Rich and Periodic-like Words*

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Abstract. In this paper we investigate the periodic structure of *rich words* (i.e., words having the highest possible number of palindromic factors), giving new results relating them with *periodic-like* words. In particular, some new characterizations of rich words and rich palindromes are given. We also prove that a periodic-like word is rich if and only if the square of its fractional root is also rich.

1 Introduction

In the study of the structural properties of finite or infinite words, a relevant role is played by palindromes, i.e., words which can be read without distinction from left to right or from right to left. Indeed, many important classes of words enjoy remarkable properties regarding their palindromic factors.

A well-known example was given in [1]: any factor w of an episturmian word has the maximum number of distinct palindromic factors (that is, $|w|+1$ counting the empty word). Recently, such a property was also found in different contexts (cf. [2, 3]), so in [4] a more systematic study of *rich words* (words w having $|w|+1$ distinct palindromic factors) was initiated; more recent results on rich words can be found in [5–7].

In this paper we study rich words in relation with other structural properties of finite words, introduced by Carpi and the second author in [8, 9] in the frame of a suitable classification of words with respect to their periods.

In the next section we recall some basic definitions and notation. In Sect. 3 we deal with periodic-like words and prove some new results which are useful for the sequel. In Sect. 4 we shall review some basic results on rich words. In the last section we give our main results relating periodic-like and rich words. In particular, we give a new characterization of rich palindromes, and we prove that a palindrome (or a periodic-like word) is rich if and only if the square of its fractional root is also rich.

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2 Preliminaries

In the following, A will denote a finite *alphabet* and A^* the *free monoid* of words over A with the natural concatenation operation. The identity of A^* is the *empty word* ε. Any other word w of A^* can be uniquely written as $w = a_1 \cdots a_n$ with $a_i \in A$ for $i = 1, \ldots, n$. The integer n, denoted by $|w|$, is called the *length* of w; naturally one sets $|\varepsilon| = 0$.

A *factor* of $w \in A^*$ is any word u such that $w = rus$ for some $r, s \in A^*$. If $r = \varepsilon$ (resp. $s = \varepsilon$) then u is a *prefix* (resp. *suffix*) of w. If $|r| = |s|$, then u is a *median* factor of w.

Two words $u, v \in A^*$ are *conjugate* if there exist $\alpha, \beta \in A^*$ such that $u = \alpha\beta$ and $v = \beta\alpha$. It is easy to see that u is a conjugate of v if and only if $|u| = |v|$ and u is a factor of v^2.

The *reversal* of a word $w = a_1 \cdots a_n$, with $a_i \in A$ for $1 \le i \le n$, is the word $\tilde{w} = a_n \cdots a_1$; one assumes $\tilde{\varepsilon} = \varepsilon$. If $w = \tilde{w}$, then w is a *palindrome*. The set of all palindromes of A^* will be denoted by PAL. For any $w \in A^*$, $w^{(+)}$ denotes the *right palindromic closure* of w, that is, the shortest palindrome having w as a prefix. In a symmetric way, $w^{(-)}$ is the shortest palindrome having w as a suffix.

Let $w = a_1 a_2 \cdots a_n$ be a non-empty word with $a_1, \ldots, a_n \in A$. A *period* of w is any positive integer p such that for all $i, j = 1, \ldots, n$,

$$i \equiv j \pmod{p} \Longrightarrow a_i = a_j \ .$$

The minimal period of w is denoted by π_w; the word w is called *periodic* if $2\pi_w \le |w|$. The *fractional root* of w is its prefix z_w of length π_w. For technical convenience, we also set $\pi_\varepsilon = 1$ and $z_\varepsilon = \varepsilon$. A word w is called *unbordered* if $w = z_w$.

A right-infinite word over the alphabet A, called *infinite word* for short, is a mapping $x : \mathbb{N}_+ \to A$, where $\mathbb{N}_+$ is the set of positive integers. One can represent x as

$$x = x_1 x_2 \cdots x_n \cdots \ ,$$

where for any $i > 0$, $x_i = x(i) \in A$. A (finite) *factor* of x is either the empty word or any sequence $u = x_i \cdots x_j$ with $i \le j$, i.e., any block of consecutive letters of x. If $i = 1$, then u is a *prefix* of x. An infinite word x such that $x = uuu \cdots = u^\omega$ for some $u \in A^*$ is called *periodic*. The set of all infinite words over A is denoted by A^ω. We also set $A^\infty = A^* \cup A^\omega$.

Let $w \in A^\infty$. We denote respectively by $\mathrm{Fact}(w)$ and $\mathrm{Pref}(w)$ the sets of factors and prefixes of w. A factor u of w is called *right special* (resp. *left special*) if there exist two letters $x, y \in A$, $x \ne y$, such that $ux, uy \in \mathrm{Fact}(w)$ (resp. $xu, yu \in \mathrm{Fact}(w)$). Any pair $(\lambda, \mu) \in A^* \times A^\infty$ such that $w = \lambda u \mu$ is called an *occurrence* of u in w. An occurrence is *internal* if λ and μ are both non-empty. The factor u is *unioccurrent* (or *unrepeated*) in w if u has exactly one occurrence in w.

An infinite word $x \in A^\omega$ is called *episturmian* (see [1]) if $\mathrm{Fact}(x)$ is closed under reversal and x has at most one right special factor of each length. A

Sturmian word can be defined as an aperiodic episturmian word on a binary alphabet.

For all definitions and notation not explicitly given in the text, the reader is referred to [10, 11].

3 Periodic-like Words

Let w be a non-empty word of A^*. Following the notation in [8], we denote by h'_w (resp. k'_w) the longest repeated prefix (resp. suffix) of w. The word w is called *periodic-like* if h'_w is not right special in w. Conventionally, the empty word ε is also considered to be periodic-like. We denote by H_w (resp. K_w) the length of the shortest unrepeated prefix (resp. suffix) of w. The following [8] holds:

Lemma 3.1. *Let w be a non-empty periodic-like word. Then*

1. *h'_w has no internal occurrence in w,*
2. *$h'_w = k'_w$,*
3. *k'_w is not left special in w,*
4. *$\pi_w = |w| - H_w + 1 = |w| - K_w + 1$,*
5. *$w = z_w h'_w$.*

Moreover, one can prove [8] that condition 1, as well as condition 3, is equivalent to the condition of being periodic-like. If a palindrome w is periodic-like, then, by definition, h'_w is a palindrome.

Let u be a non-empty factor of a finite or infinite word w. We recall (see for instance [4]) that a *complete return* to u in w is any factor of w having exactly two occurrences of u, one as a prefix and one as a suffix. Trivially, one has that a factor v of w is a complete return to u if and only if v is a periodic-like word having $h'_v = u$.

Example 3.2. Let $w = aabcaa$. The longest repeated prefix $h'_w = aa$ is not right special, so that w is periodic-like. We have $k'_w = aa = h'_w$, $H_w = K_w = 3$, $\pi_w = 4$, and $z_w = aabc$.

For any non-empty word w, w^f (resp. w^ℓ) denotes the first (resp. last) letter of w.

Proposition 3.3. *If w is periodic-like and its longest proper median factor is a palindrome, then w is a palindrome.*

Proof. Suppose that $w = xuy$ with $u \in PAL$ and $x, y \in A$. Since w is periodic-like, h'_w is a prefix and a suffix of w, so that it begins with the letter x and terminates with the letter y. We can write $h'_w = x\beta$ where β is a prefix of the palindrome u. Therefore,

$$h'_w = x\beta = \tilde{\beta}y \ .$$

If $\beta = \varepsilon$, then $x = y$. If β is non-empty, then $\beta^\ell = y$ and $\tilde{\beta}^f = x$. Since $\beta^\ell = \tilde{\beta}^f$, one derives again $y = x$. Hence, w is a palindrome. □

We say that a word w is *strongly periodic-like* if all of its median factors are periodic-like.

Example 3.4. The word $w = aabbaa$ is strongly periodic-like. Indeed, all of its median factors are periodic-like. On the contrary, the word $aabcaa$ considered in Example 3.2 is periodic-like but not strongly, since its median factor bc is not periodic-like. Finally, the palindrome $abcbaccabcba$ is periodic-like but not strongly, as its median factor $v = cbaccabc$ has $h'_v = c$ which is right special in v, so that v is not periodic-like.

Proposition 3.5. *A strongly periodic-like word is a palindrome.*

Proof. Let w be a strongly periodic-like word. The proof is by induction on the length of w. The result is trivial when $|w| \leq 3$. Indeed, the only periodic-like words are the palindromes ε, x, xx, xyx with $x, y \in A$. Let us then write w as:

$$w = xuy$$

with $x, y \in A$. Since w is strongly periodic-like, u is strongly periodic-like and $|u| < |w|$. By induction $u \in PAL$. By Proposition 3.3 one derives that w is a palindrome. □

For a word $w \in A^*$, we denote by R_w the minimal integer p such that w has no right special factor of length p. We recall [12, 13] that a word w is *trapezoidal* if $|w| = R_w + K_w$. A finite word is called *Sturmian* if it is a finite factor of a (standard) Sturmian word (cf. [11]). Notice that any finite Sturmian word is trapezoidal, but not conversely. For example, the word $aabb$ is trapezoidal, but not Sturmian.

Proposition 3.6. *A periodic-like trapezoidal word is Sturmian.*

Proof. If w is periodic-like, then by Lemma 3.1, $\pi_w = |w| - K_w + 1$. Since w is trapezoidal, one has $|w| = R_w + K_w$, so that $\pi_w = R_w + 1$. This implies that w is Sturmian (see [14, Proposition 28]). □

We observe that the converse of the preceding proposition does not hold. Indeed, even though every finite Sturmian word is trapezoidal, not every finite Sturmian word is periodic-like. For instance, $w = aab$ is Sturmian (and hence trapezoidal), but not periodic-like (since $h'_w = a$ is right special).

4 Rich Words

Let $w \in A^*$ and denote by S_w the number of palindromic factors of w (including the empty word). As proved in [1], $S_w \leq |w| + 1$. A word w is said *rich* if $S_w = |w| + 1$. We recall (cf. [4]) that the richness property is closed by factors, as well as under the operations of reversal and palindromic closures.

An infinite word is called rich if all its factors are rich. Sturmian and episturmian words (see, for instance, [15] for an overview), are well-studied families of infinite rich words; however, the richness property has been found in wider contexts (cf. [2, 3]). A general investigation on rich words was recently carried on in [4–7].

The following characterization of rich words was first established in [1]:

Proposition 4.1. *A word $w \in A^\infty$ is rich if and only if every prefix p of w has a palindromic suffix which is unioccurrent in p.*

A characterization of rich words in terms of complete returns to their palindromic factors was given in [4, Theorem 2.14]:

Theorem 4.2. *A word $w \in A^\infty$ is rich if and only if for each palindromic factor u of w, every complete return to u in w is a palindrome.*

As a consequence, we can easily derive the following:

Proposition 4.3. *Let $w \in A^\infty$. The following conditions are equivalent:*

1. *w is rich.*
2. *For any factor v of w the longest palindromic prefix (or suffix) of v is unrepeated in v.*
3. *For any periodic-like factor v of w the longest palindromic prefix (or suffix) of v is unrepeated in v.*

Proof. 1) $\Rightarrow$ 2). Since w and its factor v are rich, if the longest palindromic prefix (or suffix) α of v is repeated in v, then by Theorem 4.2 the complete return to α in v is a palindrome. This contradicts the maximality of the length of α.

2) $\Rightarrow$ 3). Trivial.

3) $\Rightarrow$ 1). In view of Theorem 4.2, it is sufficient to show that any complete return v to a palindrome in w is a palindrome. Such a v is then a periodic-like factor of w with $h'_v \in PAL$. Since the longest palindromic prefix (or suffix) α of v is unrepeated in v it follows that $|\alpha| > |h'_v|$. If $|\alpha| < |v|$, since h'_v is a palindromic suffix (or prefix) of α, one would derive that h'_v has an internal occurrence in v which is a contradiction. Hence, $\alpha = v$, so that v is palindrome. □

The following proposition summarizes two further useful results proved in [5].

Proposition 4.4. *Let $w \in A^\infty$ be a rich word. For any $v \in \mathrm{Fact}(w)$, the following holds:*

1. *Any factor of w beginning with v and ending with $\tilde{v}$, having no internal occurrences of v nor of $\tilde{v}$, is a palindrome.*
2. *If $v \notin PAL$, then $\tilde{v}$ is a unioccurrent factor of any complete return to v in w.*

5 Main Results

Let us observe that in general a periodic-like word is not rich. For instance, the word *aabcaa* is periodic-like but not rich. Conversely, there are rich words such as *abc* and *aabaaca* which are not periodic-like. However, in the palindromic case we have the following:

Theorem 5.1. *A word is a rich palindrome if and only if it is strongly periodic-like.*

Proof. Let w be a rich palindrome. The longest proper palindromic prefix α of w occurs only as a prefix and as a suffix of w. In fact, if α had an internal occurrence in w, there would exist a complete return to α, which would be a palindrome in view of Theorem 4.2, and thus a proper palindromic prefix of w longer than α, contradicting the maximality of $|\alpha|$. Thus $|\alpha| \leq |h'_w|$ so that h'_w has no internal occurrence in w, that is w is periodic-like (and moreover $h'_w = \alpha$). Since all median factors of w are also rich palindromes, the "only if" part follows.

Conversely, suppose that w is strongly periodic-like. By Proposition 3.5, w is a palindrome. If $w = \varepsilon$, then it is clearly rich, so let us suppose by induction that $w = aua$ with $a \in A$ and u a rich palindrome. Since $|u| = |w| - 2$ and $S_u = |u| + 1$, it suffices to show that h'_w is the only proper palindromic factor of w which does not occur in u. Indeed, since w is a periodic-like palindrome, h'_w is a palindrome, and in fact the longest proper palindromic prefix of w, as h'_w cannot occur in u by Lemma 3.1. Any palindromic factor v of w with $|v| < |h'_w|$ must be a factor of u. This is trivial if v is not a prefix of w. If v is a prefix of w, then v is a prefix and also a suffix of h'_w and hence a factor of u. □

Let us observe that a different characterization of rich palindromes in terms of *palindromic* and *factor complexity* was recently obtained in [6].

A rich word which is periodic-like (but not strongly) need not be a palindrome: for instance, the word *abacdcabac* is rich and periodic-like.

Corollary 5.2. *A palindrome is rich if and only if all its palindromic factors are periodic-like.*

Proof. If a palindrome is rich, then all its palindromic factors are rich. By Theorem 5.1 it follows that they are periodic-like. Conversely, if all palindromic factors of w are periodic-like, then all median factors of w will be periodic-like so that w is strongly periodic-like and by Theorem 5.1, w is a rich palindrome. □

We remark that there exist non-palindromic words whose palindromic factors are all rich (and then periodic-like), but are not themselves rich. This is the case, for instance, of the word $w = abcab$.

Corollary 5.3. *All palindromic factors of an episturmian word are periodic-like.*

Proof. Any factor of an episturmian word is rich (cf. [1]), so that the result follows from Theorem 5.1. □

Remark 5.4. From the preceding corollary one has in particular that all palindromic factors of Sturmian words are periodic-like. Moreover, in the case of *central* Sturmian words, i.e., the palindromic prefixes w of all standard Sturmian words, one has that these words are *semiperiodic* [12], i.e., $R_w < H_w$. We recall [9] that a semiperiodic word is periodic-like, whereas the converse is not in general true.

Remark 5.5. From the previous results one easily obtains that a trapezoidal palindrome is Sturmian [6]. Indeed, any trapezoidal word is rich [6] so that the result follows by Corollary 5.2 and Proposition 3.6.

Remark 5.6. Notice that the converse of Corollary 5.3 does not hold: indeed there exist periodic-like finite Sturmian and episturmian words that are not palindromes such as *abab* and *abacab*.

The following proposition holds:

Proposition 5.7. *A word $w \in A^\infty$ is rich if and only if every periodic-like factor v of w with $h'_v \in PAL$ is strongly periodic-like.*

Proof. If w is rich, then by Theorem 4.2 every periodic-like factor v of w with $h'_v \in PAL$ is a palindrome. Since v is rich, by Theorem 5.1 it is strongly periodic-like. Conversely, suppose that every periodic-like factor v of w with $h'_v \in PAL$ is strongly periodic-like. By Proposition 3.5, v is a palindrome and the result follows by Theorem 4.2. □

Proposition 5.8. *A word $w \in A^*$ is rich if and only if all palindromic factors of $w^{(+)}$ are periodic-like.*

Proof. If w is rich, then by [4, Proposition 2.6] the right palindrome closure $w^{(+)}$ is rich. By Corollary 5.2, all palindromes in $w^{(+)}$ are periodic-like. Conversely, if all palindromic factors of $w^{(+)}$ are periodic-like, one has by Corollary 5.2 that $w^{(+)}$ is rich. Since $w^{(+)}$ begins with w and the richness property is closed by factors one derives that w is rich. □

Corollary 5.9. *A word $w \in A^*$ is rich if and only if all palindromic factors of $w^{(-)}$ are periodic-like.*

Proof. It is sufficient to observe that $w^{(-)} = \tilde{w}^{(+)}$ and that the richness property is closed under reversal. □

We recall (cf. [16]) that a word w is called *symmetric* if $w \in PAL^2$. The following result was proved in [4, Theorem 3.1].

Proposition 5.10. *Let $w \in A^*$. Then the following conditions are equivalent:*

1. *w^2 is rich,*
2. *w^ω is rich,*
3. *w is symmetric and all of its conjugates are rich.*

A useful corollary of this proposition is:

Corollary 5.11. *Let $w \in A^*$ and let z_w be its fractional root. If z_w^2 is rich, then w is rich.*

Proof. If z_w^2 is rich, then by Proposition 5.10 the infinite word z_w^ω is also rich, so that its prefix w is rich as well. □

It is worth noting that if the fractional root of a word w is rich, in general w need not be rich itself. For example, the word $w = abca$ is not rich even if $z_w = abc$ is rich.

As an easy consequence of Corollary 5.11 we obtain:

Proposition 5.12. *Let w be a finite periodic word. Then w is rich if and only if z_w^2 is rich.*

Proof. Suppose w is rich. Then z_w^2 is rich, since it is a prefix of w by the definition of periodic word. The converse follows from Corollary 5.11. □

Theorem 5.13. *Let w be a palindrome. Then w is rich if and only if z_w^2 is rich.*

Proof. Since w is a rich palindrome, by Theorem 5.1 it is periodic-like, so that by Lemma 3.1 we can write:

$$w = z_w h'_w = h'_w \tilde{z}_w \ ,$$

where h'_w is the longest repeated prefix of w. Therefore, from the preceding equation, by the classic lemma of Lyndon and Schützenberger (cf. [10]) there exist words α and β and $n \geq 0$ such that:

$$z_w = \alpha\beta, \ \tilde{z}_w = \beta\alpha, \ h'_w = (\alpha\beta)^n \alpha \ .$$

Hence, as $w \in PAL$

$$w = (\alpha\beta)^{n+1}\alpha = (\tilde{\alpha}\tilde{\beta})^{n+1}\tilde{\alpha} \ .$$

Since $n \geq 0$ one has that $\alpha, \beta \in PAL$.

If $n > 0$, then z_w^2 is a prefix of w, so that it is rich. If $n = 0$, then $h'_w = \alpha$ and

$$w = h'_w \beta h'_w \ .$$

We will show that any prefix p of z_w^2 has a unioccurrent palindromic suffix. This is certainly true for $|p| \leq |w|$ by Proposition 4.1, since w is rich.

Suppose $|p| > |w|$. We can write $p = w\delta = h'_w \beta h'_w \delta$, with $\delta \in \mathrm{Pref}\,(\beta)$. Since $\beta, h'_w \in PAL$, the prefix p has the palindromic suffix $\tilde{\delta} h'_w \delta$. In fact, this suffix is unioccurrent in p, since otherwise h'_w would have an internal occurrence in w, which is absurd in view of Lemma 3.1. Thus z_w^2 is rich by Proposition 4.1.

The converse follows from Corollary 5.11. □

We remark that if w is a non-palindromic rich word, in general z_w^2 may be not rich, as in the case of $w = abc = z_w$, which is rich whereas $z_w^2 = abcabc$ is not.

Corollary 5.14. *A word $w \in A^*$ is rich if and only if $z^2_{w^{(+)}}$ is rich.*

Proof. The proof is an immediate consequence of the preceding theorem together with the fact that a word w is rich if and only if $w^{(+)}$ is rich. □

We recall [16] that if a word w is a palindrome, then its fractional root z_w is symmetric. Moreover, $z_w = z_{w^{(+)}}$ if and only if z_w is symmetric. Thus a more general result than Theorem 5.13 is the following:

Theorem 5.15. *Let w be a word such that z_w is symmetric. Then w is rich if and only if z^2_w is rich.*

Proof. It is enough to observe that $z_w \in PAL^2$ implies $z_w = z_{w^{(+)}}$, so that $z^2_w = z^2_{w^{(+)}}$ and the result follows from Corollary 5.14. □

Example 5.16. Consider the rich word $w = abacdcabac \notin PAL$, whose root $z_w = abacdc \in PAL^2$. One easily verifies that z^2_w is rich.

Corollary 5.17. *Let w be an unbordered symmetric word. Then w is rich if and only if w^2 is rich.*

Proof. Since w is unbordered and symmetric, we have $w = z_w \in PAL^2$. As $w^2 = z^2_w$, the result follows from Theorem 5.15. □

We remark that from the preceding corollary and Proposition 5.10, one has that if w is a symmetric and unbordered rich word, then all conjugates of w are rich.

Theorem 5.18. *Let $w \in A^*$ be a periodic-like word. Then w is rich if and only if z^2_w is rich.*

Proof. If w is periodic and rich, then z^2_w is rich by Proposition 5.12. Let us then assume that w is not periodic. Since w is periodic-like and $2\pi_w > |w|$, we can write

$$w = h'_w u h'_w$$

for some $u \neq \varepsilon$. Let us set $u = xu'$ with $x \in A$ and $u' \in A^*$.

We prove that wx is periodic-like and rich, with $z_{wx} = z_w$. We can write

$$wx = h'_w x u' h'_w x \ .$$

We have that $h'_{wx} = h'_w x$; indeed if $|h'_{wx}| > |h'_w x|$, then h'_w would have an internal occurrence in w, which is absurd since w is periodic-like. By Lemma 3.1 it follows that $z_{wx} = h'_w u = z_w$.

If $h'_w \in PAL$, then since $w = h'_w x u' h'_w$ is rich and is a complete return to h'_w, by Theorem 4.2 we obtain $w \in PAL$. This implies, by Theorem 5.13, that z^2_w is rich, so that its prefix wx is also rich.

Now let us suppose $h'_w \notin PAL$. By Proposition 4.4, since w is rich and is a complete return to h'_w, the word $\tilde{h}'_w$ is an internal unioccurrent factor of w. Hence w has a prefix γ which begins with $h'_w x$ and ends with $\tilde{h}'_w$, and has no other occurrences of h'_w or $\tilde{h}'_w$. By Proposition 4.4, γ is a palindrome, so that it ends with $x\tilde{h}'_w$. We can then write

$$w = \xi x \tilde{h}'_w \xi'$$

for some $\xi, \xi' \in A^*$. The suffix $\tilde{h}'_w \xi'$ ends with h'_w and has no internal occurrences of h'_w nor of $\tilde{h}'_w$, so that by Proposition 4.4 it is a palindrome. Thus the word wx has the palindromic suffix

$$x\tilde{h}'_w \xi' x = x\tilde{\xi}' h'_w x \ ,$$

which is unioccurrent because otherwise h'_w would have an internal occurrence in w. Since w is rich, by Proposition 4.1 every prefix of w has a unioccurrent palindromic suffix; hence all prefixes of wx have a unioccurrent palindromic suffix, so that wx is rich by Proposition 4.1.

We have proved that wx is periodic-like and rich, with $z_{wx} = z_w$. By iterating this argument, we eventually obtain that $z_w^2 = wu$ is rich.

The converse follows from Corollary 5.11. □

Remark 5.19. Since a rich palindrome is periodic-like by Theorem 5.1, the preceding theorem is an extension of Theorem 5.13. However, Theorem 5.13 is used in the proof of Theorem 5.18.

As a consequence of Theorem 5.18 and of Proposition 5.10, one has that if w is a periodic-like rich word, then its fractional root z_w is symmetric.

In conclusion, we mention that the importance of considering rich periodic-like words is also due to the fact that, as proved in [8], any word w can be canonically decomposed in (overlapping) periodic-like factors $w_1, \ldots, w_n$, with the property that

$$\pi_w = \sum_{i=1}^{n} \pi_{w_i} \ . \tag{1}$$

Thus any rich word w can be canonically decomposed in rich periodic-like factors w_i $(i = 1, \ldots, n)$ having symmetric roots and satisfying (1).

Example 5.20. Let w be the rich word *abaccabacabaadaab*. Following [8], the canonical decomposition of w in rich periodic-like words is given by (w_1, w_2, w_3) with

$$w_1 = abaccabac\,, \quad w_2 = abacaba\,, \quad w_3 = abaadaab \ ,$$

whose minimal periods are respectively 5, 4, and 6. We have $\pi_w = 5+4+6 = 15$, and the roots $z_{w_1} = abacc$, $z_{w_2} = abac$, and $z_{w_3} = abaada$ are all symmetric.

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