

Proof. Let $w = sQ$ with $Q = Q_R(w)$. Then by Proposition 3.2, $w^{(+)} = sQ\tilde{s}$, so that by Lemma 7.2,

$$\mu(w^{(+)}) = \mu(s)\mu(Q)\mu(\tilde{s}) = \mu(s)\mu(Q)\overline{\mu(s)} .$$

By Corollary 7.3, $\mu(Q)$ is a ϑ -palindromic suffix of $\mu(w)$. Let us prove that it is the longest one. Indeed, suppose by contradiction that λ is a ϑ -palindromic suffix of $\mu(w)$, with $|\lambda| > |\mu(Q)|$. By Proposition 7.5, $\lambda = \mu(v)$ for some $v \in PAL \cap \text{Suff}(w)$. This is a contradiction, because $|v| > |Q|$. Thus (15) is proved.

By (3), $w^{(-)} = \tilde{w}^{(+)}$ so that by (15) one has

$$\mu(w^{(-)}) = \mu(\tilde{w}^{(+)}) = (\mu(\tilde{w}))^\oplus .$$

By Lemma 7.2, $\mu(\tilde{w}) = \overline{\mu(w)}$. Therefore, since by (3)

$$\left(\overline{\mu(w)}\right)^\oplus = (\mu(w))^\ominus ,$$

equation (16) is proved. \square

Corollary 7.7. *Let $w \in A^*$ and $a \in A$. The following holds:*

$$(\mu(w)a)^\oplus = \mu((wa)^{(+)}) .$$

Proof. From the preceding theorem, one has $(\mu(wa))^\oplus = \mu((wa)^{(+)})$. Therefore, it suffices to prove that

$$(\mu(w)a)^\oplus = (\mu(wa))^\oplus = (\mu(w)\mu(a))^\oplus . \quad (17)$$

If $a \in PAL_\vartheta$, then $a = \mu(a)$ and (17) follows. Then assume $a \notin PAL_\vartheta$, so that (17) can be rewritten as

$$(\mu(w)a)^\oplus = (\mu(w)a\bar{a})^\oplus .$$

In view of Lemma 3.8, it suffices to show that $\mu(w)a\bar{a}$ is a prefix of $(\mu(w)a)^\oplus$.

Suppose first that $\bar{a}PAL_\vartheta \cap \text{Suff}(\mu(w)) = \emptyset$. Then $Q_\vartheta(\mu(w)a) = \varepsilon$, so that by Proposition 3.2,

$$(\mu(w)a)^\oplus = \mu(w)a\bar{a}\overline{\mu(w)}$$

and we are done.

If $\bar{a}PAL_\vartheta \cap \text{Suff}(\mu(w))$ is nonempty, then let $\bar{a}\lambda$ be its longest element. It is easy to see that $\bar{a}\lambda a$ is the longest ϑ -palindromic suffix of $\mu(w)a$. Moreover, by Proposition 7.5 there exists $v \in PAL \cap \text{Suff}(w)$ such that $\lambda = \mu(v)$. If $w = uv$, since $\bar{a}\mu(v)$ is a suffix of $\mu(w) = \mu(u)\mu(v)$, one derives that $u = u'a$ for some word u' . Hence

$$(\mu(w)a)^\oplus = \mu(u')a\bar{a}\mu(v)a\bar{a}\overline{\mu(u')} = \mu(w)a\bar{a}\overline{\mu(u')} ,$$

which concludes the proof. \square

We are in the position of proving the main theorem.

Proof of Theorem 7.1. Equation (11) is trivially satisfied for $w = \varepsilon$. By induction, let us assume (11) holds for some $w \in A^*$, and prove it for wa with $a \in A$. Indeed,

$$\psi_\vartheta(wa) = (\psi_\vartheta(w)a)^\oplus = (\mu(\psi_R(w))a)^\oplus = \mu\left((\psi_R(w)a)^{(+)}\right) = \mu(\psi_R(wa)) ,$$

where the third equality is a consequence of Corollary 7.7.

The case $w \in A^\omega$ is easily dealt with. \square

For any letter $a \in A$, we define the morphism $\mu_a : A^* \rightarrow A^*$ by $\mu_a(a) = a$ and $\mu_a(b) = ab$, for any $b \neq a$. Moreover, we set $\mu_\varepsilon = \text{id}$ and, for any $w = a_1a_2 \cdots a_n \in A^+$,

$$\mu_w = \mu_{a_1} \circ \mu_{a_2} \circ \cdots \circ \mu_{a_n} .$$

As a consequence of Theorem 7.1 and of a result of Justin [13], we derive the following proposition which allows one to compute $(\psi_\vartheta(w)a)^\oplus$ for any $w \in A^*$ and $a \in A$, starting from its prefix (suffix) $\psi_\vartheta(w)$, by using the morphisms μ and μ_w .

Proposition 7.8. *For any $w \in A^*$ and $a \in A$,*

$$\psi_\vartheta(wa) = (\mu \circ \mu_w)(a) \psi_\vartheta(w) .$$

Proof. We use the result of Justin [13] stating that for any $v, w \in A^*$,

$$\psi_R(wv) = \mu_w(\psi_R(v))\psi_R(w) .$$

Therefore, for $v = a \in A$ one gets $\psi_R(wa) = \mu_w(a)\psi_R(w)$. By Theorem 7.1,

$$\begin{aligned} \psi_\vartheta(wa) &= \mu(\psi_R(wa)) \\ &= \mu(\mu_w(a))\mu(\psi_R(w)) \\ &= (\mu \circ \mu_w)(a)\psi_\vartheta(w) \end{aligned}$$

as desired. \square

Example 7.2. Let $A = \{a, b\}$, $\vartheta = e$, and $w = aba$. One has $\psi_\vartheta(aba) = abbaababbaab$ and $\mu_w(a) = aba$. Hence, $\mu(\mu_w(a)) = abbaab$ and

$$\psi_\vartheta(abaa) = (abbaab)(abbaababbaab) .$$

8 A generalization of pseudostandard words

Let \mathcal{I} be the set of all involutory antimorphisms of A^* , and \mathcal{I}^ω be the set of infinite sequences over \mathcal{I} .

Let $\Theta = \vartheta_1\vartheta_2\cdots\vartheta_n\cdots \in \mathcal{I}^\omega$ and let \oplus_i be the ϑ_i -palindromic closure operator, for all $i \geq 1$. We define inductively an operator ψ_Θ by setting $\psi_\Theta(\varepsilon) = \varepsilon$, and

$$\psi_\Theta(x_1x_2\cdots x_{n+1}) = (\psi_\Theta(x_1\cdots x_n)x_{n+1})^{\oplus_{n+1}}$$

whenever $x_i \in A$ for $i \geq 1$. With this notation, ψ_{ϑ^ω} is just the operator ψ_ϑ considered in the preceding section.

If $x = x_1x_2\cdots x_n\cdots \in A^\omega$, $x_i \in A$ for $i \geq 1$, then $\psi_\Theta(x_1\cdots x_i)$ is a prefix of $\psi_\Theta(x_1\cdots x_{i+1})$ for any i , so that the infinite word

$$\psi_\Theta(x) = \lim_{n \rightarrow \infty} \psi_\Theta(x_1\cdots x_n)$$

is well defined. We call $\psi_\Theta(x)$ a *generalized pseudostandard word*. The pair (x, Θ) which determines $\psi_\Theta(x)$ can be called the *directive bi-sequence* of $\psi_\Theta(x)$. With a suitable choice of the Θ -sequences one can construct all standard episturmian words ($\Theta = R^\omega$), as well as all ϑ -standard words ($\Theta = \vartheta^\omega$). Theorem 8.1 below shows a less trivial example.

In the following, we shall assume $A = \{a, b\}$, $\oplus = \oplus_e$, and $\mu = \mu_e$, where e is the exchange antimorphism of A^* .

Theorem 8.1. *The following holds:*

$$\psi_{(eR)^\omega}(ab^\omega) = \mu^\omega(a) ,$$

i.e., the Thue-Morse word can be obtained via a ψ_Θ operator.

We need two lemmas.

Lemma 8.2. $PAL \cap b\mu(A^*) = b(ab)^*$.

Proof. The “ \supseteq ” inclusion is trivial. Let us prove the inverse inclusion. Since $PAL \cap b\mu(A^0) = \{b\} \subseteq b(ab)^*$, we assume by induction that

$$PAL \cap b\mu(A^k) \subseteq b(ab)^* \tag{18}$$

for all k less than some $n > 0$, and prove (18) for $k = n$.

Let $w \in PAL \cap b\mu(A^n)$. Since $n > 0$, w has to end with b and therefore with ab . Thus $w = bw'b$ with $w' \in PAL \cap \mu(A^{n-1})a$. If $n = 1$, then $w' = a$ and so $w = bab \in b(ab)^*$. If $n > 1$, w' has to begin with ab , so that $w' = aw''a$ with $w'' \in PAL \cap b\mu(A^{n-2}) \subseteq b(ab)^*$. Hence $w = baw''ab \in b(ab)^*$. \square

Lemma 8.3. *For any $n \geq 0$,*

$$PAL \cap \text{Suff}(\mu^{2n+1}(a)) = \{\varepsilon\} \cup \{\mu^{2k}(b) \mid 0 \leq k \leq n\} .$$

Proof. Since $PAL \cap \text{Suff}(\mu(a)) = \{\varepsilon, b\}$, it suffices to show that for any $n > 0$,

$$PAL \cap \text{Suff}(\mu^{2n+1}(a)) = \{b\} \cup \mu^2(PAL \cap \text{Suff}(\mu^{2n-1}(a))) . \tag{19}$$

Since $\mu^{2n+1}(a)$ ends with aab for all $n > 0$, the preceding lemma shows that all palindromic suffixes of $\mu^{2n+1}(a)$ different from b have even length. Indeed,

suppose that q is a palindromic suffix of $\mu^{2n+1}(a)$ of odd length. Since q has to begin with b , one can write $q = b\mu(u)$ with $u \in \text{Suff}(\mu^{2n}(a))$. From the preceding lemma, $q \in b(ab)^*$ so that if $q \neq b$, q and $\mu^{2n+1}(a)$ end with bab , which is a contradiction. Therefore, all palindromic suffixes of $\mu^{2n+1}(a)$ different from b are in $\mu(\text{Suff}(\mu^{2n}(a)))$.

If w is a word with odd length, then $\mu(w)$ cannot be a palindrome, because its minimal (nonempty) median factor is ab or ba . This implies

$$\mu(\text{Suff}(\mu^{2n}(a))) \cap PAL = \mu^2(\text{Suff}(\mu^{2n-1}(a))) \cap PAL .$$

By Corollary 7.3, $w \in PAL \iff \mu^2(w) \in PAL$, so that

$$\mu^2(\text{Suff}(\mu^{2n-1}(a))) \cap PAL = \mu^2(\text{Suff}(\mu^{2n-1}(a) \cap PAL)) .$$

This proves (19). □

Proof of Theorem 8.1. It suffices to show that, for any $n \geq 0$,

$$\mu^{2n+2}(a) = (\mu^{2n+1}(a)b)^{(+)} , \quad (20)$$

$$\mu^{2n+1}(a) = (\mu^{2n}(a)b)^{\oplus} . \quad (21)$$

Let us first prove that (20) is equivalent to the statement

$$Q_R(\mu^{2n+1}(a)b) = bb . \quad (22)$$

Indeed, suppose that (20) is satisfied. Since $|\mu^{2n+2}(a)| = 2|\mu^{2n+1}(a)|$, one derives that (22) holds. Conversely, suppose that (22) is satisfied. Since $\mu^{2n+1}(a)$ ends with b , one can write $\mu^{2n+1}(a) = ub$ with $u \in A^*$, so that

$$(\mu^{2n+1}(a)b)^{(+)} = ubb\tilde{u} = \mu^{2n+1}(a)\widetilde{\mu^{2n+1}(a)} .$$

As is well known (cf. [15]), for all $n \geq 0$ one has $\widetilde{\mu^{2n+1}(a)} = \mu^{2n+1}(b)$. Therefore,

$$(\mu^{2n+1}(a)b)^{(+)} = \mu^{2n+1}(a)\mu^{2n+1}(b) = \mu^{2n+2}(a) .$$

Equation (22) can be equivalently restated saying that any nonempty palindromic suffix of $\mu^{2n+1}(a)$ is preceded by a . By Lemma 8.3, the set of nonempty palindromic suffixes of $\mu^{2n+1}(a)$ is $\{\mu^{2k}(b) \mid 0 \leq k \leq n\}$. Since

$$\mu^{2n+1}(a) = \mu^{2n}(a)\mu^{2n}(b) = \mu^{2n}(a)\mu^{2n-1}(b)\mu^{2n-1}(a) ,$$

by iterating this formula one has that for any $k \leq n$ the suffix $\mu^{2k}(b)$ is preceded by the word $\mu^{2k}(a)$, which ends with a . This proves (20).

By Corollary 7.7 and equation (20), one has

$$(\mu^{2n}(a)b)^{\oplus} = \mu\left((\mu^{2n-1}(a)b)^{(+)}\right) = \mu(\mu^{2n}(a)) = \mu^{2n+1}(a)$$

which proves (21). □

9 Concluding remarks

In the previous sections, we have considered two extensions of the family of standard (epi)Sturmian words. The first one is the family of pseudostandard words, which is obtained by replacing the reversal operator with an arbitrary involutory antimorphism of A^* . Actually, a pseudostandard word is a morphic image of the episturmian word having the same directive word, as shown in Section 7. A second family is obtained when the involutory antimorphism ϑ is not fixed in the process of iterative ϑ -palindrome action. This is a larger class of words, called generalized pseudostandard words, which includes the Thue-Morse word on two letters.

We remark that a further extension of the above families of words can be obtained when the process of iterative application of ϑ -palindrome closure operators starts with a seed which is a nonempty word. In Section 5 we have seen that a standard Sturmian word is an R -standard word whose directive word $x \in \{a, b\}^\omega$ has infinitely many occurrences of both letters. This is not in general true for a nonempty seed. For instance, if the directive word is $x = (ab)^\omega$ and the seed u is $aabb$, one yields the infinite word

$$D_{(ab)^\omega}(aabb) = aabbaabbaaabbbaabbaa \dots$$

which, trivially, is not Sturmian. It would be interesting to study the combinatorial properties of this wider class of words, at least in the special case $\vartheta = R$. We conjecture that if t is a word of this class, then for any sufficiently large n there exists at most one right special factor of t of length n .

References

- [1] V. Anne, L. Q. Zamboni, and I. Zorca. Palindromes and pseudo-palindromes in episturmian and pseudo-palindromic infinite words. In S. Brlek and C. Reutenauer, editors, *Words 2005*, number 36 in Publications du LaCIM, pages 91–100, 2005.
- [2] J. Berstel and P. Séébold. Sturmian words. In M. Lothaire, editor, *Algebraic Combinatorics on Words*. Cambridge University Press, Cambridge UK, 2002. Chapter 2.
- [3] S. Brlek, S. Hamel, M. Nivat, and C. Reutenauer. On the palindromic complexity of infinite words. *International Journal of Foundations of Computer Science*, 15:293–306, 2004.
- [4] A. de Luca. On some combinatorial problems in free monoids. *Discrete Mathematics*, 38:207–225, 1982.
- [5] A. de Luca. Sturmian words: structure, combinatorics, and their arithmetics. *Theoretical Computer Science*, 183:45–82, 1997.
- [6] A. de Luca. On the combinatorics of finite words. *Theoretical Computer Science*, 218:13–39, 1999.
- [7] A. de Luca and A. De Luca. Combinatorial properties of Sturmian palindromes. *International Journal of Foundations of Computer Science*, 17:557–574, 2006.

- [8] A. de Luca and A. De Luca. Some characterizations of finite Sturmian words. *Theoretical Computer Science*, 356:118–125, 2006.
- [9] A. de Luca, A. De Luca, and L. Q. Zamboni. On some problems related to palindrome closure. Manuscript, 2006.
- [10] A. de Luca and F. Mignosi. Some combinatorial properties of Sturmian words. *Theoretical Computer Science*, 136:361–385, 1994.
- [11] X. Droubay, J. Justin, and G. Pirillo. Episturmian words and some constructions of de Luca and Rauzy. *Theoretical Computer Science*, 255:539–553, 2001.
- [12] J. Justin. On a paper by Castelli, Mignosi, Restivo. *Theoretical Informatics and Applications*, 34:373–377, 2000.
- [13] J. Justin. Episturmian morphisms and a Galois theorem on continued fractions. *Theoretical Informatics and Applications*, 39:207–215, 2005.
- [14] J. Justin and G. Pirillo. Episturmian words and episturmian morphisms. *Theoretical Computer Science*, 276:281–313, 2002.
- [15] M. Lothaire. *Combinatorics on Words*. Addison-Wesley, Reading MA, 1983. Reprinted by Cambridge University Press, Cambridge UK, 1997.
- [16] N. J. A. Sloane. *The On-Line Encyclopedia of Integer Sequences*. AT&T Research, <http://www.research.att.com/~njas/sequences/>.