Tecniche di Specifica e di Verifica

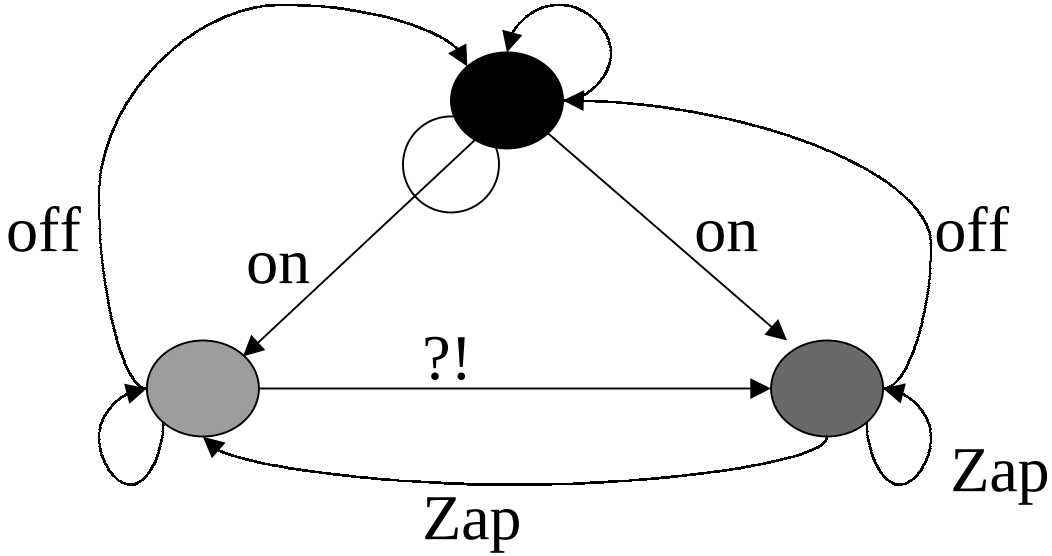
Branching Time Temporal Logics

Outline

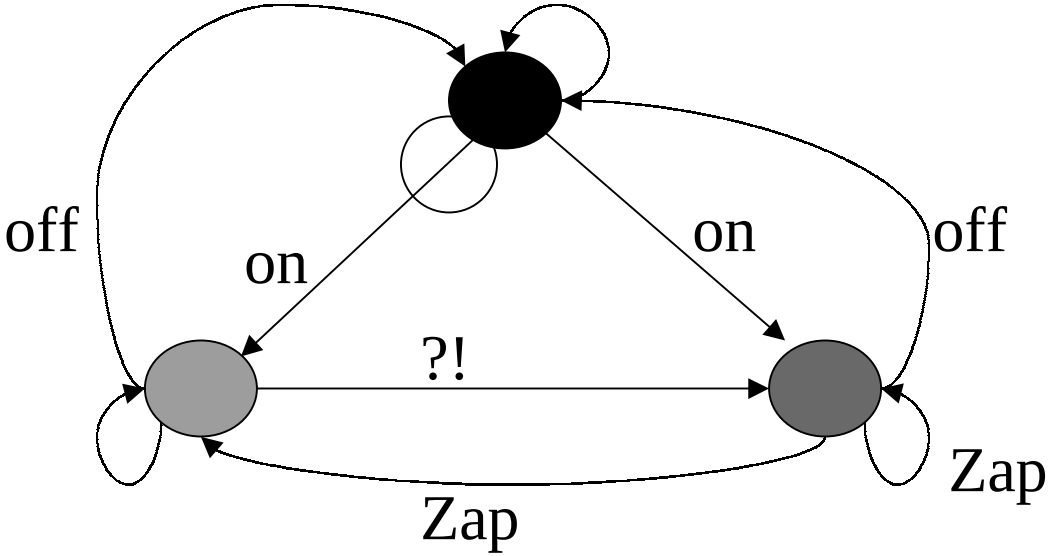
- ***CTL*** (**C**omputation **T**ree **L**ogic)
  - **Branching Time**
  - Unwindings --- computation trees
  - Syntax and semantics of CTL.

Branching Time Structures

- **Linear Time**:
  - A ***computation*** at its first state satisfies a property.
  - Property **---- LTL** formula
- **Branching Time**
  - The ***computation tree*** at its root satisfies a property.
  - Property: **CTL** (**CTL*, μ-calculus**) formula.
  - **Computation Tree**
    - **All *computations*** starting from a state ***glued together*** (to form a tree structure).
- In branching time, ***the decisions*** taken during a run are taken into account.



The TV Example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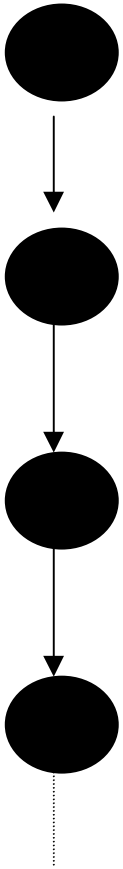

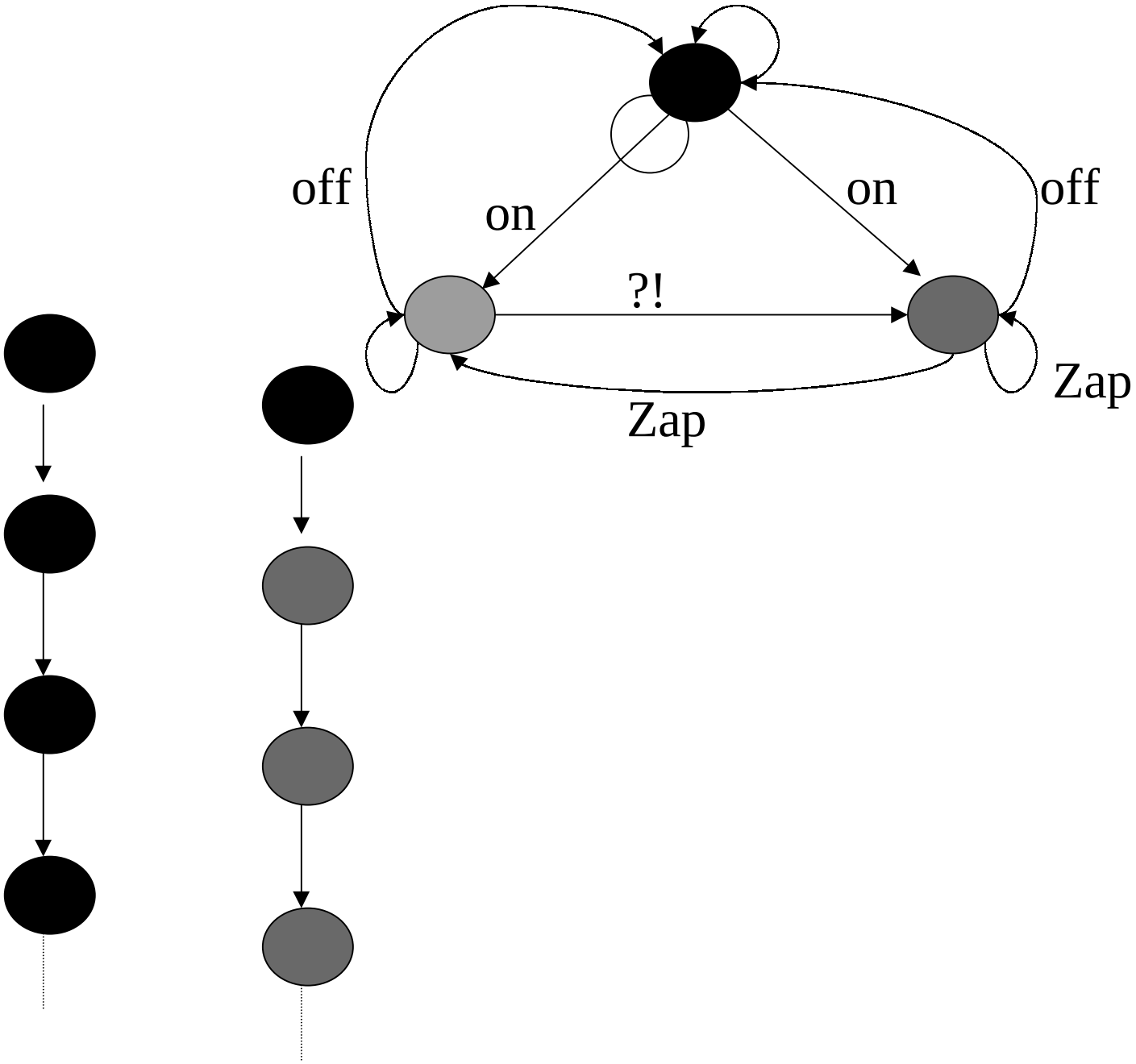
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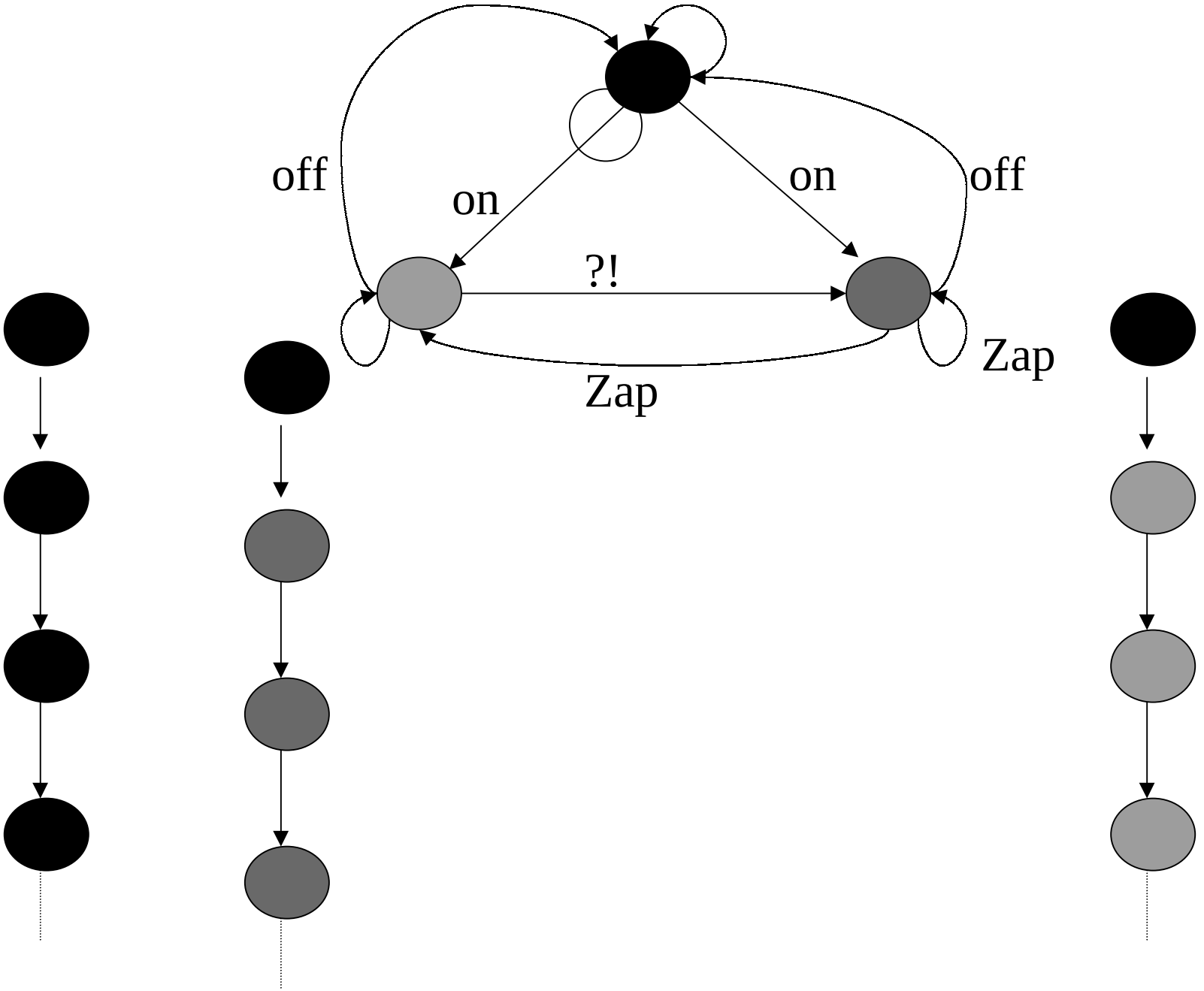
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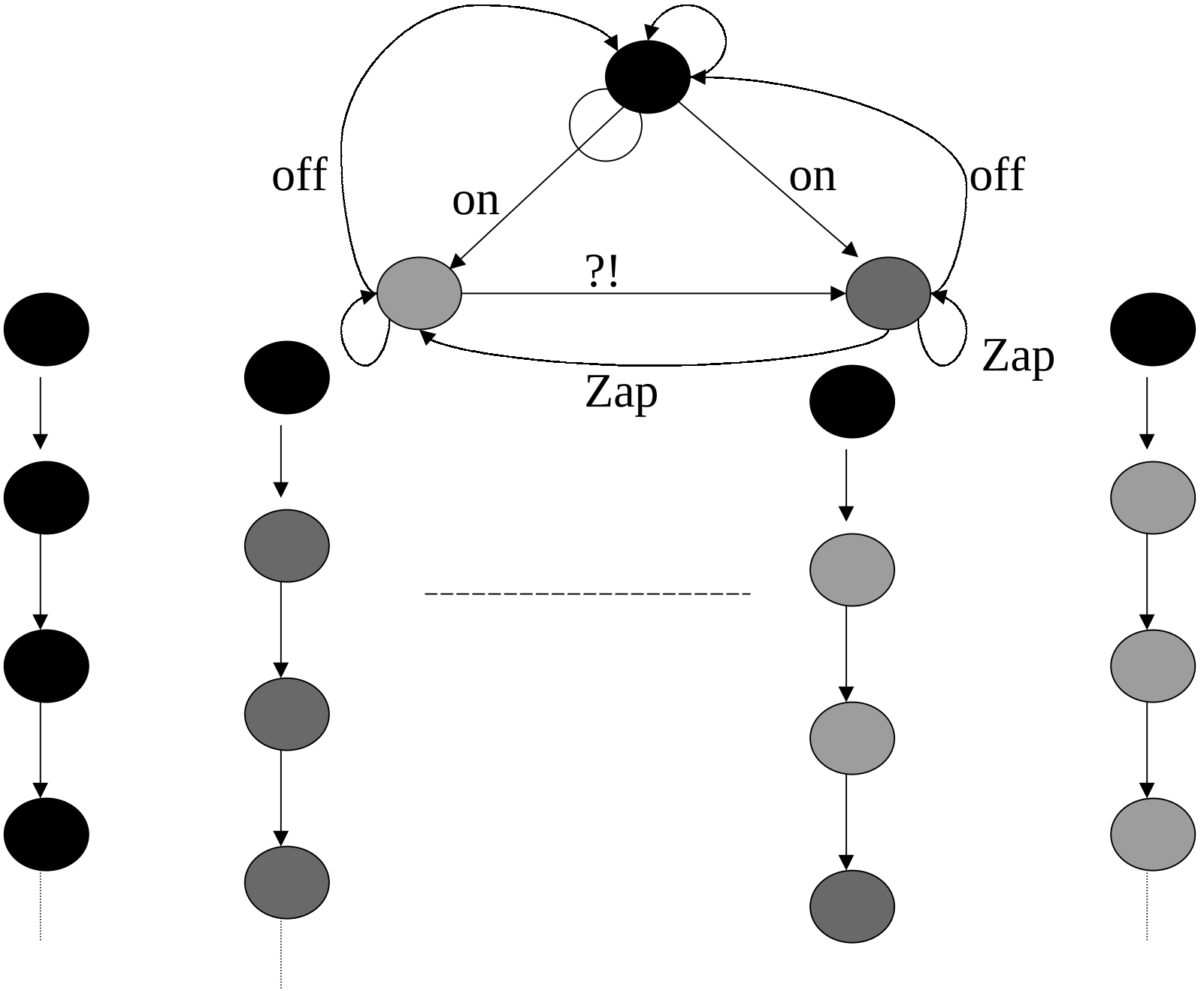
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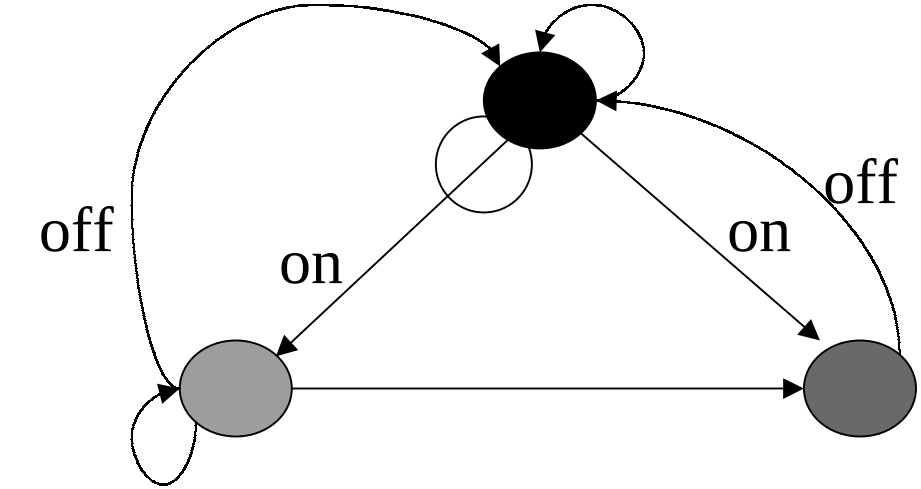
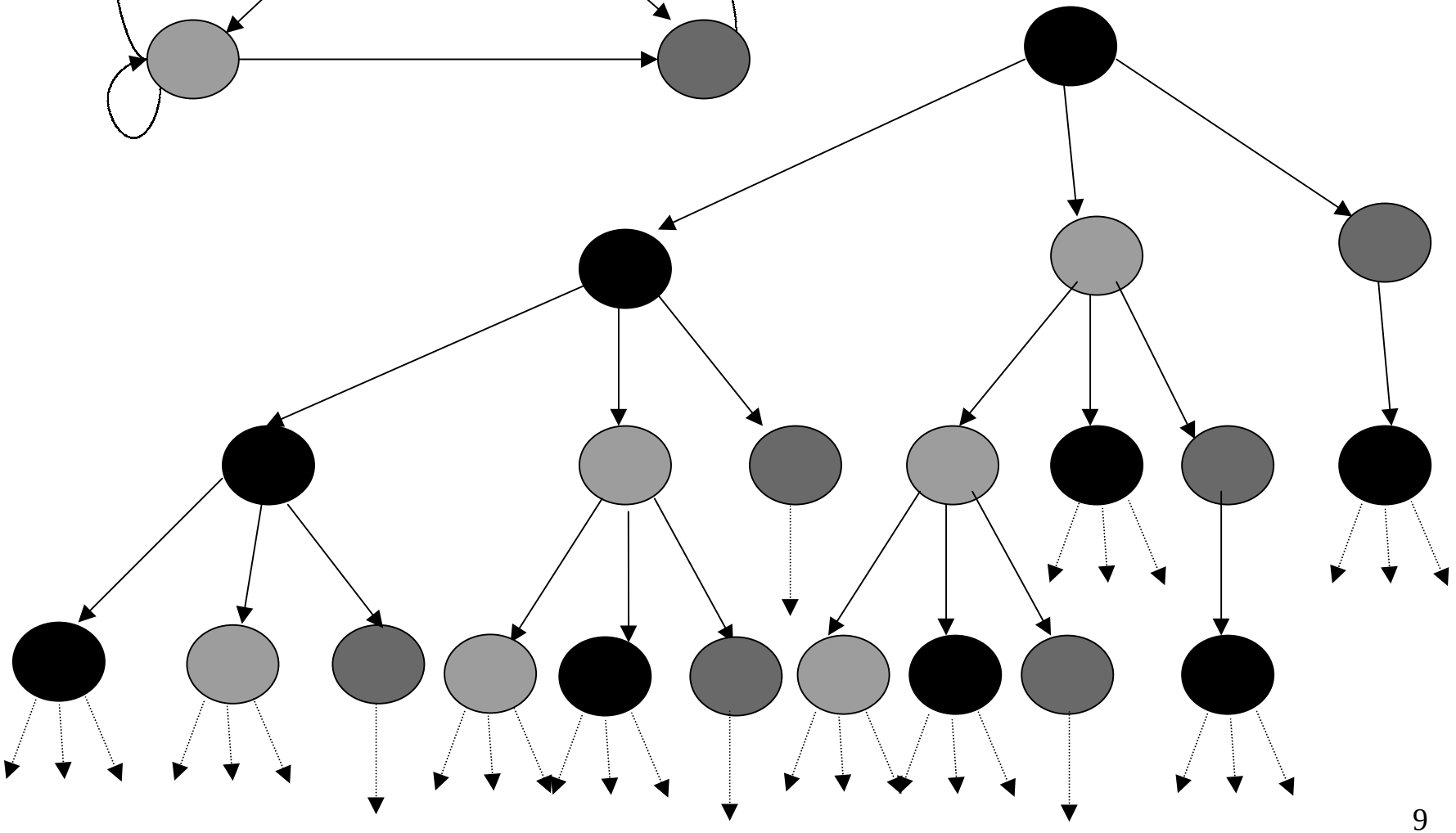
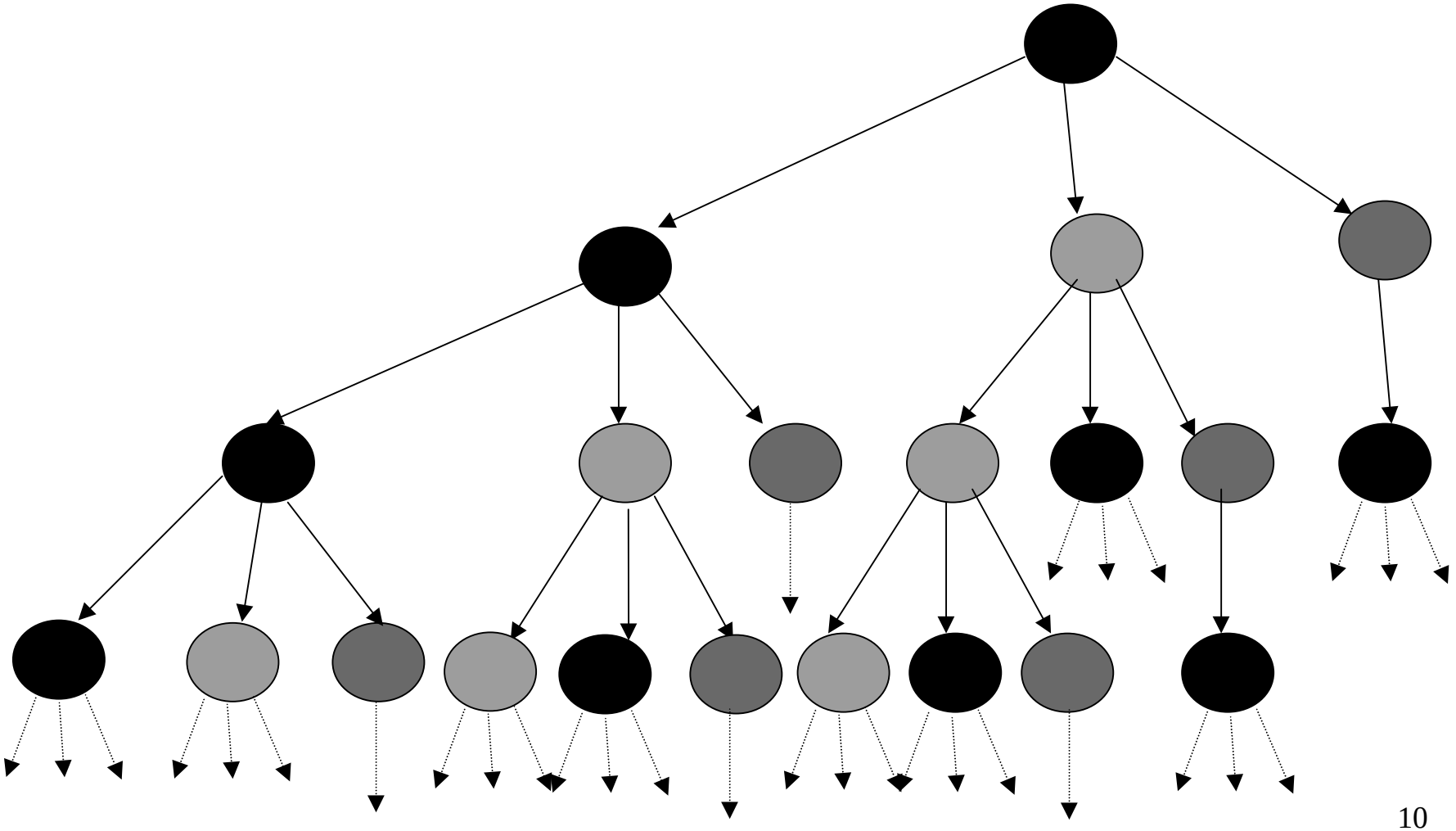
A Modified Example





For every path **π** and every state **s** on that path, there is a path **π'** starting from **s** and a state **s'** on **π'** which is **green**.



Branching Time Temporal Logic

- **K = (S, S_0, R, AP, L)**
- **K, s** $\models \psi$ -- the computation tree rooted at **s** satisfies ψ.
- **K** $\models \psi$ iff **K,** $s_0 \models \psi$ for every $s_0 \in S_0$.
- **Branching Time Temporal Logics**:
  - **CTL**
  - **CTL***
  - (The modal) μ**-calculus**

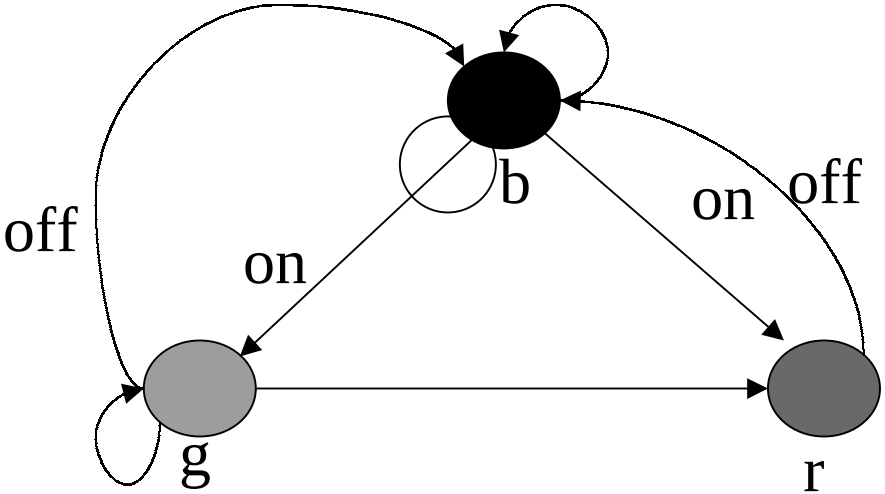
Unwinding

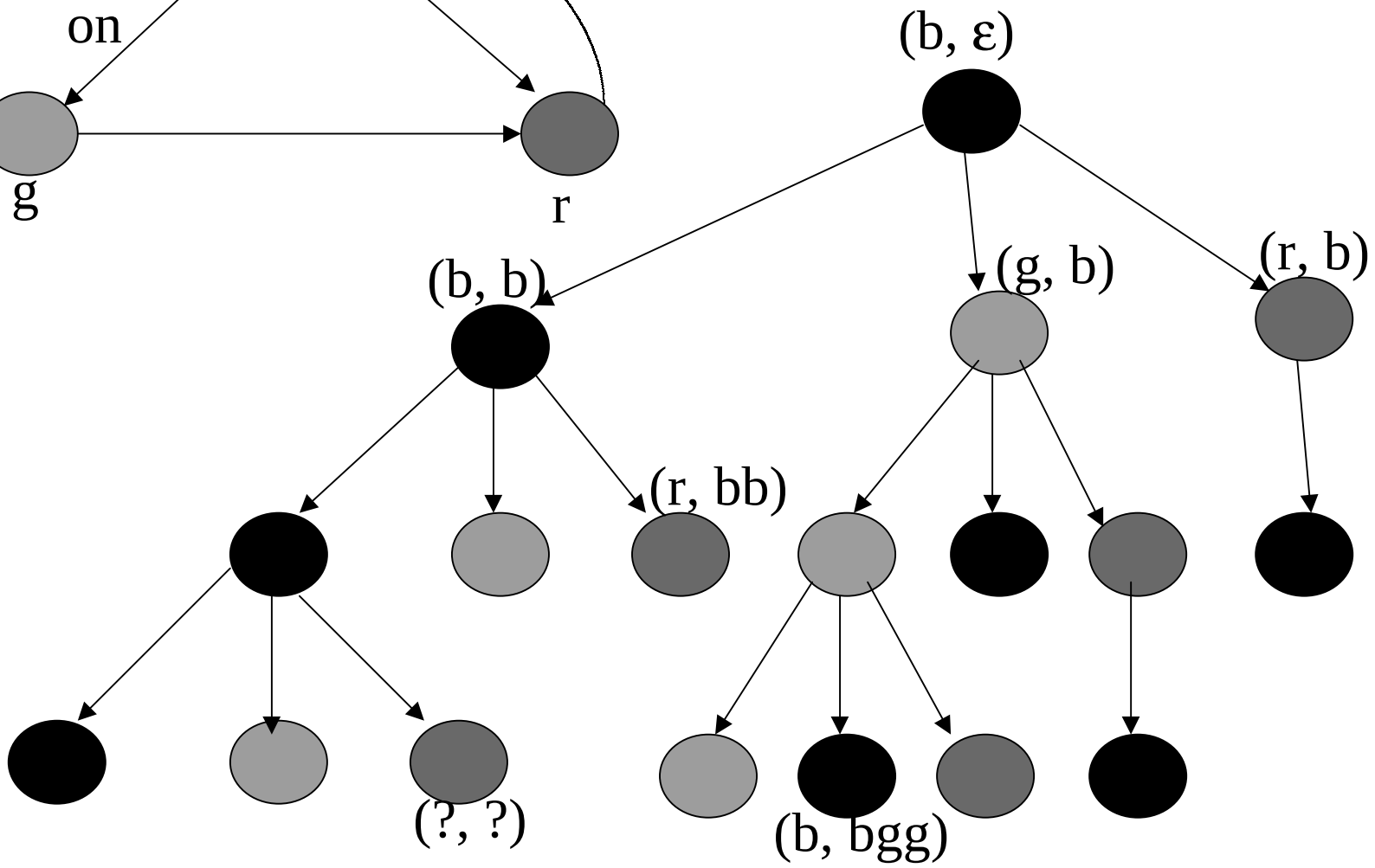
- **K = (S, S_0, R, AP, L)** **s** $\in$ **S**
- **TR(K, s)** --- The computation tree rooted at **s**.
- **TR(K, s) = ($\mathcal{S}_s$, (s,ε), $\mathcal{R}_s$, AP, $\mathcal{L}_s$).**
  - **(s, ε)** $\in \mathcal{S}_s$;
  - For any **(s', σ)** $\in \mathcal{S}_s$, **s'**$\in$ **S** and **σ = s s_1 … s_n** is a path in **K** leading from **s** to **s'** (i.e. **σ.s'** is a path from **s** to **s'**);
  - If **(s_1, σ)** $\in \mathcal{S}_s$ and **R(s_1, s_2)** then **(s_2, σ.s_1)** $\in \mathcal{S}_s$ and $\mathcal{R}_s$**((s_1, σ), (s_2, σ.s_1))** ;
  - $\mathcal{L}$**((s_1, σ)) = L(s_1)**.

Unwinding

- **TR(K, s)** is almost a Kripke structure.
  - $\mathcal{S}_s$ may be infinite
  - But $\mathcal{R}_s$ is ***tree-like***.
  - The "***graph***" of **TR(K, s)** is a tree rooted at **(s, ε)**.
- **TR(K, s)** is the ***computation tree rooted*** at **s**.

A Modified Example





Linear time Vs Branching time

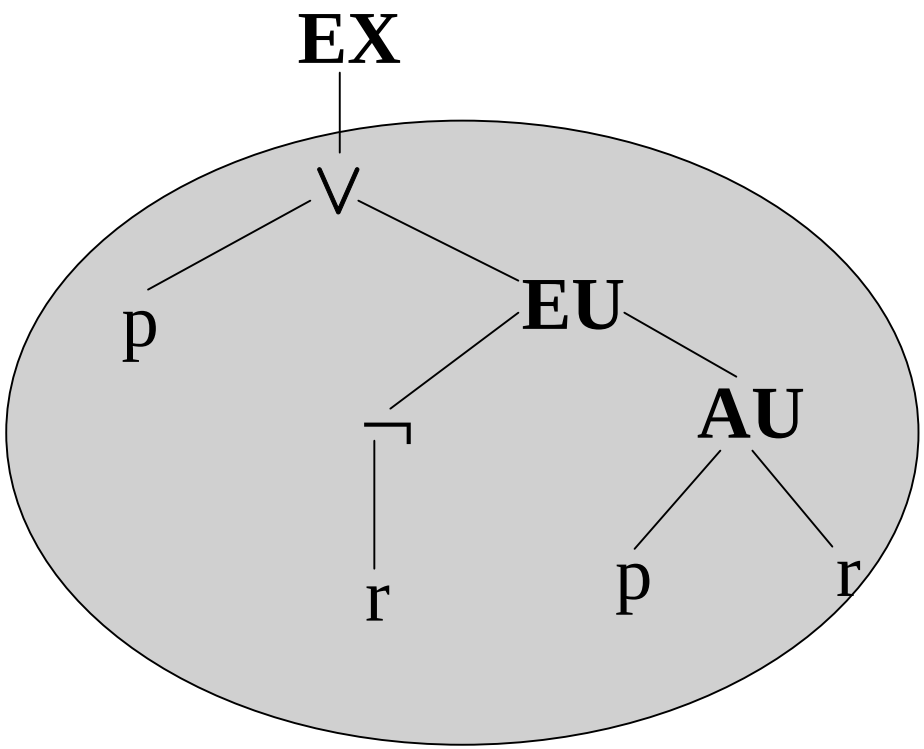
- There are ***properties*** that can be ***expressed in* LTL** but which ***can not be expressed in* CTL**. (sloppy statement!)
- There are ***properties*** that can be ***expressed in* CTL** but ***not in* LTL**.
- The **LTL *model checking problem*** can be ***converted*** into a ***restricted*** kind of a **CTL*** ***model checking problem***.

CTL

- **Syntax**
  - **AP** – a finite set of ***atomic propositions***.
  - **p** $\in$ **AP** is a formula.
  - If ψ and ψ' are formulas then so are $\neg\psi$ and $\psi \vee \psi'$.
  - If ψ is a formula then so is **EX**ψ
  - If ψ_1 and ψ_2 are formulas then so are **EU**(ψ_1, ψ_2) and **AU**(ψ_1, ψ_2).

Formulas

- **EX(p ∨ EU(¬r, AU(p, r)))**



Semantics

- **K = (S, S_0, R, AP, L)**
  - **L : S $\rightarrow$ 2^{AP}**
- ψ a **CTL** formula **s** $\in$ **S**
- **K, s** $\models$ ψ
- ψ (holds) ***is satisfied*** at **s**.
- **FACT**:

  K, s $\models$ ψ iff **TR(K, s), (s,** ε**)** $\models$ ψ.

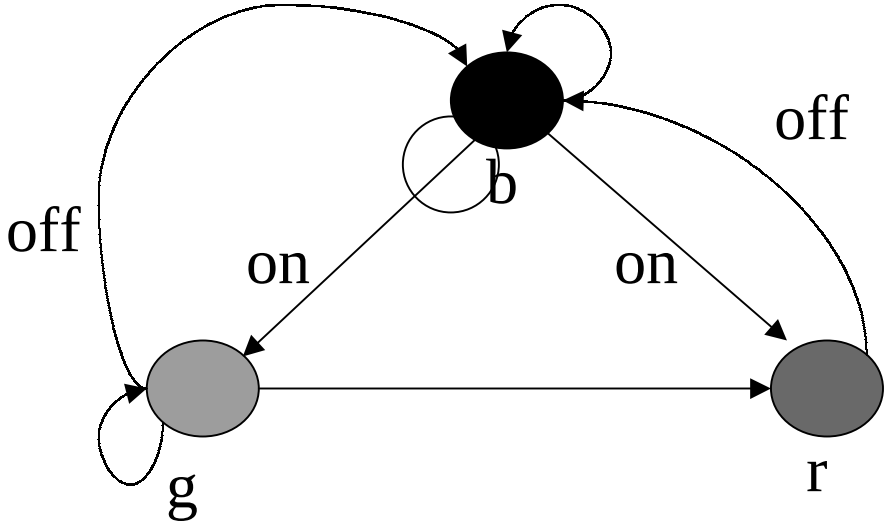
Semantics

- **CTL ::= p | $\neg \psi$ | $\psi_1 \vee \psi_2$ | EX(ψ) |**
  | EU(ψ_1, ψ_2) | AU(ψ_1, ψ_2)
- **K = (S, S_0, R, AP, L); L: $\rightarrow$ 2^{AP} ; s $\in$ S**
- **K, s $\models$ p** iff **p $\in$ L(s).**
- **K, s $\models$ $\neg\psi$** iff **K, s $\not\models$ ψ**
- **K, s $\models$ $\psi_1 \vee \psi_2$** iff
  K, s $\models$ ψ_1 or **K, s $\models$ ψ_2.**

Semantics

- **CTL ::= p** $\mid \neg\psi \mid \psi_1 \vee \psi_2 \mid$ **EX(**ψ**)** $\mid$
  $\mid$ **EU(**ψ_1**,** ψ_2**)** $\mid$ **AU(**ψ_1**,** ψ_2**)**
- **K = (S, S$_0$, R, AP, L)** ; **L:** $\rightarrow$ 2^{AP} ; $s \in S$
- **K, s** $\models$ **EX(**ψ**)** iff there exists **s'** such that:
  - **s** $\rightarrow$ **s'** (i.e. **R(s, s')**) and **K, s'** $\models \psi$

  s has a successor state **s'** at which ψ holds.



AP = {n, h, uh}

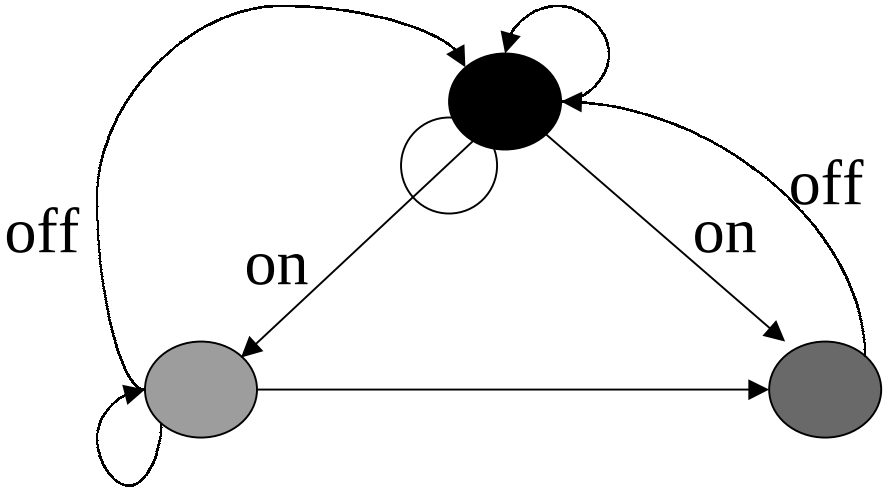
K, b ⊨ EX(uh) ? K, b ⊨ EX(¬uh) ?

K, g ⊨ EX(uh) ?

K, r ⊨ EX(h) ?

Semantics

- $\mathbf{K = (S, S_0, R, AP, L)\ ;\ L: \rightarrow\ 2^{AP}\ ;\ s \in S}$
- ***A path from*** **s** is a (infinite) sequence of states $\pi = \mathbf{s_0, s_1, s_2, \ldots, s_i, s_{i+1}, \ldots}$ s.t:
  - $\mathbf{s = s_0}$
  - $\mathbf{s_i \rightarrow\ \ s_{i+1}}$ (i.e. $\mathbf{R(s_i, s_{i+1})}$) for every **i**.
- $\mathbf{\pi(i) = s_i}$ the **i**-th element of π.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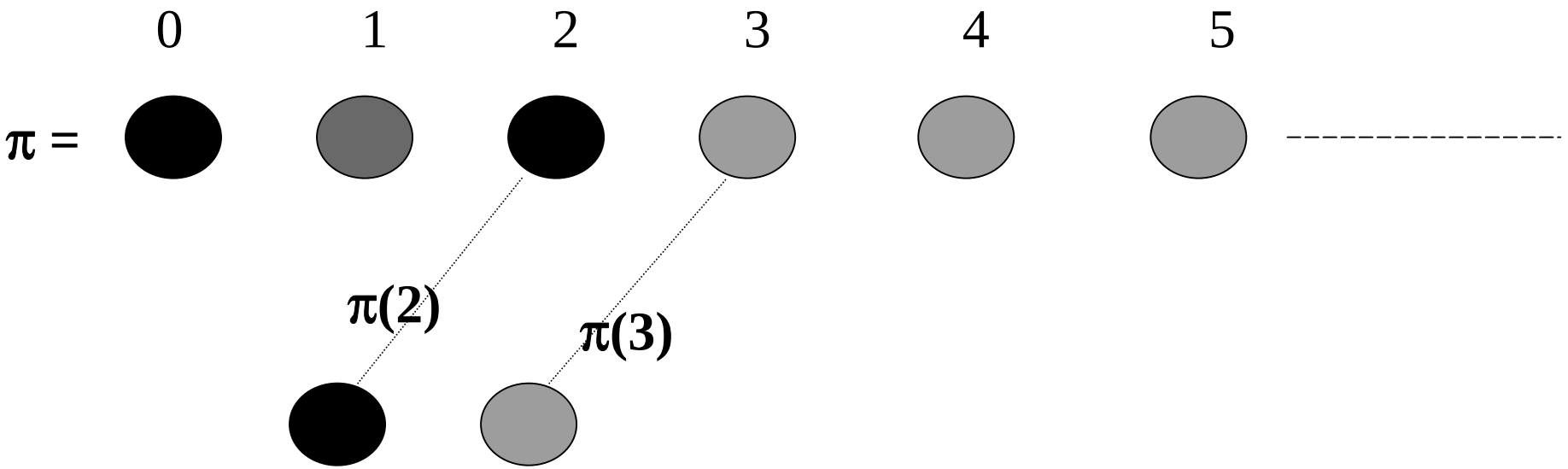
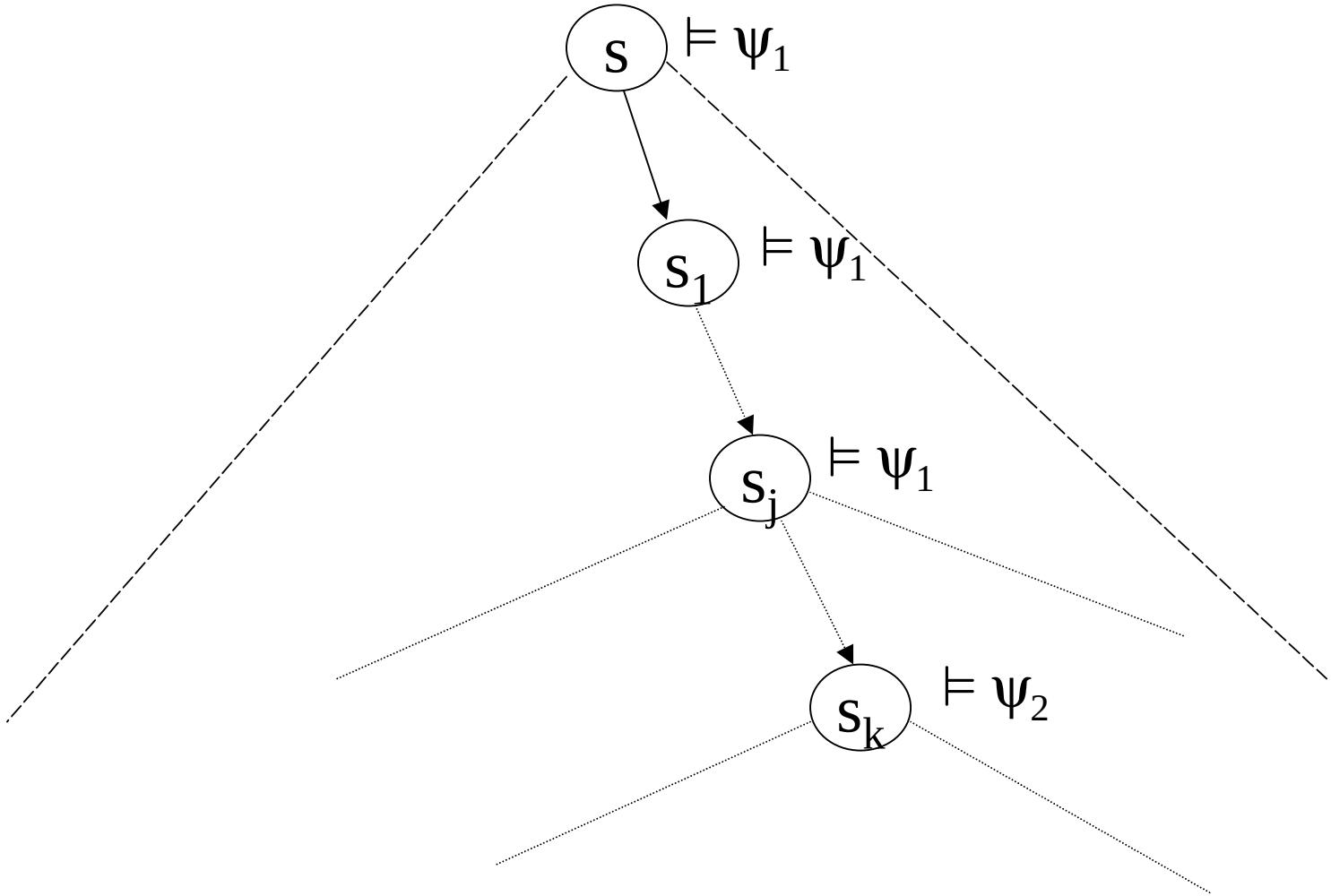

0
1
2
3
4
5
π =
π(2)
π(3)

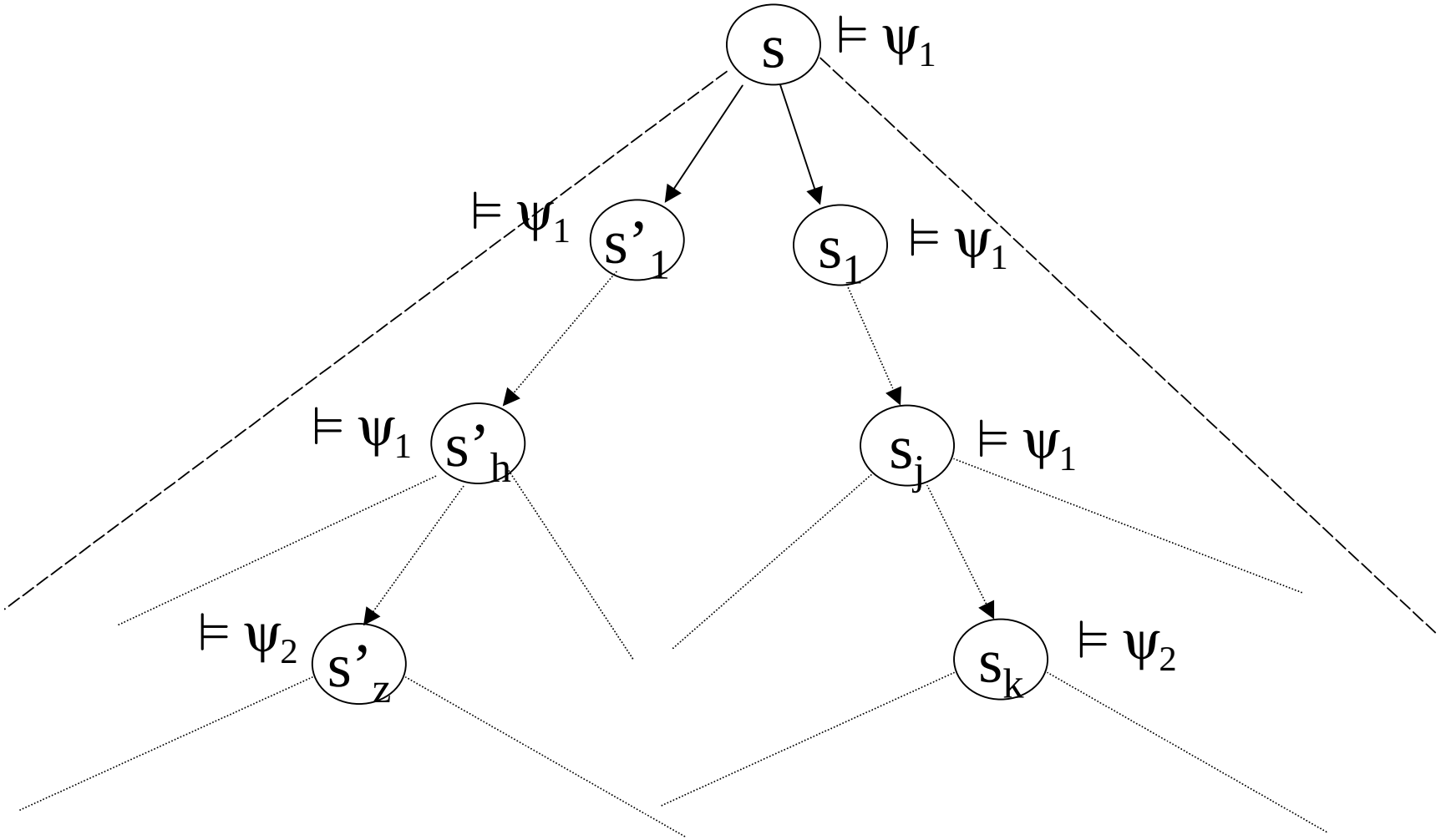
Semantics

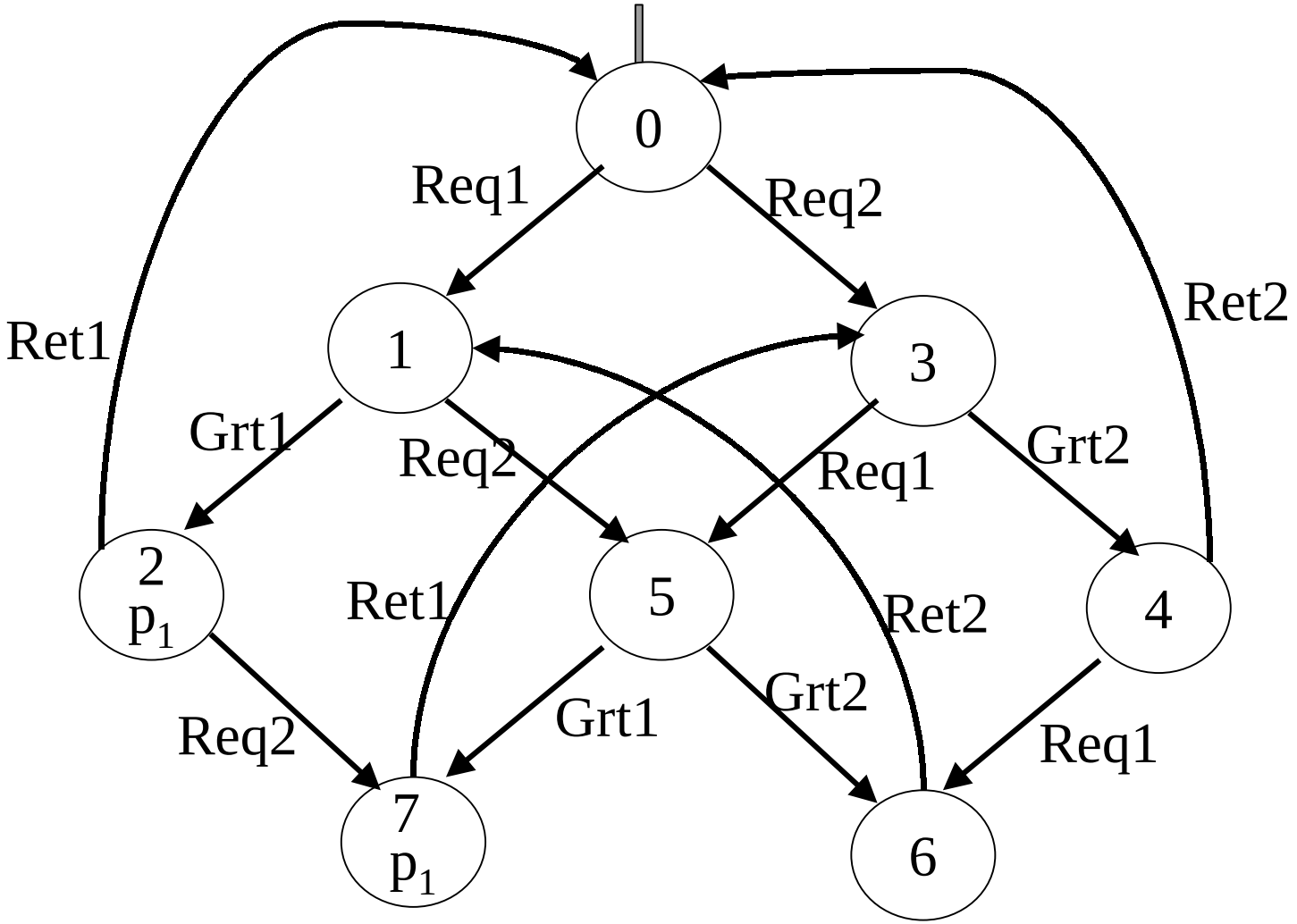
- **CTL ::= p** $\mid \neg\psi \mid \psi_1 \vee \psi_2 \mid$ **EX(**ψ**)** $\mid$
  $\mid$ **EU(**ψ_1**,** ψ_2**)** $\mid$ **AU(**ψ_1**,** ψ_2**)**
- **K = (S,** S_0**, R, AP, L)** ; **L:** $\rightarrow$ 2^{AP} ; $s \in S$
- **K, s** $\models$ **EU(**ψ_1**,** ψ_2**)** iff ***there exists a path*** $\pi = s_0, s_1, \ldots$ from **s** (i.e. s_0=**s**) and $k \geq 0$ such that:
  - **K,** π**(k)** $\models \psi_2$
  - **K,** π**(j)** $\models \psi_1$, for all $0 \leq j < k$.


s
⊨ ψ1
s1
⊨ ψ1
sj
⊨ ψ1
sk
⊨ ψ2

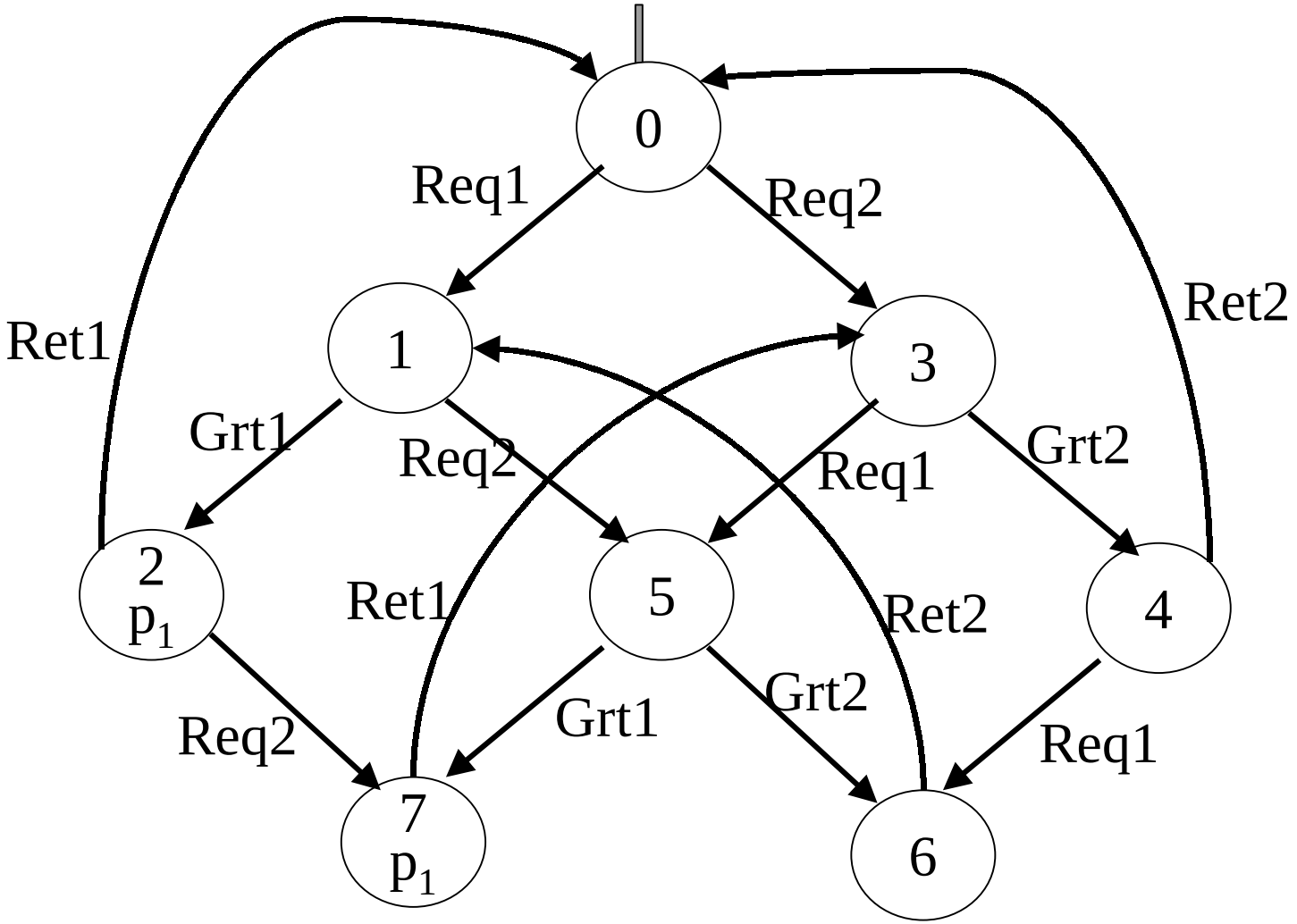
Semantics

- **CTL ::= p** $\mid \neg\psi \mid \psi_1 \vee \psi_2 \mid$ **EX(**ψ**)** $\mid$
  $\mid$ **EU(**ψ_1**,** ψ_2**)** $\mid$ **AU(**ψ_1**,** ψ_2**)**
- **K = (S,** S_0**, R, AP, L)** ; **L:** $\rightarrow 2^{AP}$; $s \in S$
- **K,** $s \models$ **AU(**ψ_1**,** ψ_2**)** iff ***for every path*** $\pi = s_0, s_1, \ldots$ from **s** there exists $k \geq 0$ such that:
  - **K,** $\pi(k) \models \psi_2$
  - **K,** $\pi(j) \models \psi_1$, for all $0 \leq j < k$.


s
⊨ ψ1
⊨ ψ1
s'1
s1
⊨ ψ1
⊨ ψ1
s'h
sj
⊨ ψ1
⊨ ψ2
s'z
sk
⊨ ψ2


0
Req1
Req2
Ret1
Ret2
1
3
Grt1
Req2
Req1
Grt2
2
p1
5
4
Ret1
Ret2
Grt2
Req2
Grt1
Req1
7
p1
6

M, 0 ⊨ EU(⊤, p_1) ?


0
Req1
Req2
Ret1
Ret2
1
3
Grt1
Req2
Req1
Grt2
2
p1
5
4
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Grt2
Req1
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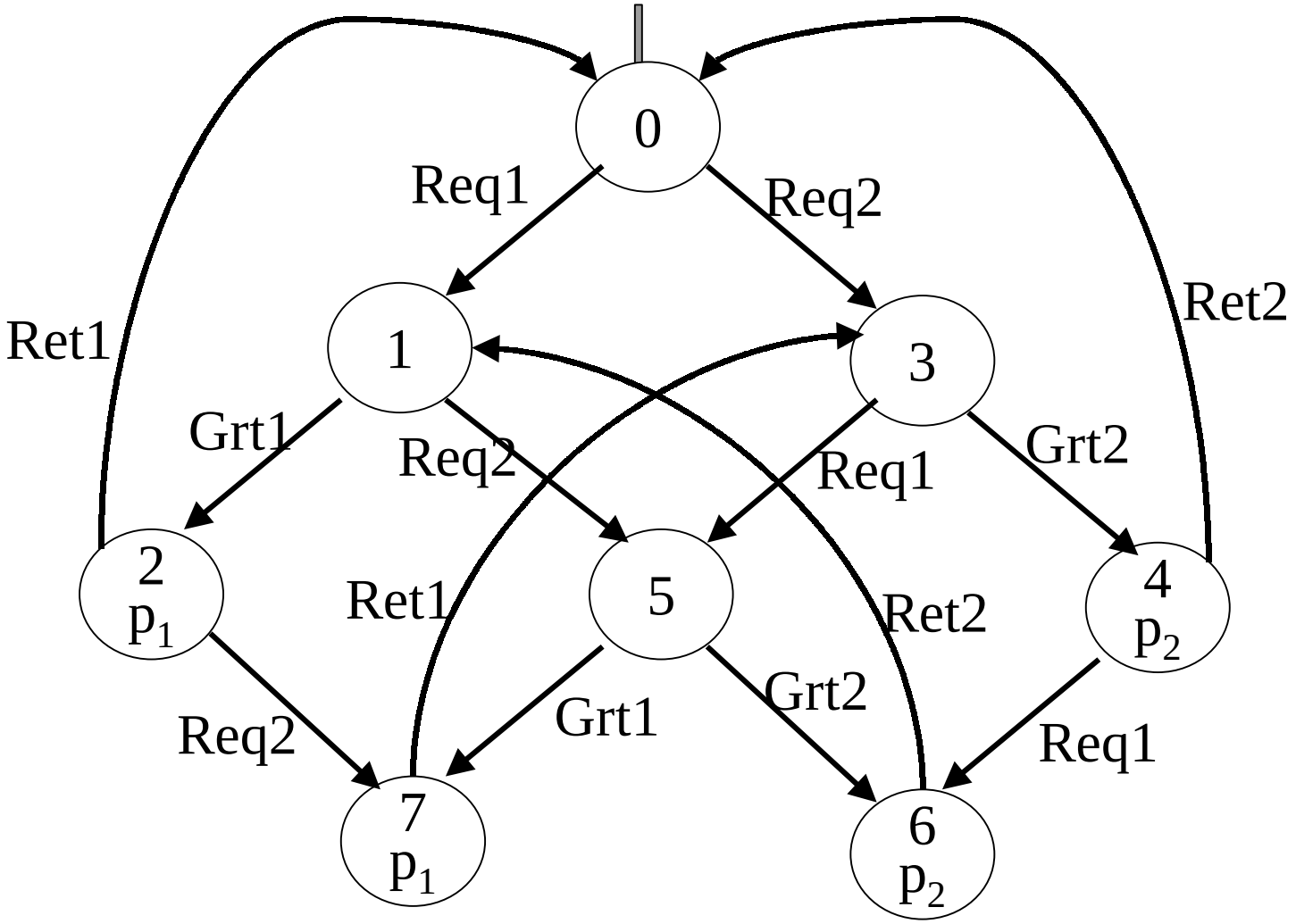
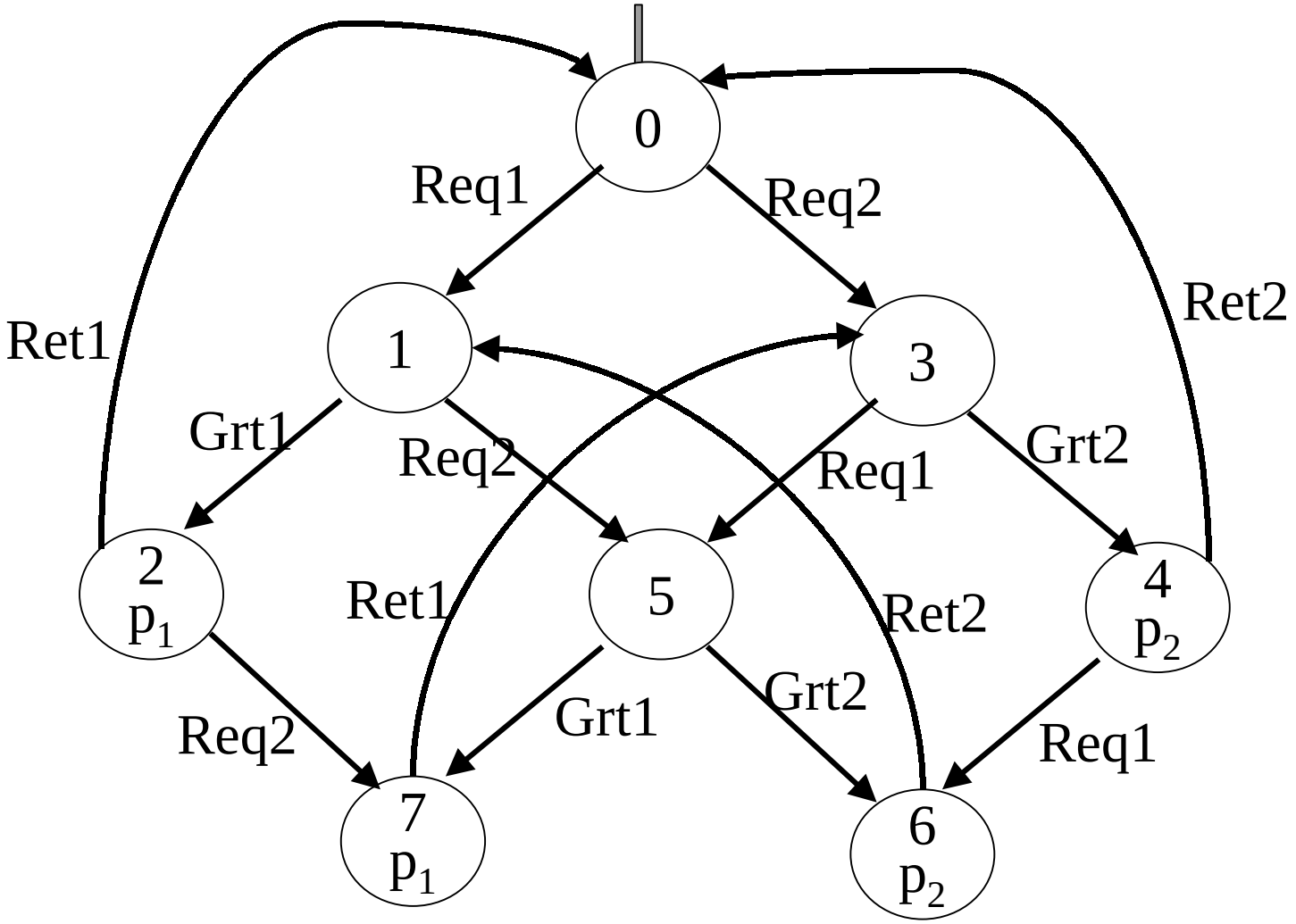
p1
6

M, 0 ⊨ AU(⊤, p_1) ?

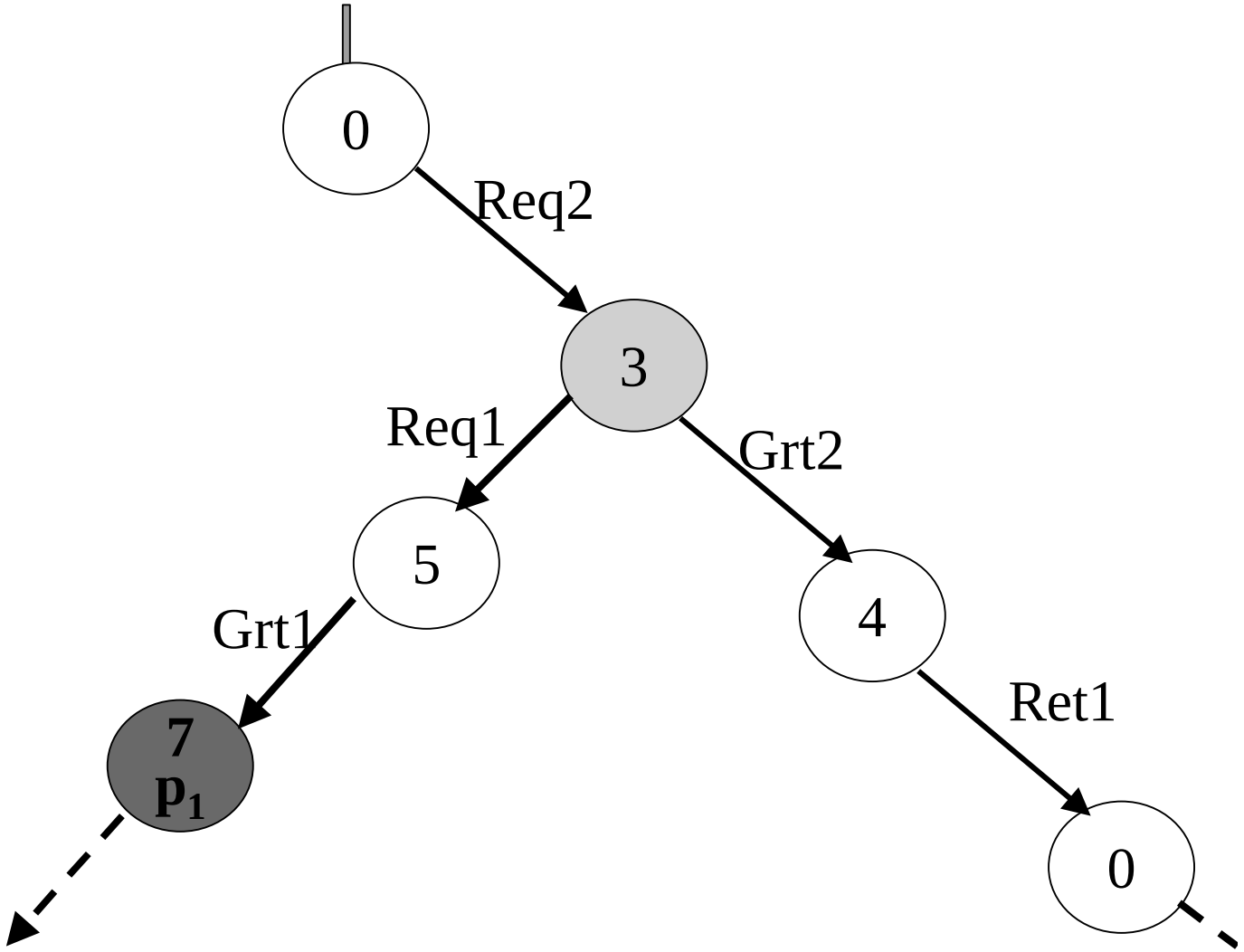


M, 0 ⊨ AU(⊤, $p_1 \vee p_2$) ?



M, 0 ⊨ AU(⊤, EU(⊤, p_1)) ?

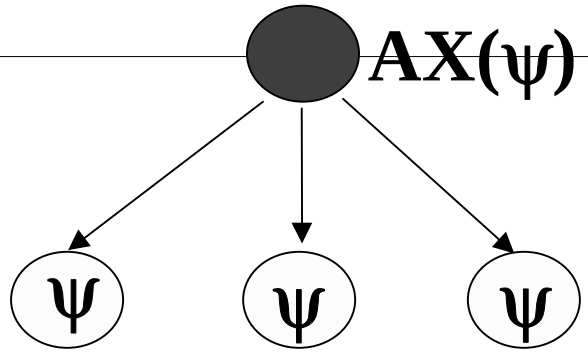
From s_0, ***all*** the computations will reach a point, where it is ***possible*** for 1 to print ***eventually.***



M, 0 ⊨ AU(⊤, EU(⊤, p_1)) ?

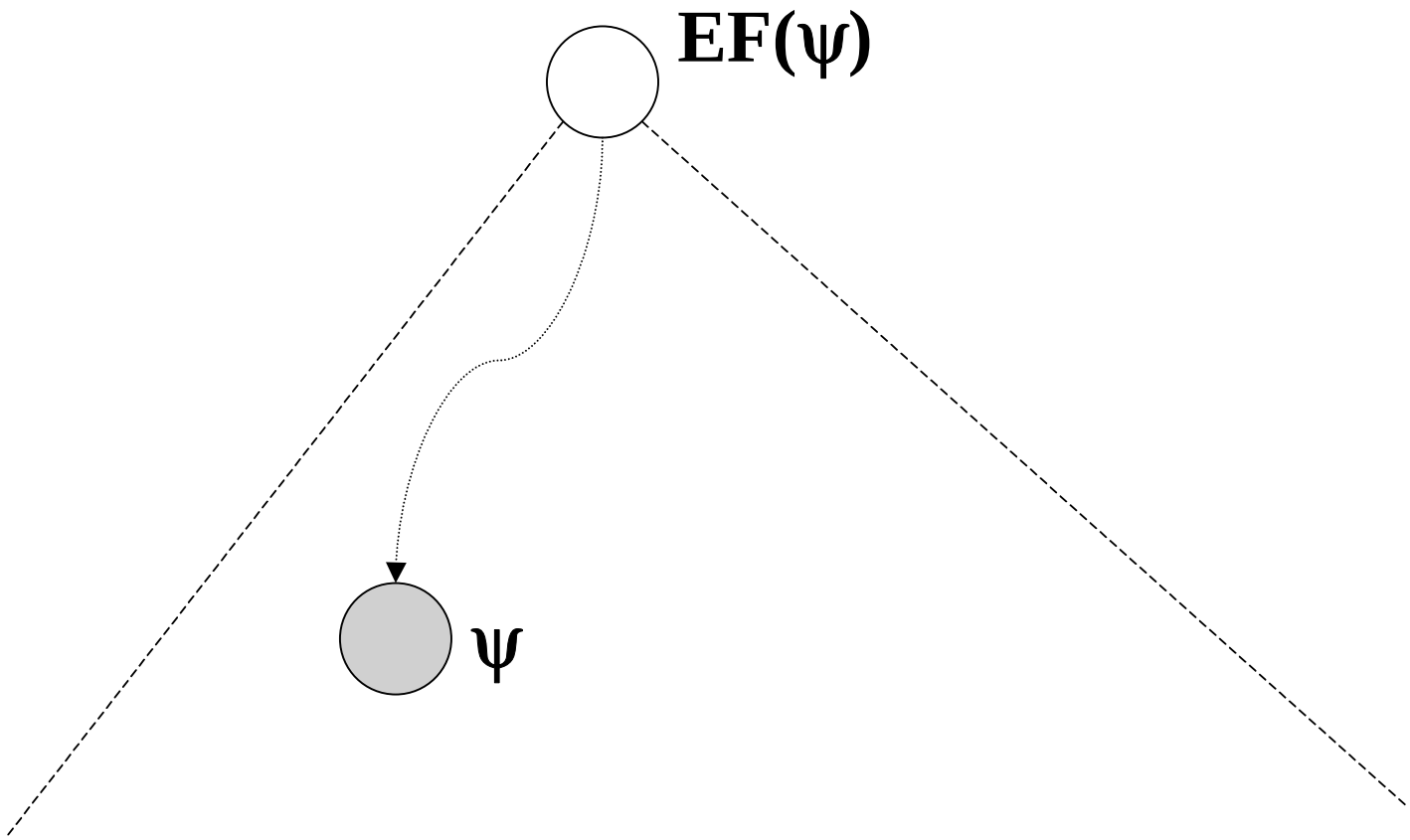
Derived Operators

- **AX(ψ) = ¬EX(¬ψ)**
  - It is not the case there exists a next state at which ψ does not hold.
  - ***For every next state*** ψ holds.



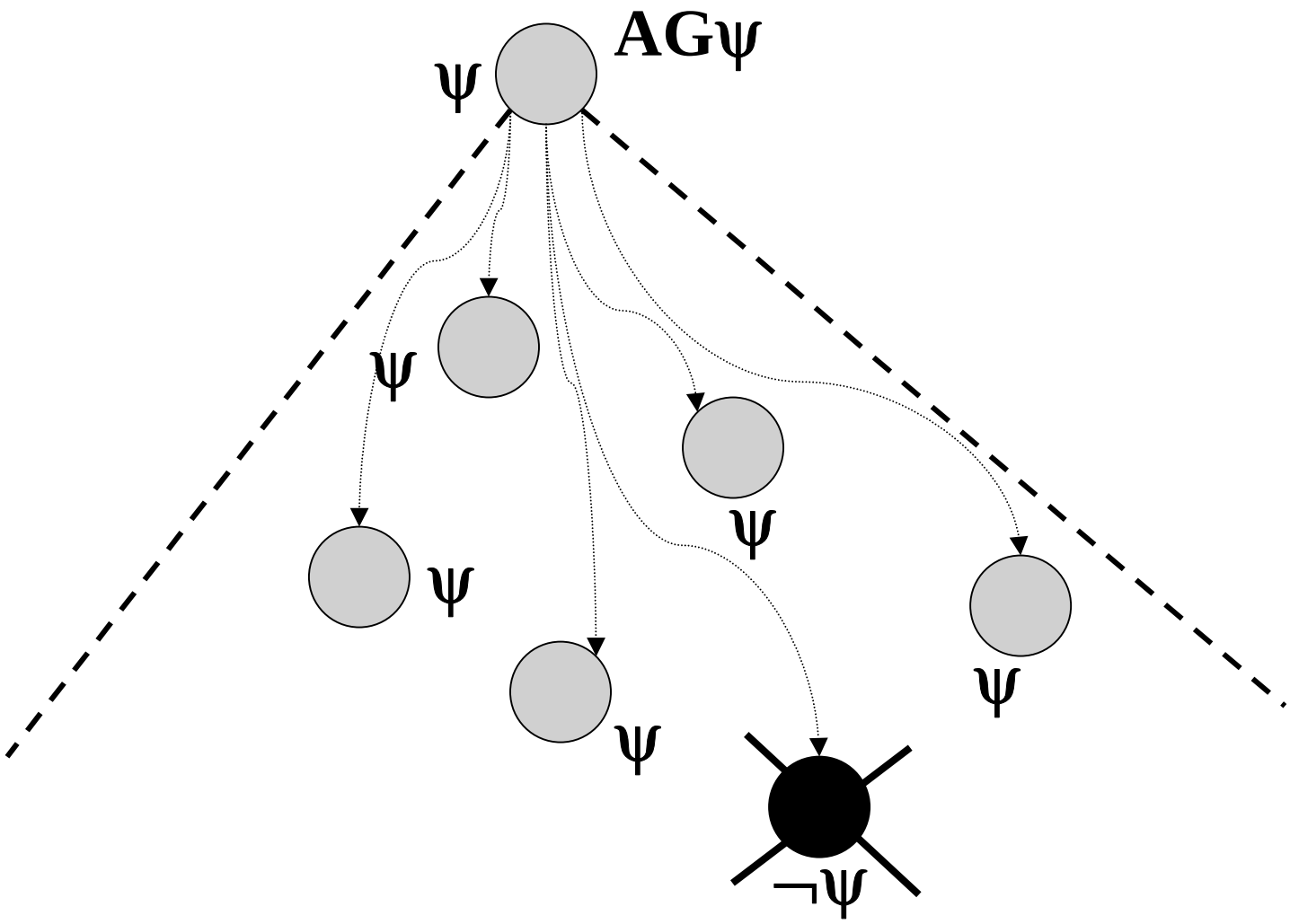
Derived Operators

- **K, s $\models$ EF(ψ)**
- **EF(ψ) = EU($\top$, ψ)**
  - There exists a path π (from **s**) and **k** $\geq$ **0** such that:
    - **K, π(k) $\models$ ψ.**


EF(ψ)
ψ

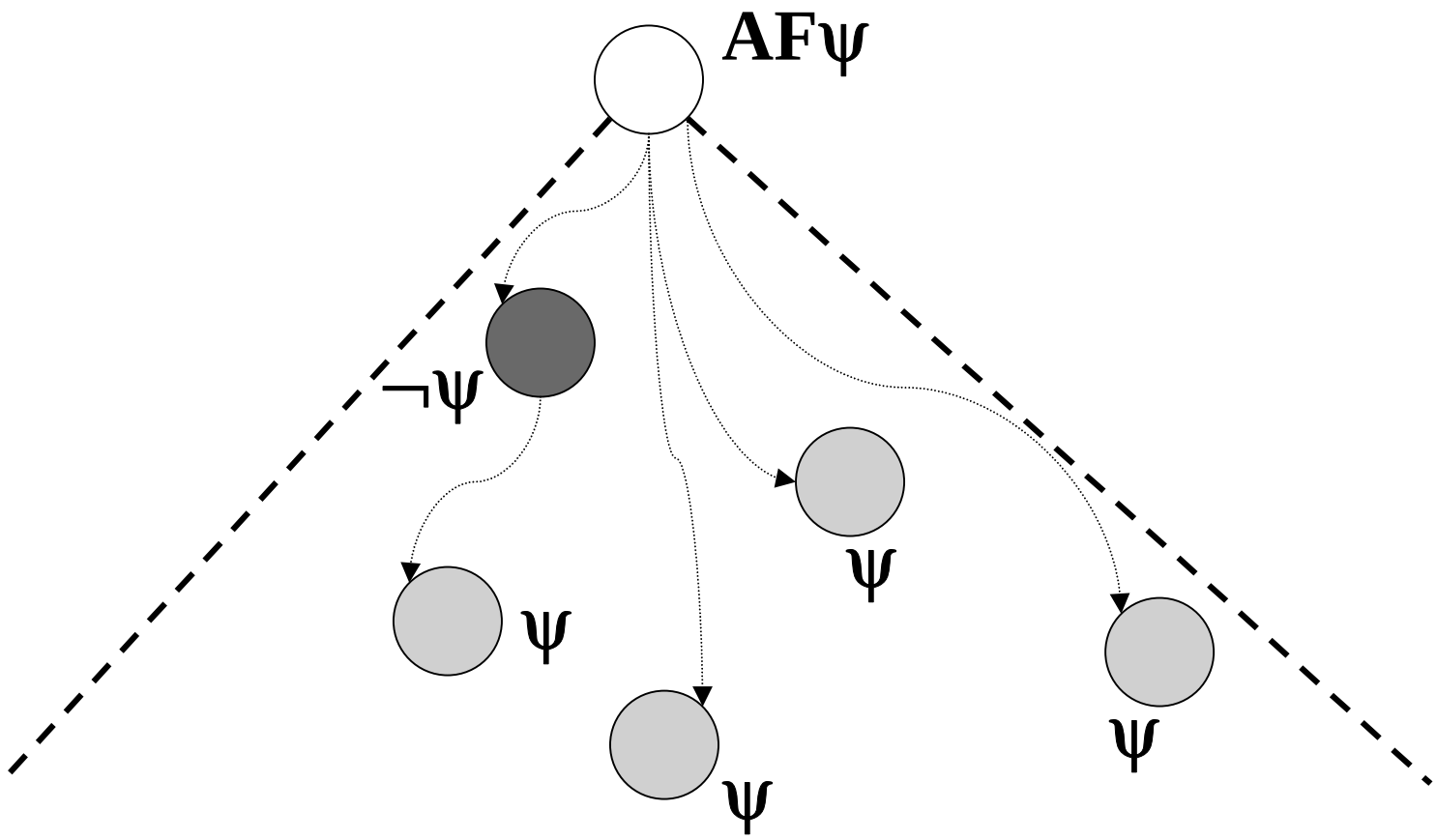
Derived Operators

- **K, s ⊨ AG(ψ)**
- **AG(ψ) = ¬EF(¬ψ)**
  - It is ***not*** the case ***there exists a path π*** (from **s**) and **k ≥ 0** such that:
    - **K, π(k) ⊨ ψ**
  - ***For every path π*** (from **s**) and ***every*** **k ≥ 0**:
    - **K, π(k) ⊨ ψ**


AGψ
ψ
ψ
ψ
ψ
ψ
ψ
¬ψ

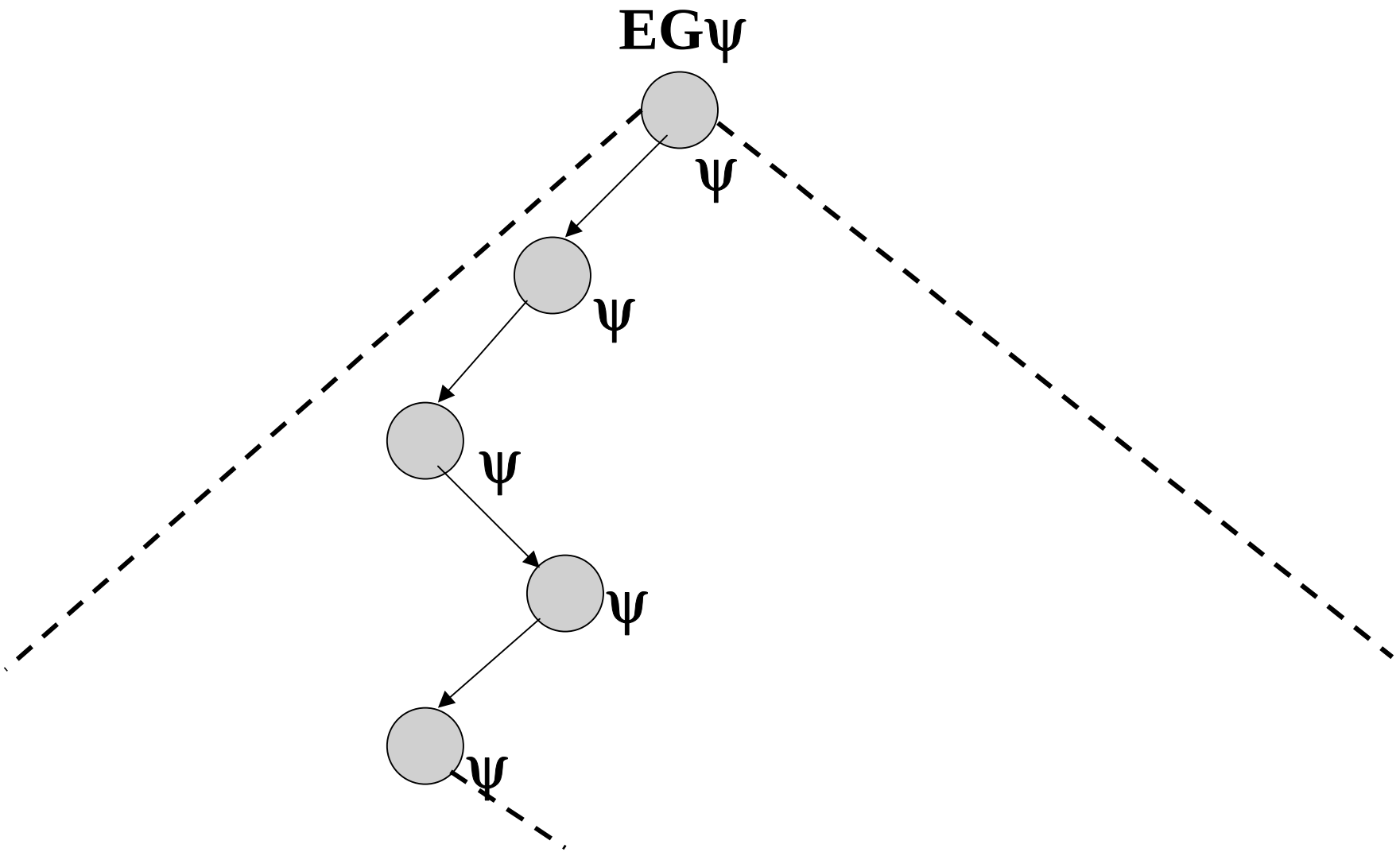
Derived Operators

- **K, s $\models$ AF(ψ)**
- **AF(ψ) = AU($\top$, ψ)**
  - ***For every path*** **π** from **s**, there ***exists*** **k $\geq$ 0** such that:
    - **K, π(k) $\models$ ψ.**


AFψ
¬ψ
ψ
ψ
ψ
ψ

Derived Oparators

- **K, s $\models$ EG(ψ)**
- **EG(ψ) = $\neg$AF($\neg\psi$)**
  - It is **not** the case that ***for every path*** π from **s** there is a **k** $\geq$ **0** such that **K,** π**(k)** $\models$ ψ.
  - ***There exists a path*** π from **s** such that, for every **k** $\geq$ **0**:
    - **K,** π**(k)** $\models$ ψ.


EGψ
ψ
ψ
ψ
ψ
ψ

A more convenient CTL

- **NCTL ::= p** $\mid \neg\psi \mid \psi_1 \vee \psi_2 \mid$ **EX(**ψ**)** $\mid$
  $\mid$ **EU(**ψ_1, ψ_2**)** $\mid$ **EG(**ψ**)**
- **CTL ::= p** $\mid \neg\psi \mid \psi_1 \vee \psi_2 \mid$ **EX(**ψ**)** $\mid$
  $\mid$ **EU(**ψ_1, ψ_2**)** $\mid$ **AU(**ψ_1, ψ_2**)**
- **NCTL** is more convenient for model checking!
- Clearly **NCTL** can be defined in terms of **CTL**.

A more convenient CTL

- **NCTL ::= p** $\mid \neg\psi \mid \psi_1 \vee \psi_2 \mid$ **EX(**ψ**)** $\mid$
  $\mid$ **EU(**ψ_1**,** ψ_2**)** $\mid$ **EG(**ψ**)**
- **CTL ::= p** $\mid \neg\psi \mid \psi_1 \vee \psi_2 \mid$ **EX(**ψ**)** $\mid$
  $\mid$ **EU(**ψ_1**,** ψ_2**)** $\mid$ **AU(**ψ_1**,** ψ_2**)**
- **CTL** can be defined in terms of **NCTL**!
- The semantics of **NCTL** is given in the obvious way.

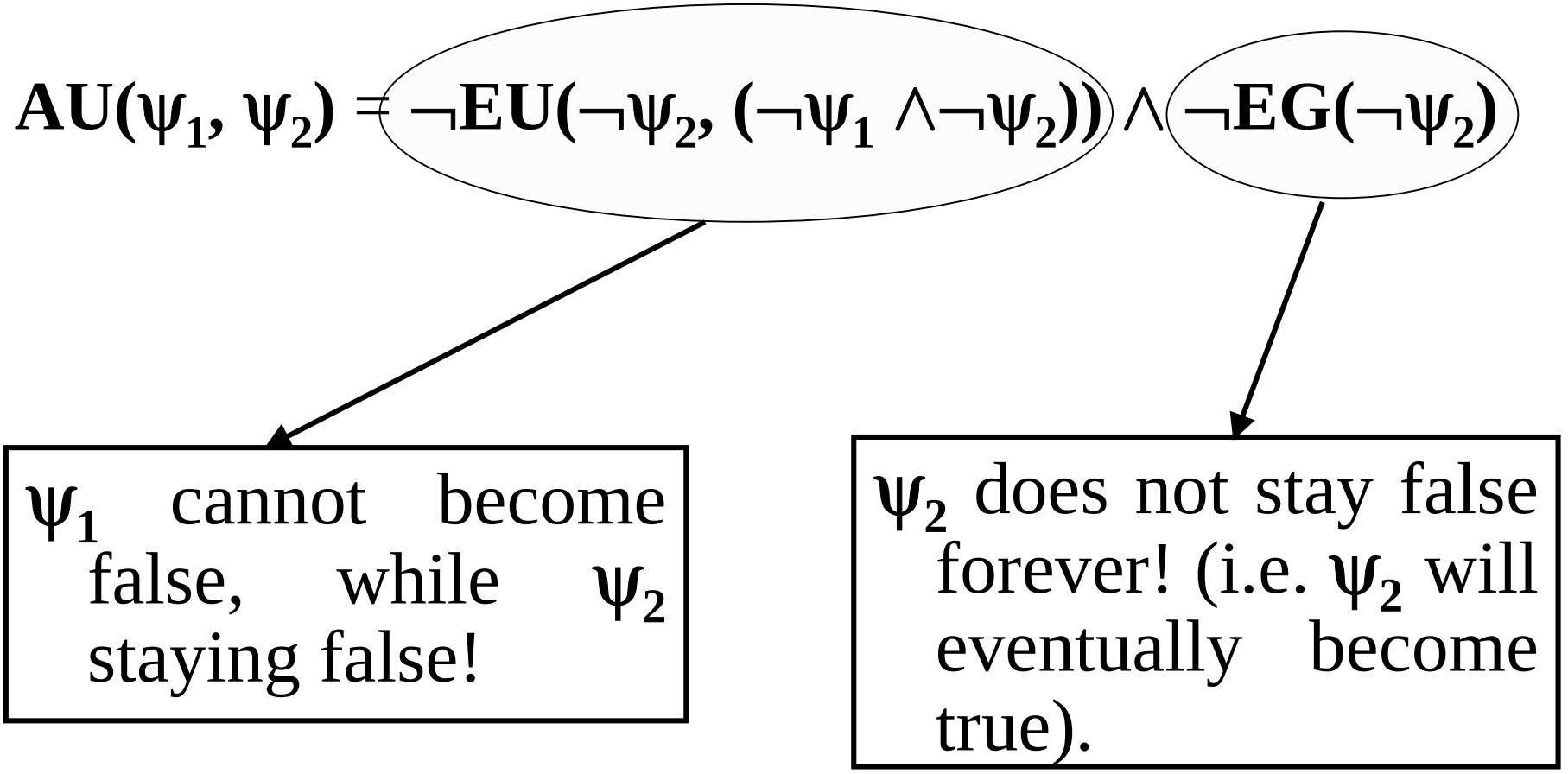
A more convenient CTL

- **NCTL ::= p** | $\neg\psi$ | $\psi_1 \vee \psi_2$ | **EX(ψ)** |
  | **EU(ψ_1, ψ_2)** | **EG(ψ)**
- **K, s** $\models$ **EG(ψ)** iff ***there exists a path*** π from **s** such that for every **k** $\geq$ **0**:
  - **K,** π**(k)** $\models$ ψ

A more convenient CTL

- **NCTL ::= p | $\neg\psi$ | $\psi_1 \vee \psi_2$ | EX(ψ) |**
  | EU(ψ_1, ψ_2) | EG(ψ)
- **CTL ::= p | $\neg\psi$ | $\psi_1 \vee \psi_2$ | EX(ψ) |**
  | EU(ψ_1, ψ_2) | AU(ψ_1, ψ_2)
- **AU(ψ_1, ψ_2) = $\neg$EU($\neg\psi_2$, ($\neg\psi_1 \wedge \neg\psi_2$)) $\wedge$ $\neg$EG($\neg\psi_2$)**

A more convenient CTL



A more convenient CTL

$\mathbf{AU}(\psi_1, \psi_2) = \neg\mathbf{EU}(\neg\psi_2, (\neg\psi_1 \wedge \neg\psi_2)) \wedge \neg\mathbf{EG}(\neg\psi_2)$

$\Rightarrow$ Assume $\mathbf{K, s} \models \mathbf{AU}(\psi_1, \psi_2)$

- Let π be a path from **s**. Then there exists $\mathbf{k} \geq \mathbf{0}$ with:
  - $\mathbf{K, s} \models \psi_2$
- Hence, $\mathbf{K, s} \not\models \mathbf{EG}(\neg\psi_2)$
- Hence, $\mathbf{K, s} \models \neg\mathbf{EG}(\neg\psi_2)$

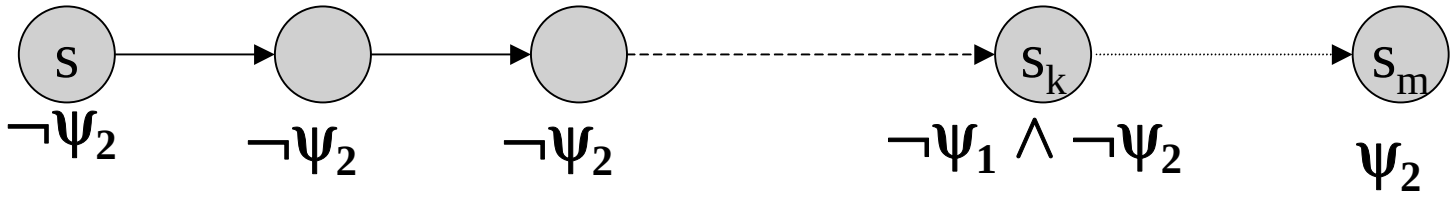
A more convenient CTL

- $\mathbf{AU}(\psi_1, \psi_2) = \mathbf{MagicAU}(\psi_1, \psi_2) =$

  $\neg\mathbf{EU}(\neg\psi_2, (\neg\psi_1 \wedge \neg\psi_2)) \wedge \neg\mathbf{EG}(\neg\psi_2)$

- Clearly $\mathbf{K, s} \models \mathbf{AU}(\psi_1, \psi_2)$ implies $\mathbf{K, s} \models \neg\mathbf{EG}(\neg\psi_2)$

- Let $\mathbf{K, s} \models \mathbf{AU}(\psi_1, \psi_2)$
  - Suppose now $\mathbf{K, s} \models \mathbf{EU}(\neg\psi_2, \neg\psi_1 \wedge \neg\psi_2)$
  - Let π be any path from **s** satisfying the above:
  - Let now **k** ***be the least integer*** such that:
    - $\mathbf{K}, \pi(\mathbf{k}) \models \neg\psi_1 \wedge \neg\psi_2$
    - $\mathbf{K}, \pi(\mathbf{j}) \models \neg\psi_2$ for $\mathbf{0 \leq j < k}$.



- Suppose **K, π(m) ⊨ ψ_2,** required by **K, s ⊨ AU(ψ_1,ψ_2)**
- Take **m** to be the least such number.
- Then **m > k**, since **K, s ⊨ EU($\neg\psi_2$, $\neg\psi_1 \wedge \neg\psi_2$)**
- But **0 ≤ k < m** and **K, π(k) ⊨ $\neg\psi_1$**
- Hence **K, s ⊭ AU(ψ_1, ψ_2).** ***Contradiction!***
- Thus **K, s ⊨ AU(ψ_1, ψ_2)** also implies:
  - **K, s ⊨ $\neg$EU($\neg\psi_2$, $\neg\psi_1 \wedge \neg\psi_2$)**
- So **K, s ⊨ AU(ψ_1, ψ_2)** implies **K, s ⊨ MagicAU(ψ_1, ψ_2)**

From CTL to NCTL

- In a similar way we can argue that:

> ***if*** **K, s $\models$ MagicAU(ψ_1, ψ_2)**
>
> ***then*** **K, s $\models$ AU(ψ_1, ψ_2).**

- Hence ***CTL*** can be expressed in terms of ***NCTL***.

A more convenient CTL

- **NCTL ::= p | $\neg\psi$ | $\psi_1 \vee \psi_2$ | EX(ψ) |**
  | EU(ψ_1, ψ_2) | EG(ψ)
- **CTL ::= p | $\neg\psi$ | $\psi_1 \vee \psi_2$ | EX(ψ) |**
  | EU(ψ_1, ψ_2) | AU(ψ_1, ψ_2)
- **AU(ψ_1, ψ_2) = MagicAU(ψ_1, ψ_2) =**

  $\neg$(EU($\neg\psi_2$, ($\neg\psi_1 \wedge \neg\psi_2$)) $\wedge$ AF(ψ_2)

- **Magic$_1$ = $\neg$ EU($\neg\psi_2$, ($\neg\psi_1 \wedge \neg\psi_2$)**
- **Magic$_2$ = AFψ_2**

$\neg$EG$\neg\psi_2$ = AFψ_2

From CTL to NCTL

- Let $\mathbf{K = (S, S_0, R, AP, L)}$ and $\mathbf{s \in S}$.
- We need to argue:
  - $\mathbf{K, s \models AU(\psi_1, \psi_2)}$ iff

    $\mathbf{K, s \models Magic_1 \wedge Magic_2}$
- We argued last time:
  - If $\mathbf{K, s \models AU(\psi_1, \psi_2)}$ then

    $\mathbf{K, s \models Magic_1 \wedge Magic_2}$

From CTL to NCTL

$$\mathbf{AU}(\psi_1, \psi_2) = \neg\mathbf{EU}(\neg\psi_2, (\neg\psi_1 \wedge \neg\psi_2)) \wedge \neg\mathbf{EG}(\neg\psi_2)$$

$\Leftarrow$ We need to argue that:

– If $\mathbf{K, s} \models \mathbf{Magic}_1 \wedge \mathbf{Magic}_2$ then

$\mathbf{K, s} \models \mathbf{AU}(\psi_1, \psi_2)$

- So assume $\mathbf{K, s} \models \mathbf{Magic}_1 \wedge \mathbf{Magic}_2$.

- $\mathbf{Magic}_1 = \neg\mathbf{EU}(\neg\psi_2, (\neg\psi_1 \wedge \neg\psi_2))$.
- $\mathbf{Magic}_2 = \neg\mathbf{EG}\neg\psi_2 = \mathbf{AF}\psi_2$

From CTL to NCTL

- Let π be some path from **s**.
- We need to show that there exists $\mathbf{k \geq 0}$ such that:
  - $\mathbf{K, \pi(k) \models \psi_2}$
  - $\mathbf{K, \pi(j) \models \psi_1}$ if $\mathbf{0 \leq j < k}$.
- But $\mathbf{K, s \models AF\ \psi_2}$ implies there exists $\mathbf{k \geq 0}$ such that:
  - $\mathbf{K, \pi(k) \models \psi_2}$
- Assume **k** is the ***least*** such number.

From CTL to NCTL

Now consider **m** such that $\mathbf{0 \leq m < k}$.

CLAIM: $\mathbf{K, \sigma(m) \models \psi_1}$

- If the **CLAIM** is true then we are done.
- Suppose $\mathbf{K, \sigma(m) \models \neg\psi_1}$.
  - Then $\mathbf{K, \sigma(m) \models \neg\psi_1 \wedge \neg\psi_2}$ $(\mathbf{m < k})$ ***WHY*?**
  - Further $\mathbf{K, \sigma(j) \models \neg\psi_2}$ if $\mathbf{0 \leq j < m}$ since $\mathbf{j < m < k}$.
  - Hence $\mathbf{K, \sigma(0) \models EU(\neg\psi_2, \neg\psi_1 \wedge \neg\psi_2)}$
  - Hence $\mathbf{K, s \not\models Magic_1}$ and ***Contradiction***!

CTL Model Checking

- The ***CTL model checking problem***.
  - **K = (S, S_0, R, AP, L)** (system model)
  - ψ a ***CTL*** formula (specification of the property)
- **K** $\models \psi$ ***iff***
  - **K, s_0** $\models \psi$ for every $s_0 \in S_0$.
- Given **K** and ψ ***determine whether or not*** **K** $\models \psi$.

CTL Model Checking

- The actual model checking problem:
  - Given **K = (S, S_0, R, AP, L)**
  - Given **s** $\in$ **S**
  - Given ψ, **an NCTL formula**.
  - Determine:
    - **K, s** $\models \psi$

The Sub-formulas of ψ

- **SF(ψ)** is the ***least set of formulas*** satisfying:
  - $\psi \in$ **SF(ψ)**
  - If $\neg\alpha \in$ **SF(ψ)** then $\alpha \in$ **SF(ψ)** .
  - If $\alpha \vee \beta \in$ **SF(ψ)** then $\alpha, \beta \in$ **SF(ψ)**
  - If **EX**$\alpha \in$ **SF(ψ)** then $\alpha \in$ **SF(ψ)** .
  - If **EU**$(\alpha, \beta) \in$ **SF(ψ)** then $\alpha, \beta \in$ **SF(ψ)**
  - If **EG**$\alpha \in$ **SF(ψ)** then $\alpha \in$ **SF(ψ)** .
- **SF(ψ) ----** The ***set of sub-formulas*** of ψ.

The Labeling Procedure.

- **K = (S, S_0, R, AP, L)**
  - **s** $\in$ **S**
  - ψ a ***NCTL*** formula (built out of **AP**).

- **Strategy**:
  - Construct **Labels**: **S** $\rightarrow$ $\mathbf{2^{SF(\psi)}}$
  - $\mathbf{2^{SF(\psi)}}$, the set of subsets of **SF(ψ)**.
  - Each state of **K** is assigned a subset of a **SF(ψ)** by the Labels function.
- **K, s** $\vDash \psi$ ***iff*** $\psi \in$ **Labels(s)**.

The Labels function

- For every **s** $\in$ **S**:
- **Stage 1**:
  - **Labels(s) = L(s)** (**K =(S, S_0, R, AP, L)**)
- ….
- Assume we have done up to stage **i**.
- **Stage i +1**:
  - If $\alpha = \neg\beta$ then

    $\alpha \in$ **Labels(s)** ***iff*** $\beta \notin$ **Labels(s)**.

The Labels function

- For every **s** $\in$ **S**:
- **Stage i +1**:
  - If $\alpha = \beta_1 \vee \beta_2$ then

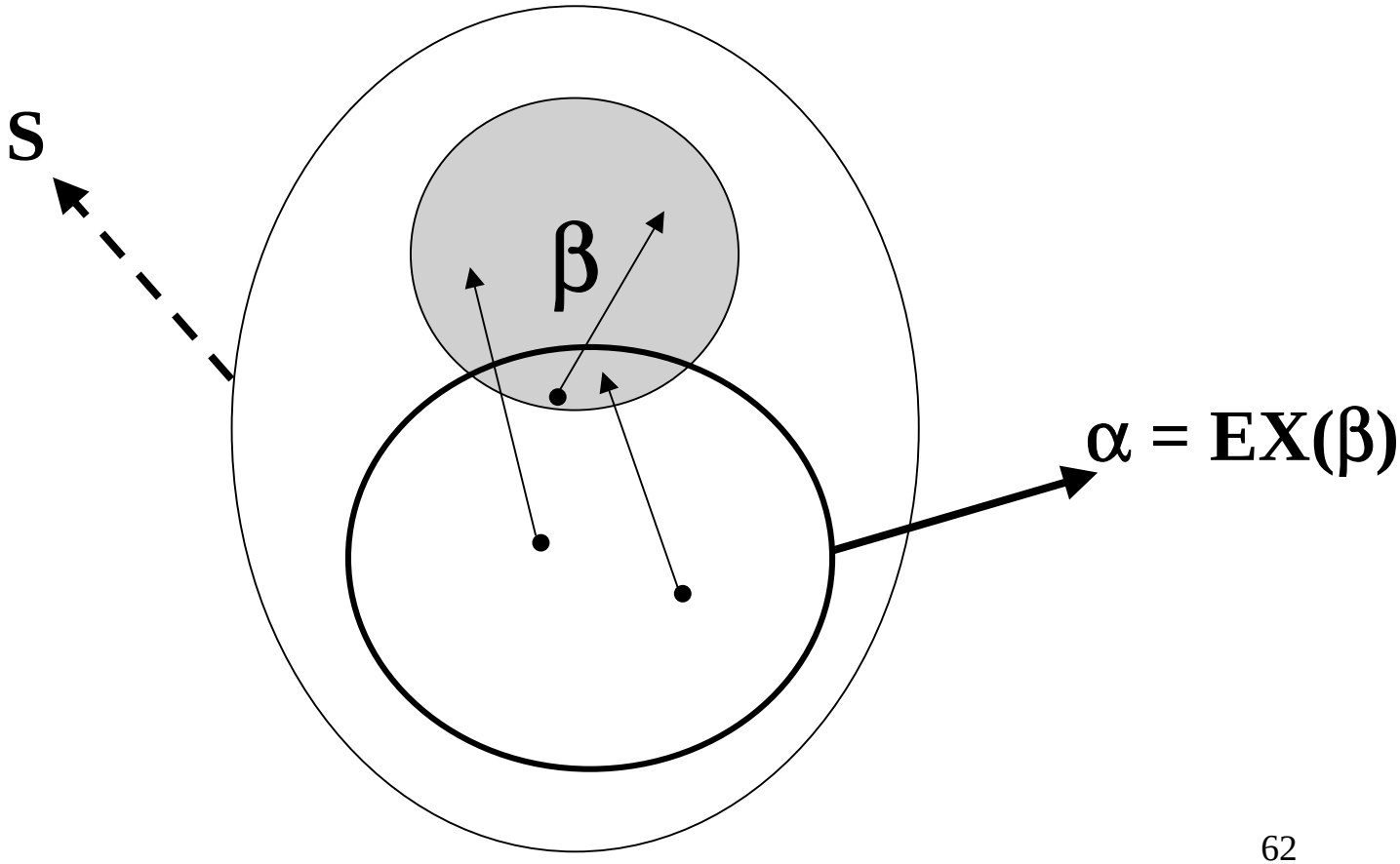
    $\alpha \in$ **Labels(s)** ***iff*** $\beta_1 \in$ **Labels(s) or** $\beta_2 \in$ **Labels(s)**

  - If $\alpha =$ **EX**β then

    $\alpha \in$ **Labels(s)** ***iff*** **there exists t** $\in$ **S such that**

    $\beta \in$ **Labels(t)** and **R(s, t)**

The Labels Function



Computing the labeling for EX(β)

Complexity: O(|M|)

Algorithm Check_EX(β)

T := {**s** | β ∈ **Labels(s)**};

while **T** ≠ ∅ do

choose **s** ∈ **T**;

T := **T** \{**s**};

forall **t** ∈ **S** such that (**t**,**s**) ∈ **R** do

Labels(**t**) := **Labels**(**t**) ∪ {**EX** β};

The Labels Function

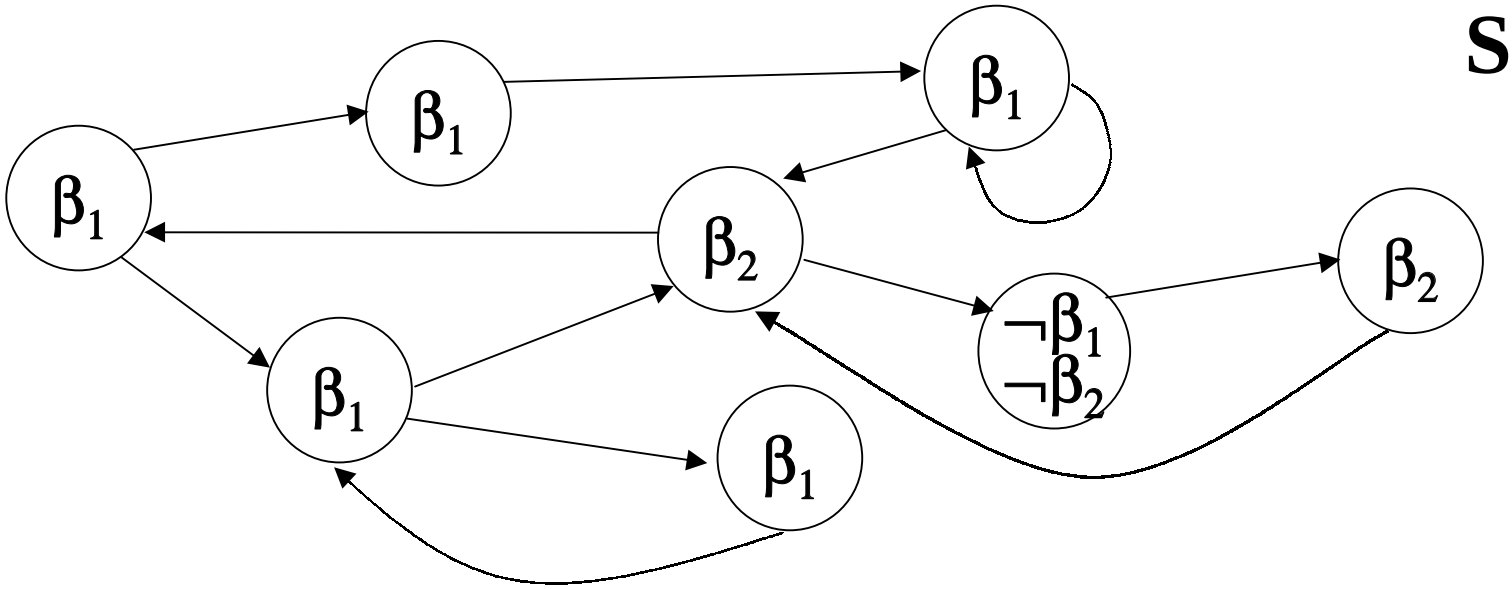
- For every **s** $\in$ **S**:
- **Stage i +1**:
  - If α = **EU**(β_1,β_2) then

    $\alpha \in$ **Labels(s)** ***iff***
    - $\beta_2 \in$ **Labels(s) or**
    - $\beta_1 \in$ **Labels(s) and EU**(β_1,β_2) $\in$ **Labels(t) for some t with R(s,t).**

The Labels Function

- Collect in **T** all the states satisfying β_2
  - **all these states do also satisfy EU(β_1,β_2)**.
- Traversing backward **R** from states in **T** and label with **EU(β_1,β_2)** those states satisfying β_1 and reaching at least a state **s** labeled with **EU(β_1,β_2)**.

If s $\in$ T and t with R(t,s) and $\beta_1 \in$ Labels(t)
then EU(β_1,β_2) $\in$ Labels(t)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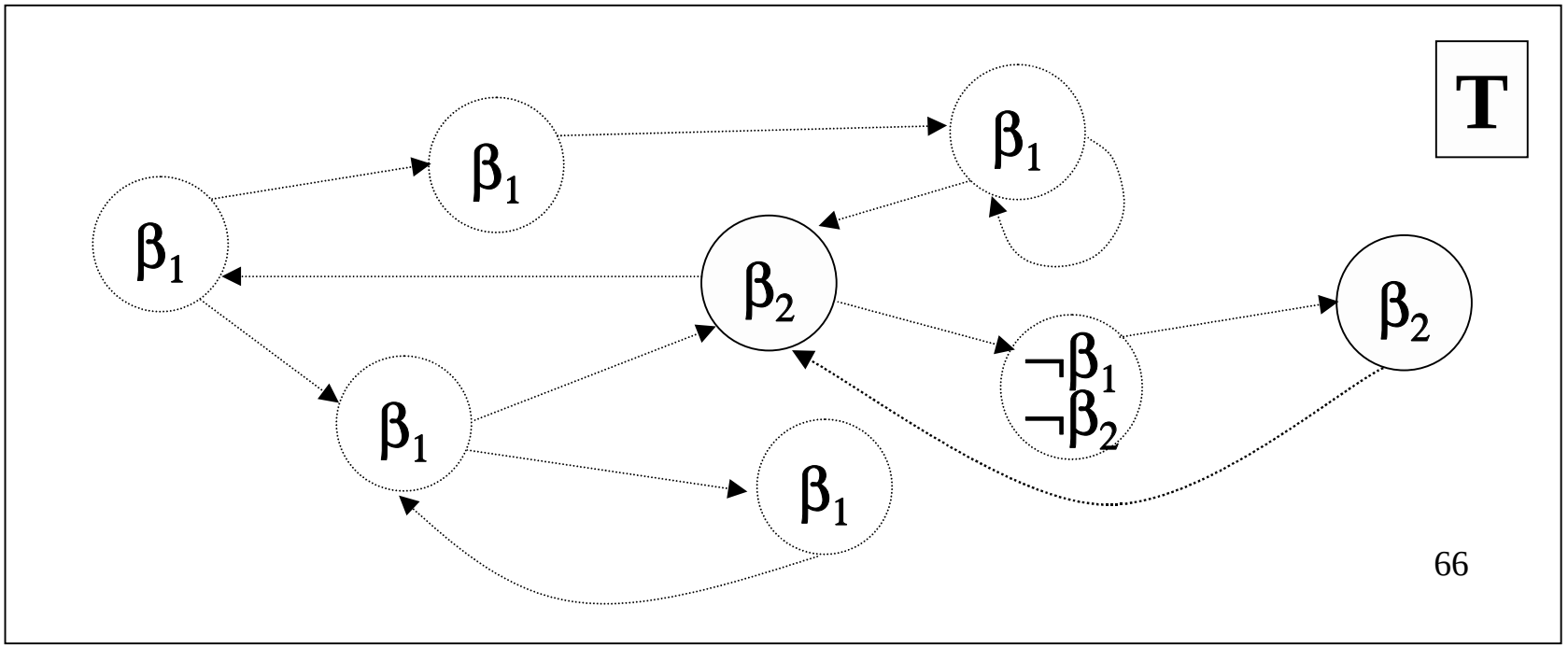
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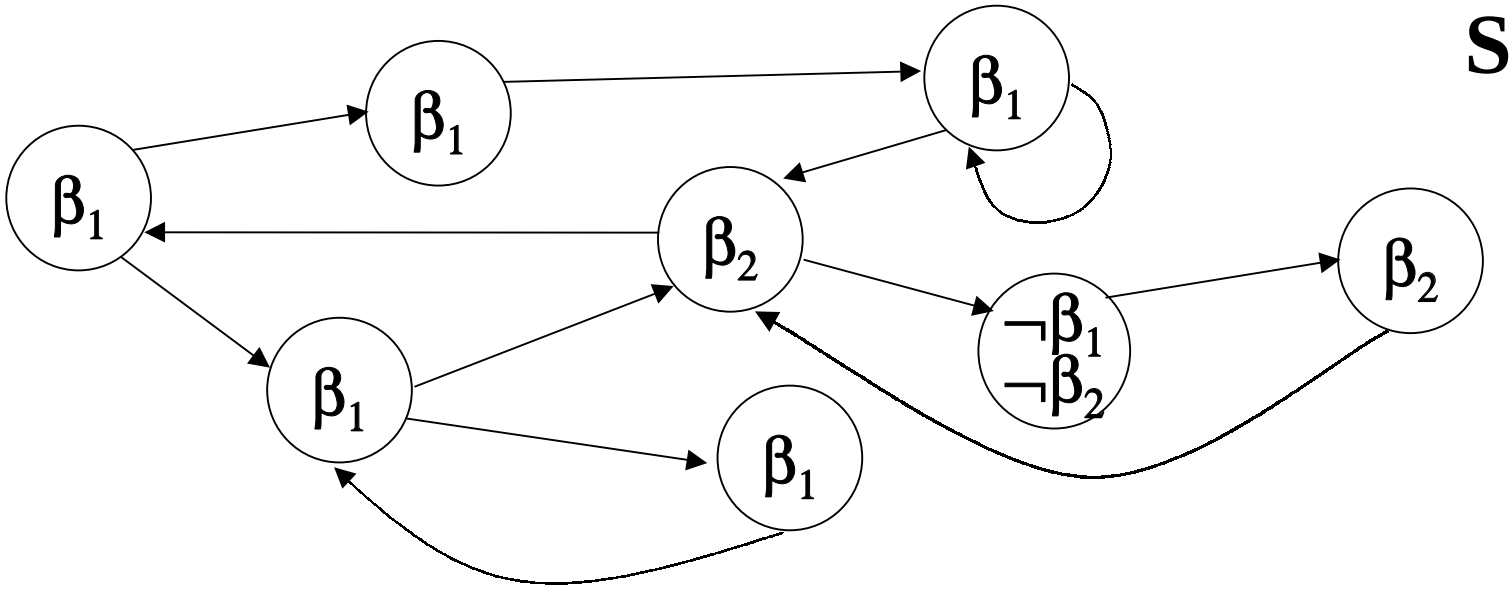
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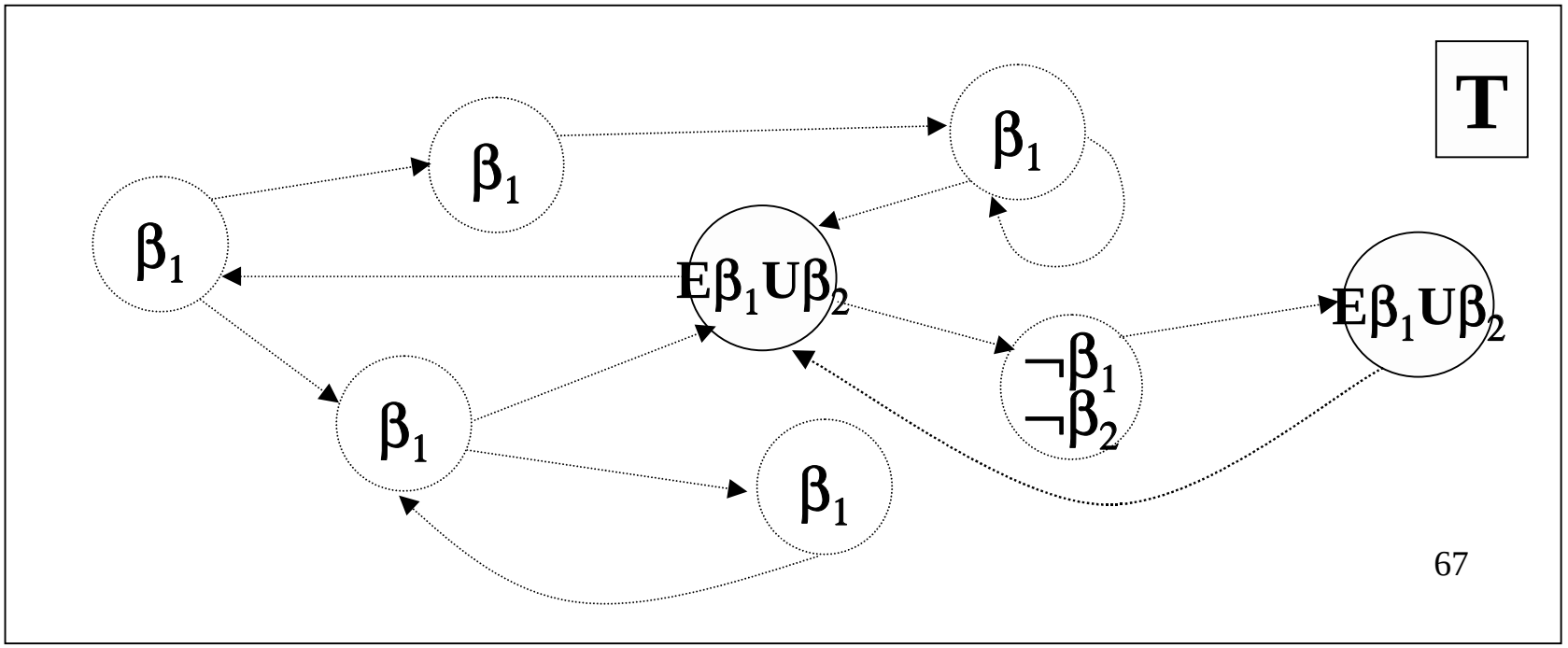
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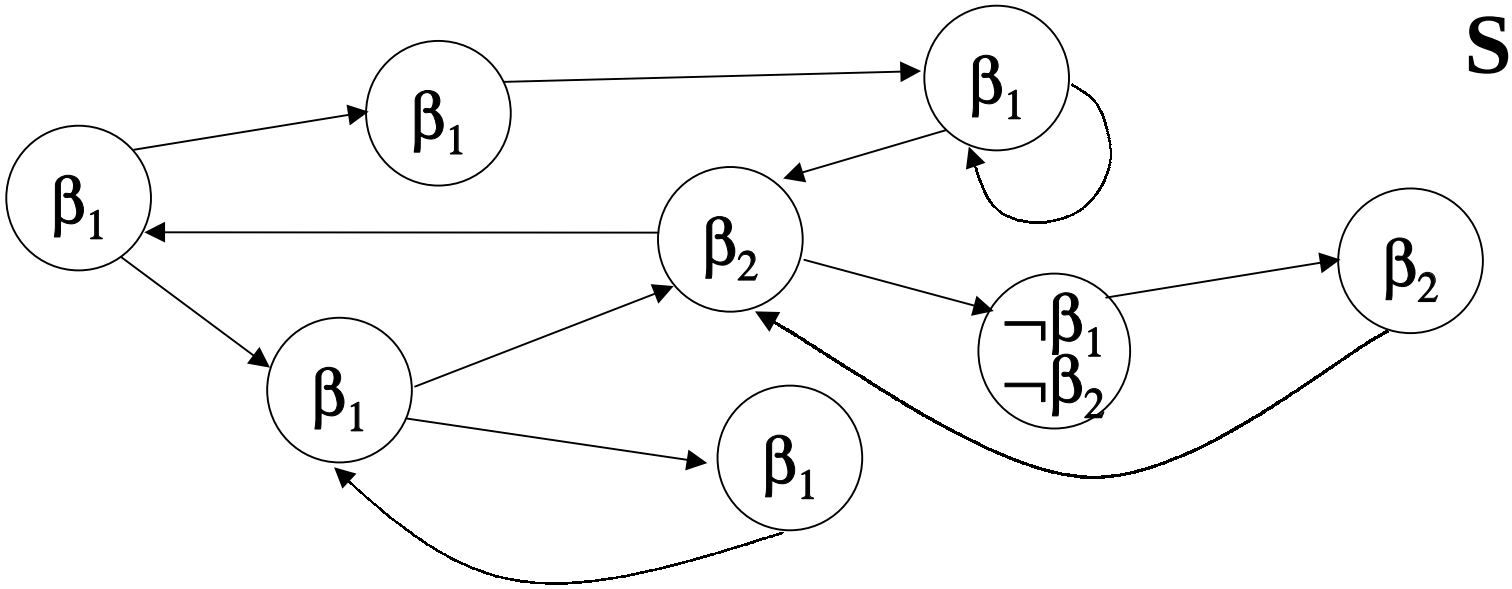
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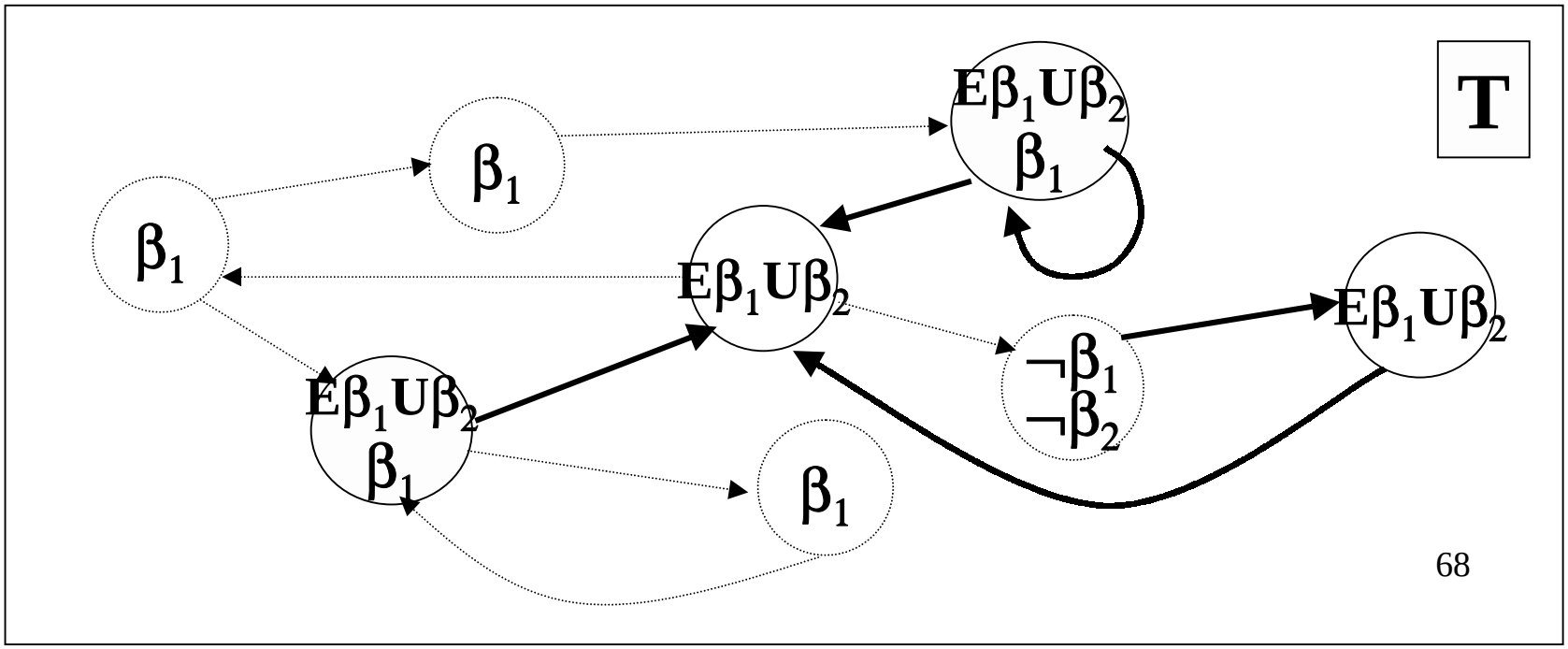
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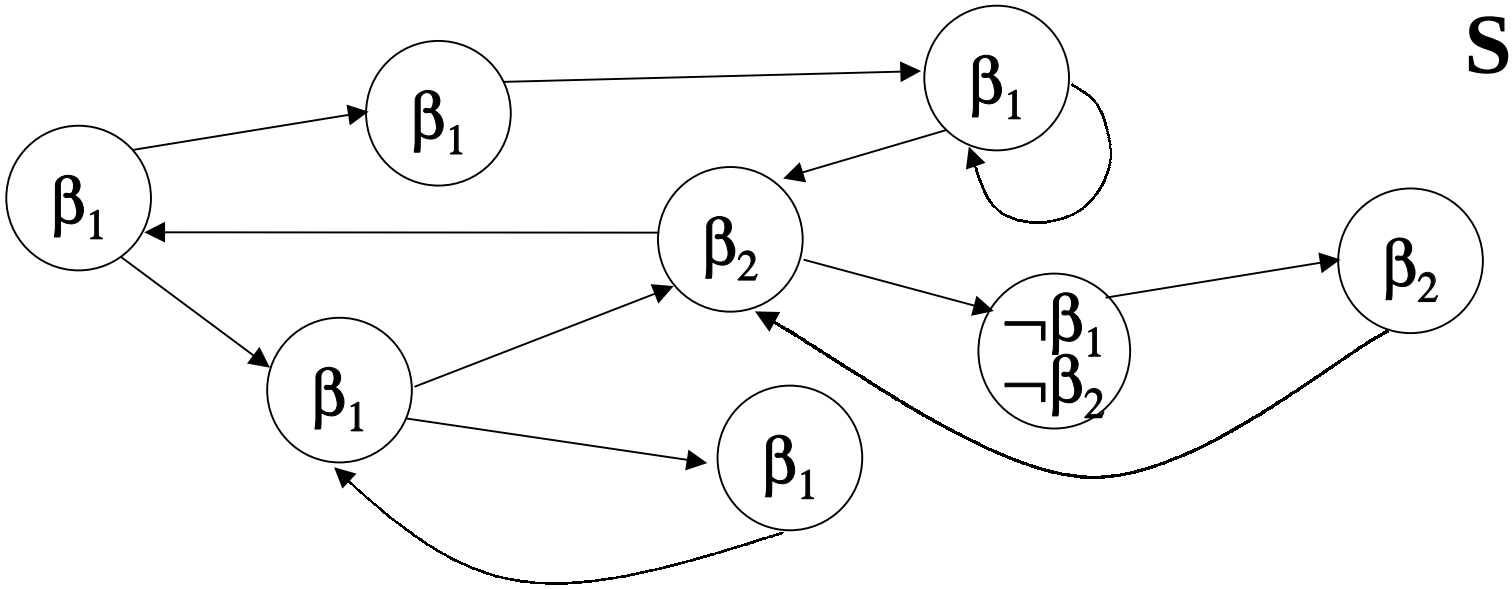
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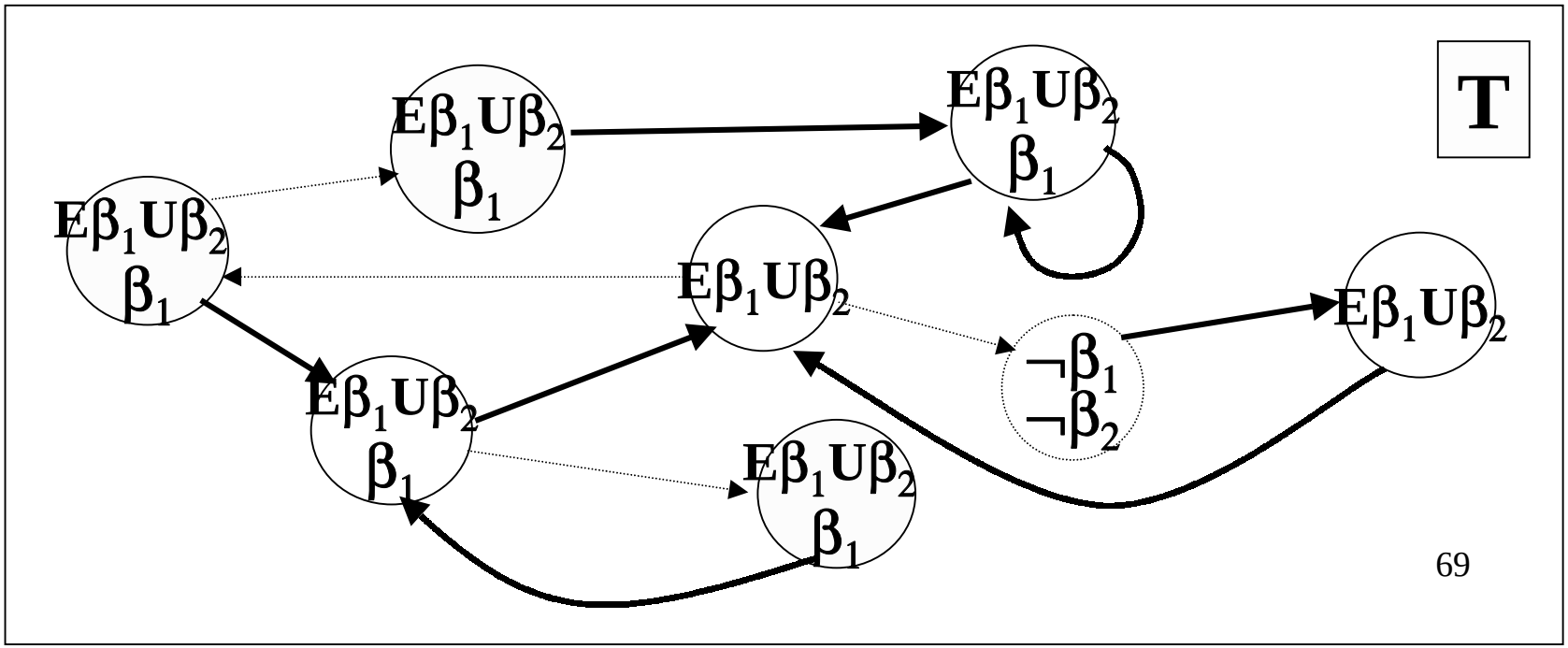
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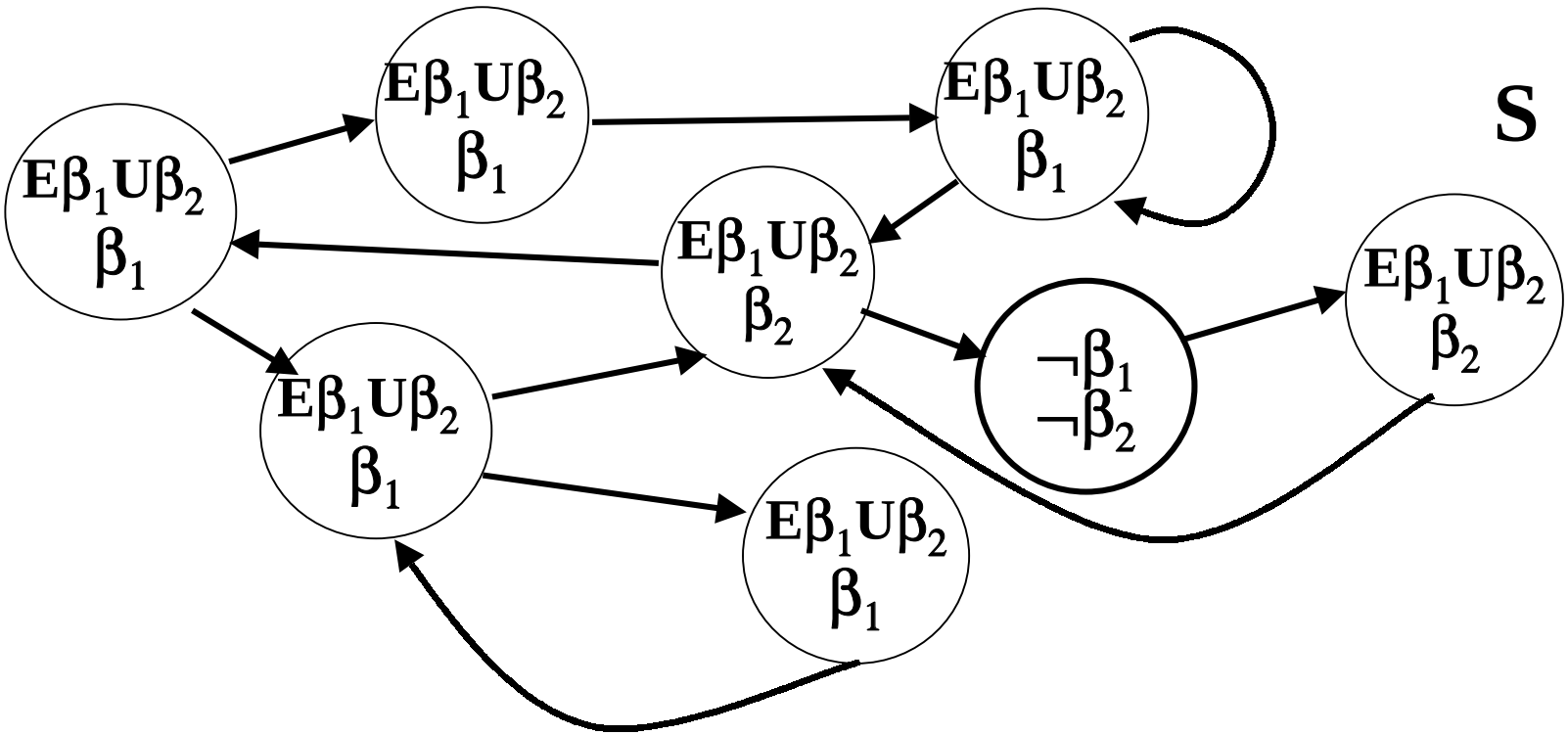
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Computing the labeling for $EU(\beta_1,\beta_2)$

Algorithm Check_$EU(\beta_1,\beta_2)$

T := {**s** | $\beta_2 \in$ Labels(**s**)};

forall **s** $\in$ **T** do

Labels(**s**) := Labels(**s**) $\cup$ {**EU**(β_1,β_2)};

while **T** $\neq \emptyset$ do

chose **s** $\in$ **T**;

T := **T** \{**s**};

forall **t** $\in$ **T** with **(t,s)** $\in$ **R** do

if **EU**(β_1,β_2) $\notin$ Labels(**t**) and $\beta_1 \in$ Labels(**t**) then

Labels(**t**) := Labels(**t**) $\cup$ {**EU**(β_1,β_2)};

T := **T** $\cup$ {**t**};

Complexity: O(|M|)

The Labels Function

- For every **t** $\in$ **S**:
- **Stage i +1**:
  - If α = **EG(**β **)** then

    $\alpha \in$ **Labels(t)** ***iff***

    - $\beta \in$ **Labels(t) and EG(**β**)** $\in$ **Labels(s) for some s with R(t,s).**

Property of EG(β)

Let **M' = (S',R',L')** be the sub-graph of **M** where

- **S' = { s | M ,s ⊨ β }**
- **R' = $R|_{S' \times S'}$** and **L' = $L|_{S'}$**

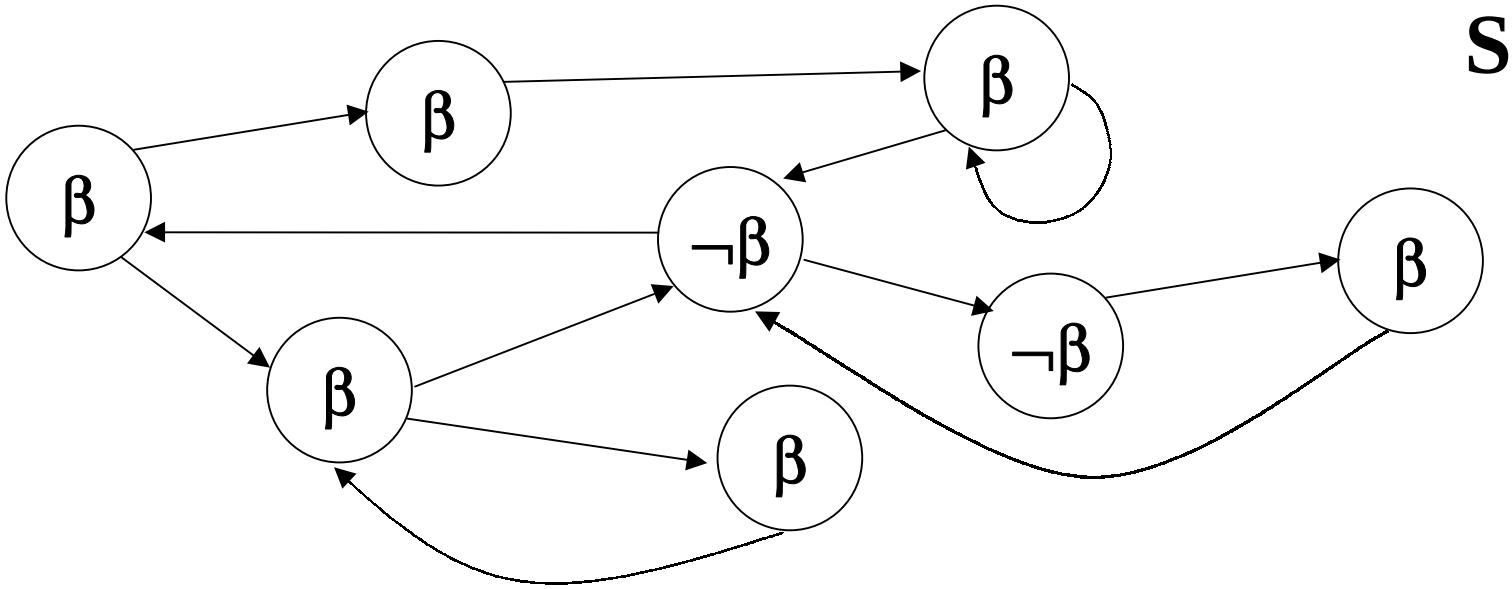
Lemma: M,s ⊨ EG(β) ***iff*** the following two conditions hold:

- **s ∈ S' and**
- ***there exists a path*** in **M'** leading from **s** to a ***non-trivial strongly connected component*** **C** of the graph **(S',R').**

The Labels Function

- Compute the ***non-trivial strongly connected components*** of the subgraph **S'** whose states all satisfy β
  - **all the states in these components do satisfy EG(β).**
- Traverse backward **R** in **M'** and label with **EG(β)** the ***states reaching at least a state*** **s** labeled with **EG(β)**.

If t ∈ S' and R(t,s) then EG(β) ∈ Labels(t)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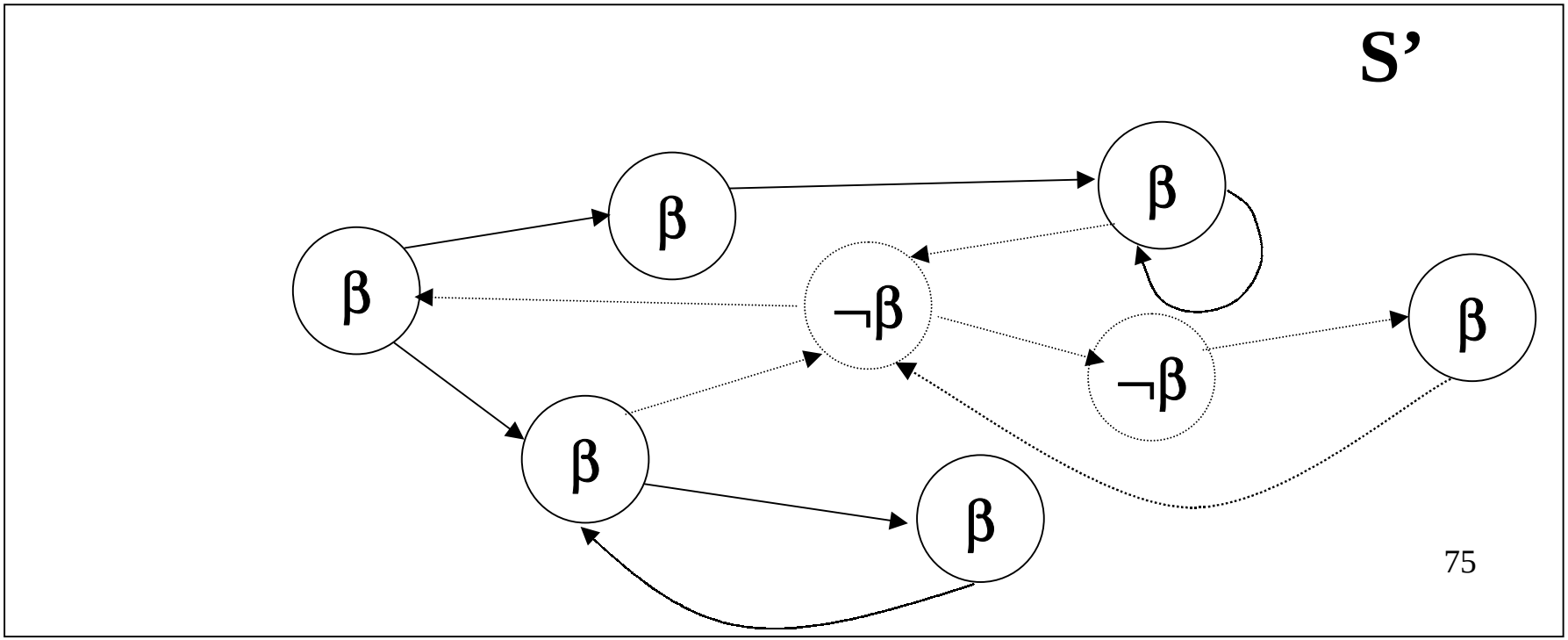
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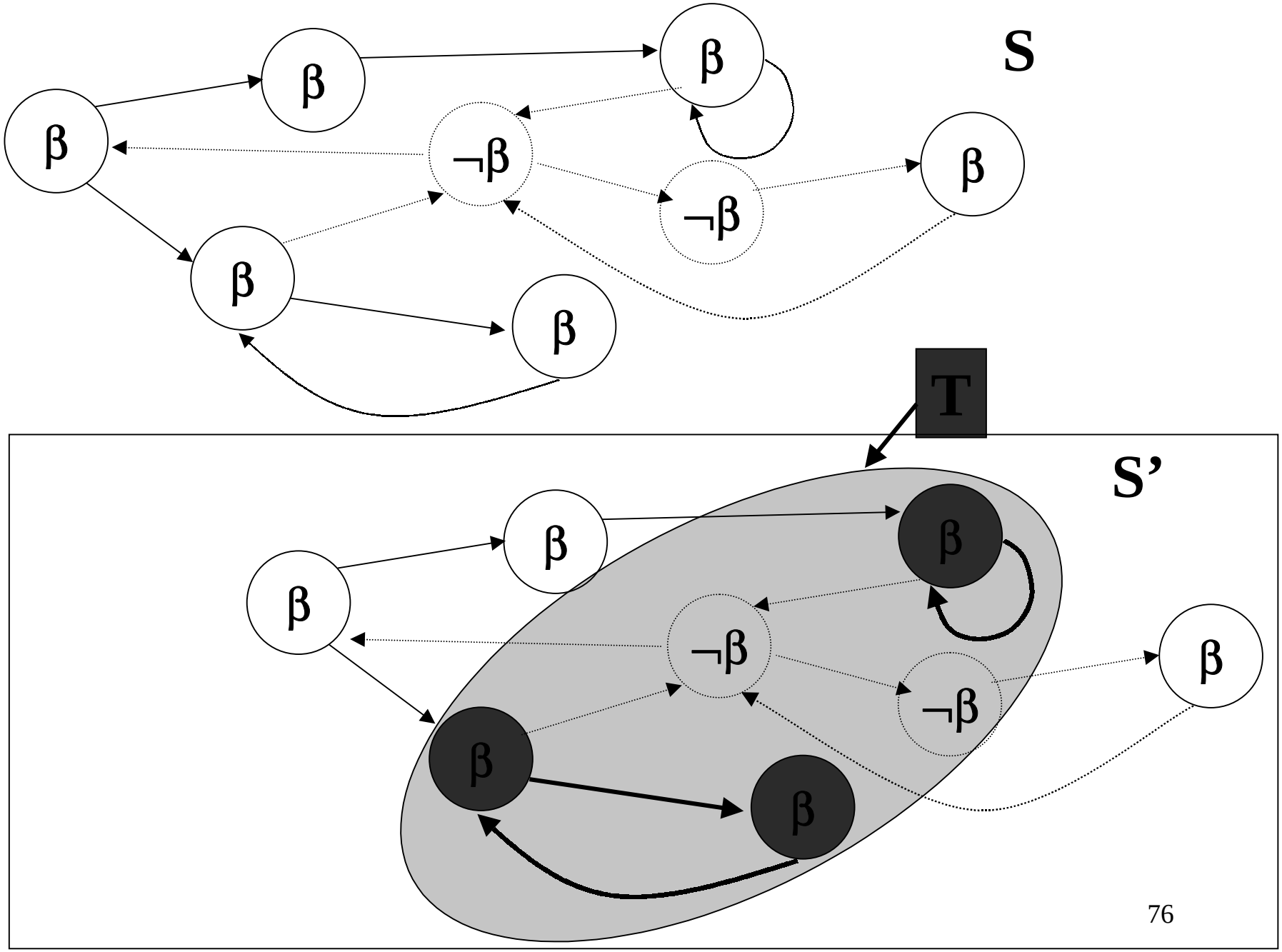
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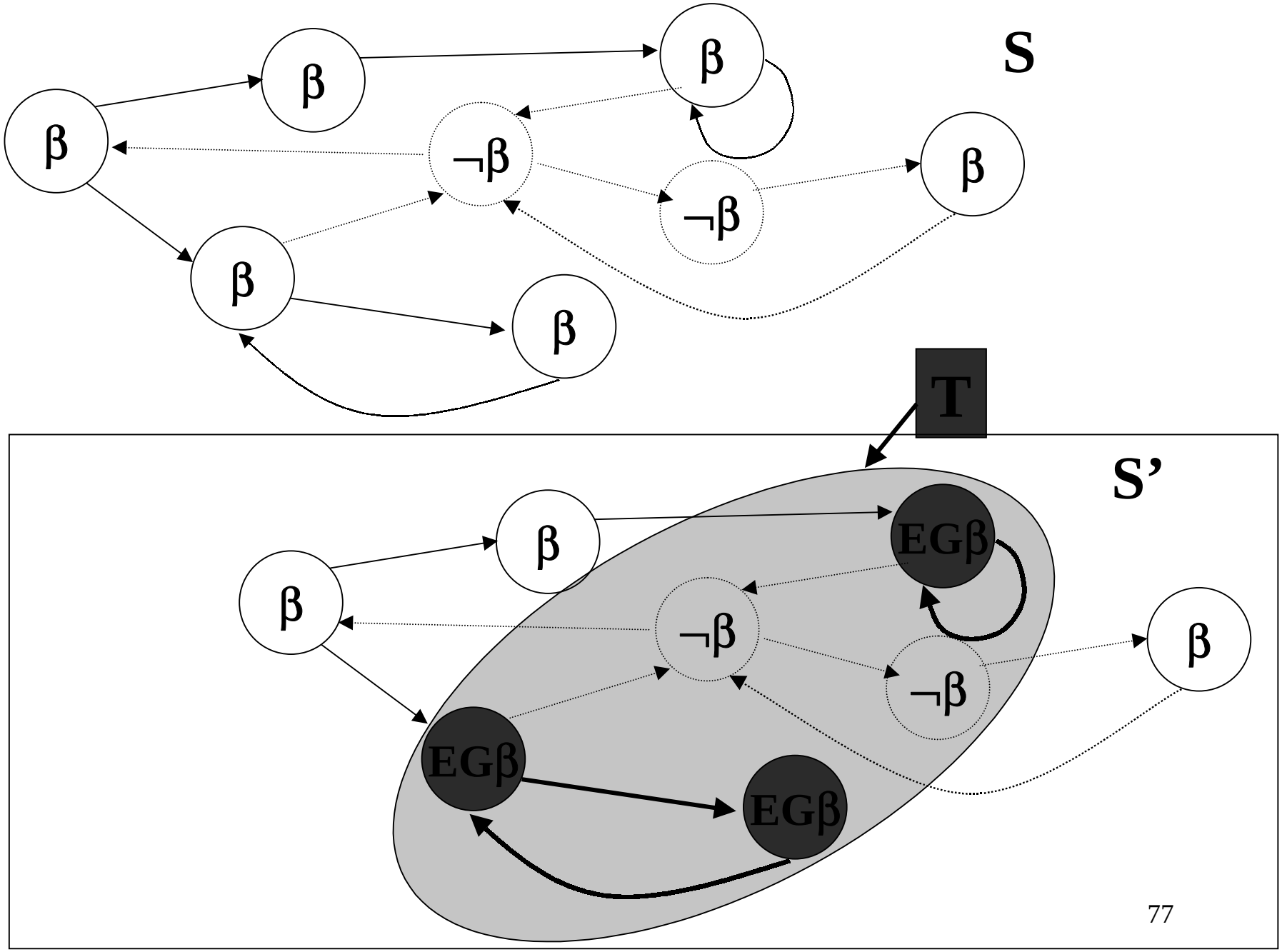
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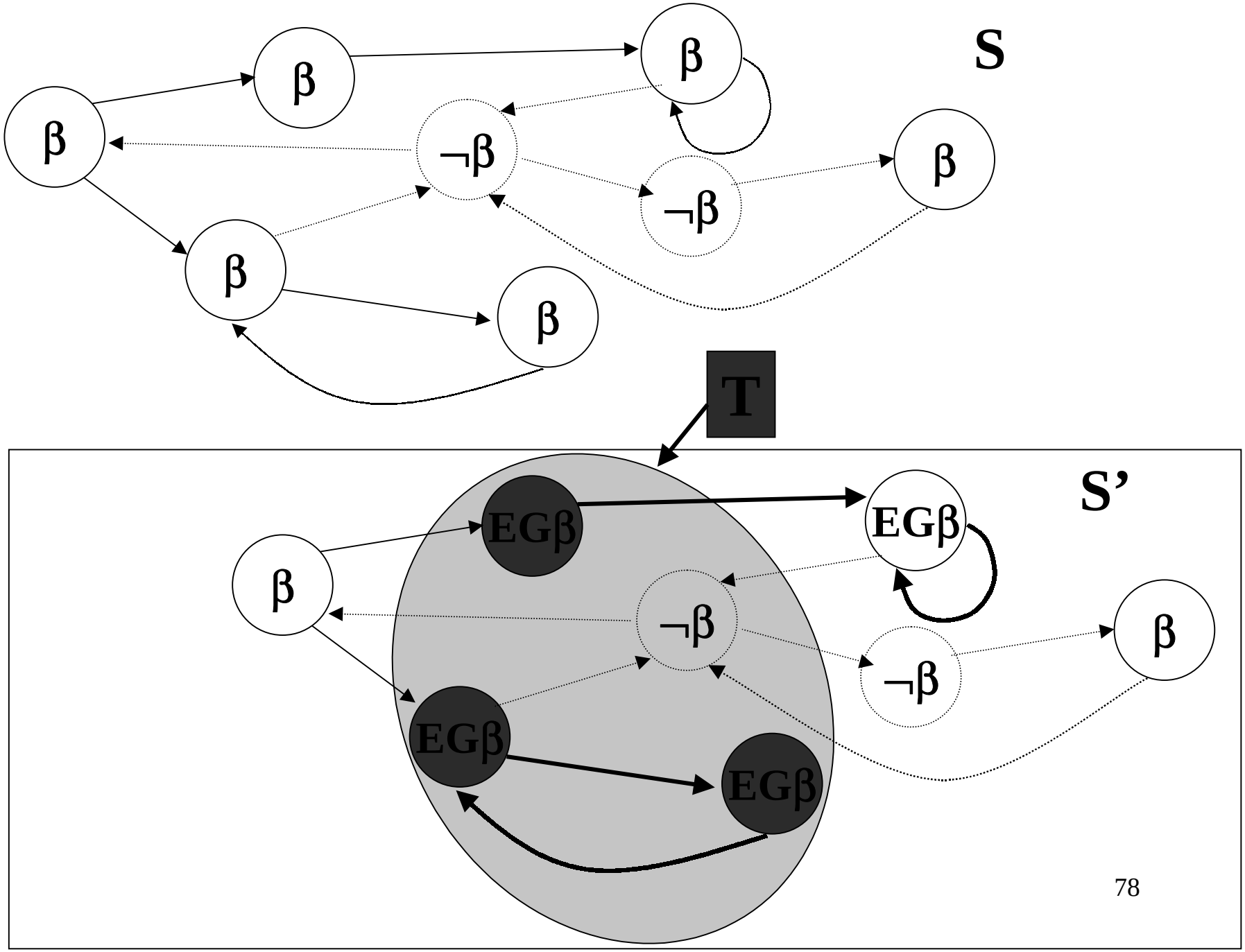
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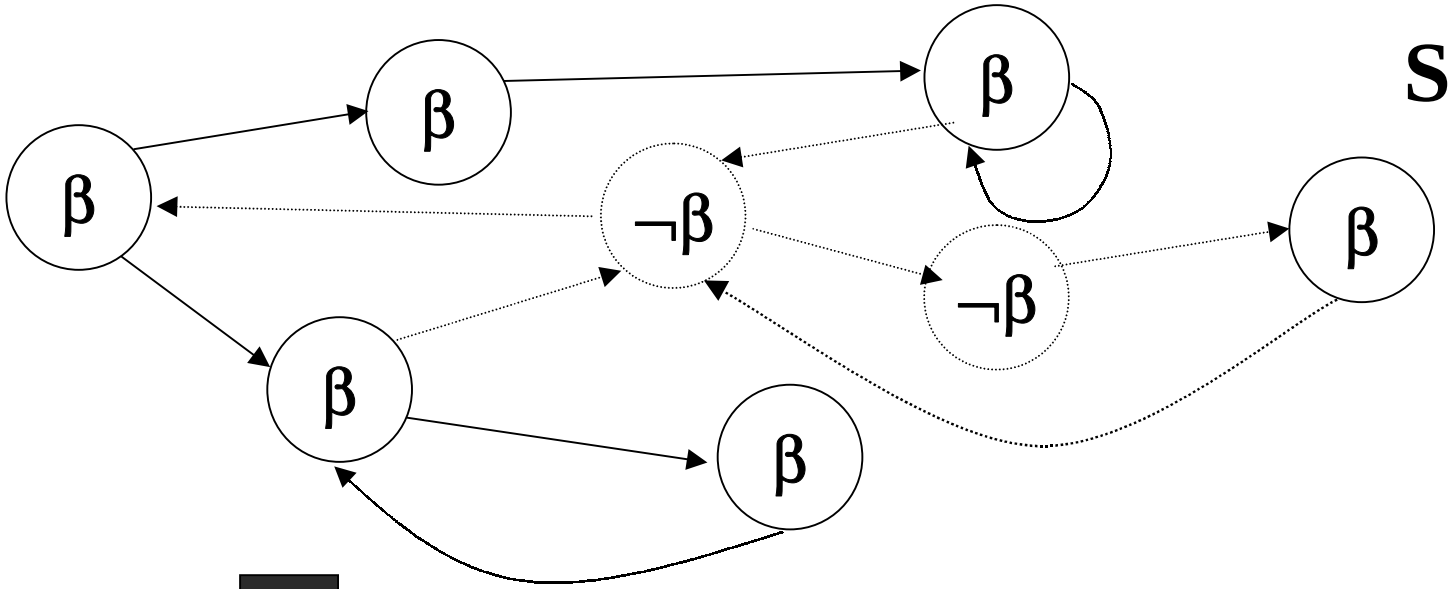
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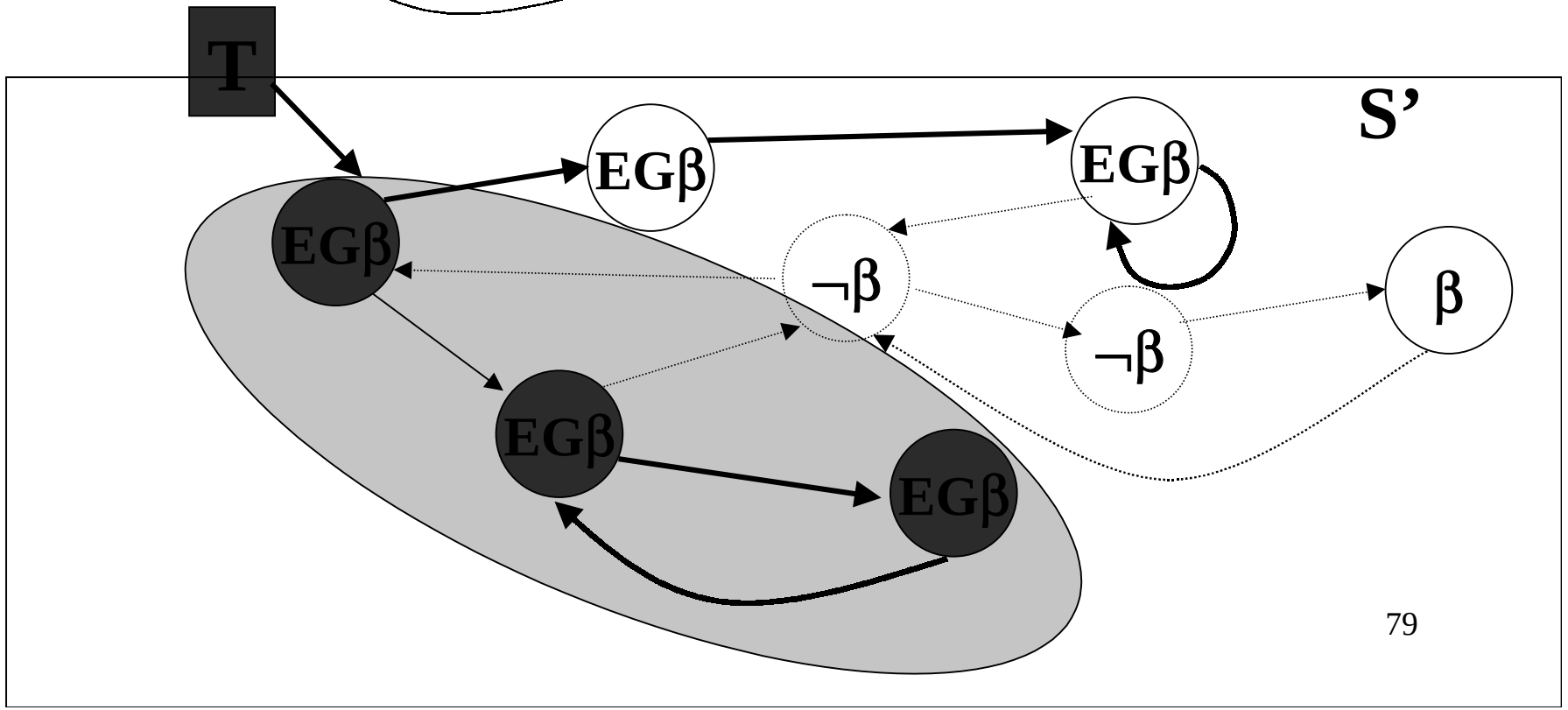
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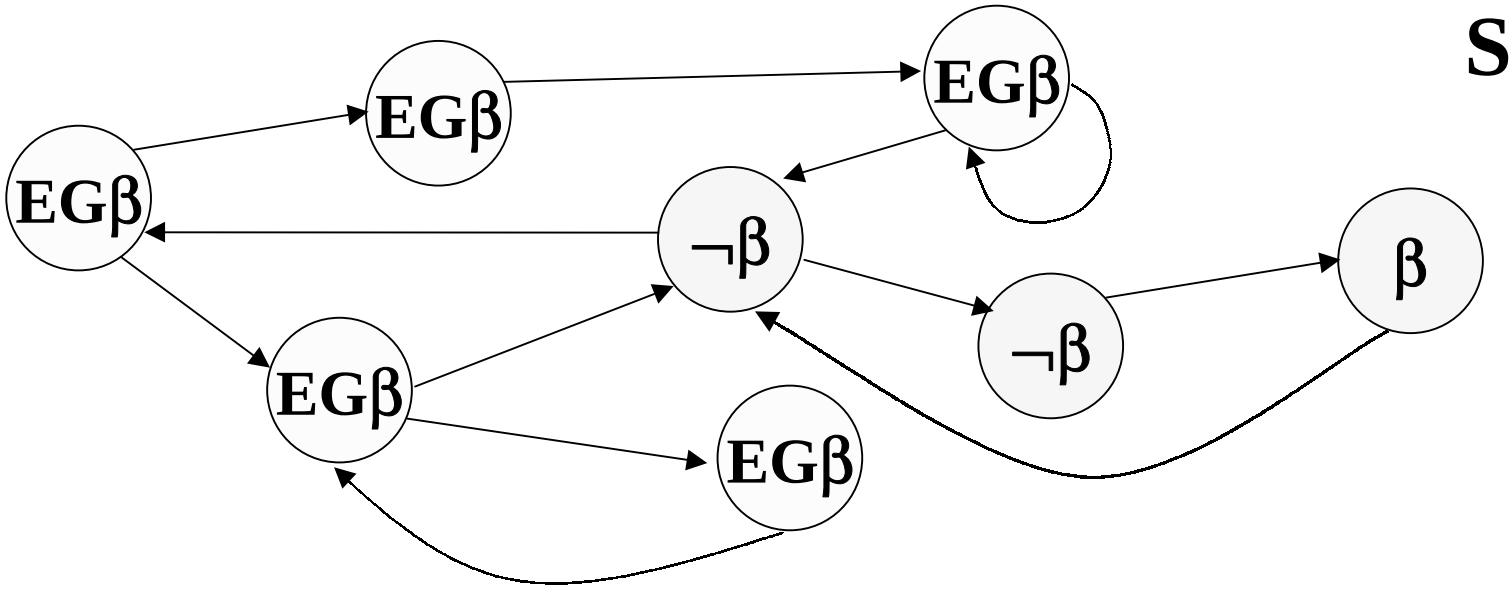
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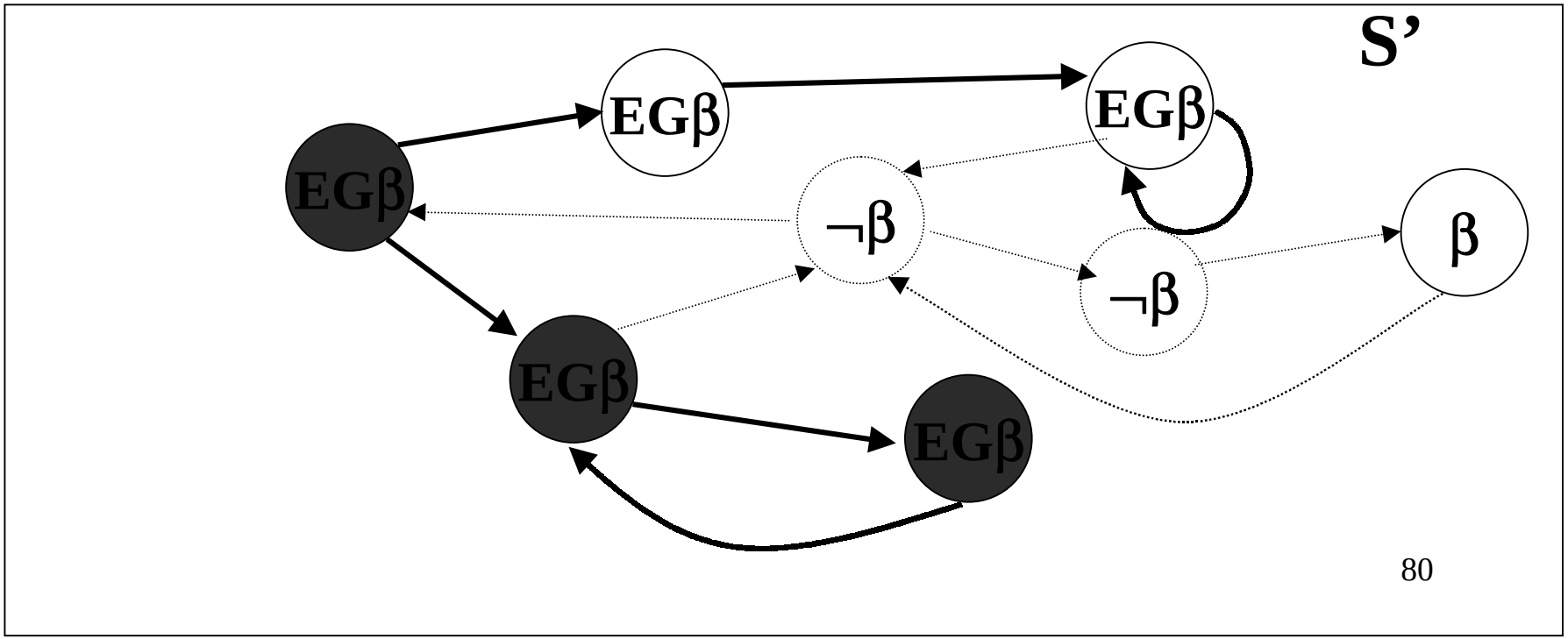
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Computing the labeling for EG(β)

Algorithm Check_EG(β)

Complexity: O(\|M\|)

S' := {**s** | β ∈ Labels(**s**)};

SCC := {**C** | **C** is a non trivial **SCC** of **S'**};

T := $\bigcup_{C \in SCC}$ {**s** | **s** ∈ **C**};

forall **s** ∈ **T** do Labels(**s**) := Labels(**s**) ∪ {**EG(**β)};

while T ≠ ∅ do

chose **s** ∈ T;

T := T \{**s**};

forall **t** ∈ **S'** with **(t,s)**∈ **R** do

if **EG(**β) ∉ Lables**(t**) then

Labels(**t**) := Labels(**t**) ∪ {**EG(**β)};

T := T ∪ {**t**};

CTL model checking

- The algorithms just presented show that the ***model checking problem*** for ***CTL*** can be solved in ***time linear*** in the size of System **M** and the size of the Property ϕ, namely:

in time **O(|M|·|ϕ|)**

where **|M|** is the size of the graph underlying **M** and |ϕ| is the number of subformulae of ϕ.

Fixed point characterization

- We will redefine the labeling function in terms of ***fixed point computation***.
- This is a ***nice*** and ***elegant*** algorithmic account.
- It will be used when ***efficient symbolic approach*** will be introduced.

Partial Orders

- A binary relation $\sqsubseteq$ on a set **A** is a ***partial order*** iff $\sqsubseteq$ is ***reflexive***, ***anti-symmetric*** and ***transitive***.
- The pair $\langle \mathbf{A}, \sqsubseteq \rangle$ is called a ***partially ordered set*** (or ***poset***).
- **Example:** If **S** is any set and $\subseteq$ is the ordinary subset relation, then $\langle \mathbf{2^S}, \subseteq \rangle$ is a *partially ordered set.*

Upper Bounds

Given <**A, ⊑**> and **A’ ⊆ A**

- **a∈ A** is an ***upper bound*** of **A’** iff ∀**a’∈ A’**, **a’ ⊑ a**
- **a∈ A** is a ***least upper bound*** (***lub***) of **A’**, written **⊔A’**, iff
  - **a** is an ***upper bound*** of **A’** and
  - ∀**a’∈ A**, if **a’** is an ***upper bound*** of **A’**, then **a ⊑ a’**

Lower Bounds

Given <**A, ⊑**> and **A’ ⊆ A**

- **a∈ A** is a ***lower bound*** of **A’** iff ∀**a’∈ A’**, **a ⊑ a’**
- **a∈ A** is a ***greatest lower bound*** (***glb***) of **A’**, written **⊓A’**, iff
  - **a** is a ***lower bound*** of **A’** and
  - ∀**a’∈ A**, if **a’** is a ***lower bound*** of **A’**, then **a’ ⊑ a**

Complete Lattice

A ***poset*** **<A, ⊑>** is a ***complete lattice*** if, for each **A’ ⊆ A**, the ***greatest lower bound*** **⊓A’** and the ***least upper bound*** **⊔A’** do exist.

A ***complete lattice*** **<A, ⊑>** has a unique ***least element*** **⊔A=⊥** and a unique ***greatest element*** **⊓A =⊤.**

Complete Lattice

The ***poset*** $<2^S,\subseteq>$ is a ***complete lattice*** where ***intersection*** and ***union*** correspond to $\sqcap$ and $\sqcup$ respectively.

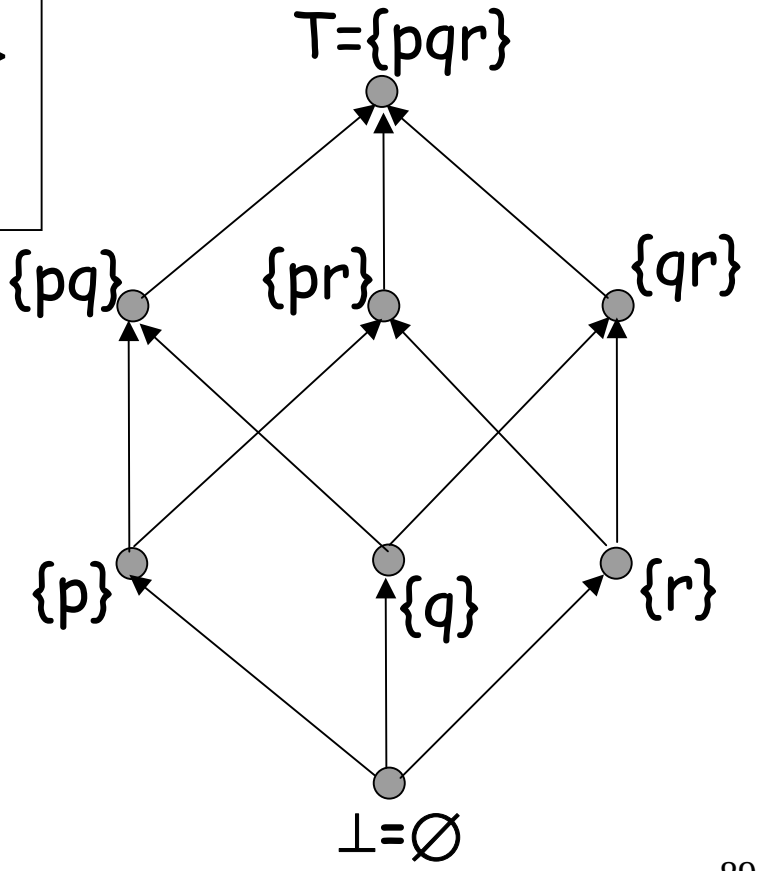
Any two subset of **S** have a ***least upper*** and a ***greatest lower bound***.

EXAMPLE: **S={a,b,c,d}**. For **{a,c}** and **{b,c}** the ***lub*** is **{c}**, while the ***glb*** is **{a,b,c}**.

The unique ***least element*** $\cup 2^S = S$ and a unique ***greatest element*** $\cap 2^S = \varnothing$.

Example of a complete lattice

The complete lattice $<2^S, \subseteq>$ given the set S={p,q,r}



Monotonic functions

- A function **F: A $\rightarrow$ A** is ***monotonic*** if for each **a,b $\in$ A**, **a $\sqsubseteq$ b** implies **F(a) $\sqsubseteq$ F(b)**.
- In other words, a function **F** is monotonic if it ***preserves the ordering*** $\sqsubseteq$.

Fixed points

- Given a function **F: A $\rightarrow$ A**, an element **a $\in$ A** is a ***fixed point*** of **F** if **F(a) = a**.
- **a $\in$ A** is called the ***least fixed point*** of **F (μx.F(x))**, if for all **a' $\in$ A** such that **F(a') = a'**, then **a $\sqsubseteq$ a'**.
- **a $\in$ A** is called the ***greatest fixed point*** of **F (νx.F(x))**, if for all **a' $\in$ A** such that **F(a') = a'**, then **a $\sqsubseteq$ a'**.

Tarski's Fixed Point theorem

THEOREM: Let **<A, ⊑>** be a ***complete lattice***, and **F: A → A** a monotonic function. Then **F** has a ***least*** and a ***greatest fixed point*** given, respectively, by:

- **$\mu x.F(x) = \sqcup\{x \in A \mid x \sqsubseteq F(x)\}$**
- **$\nu x.F(x) = \sqcap\{x \in A \mid F(x) \sqsubseteq x\}$**

Fixed point in finite lattices

Let $\langle \mathbf{A}, \sqsubseteq \rangle$ be a ***finite complete lattice***, and $\mathbf{F}: \mathbf{A} \rightarrow \mathbf{A}$ be a monotonic function.

The ***least element*** of **A**

Then the ***least fixed point*** for **F** is obtained as

$$\mu x.F(x) = F^m(\bot)$$

for some **m**, where $F^0(\bot) = \bot$, and $F^{n+1}(\bot) = F(F^n(\bot))$.

Moreover, the ***greatest fixed point*** for **F** is obtained as

$$\nu x.F(x) = F^k(\top)$$

for some **k**, where $F^0(\top) = \top$, and $F^{n+1}(\top) = F(F^n(\top))$.

The ***greatest element*** of **A**

Generic fixed point algorithm

Algorithm Compute_lfp(**F**:function)

$\mathbf{X_0} := \bot$;

$\mathbf{X_1} := \mathbf{F}(\mathbf{X_0})$;

j=1;

while $\mathbf{X_j} \neq \mathbf{X_{j-1}}$

$\mathbf{j} := \mathbf{j+1}$;

$\mathbf{X_j} := \mathbf{F}(\mathbf{X_{j-1}})$;

return $\mathbf{X_j}$

CTL and complete lattices

- Given a Kripke structure $\mathbf{M=<S,S_0,R,L,AP>}$. We will then consider the ***poset*** $\mathbf{<2^S,\subseteq>}$.
- $\mathbf{< 2^S , \subseteq >}$ is clearly a ***complete lattice*** (with respect to intersection and union).
- We will identify a ***CTL formula*** with the ***set of states*** which ***satisfy it***.
- In this way we can define ***temporal operators*** as ***functions*** on the ***complete lattice*** $\mathbf{<2^S,\subseteq>}$.

Denotation of a CTL formula

- Given a formula ϕ, let us define its ***denotation*** (in **M**), in symbols $|[\phi]|$, as the set of states satisfying the formula:

$$|[\phi]| = \{ s \mid M,s \models \phi \}$$

- We could then define the cpo $<\mathit{CTL}, \sqsubseteq >$ by:

$$\phi \sqsubseteq \psi \quad \textit{iff} \quad |[\phi]| \subseteq |[\psi]|$$

Denotation of a CTL formula

- Given the ***denotation*** of a formula

$$|[\phi]| = \{ s \mid M,s \vDash \phi \}$$

- We could then define the cpo $\langle \mathit{CTL}, \sqsubseteq \rangle$ by:

$$\phi \sqsubseteq \psi \quad \mathit{iff} \quad |[\phi]| \subseteq |[\psi]|$$

- Then $|[\bot]| = \varnothing$; $|[\top]| = S$;
- $|[p]| = \{ s \mid p \in L(s) \}$;
- $|[\neg\phi]| = S \setminus |[\phi]|$;
- $|[\phi \vee \psi]| = |[\phi]| \cup |[\psi]|$;
- $|[\phi \wedge \psi]| = |[\phi]| \cap |[\psi]|$;

CTL is closed under ***conjunction*** and ***disjunction***, therefore for any pair of formulae the ***upper*** and ***lower bound*** do exist.

Denotation of a CTL formula

- Given a formula ϕ, let us define its ***denotation*** (in **M**), in symbols $|[\phi]|$, as the set of states satisfying the formula:

$$|[\phi]| = \{ s \mid M,s \models \phi \}$$

- **…**
- $|[EX\phi]| = \{ s \mid \exists t. (t \in |[\phi]| \cap R(s)) \}$
- for the other **temporal operators** we would need to use **fixed points…**

Fixed point characterization of $\mathbf{EU(\beta_1,\beta_2)}$

- $\mathbf{EU(\beta_1,\beta_2) \equiv \beta_2 \vee (\beta_1 \wedge EX\ EU(\beta_1, \beta_2))}$
- $\mathbf{|[EU(\beta_1,\beta_2)]| = \mu Z.(|[\beta_2]| \cup (|[\beta_1]| \cap |[EX\ Z]|))}$
- $\mathbf{|[EU(\beta_1,\beta_2)]| =}$

  $\mathbf{\mu Z.(|[\beta_2]| \cup (|[\beta_1]| \cap \{ s \mid \exists t \in Z \cap R(s) \}))}$

Fixed point characterization of $\mathbf{EU(\beta_1,\beta_2)}$

Lemma: Let

$$\mathbf{F(Z) = (|[\beta_2]| \cup (|[\beta_1]| \cap \{\, s \mid \exists t \in Z \cap R(s) \,\}))}$$

then **F** is a ***monotonic function***, i.e.

$$\mathbf{Z_1 \subseteq Z_2 \text{ implies } F(Z_1) \subseteq F(Z_2)}$$

Fixed point characterization of $\mathbf{EU(\beta_1,\beta_2)}$

Theorem:

$$|[EU(\beta_1,\beta_2)]| = \mu Z.(|[\beta_2]| \cup (|[\beta_1]| \cap \{ s \mid \exists t \in Z \cap R(s) \}))$$

in other words:

$$\mu Z.(|[\beta_2]| \cup (|[\beta_1]| \cap \{ s \mid \exists t \in Z \cap R(s)\})) \subseteq |[EU(\beta_1,\beta_2)]|$$

and

$$|[EU(\beta_1,\beta_2)]| \subseteq \mu Z.(|[\beta_2]| \cup (|[\beta_1]| \cap \{ s \mid \exists t \in Z \cap R(s)\}))$$

Computing fixed point for $EU(\beta_1,\beta_2)$

Algorithm Compute_$EU(\beta_1,\beta_2)$

$\mathbf{X_0} := |[\bot]|$; /* i.e. $\mathbf{X_0} := \emptyset$ */

$\mathbf{X_1} := |[\beta_2]| \cup (|[\beta_1]| \cap \mathbf{X_0})$; /* i.e. $\mathbf{X_1} := |[\beta_2]|$ */

j=1;

while $\mathbf{X_j} \neq \mathbf{X_{j-1}}$

$\mathbf{j} := \mathbf{j+1}$; $\mathbf{X} := \mathbf{T} := \mathbf{X_{j-1}}$;

while $\mathbf{T} \neq \emptyset$ do

chose $\mathbf{s} \in \mathbf{T}$;

$\mathbf{T} := \mathbf{T} \setminus \{\mathbf{s}\}$;

forall $\mathbf{t} \in \mathbf{S}$ do

if $\mathbf{s} \in \mathbf{R(t)}$ then $\mathbf{X} := \mathbf{X} \cup \{\mathbf{t}\}$;

$\mathbf{X_j} := |[\beta_2]| \cup (|[\beta_1]| \cap \mathbf{X})$

This computes

$\mathbf{X_j} := \mathbf{F(X_{j-1})}$;

Computing fixed point for $\mathbf{EU(\beta_1,\beta_2)}$

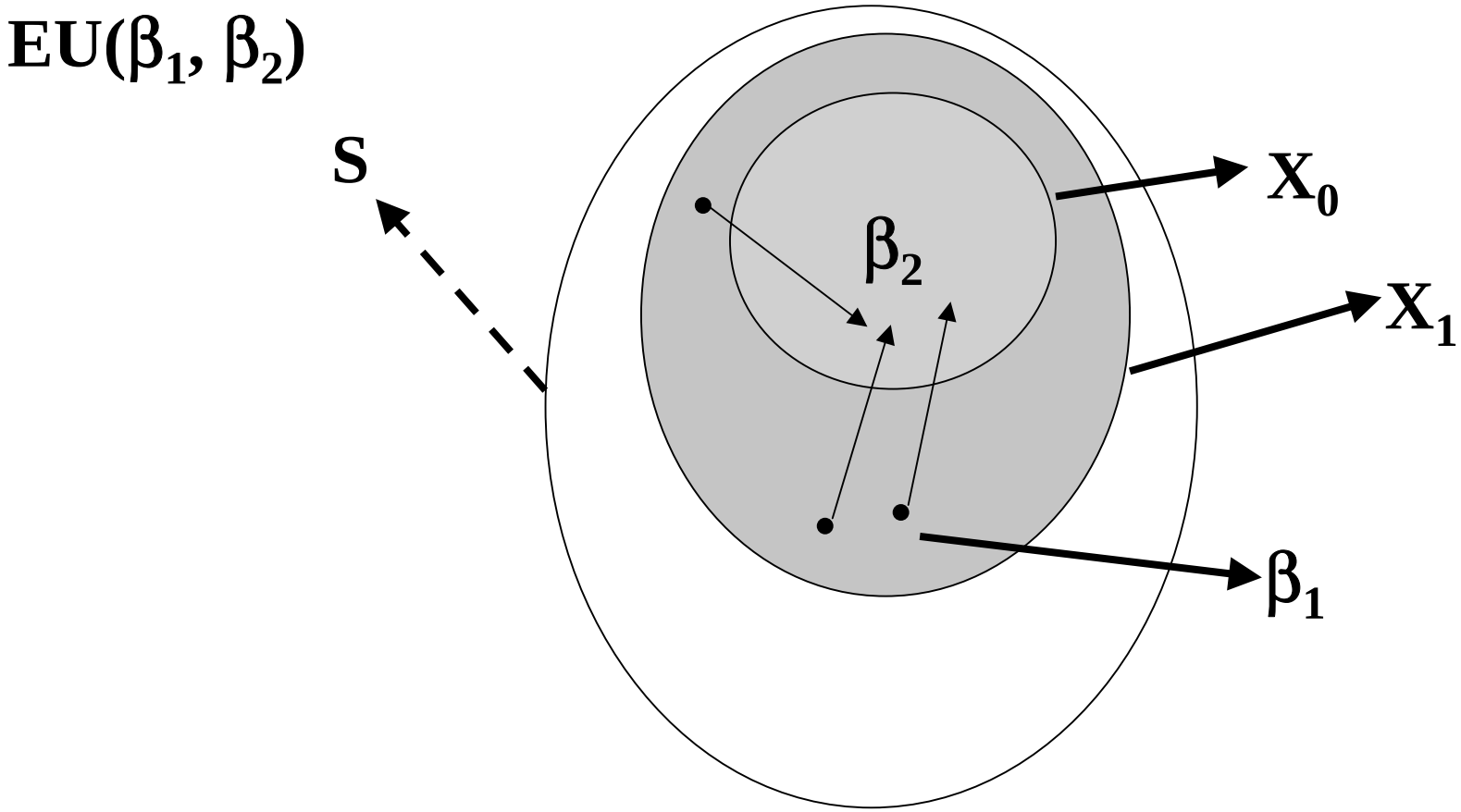
To compute $\mathbf{|[EU(\beta_1,\beta_2)]|}$ we can ***construct inductively the set*** of states $\mathbf{X_j}$ as follows:

- $\mathbf{X_1 = \{s \mid s \in |[\beta_2]|\}}$.
- $\mathbf{X_{j+1} = X_j \cup \{ s \mid s \in |[\beta_1]|}$ and $\mathbf{R(s,t)}$ for some $\mathbf{t \in X_j\}}$

$\mathbf{|[EU(\beta_1,\beta_2)]|}$ is then the set $\mathbf{X}$ such that $\mathbf{X = X_n}$ for $\mathbf{n}$ such that $\mathbf{X_{n+1} = X_n}$.

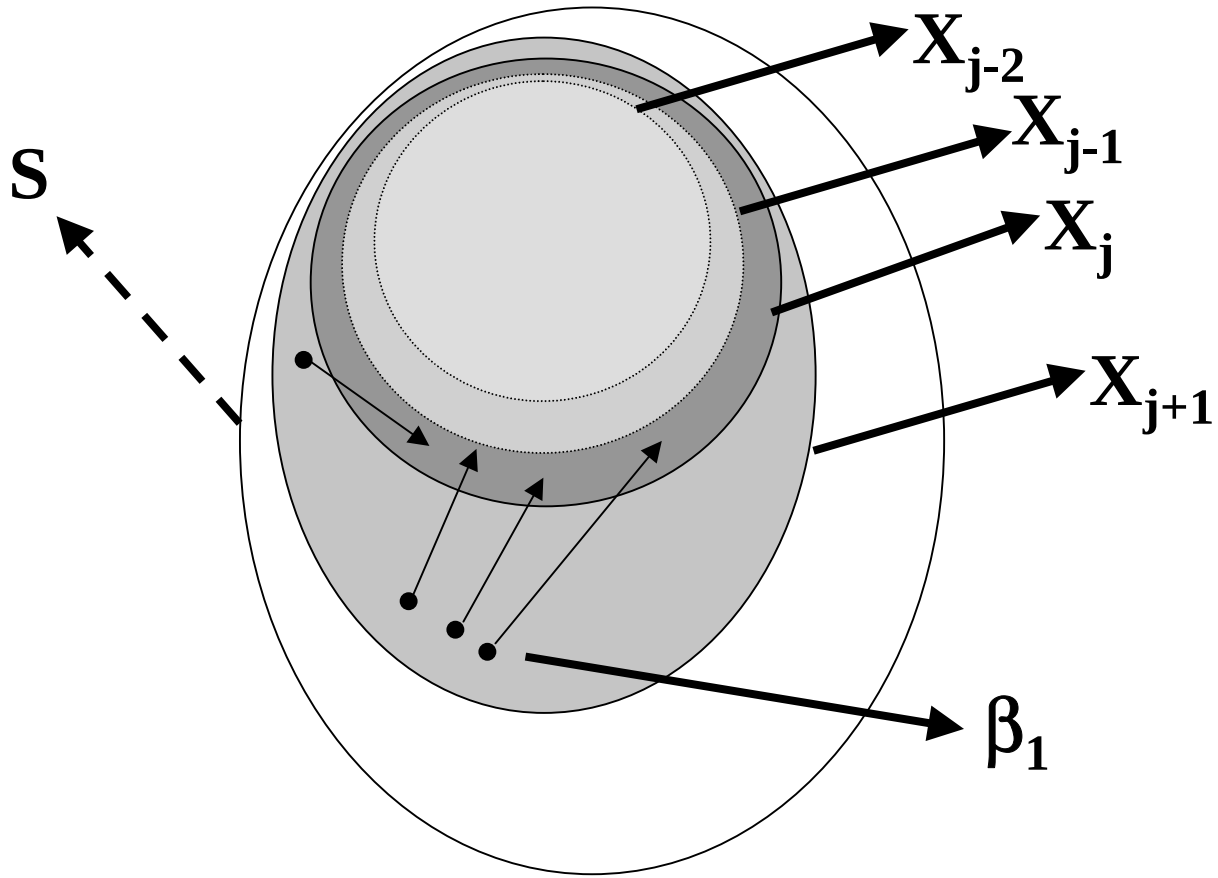
Notice that $\mathbf{n}$ ***must exist*** by ***Tarski's Theorem*** since $\mathbf{X_j \subseteq X_{j+1} \subseteq S}$.

From X_0 to X_1



From X_j to X_{j+1}

EU(β_1, β_2)



Computing fixed point for EU(β_1,β_2)

Algorithm Compute_EU(β_1,β_2)

$\mathbf{X_1} := |[\beta_2]|$;

j=1;

repeat

j := **j+1**; **X** := **T** := $\mathbf{X_{j-1}}$;

while **T** $\neq \emptyset$ do

chose **s** $\in$ **T**;

T := **T** \{**s**};

forall **t** such that **s** $\in$ **R(t)** do

if **t** $\in |[\beta_1]|$ then **X** := **X** $\cup$ {**t**};

$\mathbf{X_j}$:= **X**

until $\mathbf{X_{j-1}} = \mathbf{X_j}$

Fixed point characterization of EG(β)

- **EG(β) ≡ β ∧ EX EG(β)**
- **|[EG(β)]| = νZ.(|[β]| ∩ |[EX Z]|))**
- **|[EG(β)]| =**

  νZ.(|[β]| ∩ { s | ∃t ∈ Z ∩ R(s) })

Computing the fixed point for EG(β)

Algorithm Compute_EG(β)

$\mathbf{X_0} := \mathbf{|[\top]|}$; /* i.e. $\mathbf{X_0} := \mathbf{S}$ */

$\mathbf{X_1} := \mathbf{|[\beta]|} \cap \mathbf{X_0}$; /* i.e. $\mathbf{X_1} := \mathbf{|[\beta]|}$ */

j=1;

while $\mathbf{X_j} \neq \mathbf{X_{j-1}}$

$\mathbf{j} := \mathbf{j+1}$; $\mathbf{T} := \mathbf{X_{j-1}}$; $\mathbf{X_j} := \emptyset$;

while $\mathbf{T} \neq \emptyset$ do

chose $\mathbf{s} \in \mathbf{T}$;

$\mathbf{T} := \mathbf{T} \setminus \{\mathbf{s}\}$;

forall $\mathbf{t} \in \mathbf{S}$ do

if $\mathbf{s} \in \mathbf{R(t)}$ then $\mathbf{X_j} := \mathbf{X_j} \cup \{\mathbf{t}\}$;

$\mathbf{X_j} := \mathbf{|[\beta]|} \cap \mathbf{X_j}$

The Labels function

- To compute $|[\mathbf{EG}\beta]|$ we can ***construct inductively the set*** of states $\mathbf{X_j}$ as follows:

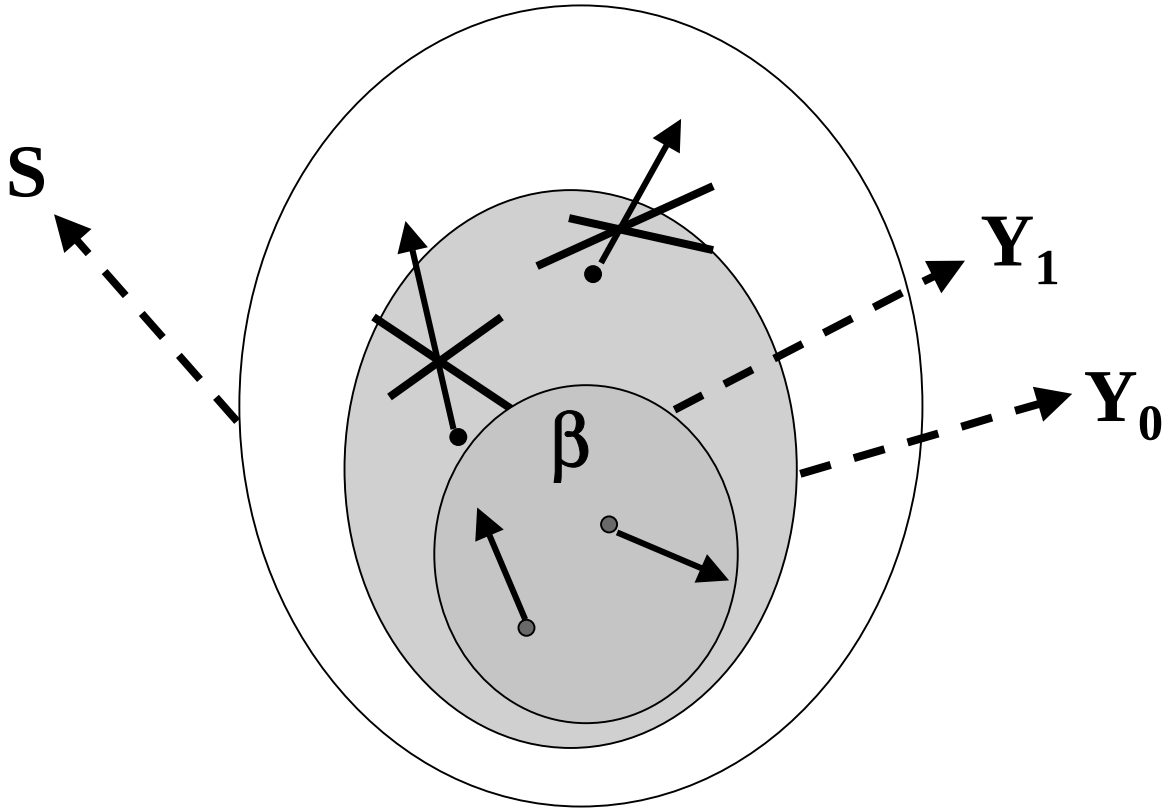
  - $\mathbf{X_1} = |[\beta]|$.
  - $\mathbf{X_{j+1}} = \mathbf{X_j} - \{ \mathbf{s} \mid \mathbf{s} \in \mathbf{X_j}$ and
    there does not exist $\mathbf{t} \in \mathbf{X_j}$
    such that $\mathbf{R(s, t)}\}$

$|[\mathbf{EG}\beta]|$ is then the set $\mathbf{X}$ such that $\mathbf{X} = \mathbf{X_n}$ for $\mathbf{m}$ such that $\mathbf{X_{n+1}} = \mathbf{X_n}$.

- Notice that $\mathbf{m}$ ***must exists*** by ***Tarski's Theorem*** since $\varnothing \subseteq \mathbf{X_{j+1}} \subseteq \mathbf{X_j}$

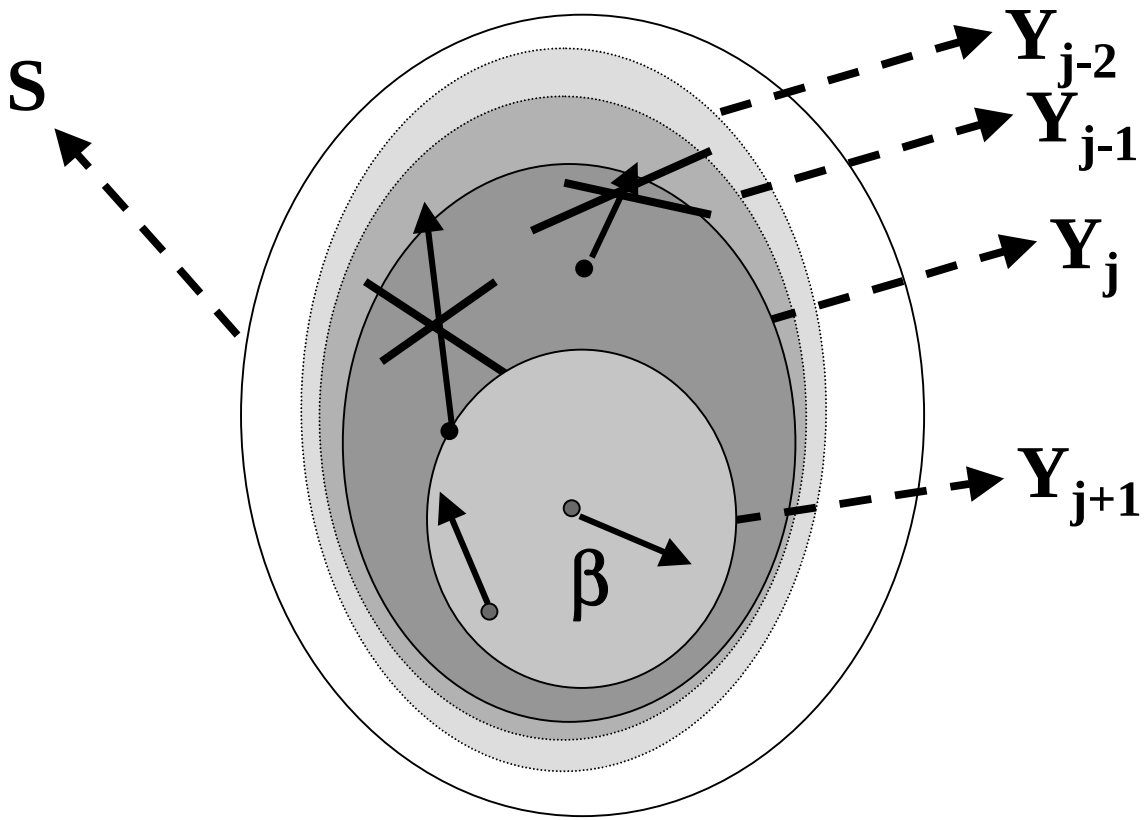
From Y_0 to Y_1

EGβ



From Y_j to Y_{j+1}

EGβ



Computing the fixed point for EG(β)

Algorithm Compute_EG(β)

$\mathbf{X_1}$:= **|[β]|**;

j=1;

repeat

j := **j+1**; T := $\mathbf{X_j}$:= $\mathbf{X_{j-1}}$;

while T ≠ ∅ do

chose **s** ∈ T;

T := T \{**s**};

if for no **t** ∈ **R(s)**, **t** ∈ $\mathbf{X_{j-1}}$ then

$\mathbf{X_j}$:= $\mathbf{X_j}$ - {**s**};

until $\mathbf{X_j}$ = $\mathbf{X_{j-1}}$;