Markov numbers, Christoffel words, and the uniqueness conjecture

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Outline

1. Characteristic matrices
   - Basics
   - General properties of $\mu(w)$

2. Frobenius’ uniqueness conjecture
   - The map $S : w \mapsto \mu(w)_{1,2}$
   - Tight bounds and uniqueness
   - The Fibonacci and Pell cases
Trace Equals 3 Times Upper Right

Definition

A matrix $M \in SL_2(\mathbb{Z})$ is characteristic if

$$\text{tr } M = 3M_{1,2}.$$ 

Example

$M = \begin{pmatrix} 17 & 10 \\ 22 & 13 \end{pmatrix}$ is characteristic, as

- $\det M = 17 \cdot 13 - 22 \cdot 10 = 1$ and
- $17 + 13 = 3 \cdot 10.$
Simple Constraints

Proposition

Let

\[ M = \begin{pmatrix} \alpha & m \\ \beta & \gamma \end{pmatrix} \]

be characteristic. Elements on the same row or column are coprime, and

\[ \alpha^2 \equiv \gamma^2 \equiv -1 \pmod{m}. \]

Also, up to switching \( \alpha \) and \( \gamma \), \( M \) is determined by any two elements.
Markov Triples = Characteristic Products

Products of char. matrices need not be characteristic, but:

**Theorem**

Let $M'$, $M''$ be characteristic and $M = M'M''$. Then $M$ is characteristic $\iff$ the upper right elements $m'$, $m''$, $m$ (of $M'$, $M''$, $M$ respectively) verify the Markov equation, i.e.,

$$(m')^2 + (m'')^2 + m^2 = 3m'm''m.$$
The Morphism $\mu : \{a, b\}^* \rightarrow SL_2(\mathbb{Z})$

Setting

$$\mu(a) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mu(b) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

defines an injective morphism $\mu : \{a, b\}^* \rightarrow SL_2(\mathbb{Z})$.

Note that $\mu(a)$ and $\mu(b)$ are characteristic…
Reversal and $\mu$

Let $\tilde{w}$ denote the reversal of $w$.
For instance, if $w = aabab$, then $\tilde{w} = babaa$.

**Lemma**

*For all* $w \in \{a, b\}^*$, $\mu(\tilde{w}) = \mu(w)^T$.

*So, $w$ is a palindrome* $\iff$ *$\mu(w)$ is symmetric.*

Let PAL denote the set of palindromes over $\{a, b\}$.
More Relations on elements

Proposition

Let \( w \in \{a, b\}^+ \), and let \( \mu(w) = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \). Then \( p > q, r \geq s \).

Moreover, \( q < r \iff w = \tilde{u} a v b u \) for suitable \( u, v \in \{a, b\}^* \).

Proposition

Let \( u \in \text{PAL} \) and \( \mu(u) = \begin{pmatrix} p & q \\ q & s \end{pmatrix} \). Then

\[
q + s \leq p \leq 2q + s
\]

with \( p = q + s \iff u \in a^* \) and \( p = 2q + s \iff u \in b^* \).
Characterizing Characteristic $\mu(w)$

Matrices $\mu(w)$ need not be characteristic; for instance, 

$$\mu(aa) = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$$

is not.

**Theorem**

Let $w \in \{a, b\}^*$. Then

$$\mu(w) \text{ is characteristic} \iff w \in \{a, b\} \cup aPALb.$$
A Meaningful Decomposition

Let $\nu : \{a, b\}^* \rightarrow SL_2(\mathbb{Z})$ be defined by

$$
\nu(a) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \nu(b) = \nu(a)^T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
$$

It is injective and a well-known tool in the study of Christoffel pairs.

It is easy to see that

$$
\mu = \nu \circ \zeta
$$

where $\zeta$ is the injective endomorphism defined by

$$
\zeta(a) = ba, \quad \zeta(b) = bbba.
$$
A Consequence

Recall the **palindromization** map $\psi$ defined by $\psi(\varepsilon) = \varepsilon$ and

$$\psi(vx) = (\psi(v)x)^{(+)\text{ }} \text{ for } v \in \{a, b\}^*, x \in \{a, b\}$$

where $w^{(+)\text{ }}$ is the right palindromic closure of $w$.

**Proposition**

*Let $w = aub$ with $u \in \text{PAL}$. Then*

$$\mu(w)_{1,2} = |a\psi(a\zeta(u)b)b|$$

i.e., it is the length of a Christoffel word whose directive word $a\zeta(u)b$ has an antipalindromic middle $\zeta(u)$.
The following independent result uses almost the same decomposition:

**Theorem (Reutenauer & Vuillon 2017)**

For all \( v \in \{a, b\}^* \),

\[
\mu(a_\psi(v)b)_{1,2} = |a_\psi(\psi_E(av))b|,
\]

where \( \psi_E = \theta \circ \psi \) is the **antipalindromization** map,

and \( \theta \) is the Thue-Morse morphism \( (\theta(a) = ab, \theta(b) = ba) \).
Definition of $S$

Define a map $S : \{a, b\}^* \to \mathbb{N}$ by $S(w) = \mu(w)_{1,2}$.

Since $S(w) = \begin{pmatrix} 1 & 0 \end{pmatrix} \mu(w) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, when viewed as a formal series,

$$S = \sum_{w \in \{a,b\}^*} S(w)w$$

is rational.
S as a Series

\[ S = (2a + 5b + (a + 2b)(a + b)^* (a + 2b))^* (a + 2b)(a + b)^* \]
Not Injective in General

**Proposition**

For all $u \in \{a, b\}^*$,

- $S(aub) = S(a\tilde{u}b)$,
- $S(a\theta(u)b) = S(a\theta(\tilde{u})b)$,

where $\theta$ is the Thue-Morse morphism.

**Example**

- $S(aabb) = 75 = S(abab)$;
- $S(aabbab) = 1130 = S(abaabb)$.

Thus, even the restriction of $S$ to $\{a, b\} \cup aPAlb$ is not injective...
What about Christoffel?

Let CH be the set of (lower) Christoffel words over \( \{a, b\} \).

**Theorem (Borel, Laubie 1993, etc.)**

The following holds:

\[
CH = \{a, b\} \cup \left( aPALb \cap (\{a, b\} \cup aPALb)^2 \right).
\]

That is, Christoffel words can be recursively defined:

\( w \in CH \iff w \) is either a letter or a word of \( aPALb \)

that is the concatenation of two shorter Christoffel words.
Hence, Markov Triples

We have seen that:

1. \( w \in \text{CH} \iff w \in \{a, b\} \) or
   \( w \in a\text{PAL}b \) and \( w = w'w'' \) with \( w', w'' \in \{a, b\} \cup a\text{PAL}b \);

2. \( w \in \{a, b\} \cup a\text{PAL}b \iff \mu(w) \) is characteristic;

3. Characteristic matrices \( M', M'', M'M'' \) correspond
   (by their upper right elements) to Markov triples.

Corollary (see Cohn 1972, Reutenauer 2009, etc.)

\( S \) maps Christoffel pairs to (nonsingular) Markov triples, i.e.,
if \( w = w'w'', w, w', w'' \in \text{CH} \), then

\[
S(w')^2 + S(w'')^2 + S(w)^2 = 3S(w')S(w'')S(w).
\]
The Conjecture: $S$ Injective on Christoffel

The previous map is actually a bijection, so that the

**Conjecture (Frobenius 1913)**

*Markov triples are uniquely determined by their maximal element.*

is equivalent to

**Conjecture**

*The restriction $S|_{CH}$ is injective.*
Evidence Suggesting More Conjectures

Our *limited* experiments (*Markov numbers grow fast!...*) also suggest that:

1. if \( w \neq w' \) and \( S(w) = S(w') \), then \( w, w' \in a\{a, b\}^*b; \)

2. if \( w \in a\text{PAL}b \) and \( S(w) \) is a Markov number, then \( w \in \text{CH}. \)
Proving Uniqueness via Matrices

A Markov number $m$ is unique if it is the maximal element of a unique Markov triple, or equivalently, if the set

$$S^{-1}_{\text{CH}}(m) = \{ w \in \text{CH} \mid S(w) = m \}$$

is a singleton.

Since $\mu$ is injective, and matrices $\mu(w)$ are uniquely determined by any two elements,

$$m \text{ is unique} \iff \exists! \gamma : \begin{pmatrix} \alpha & m \\ \beta & \gamma \end{pmatrix} \in \mu(\text{CH}) \text{ for suitable } \alpha, \beta.$$
(Extended) Known Bounds

Hence, as $\gamma^2 \equiv -1 \mod m$, $m$ is closer to uniqueness when this has few solutions for $\gamma$.

**Theorem**

*If* $w \in aPALb$ *and* $\mu(w) = \begin{pmatrix} \alpha & m \\ \beta & \gamma \end{pmatrix}$, *then*

$$
\left(2 - \sqrt{2}\right) m < \gamma < \frac{\sqrt{5} - 1}{2} m.
$$

$$
\left\lceil \left(2 - \sqrt{2}\right) m \right\rceil \leq \gamma \leq \left\lfloor \frac{\sqrt{5} - 1}{2} m \right\rfloor.
$$
Tight Versions

The previous bounds are tight, since for example

- \( w \in ab^* \implies \gamma = \left\lfloor \left(2 - \sqrt{2}\right)m \right\rfloor \);
- \( w \in a^*b \implies \gamma = \left\lfloor \frac{\sqrt{5} - 1}{2} m \right\rfloor \).

Further examples exist, though; for \( w = aabab \), e.g., we have
\[ \gamma = 119 = \left\lfloor \frac{\sqrt{5} - 1}{2} \times 194 \right\rfloor = \left\lfloor \frac{\sqrt{5} - 1}{2} m \right\rfloor \].
More Precise Bounds

Theorem

Let \( w \in aPALb, \; m = \mu(w)_{1,2}, \; \gamma = \mu(w)_{2,2}. \) Then

\[
2m - \sqrt{2m^2 - 1} \leq \gamma \leq \frac{-m + \sqrt{5m^2 - 4}}{2},
\]

and the lower (resp. upper) bound is attained if and only if \( w \in ab^* \) (resp. \( w \in a^*b \)).
Extended Limited Uniqueness Results

**Theorem**

Let $w = aub$, $u \in \text{PAL}$ be such that $S(w) = 2^h p^k$ (resp. $[\mu(u)] = 2^h p^k$) for an odd prime $p$ and integers $h \geq 0$, $k \geq 1$. Then for all $w' \in a\text{PAL}b$,

\[ w \neq w' \implies S(w) \neq S(w'). \]

*(Here $[M] = M_{1,1} + M_{1,2} + M_{2,1} + M_{2,2}$).*

**Remark**

It is not known whether there are infinitely many such Markov numbers!
Pell numbers are defined by:

- $P_0 = 0, P_1 = 1$;
- $P_{n+1} = 2P_n + P_{n-1}$ for $n \geq 1$.

Well-known: for all $n \geq 0$, \{1, $F_{2n+1}$, $F_{2n+3}$\} and \{2, $P_{2n+1}$, $P_{2n+3}$\} are Markov triples.

Lemma (cf. Gessel 1972)

A natural number $n$ is an odd- (resp. even-) indexed Fibonacci number if and only if $5n^2 - 4$ (resp. $5n^2 + 4$) is a perfect square.

Similarly, $n$ is an odd- (resp. even-) indexed Pell number if and only if $2n^2 - 1$ (resp. $2n^2 + 1$) is a perfect square.
Corresponding Words and Matrices

In particular,

\[
\mu(a^n b) = \begin{pmatrix}
2F_{2n+3} + F_{2n+1} & F_{2n+3} \\
2F_{2n+2} + F_{2n} & F_{2n+2}
\end{pmatrix},
\]

\[
\mu(ab^n) = \begin{pmatrix}
P_{2n+2} & P_{2n+1} \\
P_{2n+1} + P_{2n} & P_{2n} + P_{2n-1}
\end{pmatrix}.
\]

Theorem (Bugeaud, Reutenauer, Siksek 2009)

Odd-indexed Fibonacci and Pell numbers $> 5$ have no intersection. Also, when written in order, they alternate forming a Sturmian sequence.
Specialized Uniquenessness

However, it is not even known whether all odd-indexed Fibonacci and Pell numbers are unique Markov numbers in general!

**Theorem**

Let $w = a^n b$, $w' \in \text{CH} \setminus \{w\}$, and $\gamma' = \mu(w')_{2,2}$.

If $\gamma' > F_{2n+2}$ or $\gamma' \leq \frac{2(2-\sqrt{2})}{\sqrt{5}-1} F_{2n+2} \approx 0.948 F_{2n+2}$, then

$$S(w') \neq S(w).$$

**Corollary**

Let $w = a^n b$, $w' \in \text{CH} \setminus \{w\}$, and $\gamma' = \mu(w')_{2,2}$.

If $S(w') = S(w) = F_{2n+3}$, then $\gamma'$ is not a Fibonacci number.
The Fibonacci case of the uniqueness conjecture can be restated as follows:

**Conjecture**

Let $x$, $y$, $z \in \mathbb{N}$ be such that $x \leq y \leq z$, with

\[ x^2 + y^2 + z^2 = 3xyz \quad \text{and} \quad \sqrt{5z^2 - 4} \in \mathbb{N}. \]

Then $x = 1$. 
Main References


Thank You

In loving memory of Aldo (1941–2018)
mentor and friend