Corso di
Complementi di gasdinamica

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Moto su cuneo (Wedge flow)

Dalla funzione potenziale complesso:

$$z = x + jy = r(\cos\theta + j\sin\theta)$$

$$\phi(z) = \varphi(x,y) + j\psi(x,y) = Az^n = Ar^n[\cos(n\theta) + j\sin(n\theta)]$$

$$V = \underline{\nabla}\phi = \underline{\nabla} \wedge \psi \underline{k} \begin{cases} u = \varphi_x = \psi_y \\ v = \varphi_y = -\psi_x \\ V_r = \varphi_r = \dfrac{1}{r}\psi_\theta \\ V_\theta = \dfrac{1}{r}\varphi_\theta = -\psi_r \end{cases}$$

Derivando (con *m*=*n*-1) si trova:

$$V_r = nAr^{n-1}\cos(n\theta) = Kr^m \cos((m+1)\theta)$$

$$V_\theta = -nAr^{n-1}\sin(n\theta) = -Kr^m \sin((m+1)\theta)$$

Imponendo $\psi = 0$:

$$\psi = 0 \rightarrow \begin{cases} \theta = 0 \\ n\theta = \pi \end{cases}$$

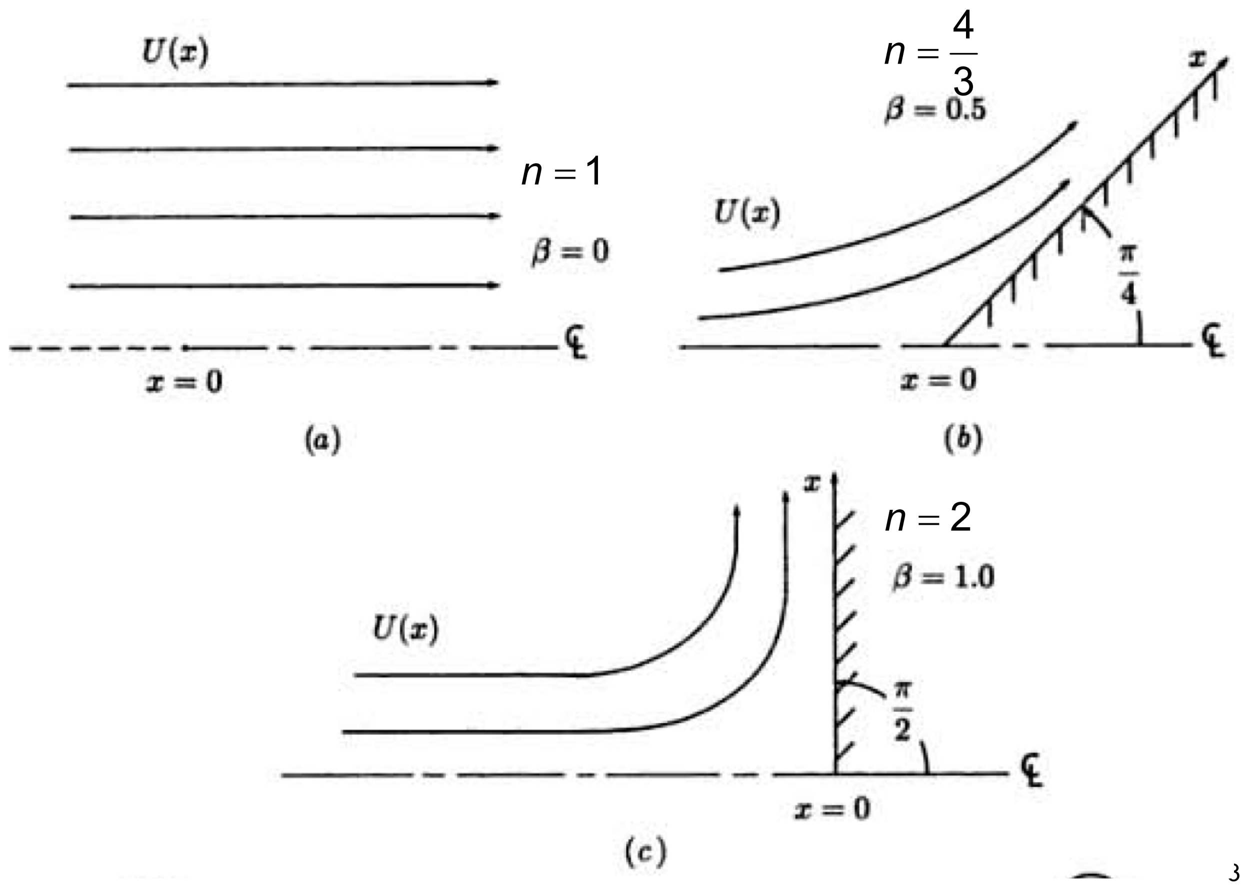
Le rette di equazione:

$$\begin{cases} \theta = 0 \\ \theta = \alpha = \dfrac{\pi}{n} \end{cases}$$

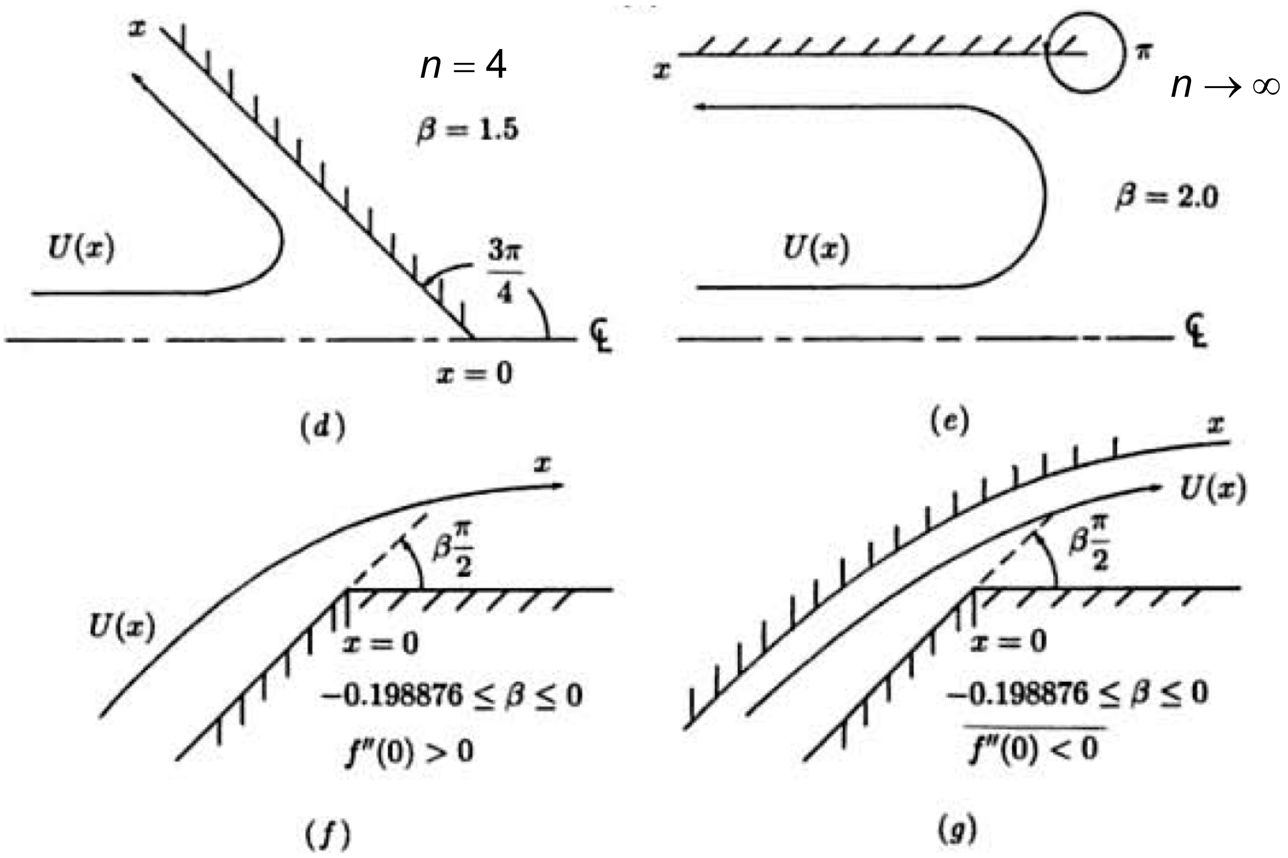
Individuano un cuneo. Considerando l'angolo supplementare ad α misurato in frazioni β di π/2 si ha:

$$\pi - \alpha = \frac{\beta}{2}\pi \rightarrow \left(1 - \frac{1}{n}\right) = \frac{\beta}{2} \rightarrow \beta = \frac{2(n-1)}{n} = \frac{2m}{m+1} \rightarrow n = \frac{2}{2-\beta}$$

Moto su cuneo



Moto su cuneo



Soluzioni simili

Le equazioni da risolvere all'interno dello strato limite sono:

$$u_x + v_y = 0$$

$$uu_x + vu_y = U_e U_{ex} + \nu u_{yy}$$

Dall'equazione di continuità si ricava:

$$v = -\frac{\partial}{\partial x}\int_0^y u\,dy$$

Imponendo:

$$u = U_e f'(\eta)$$

$$\eta = yg(x)$$

Si ha:

$$\eta_x = yg' \quad \eta_y = g$$

Per x costante: $\frac{d\eta}{g} = dy$

$$v = -\frac{\partial}{\partial x}\int_0^y U_e f'(\eta)dy = -\frac{\partial}{\partial x}\frac{U_e}{g}\int_0^\eta f'(\eta)d\eta = -\frac{U'_e}{g}f(\eta) + \frac{U_e g'}{g^2}f(\eta) - \frac{U_e}{g}f'(\eta)yg'$$

Soluzioni simili

$$u = U_e f'(\eta)$$

$$\eta = yg(x)$$

$$\eta_x = yg' \quad \eta_y = g$$

$$v = -\frac{U'_e}{g}f(\eta) + \frac{U_e g'}{g^2}f(\eta) - \frac{U_e}{g}f'(\eta)yg'$$

Le derivate di u sono:

$$u_x = U'_e f'(\eta) + U_e f''(\eta)yg'$$

$$u_y = U_e f''(\eta)g$$

$$u_{yy} = U_e f'''(\eta)g^2$$

Dall'equazione di bilancio della quantità di moto:

$$uu_x + vu_y = U_e U_{ex} + \nu u_{yy}$$

$$U_e f'(\eta)\left(U'_e f'(\eta) + U_e f''(\eta)yg'\right) + \left(-\frac{U'_e}{g}f(\eta) + \frac{U_e g'}{g^2}f(\eta) - \frac{U_e}{g}f'(\eta)yg'\right)U_e f''(\eta)g =$$

$$= U_e U'_e + \nu U_e f'''(\eta)g^2$$

Tralasciando anche la dipendenza con η:

$$U_e U'_e f'^2 + \left(-\frac{U'_e}{g}f + \frac{U_e g'}{g^2}f\right)U_e f'' g = U_e U'_e + \nu U_e f''' g^2$$

Soluzioni simili

$$U_e U_e' f'^2 + \left(-\frac{U_e'}{g} f + \frac{U_e g'}{g^2} f \right) U_e f'' g = U_e U_e' + \nu U_e f''' g^2$$

Dividendo per $\nu U_e g^2$ e risolvendo in f''':

$$f''' = \frac{U_e'}{\nu g^2}\left(f'^2 - ff'' - 1\right) + \frac{U_e g'}{\nu g^3} ff''$$

Visto che f dipende da η mentre le altre funzioni da x si deve avere:

$$\frac{U_e'}{\nu g^2} = c_1 \qquad \frac{U_e g'}{\nu g^3} = c_2 \qquad f''' = c_1\left(f'^2 - ff'' - 1\right) + c_2 ff''$$

Con c_1 e c_2 costanti arbitrarie. Facendo il rapporto fra le due equazioni precedenti si ha:

$$\frac{U_e'}{\nu g^2}\frac{\nu g^3}{U_e g'} = \frac{c_1}{c_2} \rightarrow \frac{U_e'}{U_e}\frac{g}{g'} = \frac{c_1}{c_2} \rightarrow \frac{U_e'}{U_e} = \frac{g'}{g}\frac{c_1}{c_2}$$

Che integrata imponendo $a = c_1/c_2$ fornisce: $ln(U_e) = a\,ln(g) + ln\,B$

Soluzioni simili

Relazione che risolta da: $ln(U_e) = a\,ln(g) + ln\,B$

$$U_e = Bg^a$$

Ed è soddisfatta per:

$$U_e = kx^m \qquad g = Cx^{\frac{m}{a}} \qquad \text{con} \qquad C = \left(\frac{k}{B}\right)^{\frac{1}{a}}$$

Sostituendo nelle:

$$\frac{U_e'}{\nu g^2} = c_1 \qquad \frac{U_e g'}{\nu g^3} = c_2$$

Danno:

$$\frac{mkx^{m-1}}{\nu C^2 x^{\frac{2m}{a}}} = c_1 \qquad \frac{kx^m C \frac{m}{\alpha} x^{\frac{m}{a}-1}}{\nu C^3 x^{3\frac{m}{a}}} = c_2$$

Soluzioni simili

$$\frac{mkx^{m-1}}{\nu C^2 x^{\frac{2m}{a}}} = c_1 \qquad \frac{kx^m C \frac{m}{\alpha} x^{\frac{m}{a}-1}}{\nu C^3 x^{3\frac{m}{a}}} = c_2$$

$$U_e = kx^m \qquad g = Cx^{\frac{m}{a}}$$

Quindi per gli esponenti rispettivamente per la prima e per la seconda:

$$m-1 = \frac{2m}{a} \rightarrow a = \frac{2m}{m-1}$$

$$m + \frac{m}{a} - 1 = \frac{3m}{a} \rightarrow ma + m - a = 3m \rightarrow a = \frac{2m}{m-1}$$

$$U_e = kx^m \qquad g = Cx^{\frac{m-1}{2}}$$

Mentre per i coefficienti:

$$\frac{mk}{\nu C^2} = c_1 \qquad \frac{k\frac{m}{a}}{\nu C^2} = c_2$$

Che sono equivalenti avendo imposto $a = c_1/c_2$. Risolvendo la prima:

$$C^2 = \frac{mk}{\nu c_1}$$

Soluzioni simili

$$a = \frac{c_1}{c_2} = \frac{2m}{m-1} \qquad C^2 = \frac{mk}{\nu c_1}$$

È possibile assegnare almeno una condizione a piacere, per semplificare l'equazione si impone:

$$c_1 - c_2 = 1 \qquad f''' = c_1\left(f'^2 - ff'' - 1\right) + c_2 ff''$$

$$c_1 = \beta \qquad f''' + ff'' + \beta\left(1 - f'^2\right) = 0$$

$$U_e = kx^m \qquad g = Cx^{\frac{m-1}{2}}$$

Quindi:

$$\beta - c_2 = 1 \rightarrow c_2 = \beta - 1$$

$$\frac{2m}{m-1} = a = \frac{c_1}{c_2} = \frac{\beta}{\beta - 1} \rightarrow$$

$$2m(\beta - 1) = \beta(m-1) \rightarrow \beta(m+1) = 2m \rightarrow \beta = \frac{2m}{m+1}$$

Inoltre:

$$C^2 = \frac{mk}{\nu\beta} = \frac{k}{\nu}\frac{m+1}{2}$$

In definitiva:

$$g = \sqrt{\frac{m+1}{2}\frac{k}{\nu}}\,x^{\frac{m-1}{2}} = \sqrt{\frac{m+1}{2}\frac{kx^m}{\nu x}} = \sqrt{\frac{m+1}{2}\frac{U_e}{\nu x}}$$

Soluzioni simili

$$u = U_e f'(\eta)$$
$$\eta = y g(x)$$

Dalla:

$$v = -\frac{U_e'}{g} f(\eta) + \frac{U_e g'}{g^2} f(\eta) - \frac{U_e}{g} f'(\eta) y g'$$

$$\frac{U_e'}{\nu g^2} = c_1 = \beta \qquad \frac{U_e g'}{\nu g^3} = c_2 = \beta - 1$$

$$g = \sqrt{\frac{m+1}{2}\frac{k}{\nu}}\, x^{\frac{m-1}{2}}$$

Si ha:

$$v = -\beta \nu g f(\eta) + (\beta - 1)\nu g f(\eta) - (\beta - 1)\nu g \eta f'(\eta)$$
$$v = \left(-f(\eta) + (1-\beta)\eta f'(\eta)\right)\nu g$$
$$v = \left(\frac{1-m}{m+1}\eta f'(\eta) - f(\eta)\right)\sqrt{\frac{m+1}{2}}\sqrt{\nu k}\, x^{\frac{m-1}{2}}$$

$$\beta = \frac{2m}{m+1} \rightarrow 1 - \beta = \frac{m+1-2m}{m+1} = \frac{1-m}{m+1}$$

Imponendo:

$$\hat{v} = \left(\frac{1-m}{m+1}\eta f'(\eta) - f(\eta)\right)\sqrt{\frac{m+1}{2}}$$

Infine si ha:

$$v = \hat{v}\sqrt{\nu k}\, x^{\frac{m-1}{2}} = \hat{v}\sqrt{\frac{\nu k x^m}{x}} = \hat{v}\sqrt{\frac{\nu U_e}{x}} \qquad \frac{v}{U_e} = \hat{v}\sqrt{\frac{\nu}{x U_e}}$$

Soluzioni simili

In definitiva:

$$\eta = y g(x) = y x^{\frac{m-1}{2}}\sqrt{\frac{m+1}{2}\frac{k}{\nu}} \qquad \beta = \frac{2m}{m+1}$$

$$U_e = k x^m$$

$$u = U_e f'(\eta) = k x^m f'(\eta) \qquad \frac{u}{U_e} = f'(\eta)$$

$$\hat{v} = \left(\frac{1-m}{m+1}\eta f'(\eta) - f(\eta)\right)\sqrt{\frac{m+1}{2}}$$

$$v = \hat{v}\sqrt{\nu k}\, x^{\frac{m-1}{2}} = \hat{v}\sqrt{\frac{\nu U_e}{x}} \qquad \frac{v}{U_e} = \hat{v}\sqrt{\frac{\nu}{x U_e}}$$

Si deve risolvere l'equazione di Falkner-Skan con le condizioni al contorno:

$$f''' + f f'' + \beta\left(1 - f'^2\right) = 0$$

$$y = 0 \rightarrow u = v = 0 \qquad \eta = 0 \rightarrow \begin{cases} f'(0) = 0 \\ f(0) = 0 \end{cases}$$

$$y \rightarrow \infty \rightarrow u = U_e \qquad \eta \rightarrow \infty \rightarrow f'(\eta \rightarrow \infty) = 1$$

$$\eta = y g(x) = y x^{\frac{m-1}{2}} \sqrt{\frac{m+1}{2}\frac{k}{\nu}}$$

Soluzioni simili

$$d(x) = \frac{1}{g(x)}$$

Posto $\overline{\eta}$ tale che $f'(\overline{\eta}) = 0.99$ si ha: $\delta = \frac{\overline{\eta}}{g(x)} = \overline{\eta}\sqrt{\frac{2}{m+1}\frac{\nu}{k}}x^{\frac{1-m}{2}}$

$$\frac{\delta}{x} = \overline{\eta}\sqrt{\frac{2}{m+1}\frac{\nu}{U_e x}} = \overline{\eta}\sqrt{\frac{2}{m+1}}\frac{1}{\sqrt{Re_x}} = \overline{\eta}\sqrt{\frac{2}{m+1}\frac{\nu}{k}}x^{-\frac{m+1}{2}}$$

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U_e}\right) dy = d(x)\int_0^\infty (1-f')\,d\eta = d(x)\left[\eta - f(\eta)\right]_0^\infty = d(x)\eta^*$$

$$\theta = \int_0^\infty \frac{u}{U_e}\left(1 - \frac{u}{U_e}\right) dy = d(x)\int_0^\infty f'(1-f')\,d\eta = d(x)\frac{f''(0) - \eta^*\beta}{\beta+1} = d(x)\theta^*$$

$$\tau_w = \mu u_y = \mu U_e g(x) f''(0) = \mu\sqrt{\frac{m+1}{2}\frac{k^3}{\nu}}x^{\frac{3m-1}{2}}f''(0) = \sqrt{\frac{m+1}{2}\mu\rho k^3}\,x^{\frac{3m-1}{2}}f''(0)$$

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U_e^2} = \sqrt{2(m+1)\frac{\nu}{k}}x^{-\frac{m+1}{2}}f''(0) = \sqrt{\frac{2(m+1)}{Re_x}}f''(0)$$

Soluzioni simili

Lastra piana m=0, β=0, k=U_∞ e $f'(0)$=0.46960

$$\overline{\eta} = 3.49$$

$$\frac{\delta}{x} = \overline{\eta}\sqrt{2\frac{\nu}{U_e x}} = \overline{\eta}\sqrt{2}\frac{1}{\sqrt{Re_x}} = \frac{4.93}{\sqrt{Re_x}}$$

$$\frac{\delta^*}{x} = \eta^*\sqrt{2}\frac{1}{\sqrt{Re_x}} = \frac{1.72}{\sqrt{Re_x}}$$

$$\frac{\theta}{x} = \theta^*\sqrt{2}\frac{1}{\sqrt{Re_x}} = \frac{0.664}{\sqrt{Re_x}}$$

$$\tau_w = \sqrt{\frac{1}{2}\rho\mu U_\infty^3}\,x^{-\frac{1}{2}}f''(0)$$

$$c_f = \sqrt{\frac{2}{Re_x}}f''(0) = \frac{0.664}{\sqrt{Re_x}}$$

$$\frac{v}{U_\infty} = \hat{v}\sqrt{\frac{\nu}{U_\infty}}x^{-\frac{1}{2}}$$

Soluzioni simili

Punto di ristagno piano m=1, β=1, k=U_∞ (x=1) e $f'(0)$=1.23259

$$\overline{\eta} = 2.39$$

$$\delta = \overline{\eta}\sqrt{\frac{\nu}{k}} = 2.39\sqrt{\frac{\nu}{k}}$$

$$(2) \quad f''' + ff'' + \beta\left(1 - f'^2\right) = 0$$

$$\delta^* = \eta^*\sqrt{\frac{\nu}{k}} = 0.648\sqrt{\frac{\nu}{k}}$$

$$\theta = \theta^*\sqrt{\frac{\nu}{k}} = 0.292\sqrt{\frac{\nu}{k}}$$

$$\tau_w = \sqrt{\rho\mu k^3}\, x f''(0)$$

$$c_f = 2\sqrt{\frac{\nu}{k}}\frac{f''(0)}{x} = \sqrt{\frac{\nu}{k}}\frac{2.47}{x}$$

$$\frac{v}{U_e} = \hat{v}\sqrt{\frac{\nu}{k}}x^{-1}$$

Soluzioni simili

Poiché:

$$(ff')' = f'^2 + ff'' \rightarrow f'^2 = (ff')' - ff''$$

$$f''' + ff'' + \beta\left(1 - f'^2\right) = 0$$

Dalla equazione di Falkner scan:

$$f'^2 = (ff')' + f''' + \beta - \beta f'^2 \rightarrow f'^2 = \frac{(ff')' + f''' + \beta}{\beta + 1}$$

$$\int_0^\infty f'(1 - f')d\eta = \int_0^\infty \left(f' - \frac{(ff')' + f''' + \beta}{\beta + 1}\right)d\eta = \int_0^\infty \frac{-f'''}{\beta + 1}d\eta + \frac{1}{\beta + 1}\int_0^\infty \left((\beta + 1)f' - (ff')' - \beta\right)d\eta =$$

$$= \left[\frac{-f''}{\beta + 1}\right]_0^\infty + \frac{\beta}{\beta + 1}\int_0^\infty (f' - 1)d\eta + \frac{1}{\beta + 1}\int_0^\infty \left(f' - (ff')'\right)d\eta = \frac{f''(0)}{\beta + 1} + \frac{-\beta\eta^*}{\beta + 1} + \frac{\left[f - ff'\right]_0^\infty}{\beta + 1} =$$

$$= \frac{f''(0) - \beta\eta^*}{\beta + 1}$$

Dove si è supposto $f'''(\infty) = 0$ $\quad \int_0^\infty (f' - 1)d\eta = -\eta^*$ $\quad f(\infty)(1 - f'(\infty)) = 0$

Soluzioni simili

$$c_d = \frac{\int_0^x \tau_w dt}{\frac{1}{2}\rho U_e^2 x} = \frac{\sqrt{2(m+1)\frac{\nu}{k}}f''(0)\int_0^x x^{-\frac{m+1}{2}} dt}{\frac{1}{2}\rho U_e^2 x} = \frac{\sqrt{2(m+1)\frac{\nu}{k}}f''(0)\frac{1-m}{2}x^{\frac{1-m}{2}}}{\frac{1}{2}\rho U_e^2 x} = \frac{1-m}{2}c_f$$