Tecniche di Specifica e di Verifica

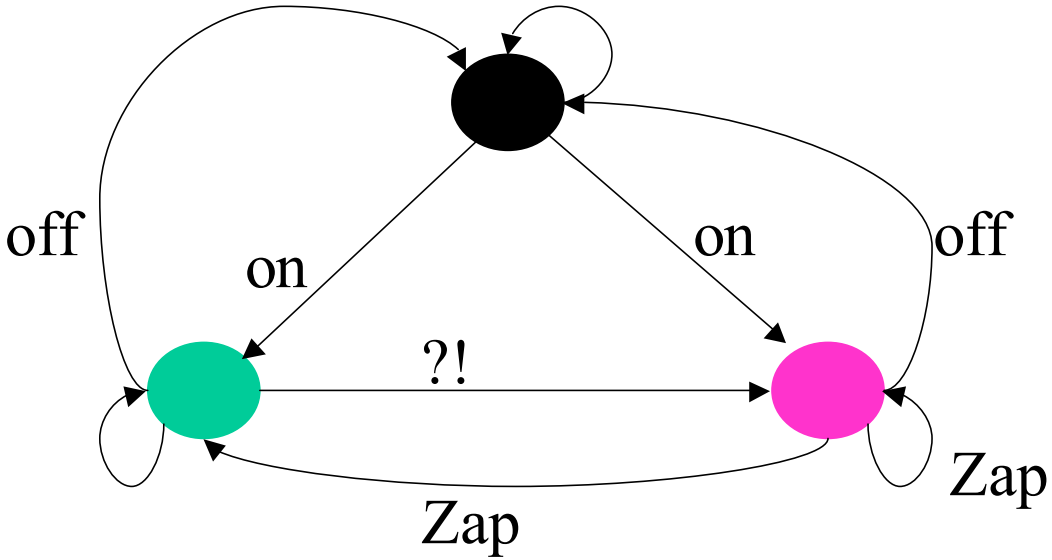
Branching Time Temporal Logics I

Outline

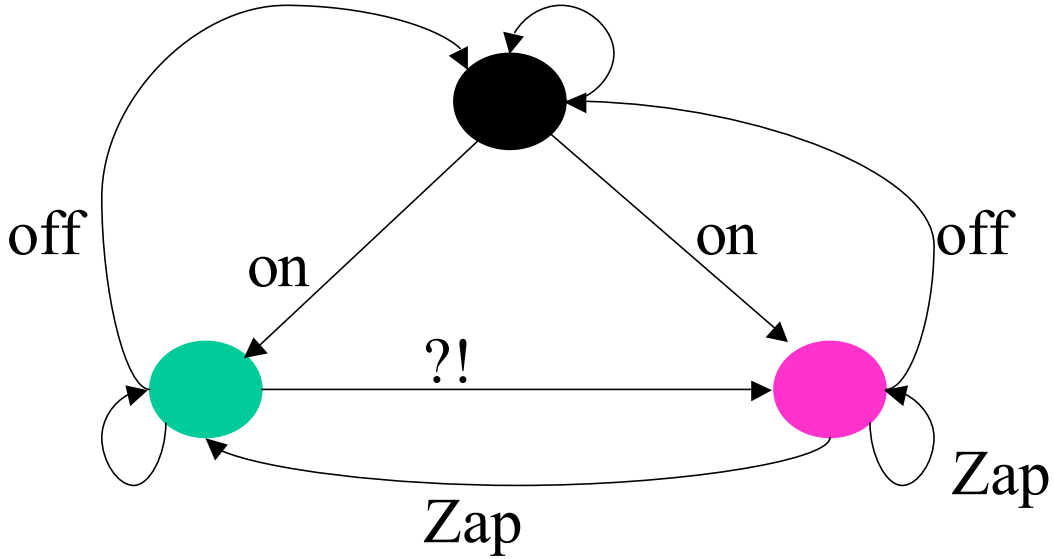
- ***CTL*** (**C**omputation **T**ree **L**ogic)
  - **Branching Time**
  - Unwindings --- computation trees
  - Syntax and semantics of CTL.

Branching Time Structures

- **Linear Time**:
  - A ***computation*** at its first state satisfies a property.
  - Property ---- **LTL** formula
- **Branching Time**
  - The ***computation tree*** at its root satisfies a property.
  - Property: **CTL** (**CTL***, **m-calculus**) formula.
  - **Computation Tree**
    - **All *computations*** starting from a state ***glued together*** (to form a tree structure).
- In branching time, ***the decisions*** taken during a run are taken into account.



The TV Example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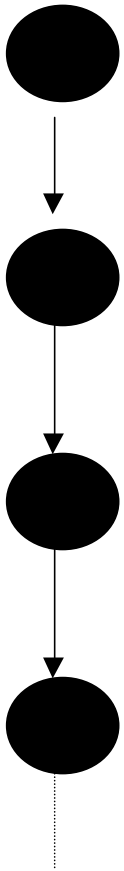

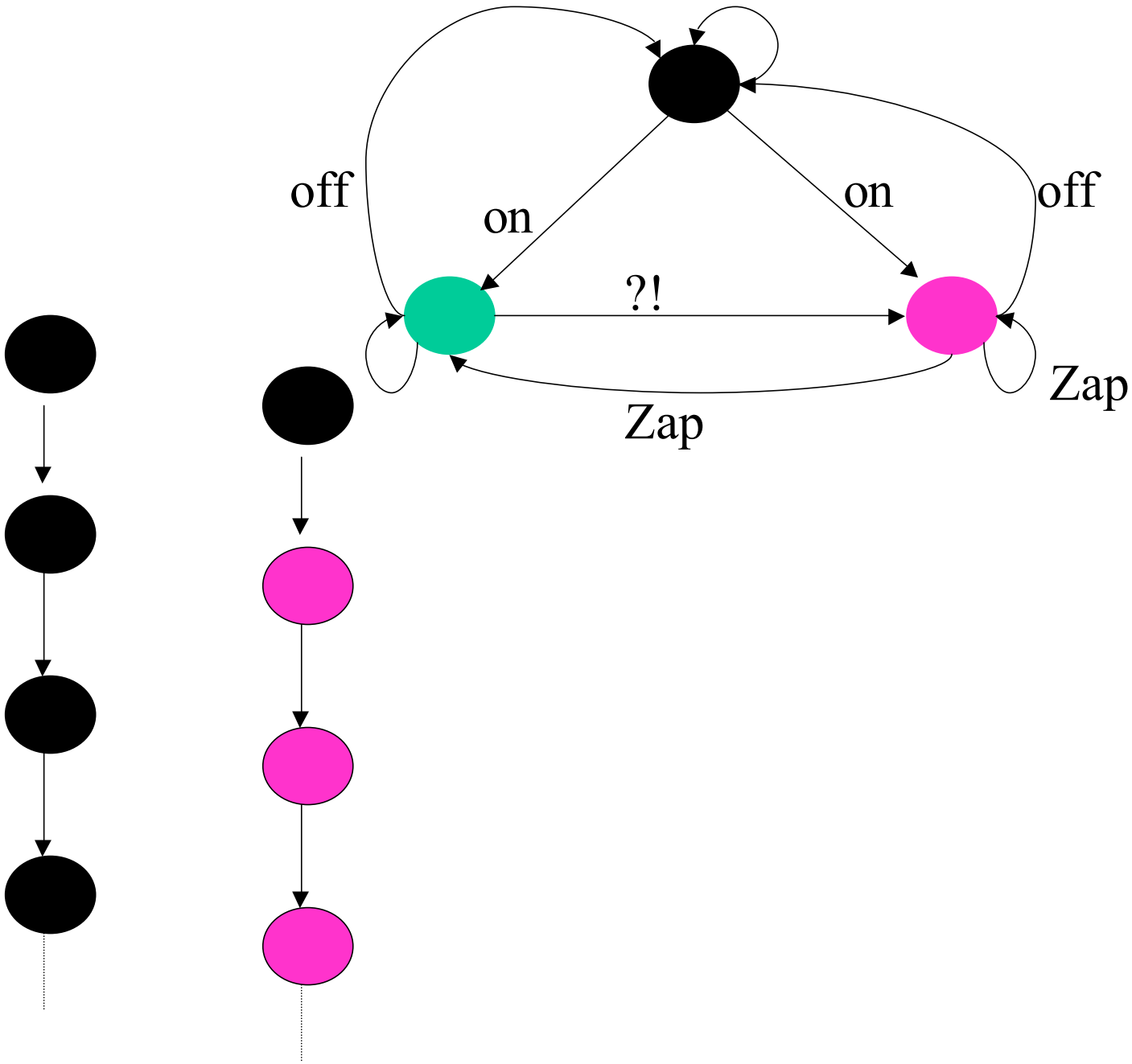
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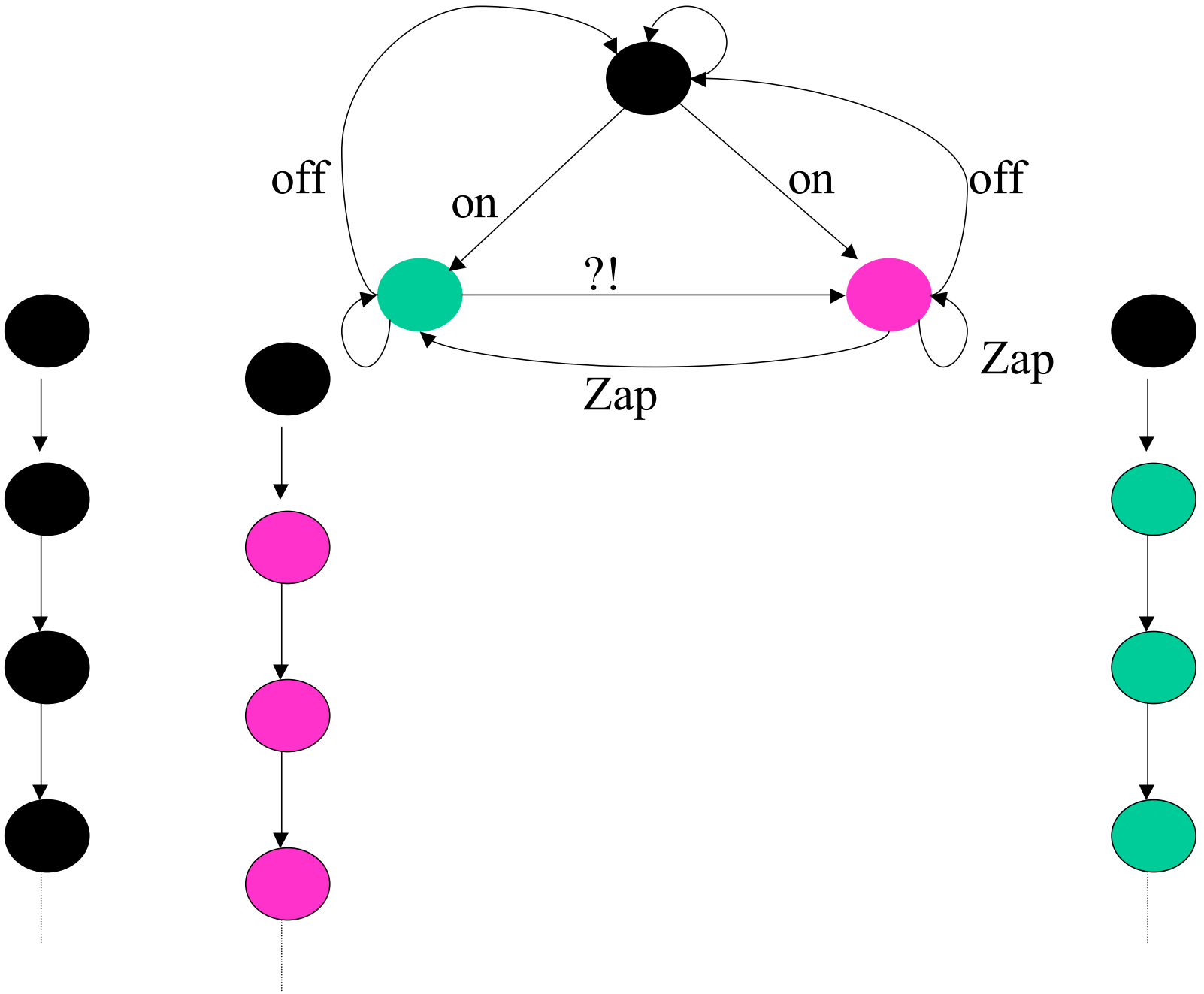
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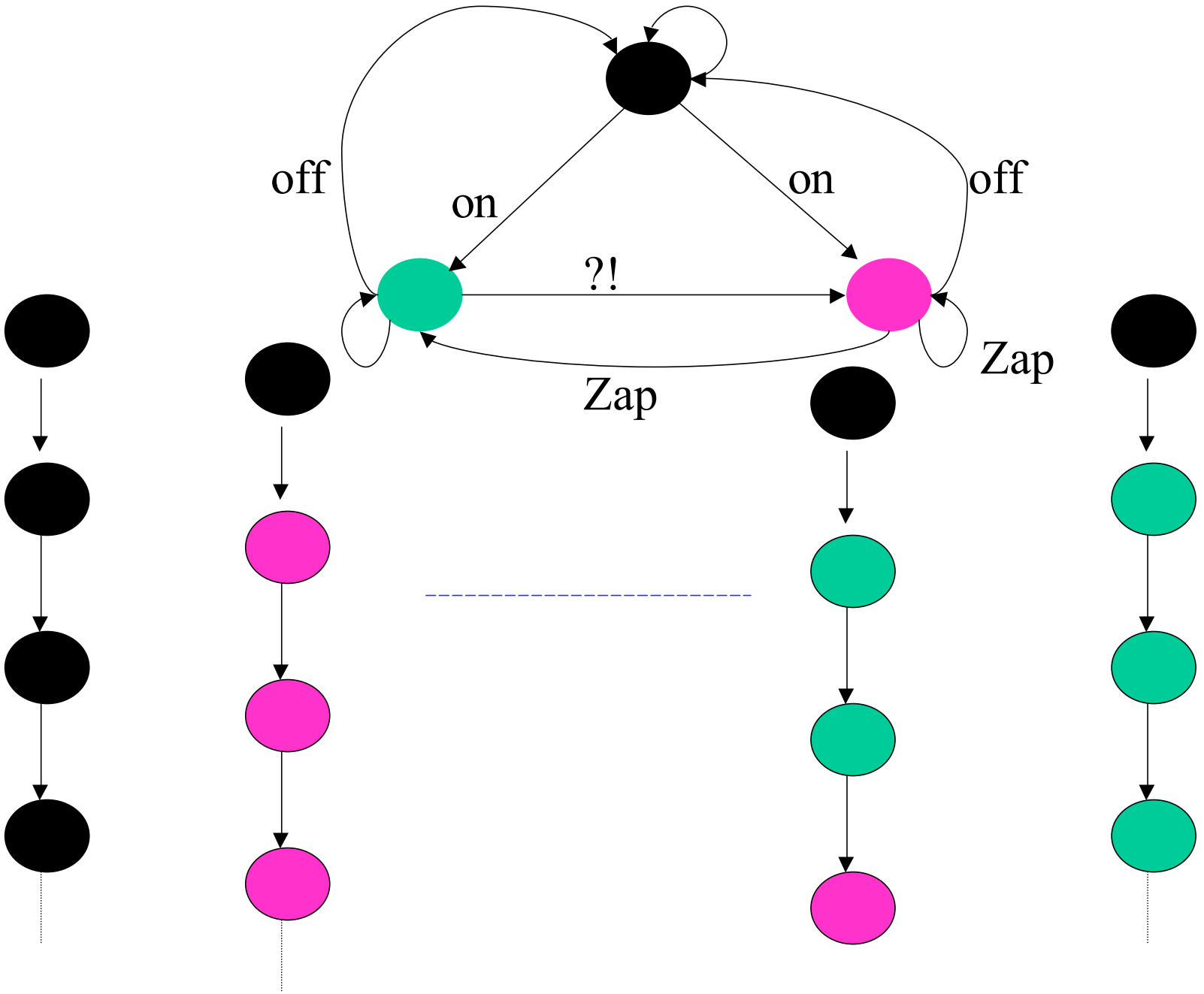
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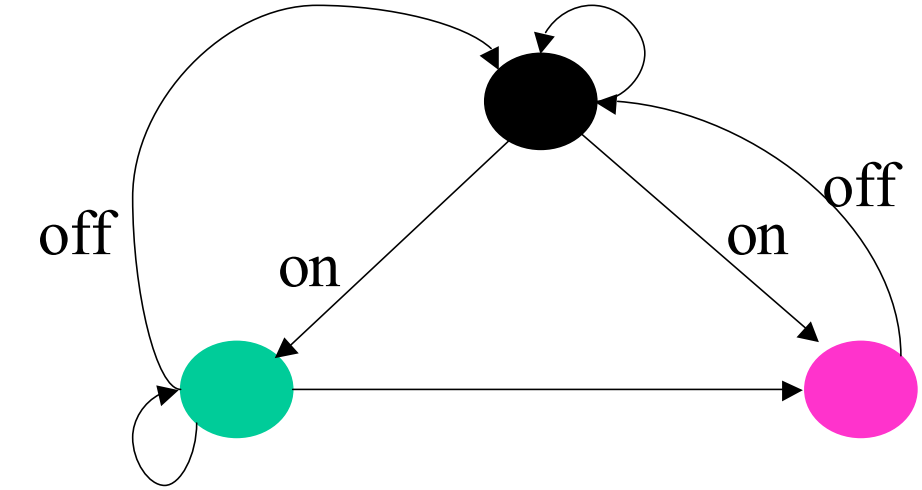
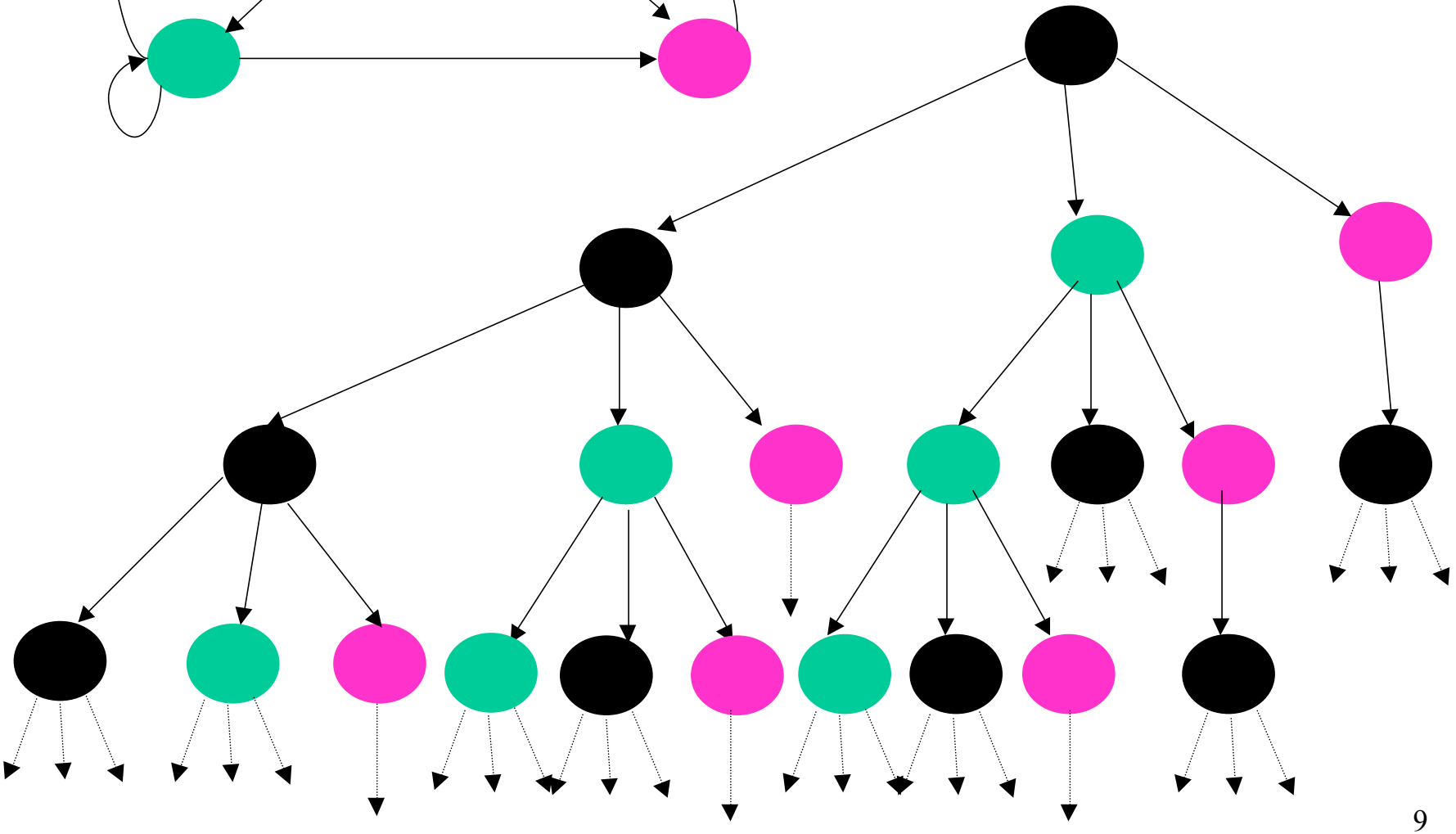
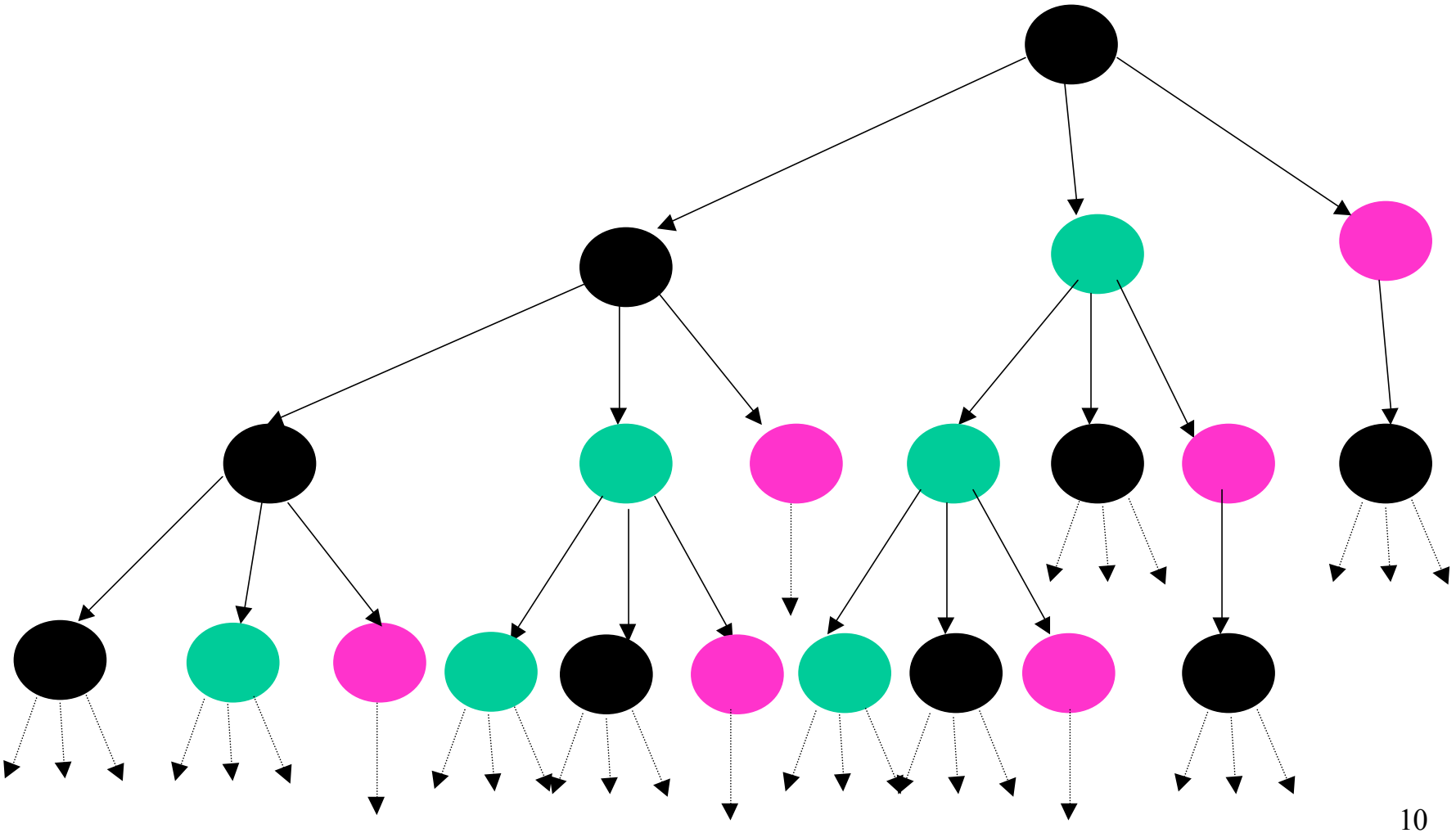
A Modified Example





For every path p and every state **s** on that path, there is a path p' starting from **s** and a state **s'** on p' which is **green**.



Branching Time Temporal Logic

- **K = (S, S_0, R, AP, L)**
- **K, s ⊨ y** -- the computation tree rooted at **s** satisfies y.
- **K ⊨ y** iff **K, s_0 ⊨ y** for every s_0 Î S_0.
- **Branching Time Temporal Logics**:
  - **CTL**
  - **CTL***
  - (The modal) **m-calculus**

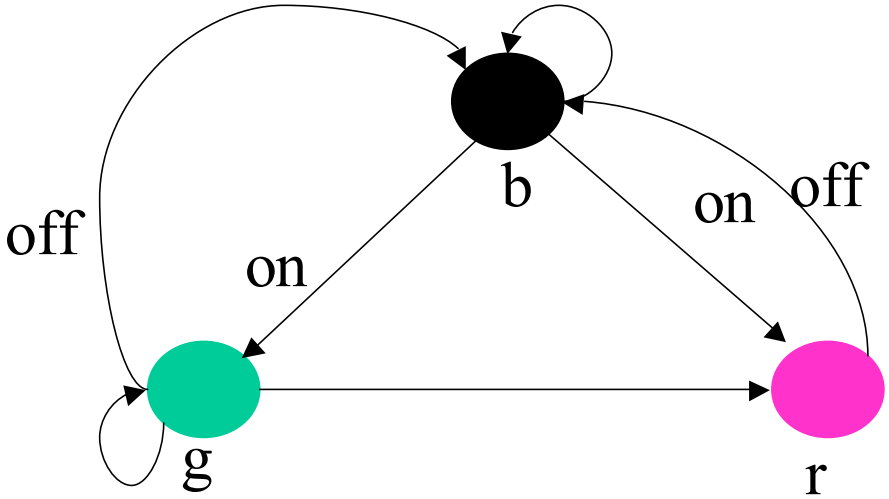
Unwinding

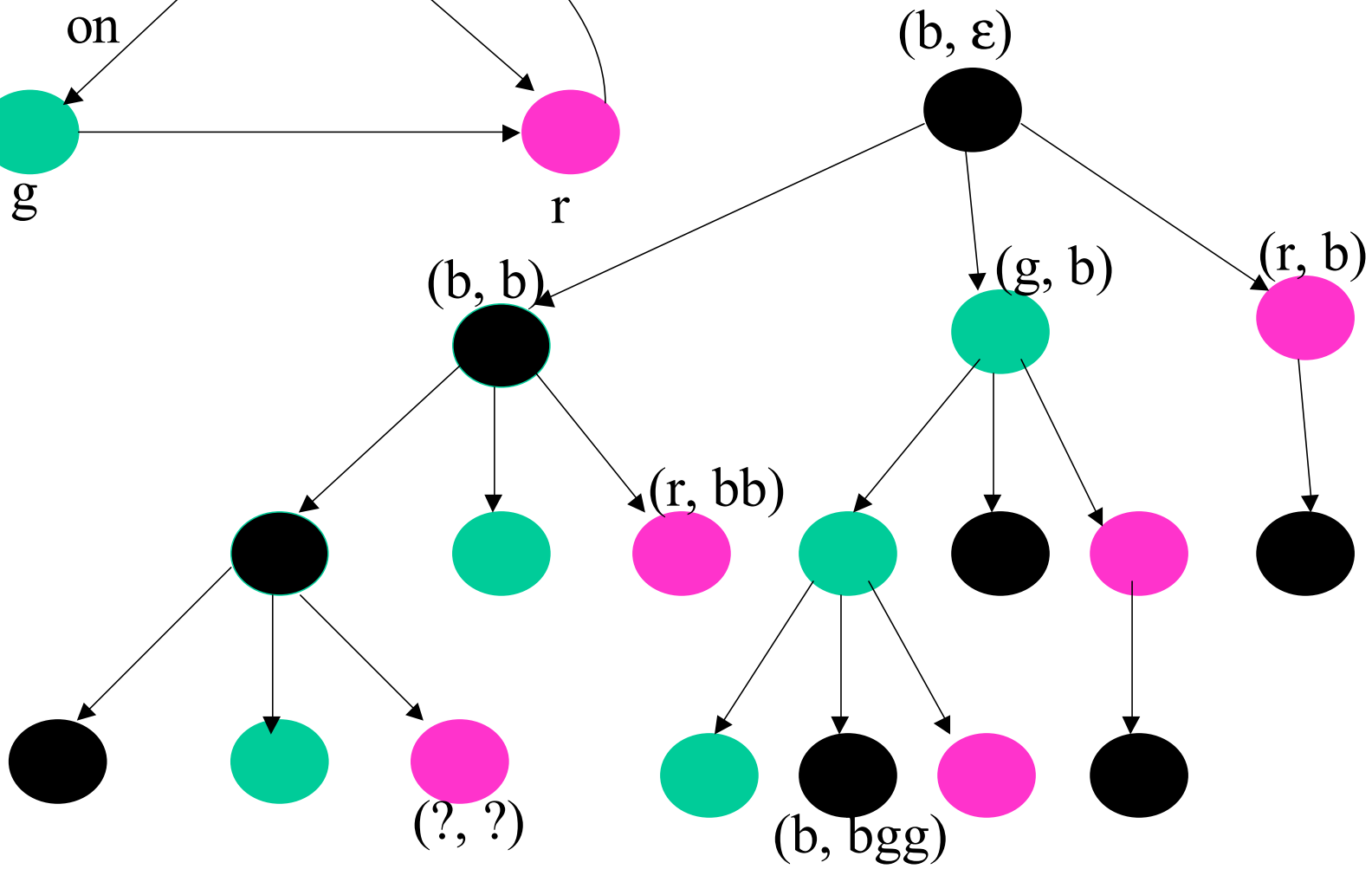
- $K = (S, S_0, R, AP, L)$ $s \in S$
- **TR(K, s)** --- The computation tree rooted at **s**.
- $TR(K, s) = (\mathcal{S}_s, (s,e), \mathcal{R}_s, AP, \mathcal{L}_s)$.
  - $(s, e) \in \mathcal{S}_s$;
  - For any $(s', s) \in \mathcal{S}_s$, $s' \in S$ and $s = s\, s_1 \ldots s_n$ is a path in **K** leading from **s** to **s'** (i.e. $s.s'$ is a path from **s** to **s'**);
  - If $(s_1, s) \in \mathcal{S}_s$ and $R(s_1, s_2)$ then $(s_2, s.s_1) \in \mathcal{S}_s$ and $\mathcal{R}_s((s_1, s), (s_2, s.s_1))$;
  - $\mathcal{L}((s_1, s)) = L(s_1)$.

Unwinding

- **TR(K, s)** is almost a Kripke structure.
  - $\mathcal{S}_s$ may be infinite
  - But $\mathcal{R}_s$ is ***tree-like***.
  - The "***graph***" of **TR(K, s)** is a tree rooted at **(s,** e**)**.
- **TR(K, s)** is the ***computation tree rooted*** at **s**.

A Modified Example





Linear time Vs Branching time

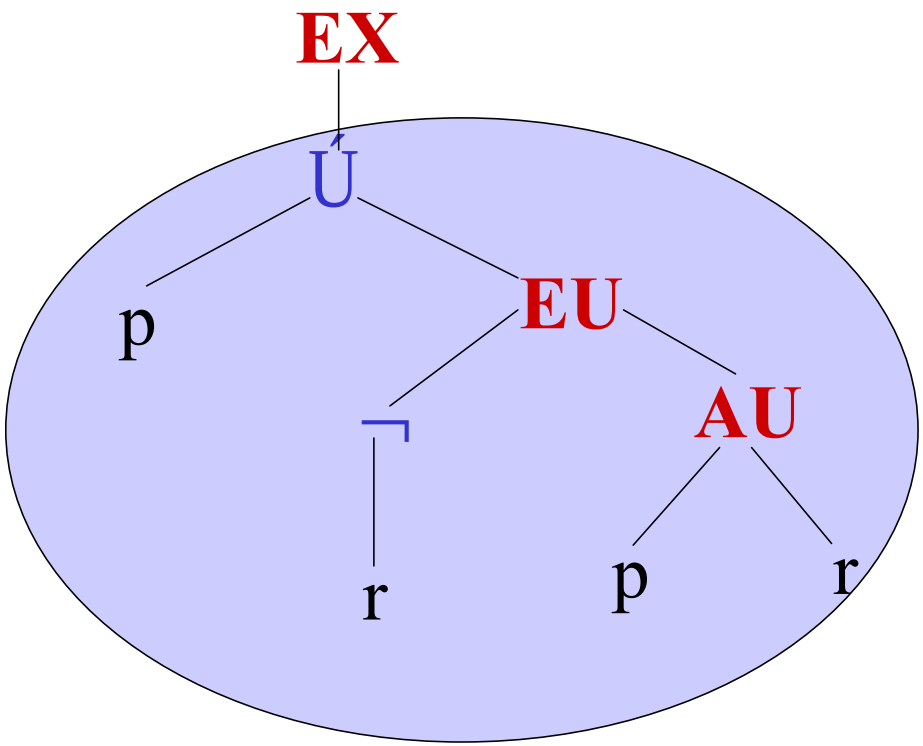
- There are ***properties*** that can be ***expressed in* LTL** but which ***can not be expressed in* CTL**. (sloppy statement!)
- There are ***properties*** that can be ***expressed in* CTL** but ***not in* LTL**.
- The **LTL *model checking problem*** can be ***converted*** into a ***restricted*** kind of a **CTL*** ***model checking problem***.

CTL

- **Syntax**
  - **AP** – a finite set of ***atomic propositions***.
  - **p Î AP** is a formula.
  - If y and y' are formulas then so are Øy and y Ú y'.
  - If y is a formula then so is **EX**y
  - If y_1 and y_2 are formulas then so are **EU**(y_1, y_2) and **AU**(y_1, y_2).

Formulas

- **EX**(p Ú **EU**(Ø**r**, **AU**(p, r)))



Semantics

- **K = (S, S_0, R, AP, L)**
  - **L : S $\longrightarrow$ 2^{AP}**
- **y** a **CTL** formula and **s Î S**
- **K, s ⊨ y**
- **y** (holds) ***is satisfied*** at **s**.
- **FACT**:

K, s ⊨ y iff **TR(K, s), (s, e) ⊨ y**.

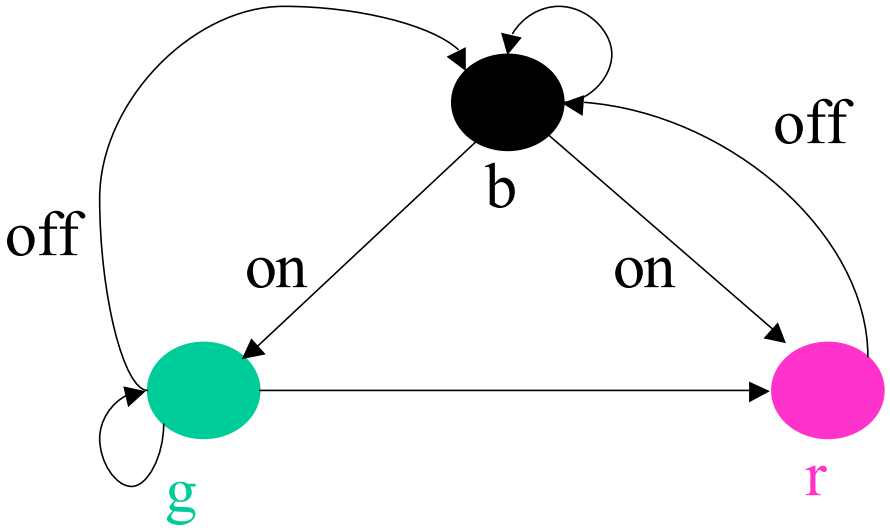
Semantics

- **CTL ::= p | Ø y | y_1 Ú y_2 | EX(y) |**
  | EU(y_1, y_2) | AU(y_1, y_2)
- **K = (S, S_0, R, AP, L) ; L: S ⟶ 2^{AP} ; s Î S**
- **K, s ⊨ p** iff **p Î L(s).**
- **K, s ⊨ Ø y** iff **not K, s ⊨ y**
- **K, s ⊨ y_1 Ú y_2** iff
  K, s ⊨ y_1 or **K, s ⊨ y_2.**

Semantics

- **CTL ::= p | Ø y | y_1 Ú y_2 | EX(y) |**
  | EU(y_1, y_2) | AU(y_1, y_2)
- **K = (S, S_0, R, AP, L) ; L: S ⟶ 2^{AP} ; s Î S**
- **K, s ⊨ EX(y)** iff there exists **s'** such that:
  - **s ⟶ s'** (i.e. **R(s, s')**) and **K, s' ⊨ y**

  s has a successor state **s'** at which **y** holds.



AP = {n, h, uh}

K, b ⊨ EX(uh) ? K, b ⊨ EX(∅uh) ?

K, g ⊨ EX(uh) ?

K, r ⊨ EX(h) ?

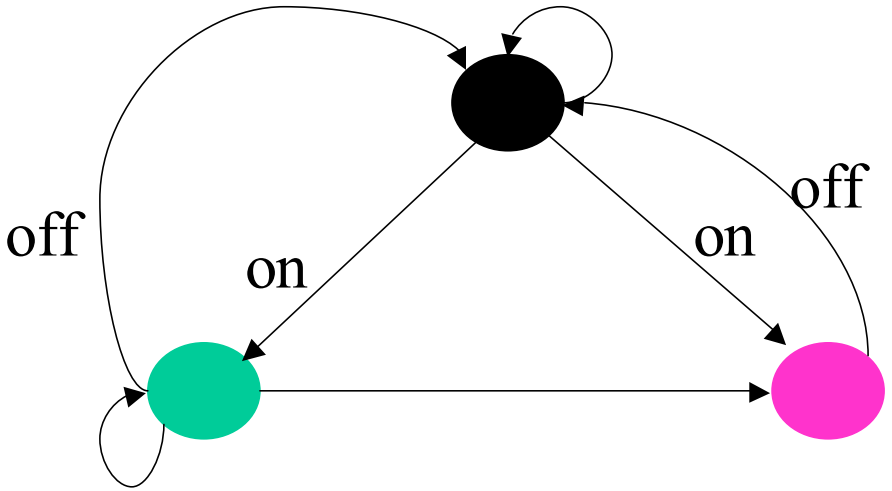
Semantics

- $K = (S, S_0, R, AP, L)$; $L: S \longrightarrow 2^{AP}$; $s \hat{I} S$
- ***A path from*** **s** is a (infinite) sequence of states

  $p = s_0, s_1, s_2, \ldots, s_i, s_{i+1}, \ldots$ s.t:

  – $s = s_0$

  – $s_i \longrightarrow s_{i+1}$ (i.e. $R(s_i, s_{i+1})$) for every **i**.
- $p(i) = s_i$ the **i**-th element of p.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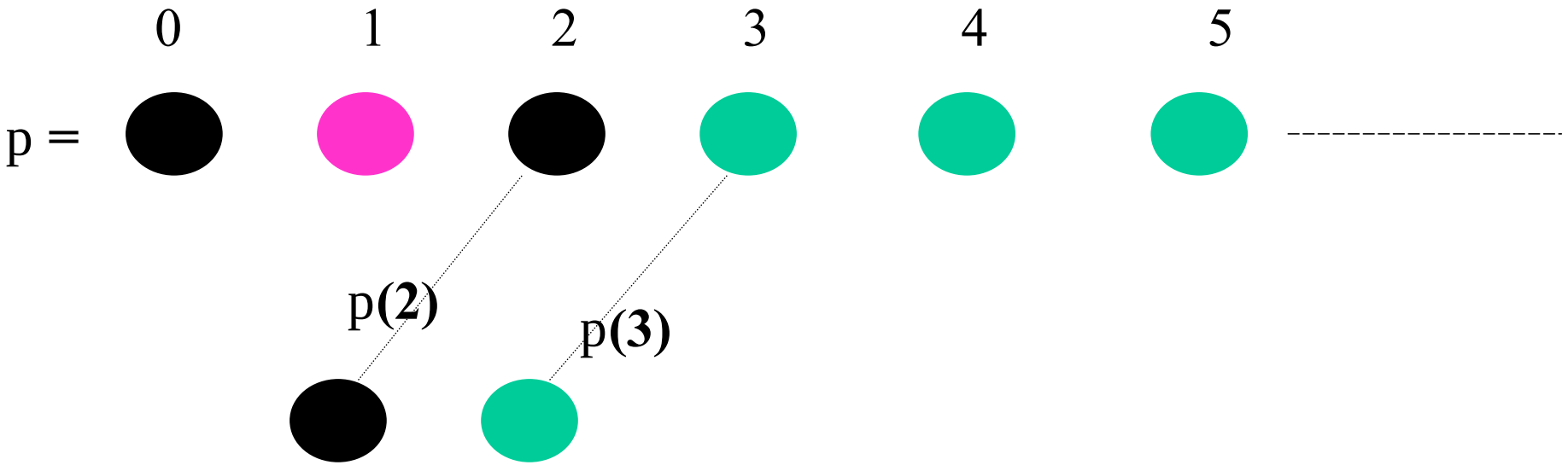
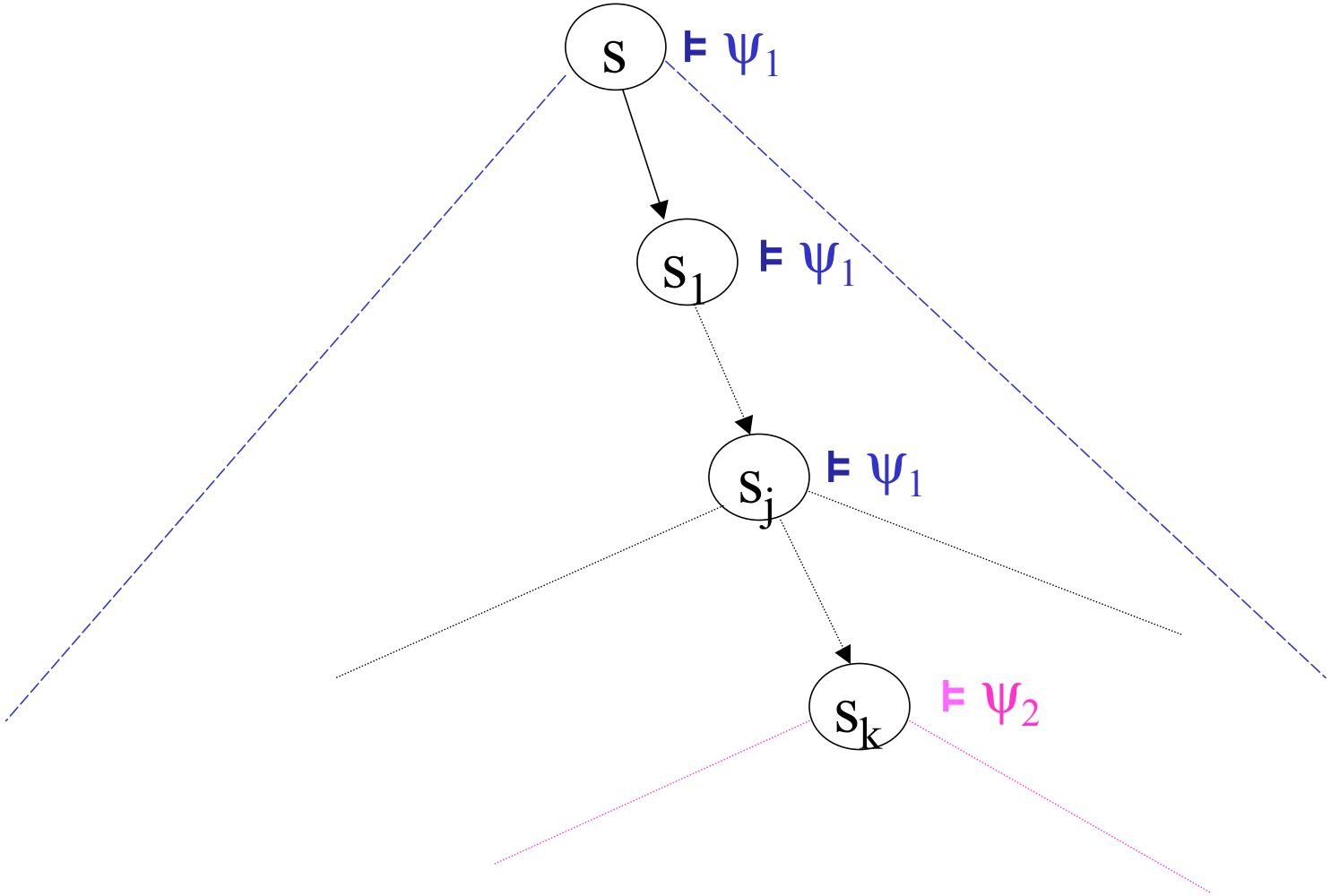

0
1
2
3
4
5
p =
p(2)
p(3)

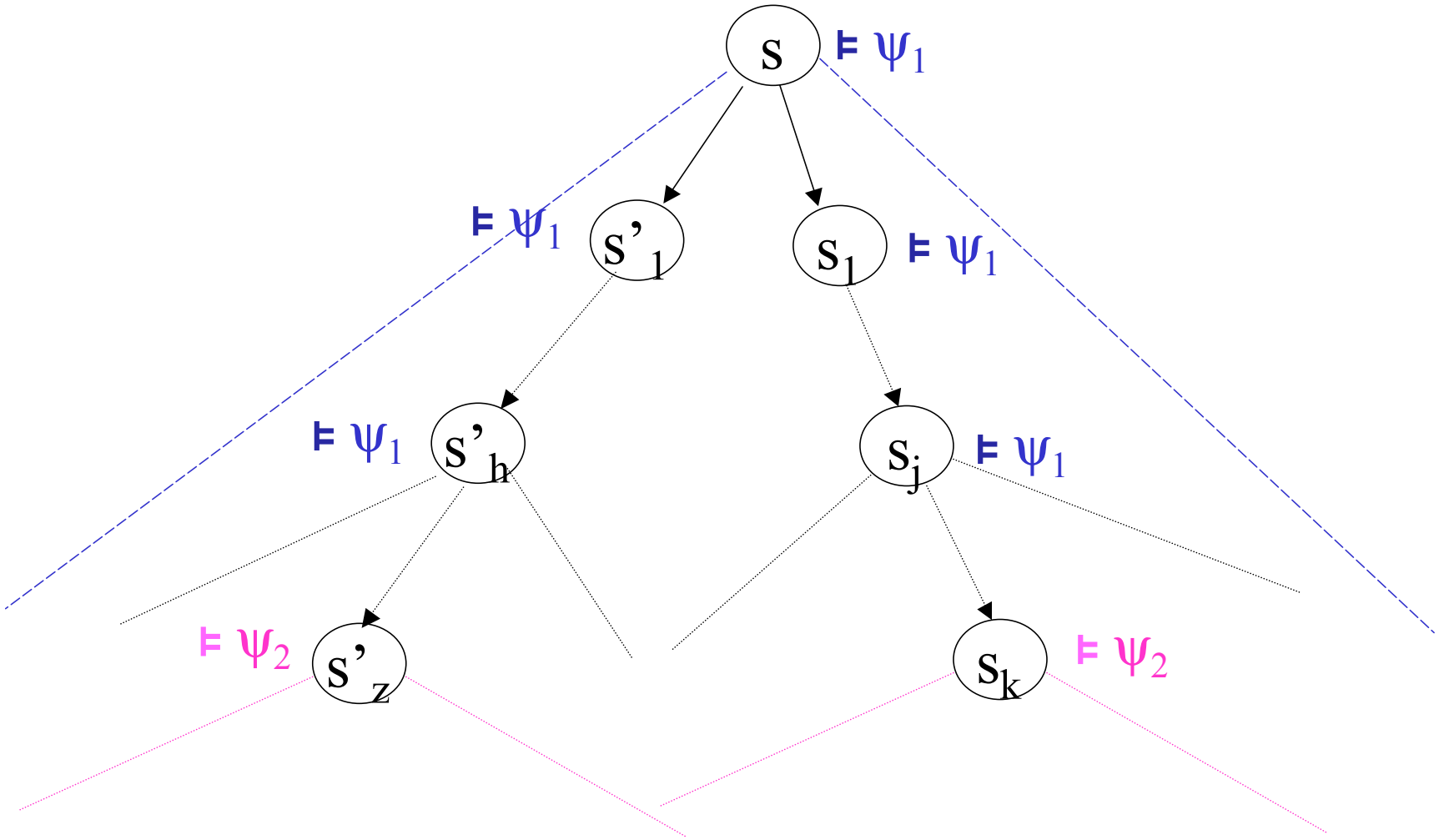
Semantics

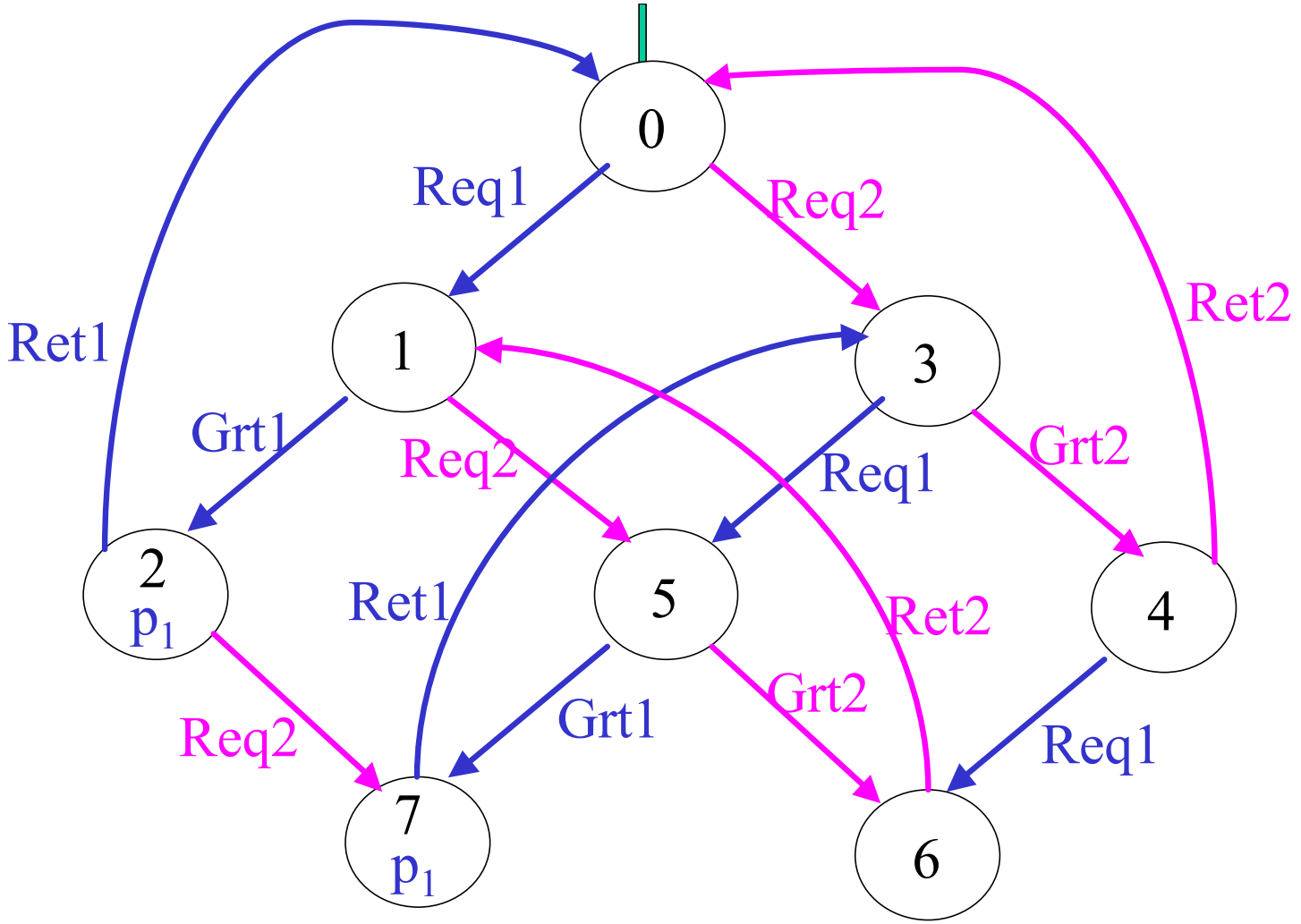
- **CTL ::= p | Ø y | y_1 Ú y_2 | EX(y) |**
  | EU(y_1, y_2) | AU(y_1, y_2)
- **K = (S, S_0, R, AP, L) ; L: S ⟶ 2^{AP} ; s Î S**
- **K, s ⊨ EU(y_1, y_2)** iff ***there exists a path*** p = **s_0, s_1, …** from **s** (i.e. **s_0=s**) and **k ³ 0** such that:
  - **K,** p**(k) ⊨** y_2
  - **K,** p**(j) ⊨** y_1 , for all **0 £ j < k**.


s
⊨ ψ1
s1
⊨ ψ1
sj
⊨ ψ1
sk
⊨ ψ2

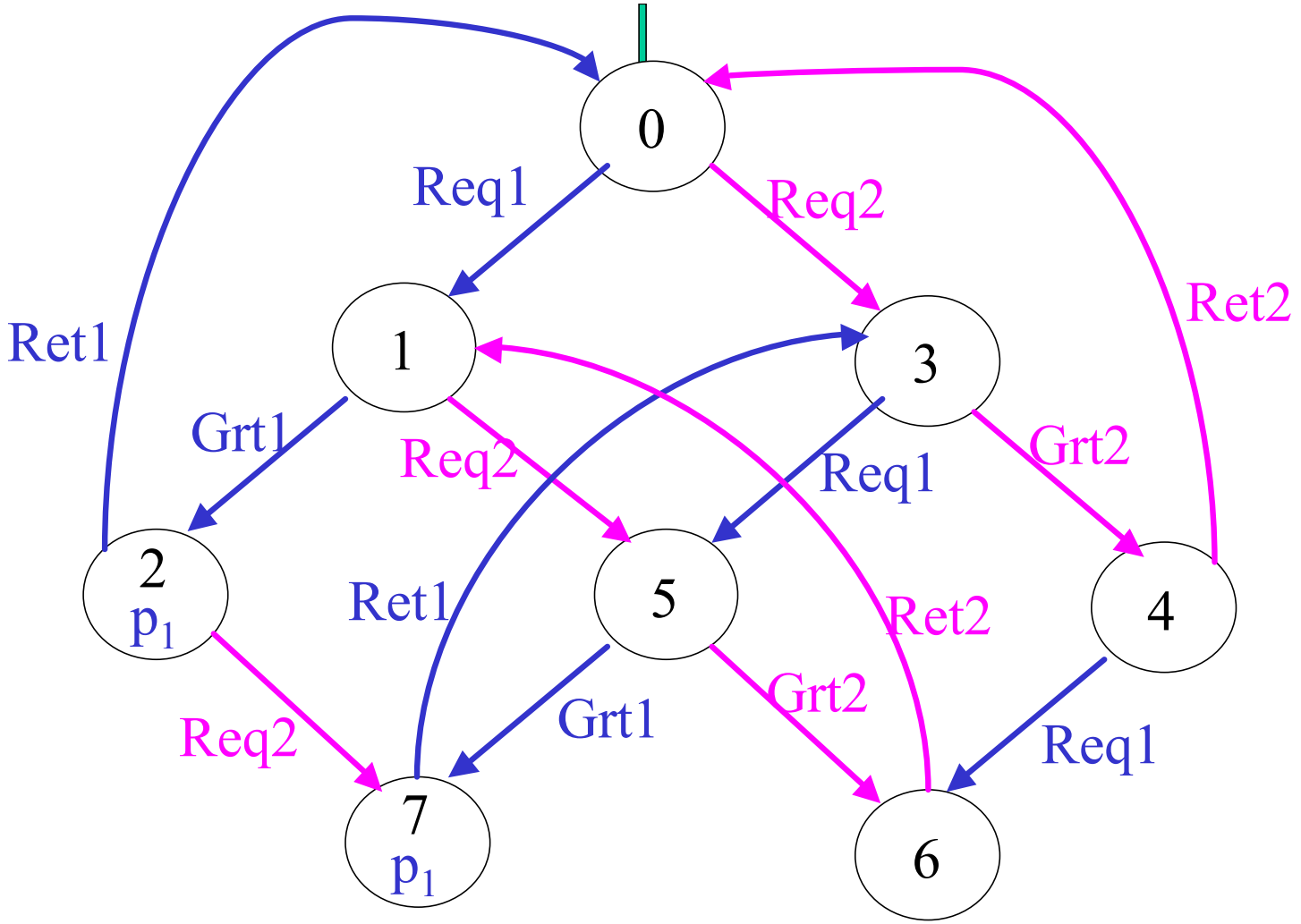
Semantics

- **CTL ::= p | Ø y | y_1 Ú y_2 | EX(y) |**
  | EU(y_1, y_2) | AU(y_1, y_2)
- **K = (S, S_0, R, AP, L) ; L: S ⟶ 2^{AP} ; s Î S**
- **K, s ⊨ AU(y_1, y_2)** iff ***for every path*** p = **s_0, s_1, …** from **s** there exists **k ³ 0** such that:
  - **K,** p**(k) ⊨** y_2
  - **K,** p**(j) ⊨** y_1, for all **0 £ j < k**.


s
⊨ ψ1
⊨ ψ1
s'1
s1
⊨ ψ1
⊨ ψ1
s'h
sj
⊨ ψ1
⊨ ψ2
s'z
sk
⊨ ψ2


0
Req1
Req2
Ret1
Ret2
1
3
Grt1
Req2
Req1
Grt2
2
p1
Ret1
5
Ret2
4
Req2
Grt1
Grt2
Req1
7
p1
6

M, 0 ⊨ EU(⊤, p_1) ?


0
Req1
Req2
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Ret2
1
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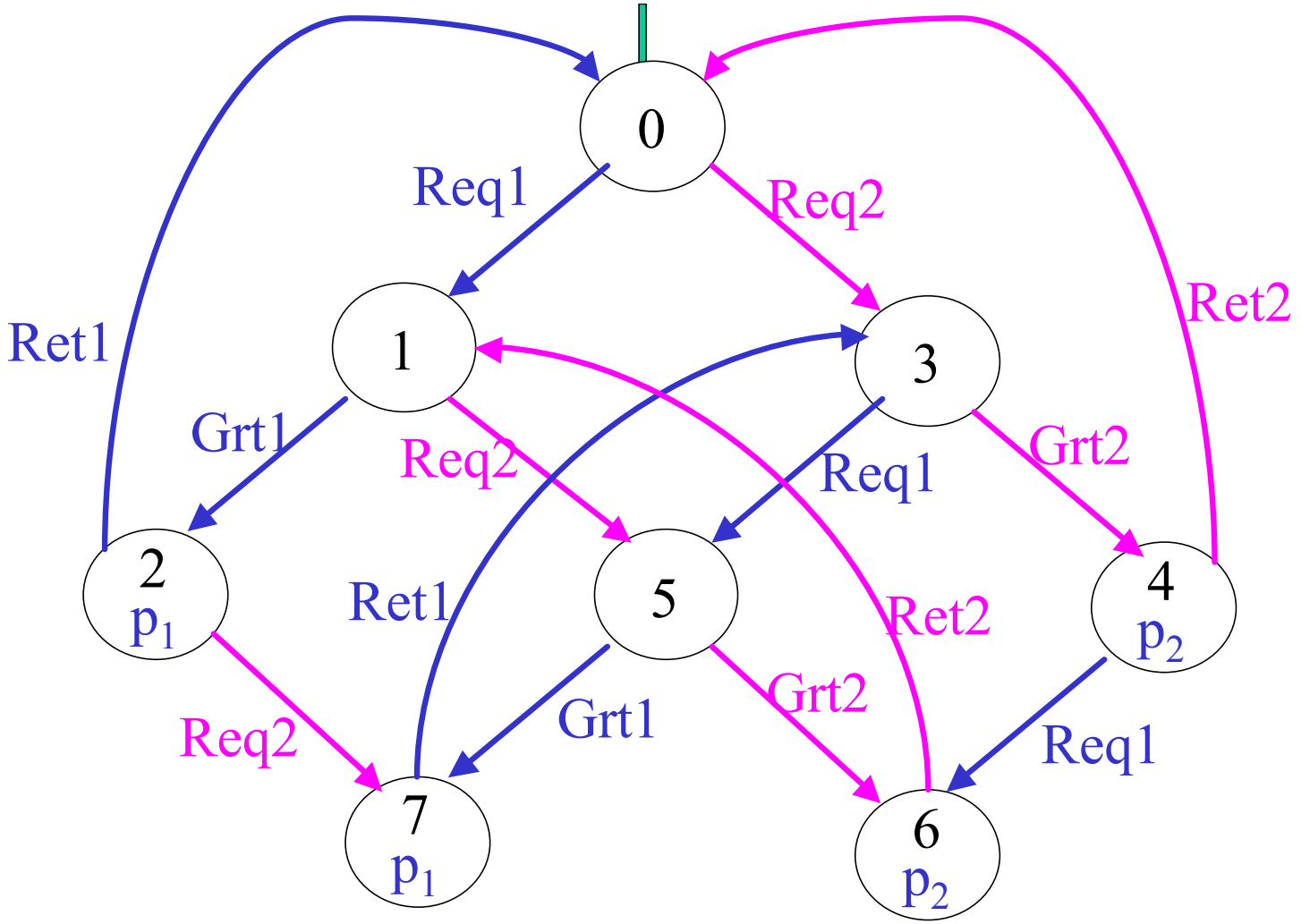
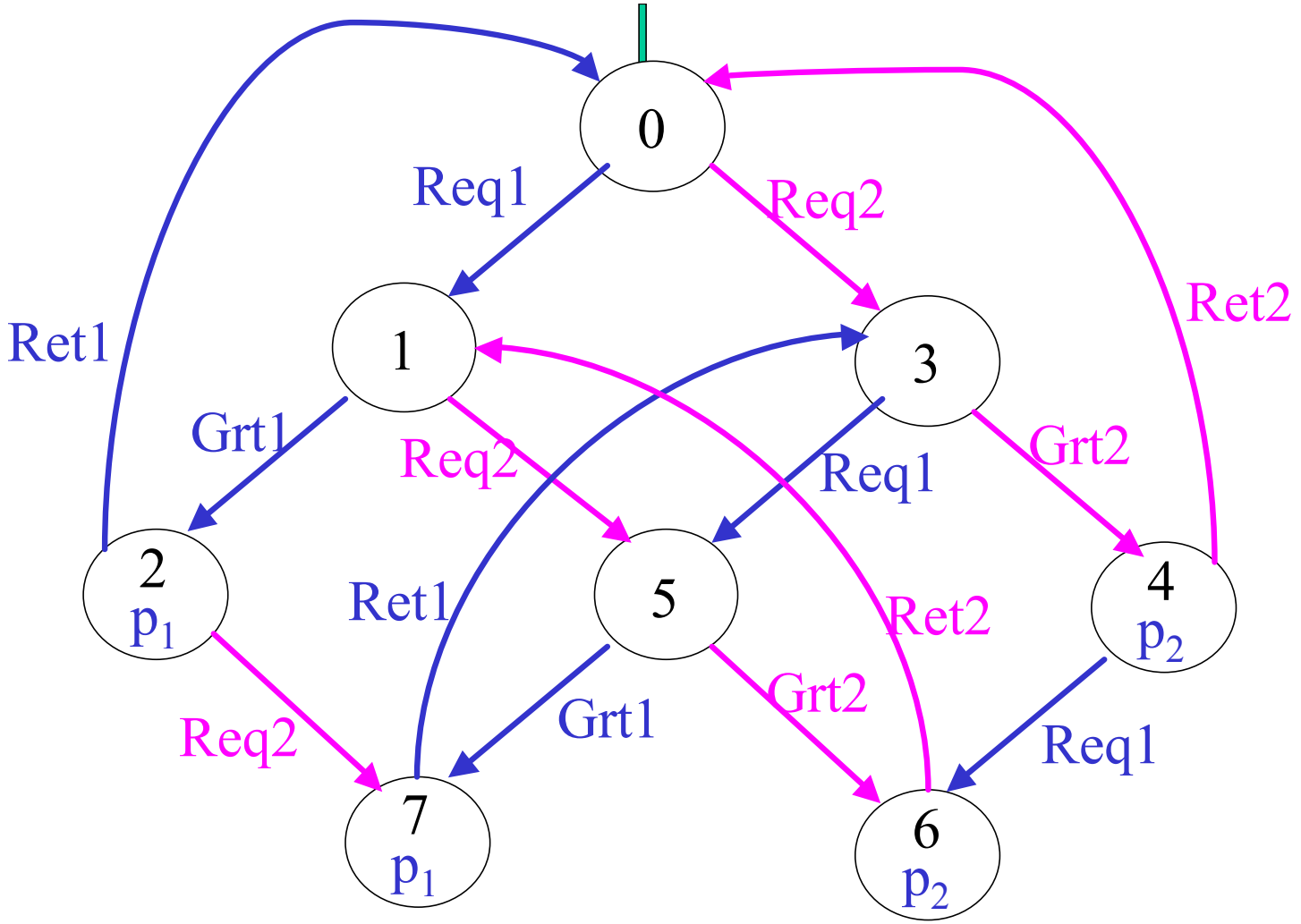
Req1
Grt2
2
p1
5
4
Ret1
Ret2
Grt1
Grt2
Req2
Req1
7
p1
6

M, 0 ⊨ AU(⊤, p_1) ?

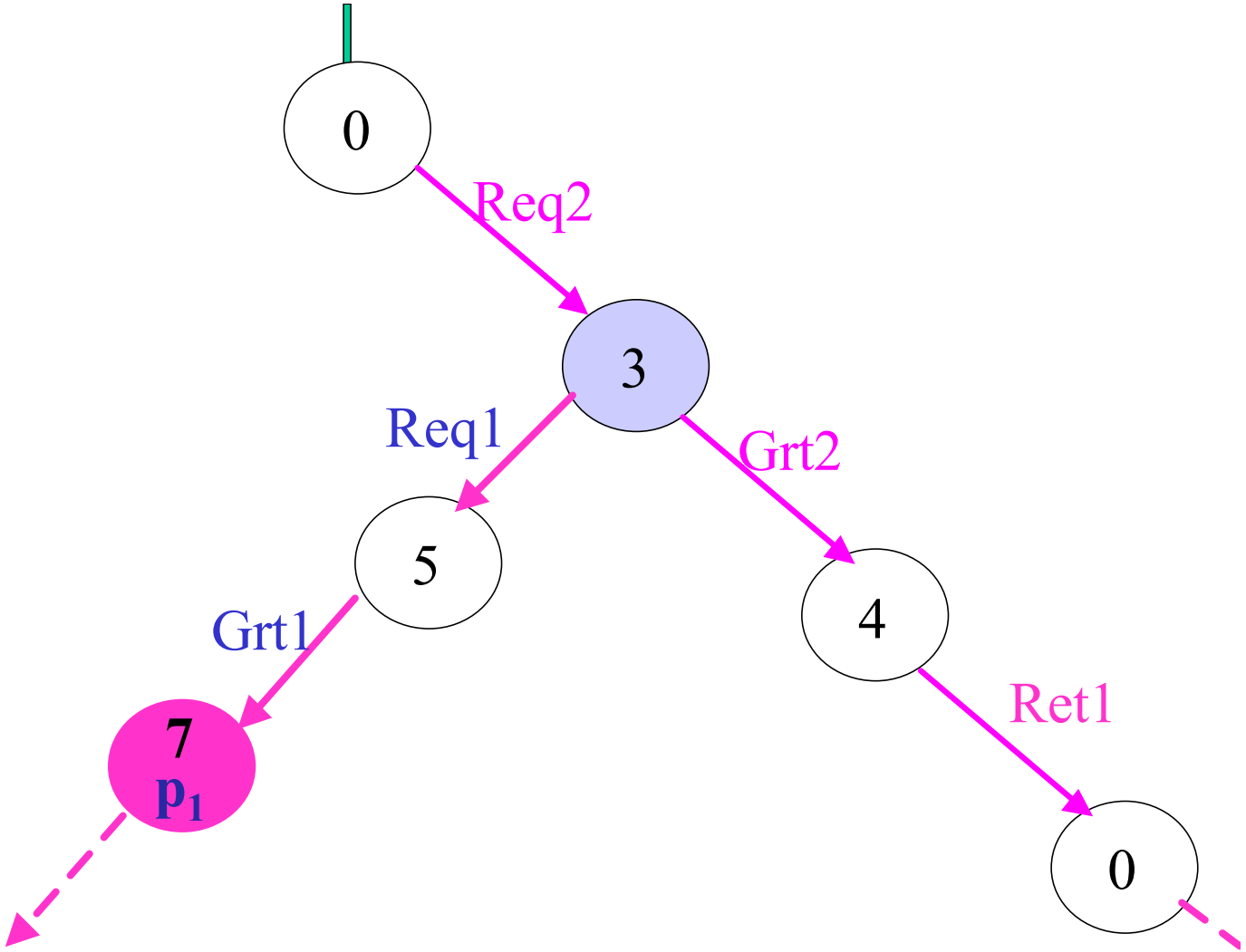

0
Req1
Req2
Ret1
Ret2
1
3
Grt1
Req2
Req1
Grt2
2
p1
Ret1
5
Ret2
4
p2
Req2
Grt1
Grt2
Req1
7
p1
6
p2

M, 0 ⊨ AU(⊤, p_1 Ú p_2) ?



M, 0 ⊨ AU(⊤, EU(⊤, p_1)) ?

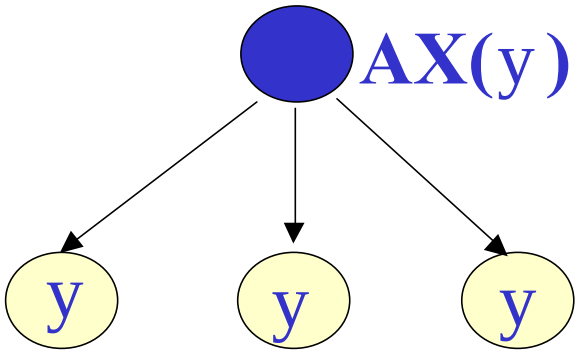
From s_0, ***all*** the computations will reach a point, where it is ***possible*** for 1 to print ***eventually.***



M, 0 ⊨ AU(⊤, EU(⊤, p_1)) ?

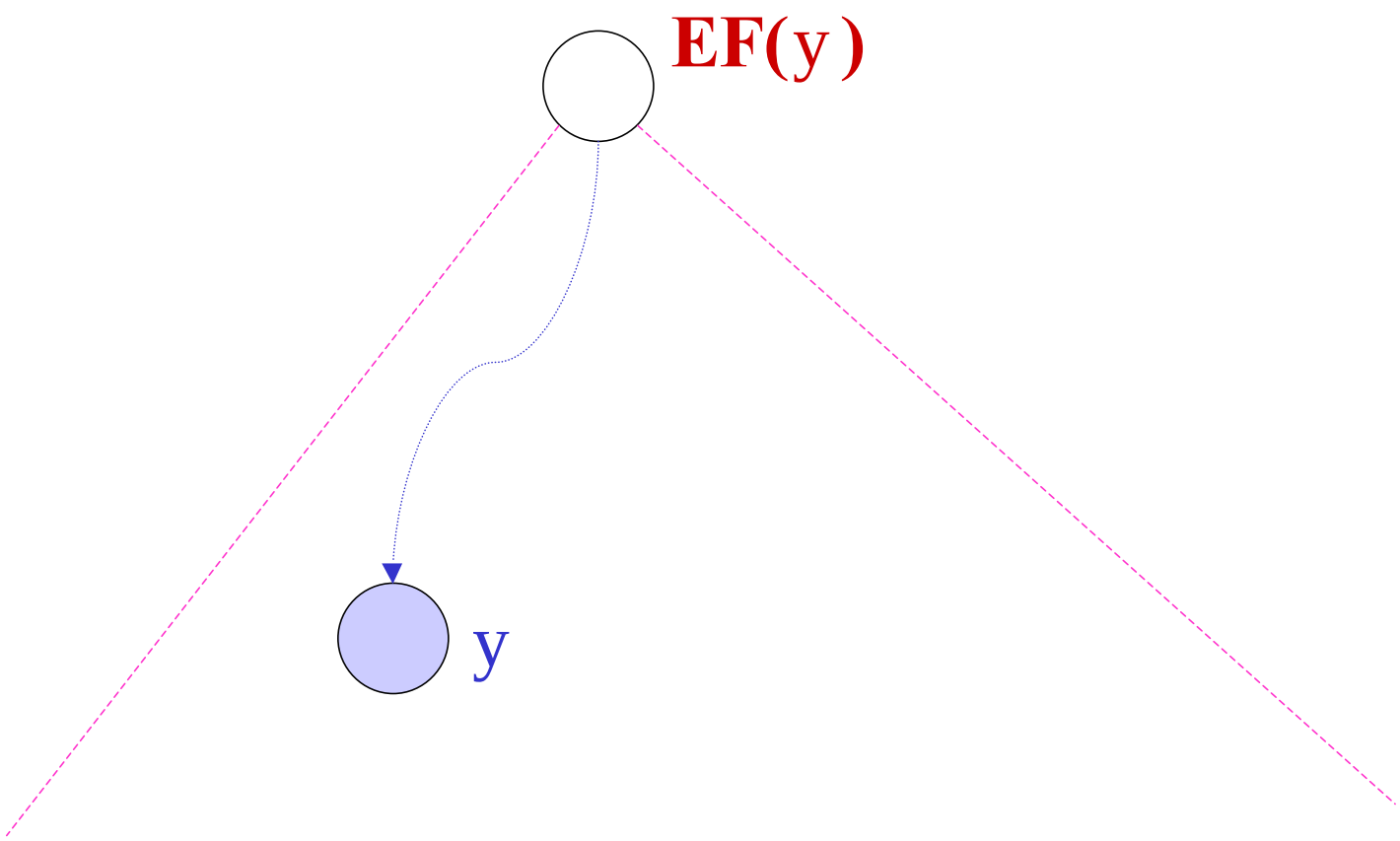
Derived Operators

- **AX(y) = ØEX(Øy)**
  - It is not the case there exists a next state at which y does not hold.
  - ***For every next state*** y holds.



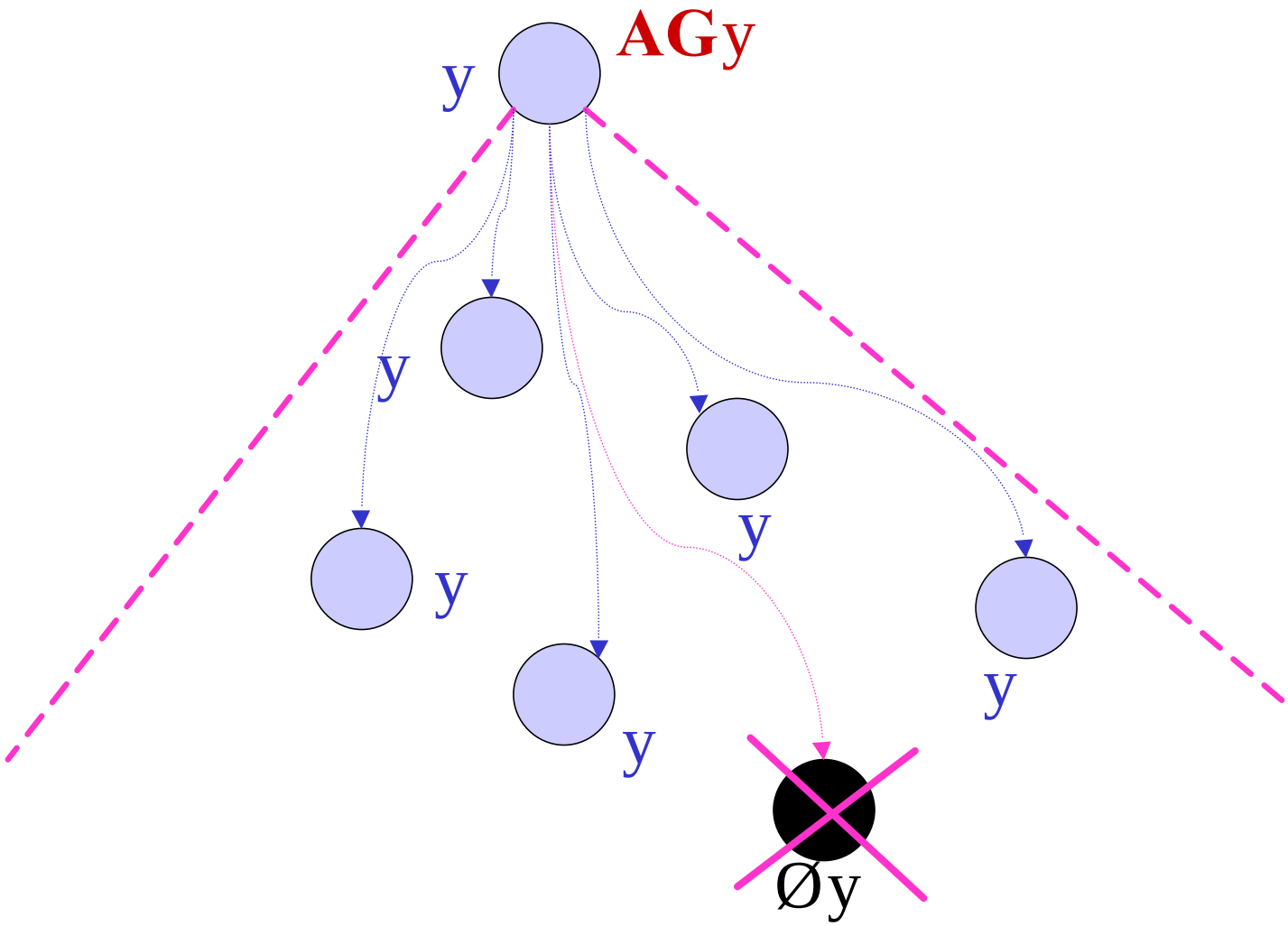
Derived Operators

- **K, s ⊨ EF(y)**
- **EF(y) = EU(⊤, y)**
  - There exists a path p (from **s**) and **k ³ 0** such that:
    - **K, p(k) ⊨ y.**


EF(y)
y

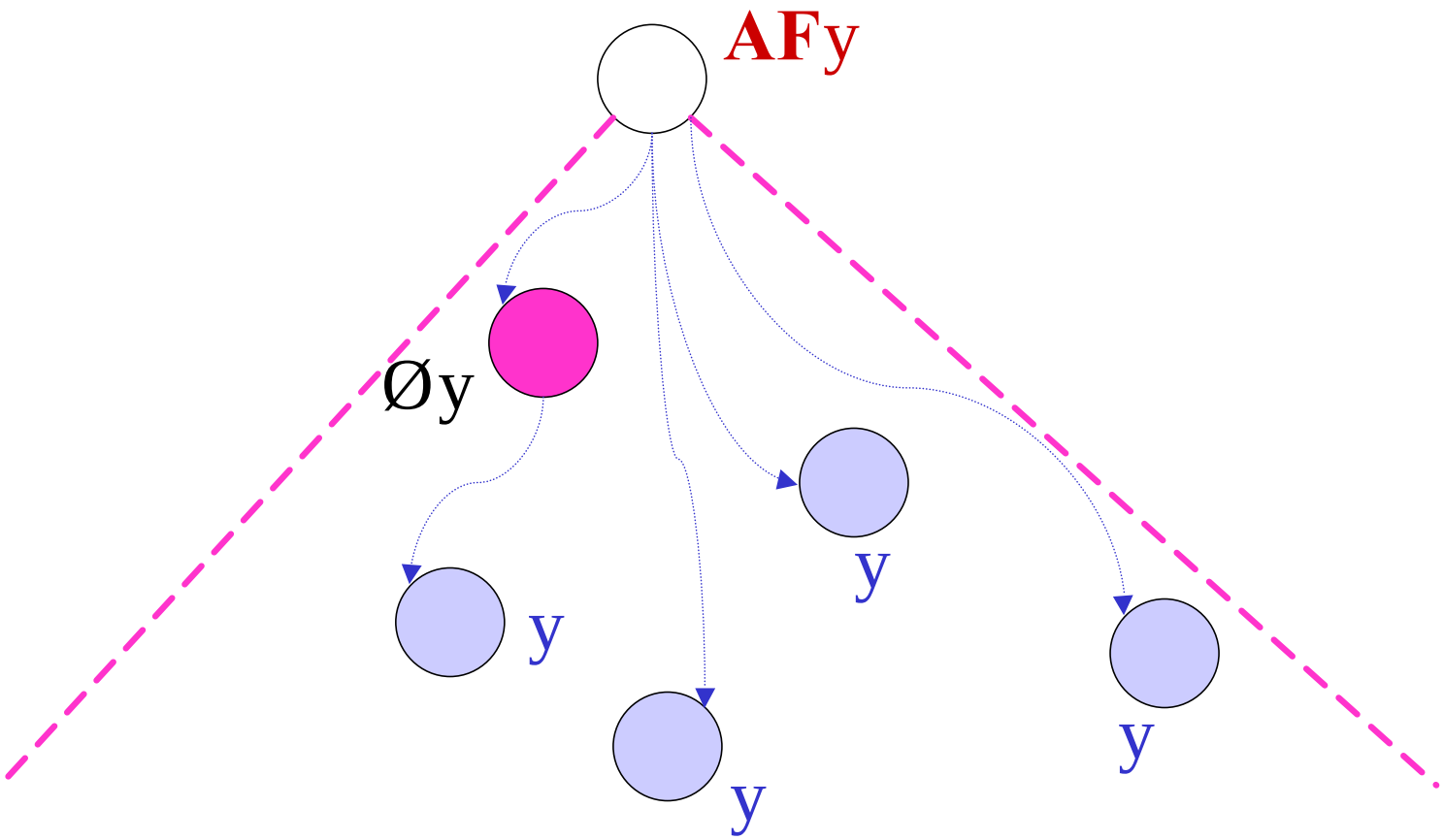
Derived Operators

- **K, s ⊨ AG(y)**
- **AG(y) = ØEF(Øy)**
  - It is ***not*** the case ***there exists a path*** p (from **s**) and **k ³ 0** such that:
    - **K,** p**(k) ⊨** y
  - ***For every path*** p (from **s**) and ***every*** **k ³ 0**:
    - **K,** p**(k) ⊨** y


AGy
y
y
y
y
y
y
Øy

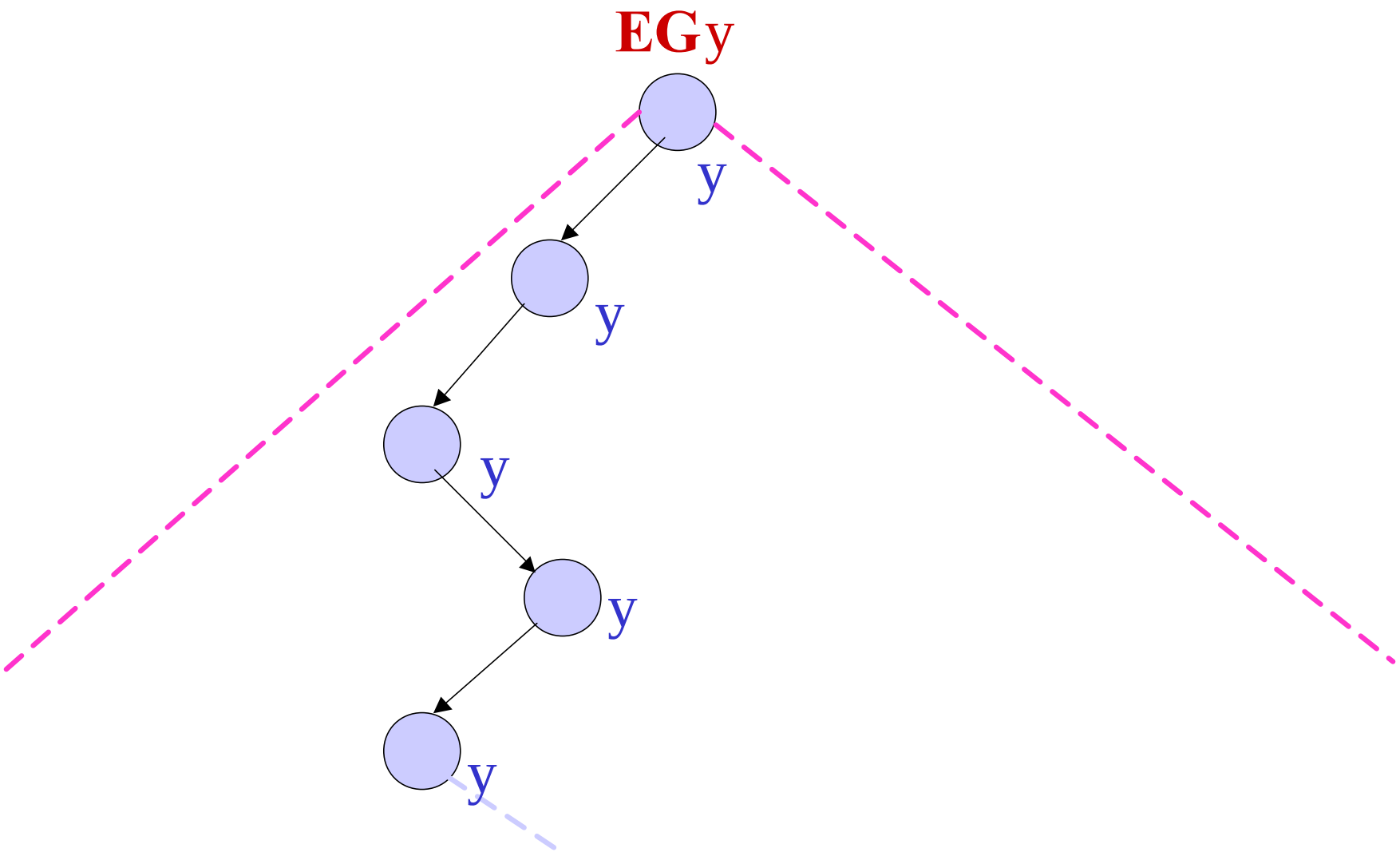
Derived Operators

- **K, s ⊨ AF(y)**
- **AF(y) = AU(⊤, y)**
  - ***For every path*** p from **s**, there ***exists*** **k ³ 0** such that:
    - **K,** p**(k) ⊨** y **.**


AFy
Øy
y
y
y
y

Derived Oparators

- **K, s ⊨ EG(y)**
- **EG(y) = ØAF(Øy)**
  - It is **not** the case that ***for every path*** p from **s** there is a **k ³ 0** such that **K,** p**(k) ⊨** y .
  - ***There exists a path*** p from **s** such that, for every **k ³ 0**:
    - **K,** p**(k) ⊨** y .


EGy
y
y
y
y
y

A more convenient CTL

- **NCTL ::= p** | Ø y | y_1 Ú y_2 | **EX(y)** |
  | **EU(y_1, y_2)** | **EG(y)**
- **CTL ::= p** | Ø y | y_1 Ú y_2 | **EX(y)** |
  | **EU(y_1, y_2)** | **AU(y_1, y_2)**
- **NCTL** is more convenient for model checking!
- Clearly **NCTL** can be defined in terms of **CTL**.

A more convenient CTL

- **NCTL ::= p** | Ø y | y_1 Ú y_2 | **EX(y)** |
  | **EU(y_1, y_2)** | **EG(y)**
- **CTL ::= p** | Ø y | y_1 Ú y_2 | **EX(y)** |
  | **EU(y_1, y_2)** | **AU(y_1, y_2)**
- **CTL** can be defined in terms of **NCTL**!
- The semantics of **NCTL** is given in the obvious way.

A more convenient CTL

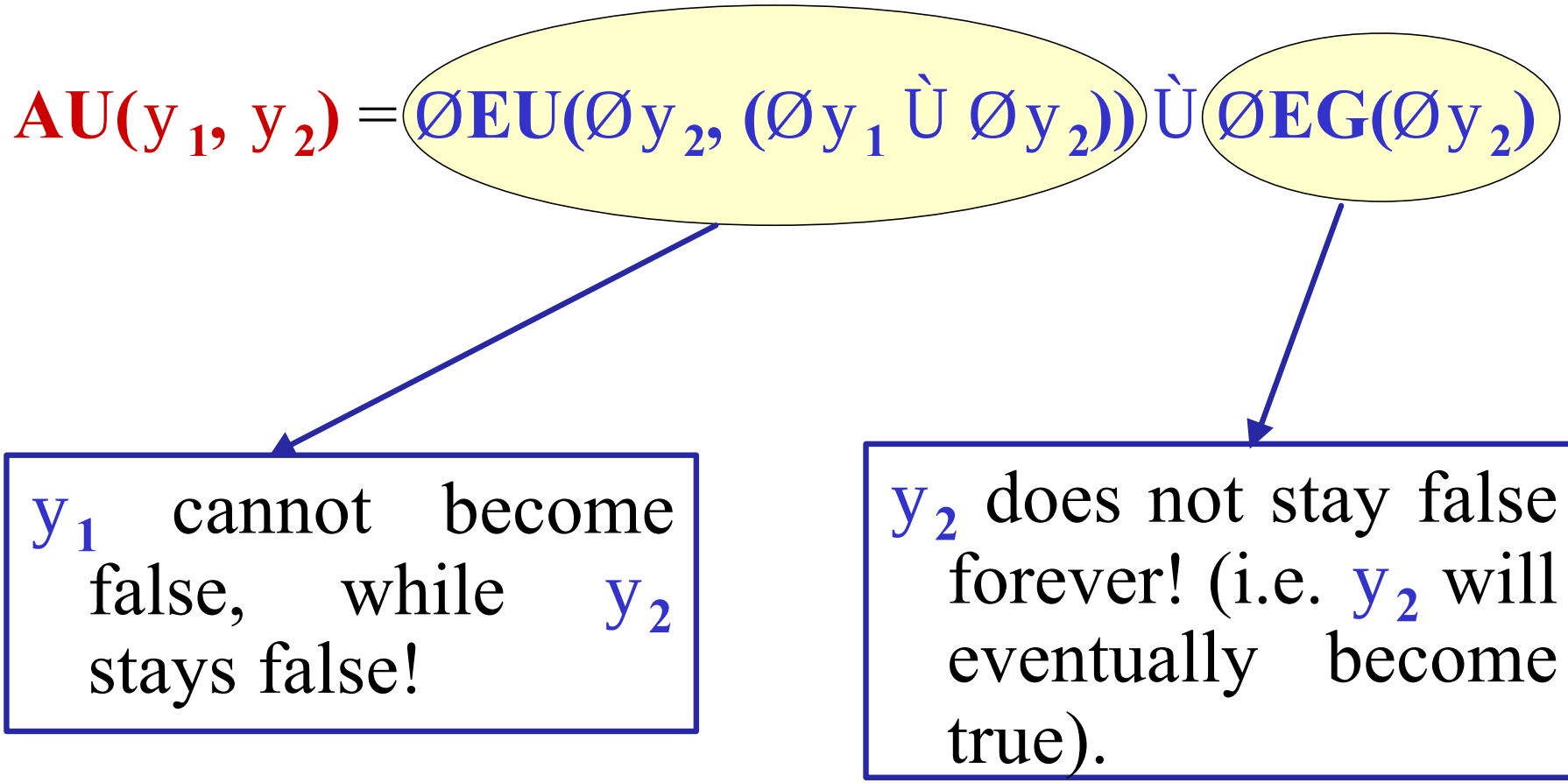
- **NCTL ::= p** | Ø y | y_1 Ú y_2 | **EX(y)** |
  | **EU(y_1, y_2)** | **EG(y)**
- **K, s ⊨ EG(y)** iff ***there exists a path*** p from **s** such that for every **k ³ 0**:
  - **K, p(k) ⊨** y

A more convenient CTL

- **NCTL ::= p | Ø y | y_1 Ú y_2 | EX(y) |**
  | EU(y_1, y_2) | EG(y)
- **CTL ::= p | Ø y | y_1 Ú y_2 | EX(y) |**
  | EU(y_1, y_2) | AU(y_1, y_2)
- **AU(y_1, y_2) = ØEU(Øy_2, (Øy_1 Ù Øy_2)) Ù ØEG(Øy_2)**

i.e., along any path: y_2 must hold eventually and (Øy_1 Ù Øy_2) can only happen after y_2 (recall the *before operator* of LTL)

A more convenient CTL



A more convenient CTL

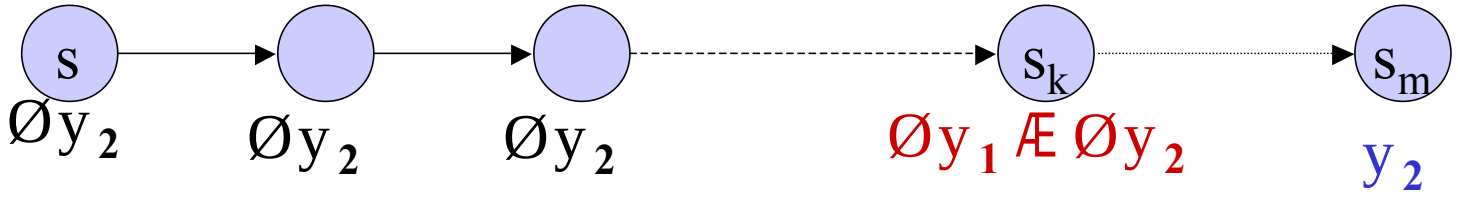
AU(y_1, y_2) = **ØEU(Øy_2, (Øy_1 Ù Øy_2))** Ù **ØEG(Øy_2)**

⇒ Assume **K, s ⊨ AU(y_1, y_2)**

- Let p be a path from **s**. Then there exists **k ³ 0** with:
  - **K, s ⊨ y_2**
- Hence, **not K, s ⊨ EG(Øy_2)**
- Hence, **K, s ⊨ ØEG(Øy_2)**

A more convenient CTL

- **AU(y_1, y_2) = NewAU(y_1, y_2) =**
  ØEU(Øy_2, (Øy_1 Ù Øy_2)) Ù ØEG(Øy_2)
- Clearly **K, s ⊨ AU(y_1, y_2)** implies **K, s ⊨ ØEG(Øy_2)**
- Let **K, s ⊨ AU(y_1, y_2)**
  - Suppose now **K, s ⊨ EU(Øy_2, Øy_1 Ù Øy_2)**
  - Let p be any path from **s** witnessing the above:
  - Let now **k *be the least integer*** such that:
    - **K, p(k) ⊨ Øy_1 Ù Øy_2**
    - **K, p(j) ⊨ Øy_2** for **0 £ j < k.**



- Suppose **K, p(m) ⊨ y 2**, required by **K, s ⊨ AU(y 1, y 2)**
- Take **m** to be the least such number.
- Then **m > k**, since **K, s ⊨ EU(Øy 2, Øy 1 Ù Øy 2)**
- But **0 £ k < m** and **K, p(k) ⊨ Øy 1**
- Hence **not K, s ⊨ AU(y 1, y 2)**. ***Contradiction!***
- Thus **K, s ⊨ AU(y 1, y 2)** also implies:
  - **K, s ⊨ ØEU(Øy 2, Øy 1 Ù Øy 2)**
- So **K, s ⊨ AU(y 1, y 2)** implies **K, s ⊨ NewAU(y 1, y 2)**

From CTL to NCTL

- In a similar way we can argue that:

> ***if*** **K, s ⊨ newAU(y_1, y_2)**
>
> ***then*** **K, s ⊨ AU(y_1, y_2).**

- Hence ***CTL*** can be expressed in terms of ***NCTL***.

A more convenient CTL

- **NCTL ::= p** $\mid \neg y \mid y_1 \vee y_2 \mid$ **EX(y)** $\mid$
  $\mid$ **EU(y_1, y_2)** $\mid$ **EG(y)**
- **CTL ::= p** $\mid \neg y \mid y_1 \vee y_2 \mid$ **EX(y)** $\mid$
  $\mid$ **EU(y_1, y_2)** $\mid$ **AU(y_1, y_2)**
- **AU(y_1, y_2) = NewAU(y_1, y_2) =**
  $\neg$(EU($\neg y_2$, ($\neg y_1 \wedge \neg y_2$)) $\wedge$ **AF(y_2)**
- **NewAU$_1$ = $\neg$ EU($\neg y_2$, ($\neg y_1 \wedge \neg y_2$)**
- **NewAU$_2$ = AF y_2**

$\neg$EG$\neg y_2$ = AF y_2

From CTL to NCTL

- Let **K = (S, S_0, R, AP, L)** and **s Î S**.
- We need to argue:
  - **K, s ⊨ AU(y_1, y_2)** iff

    K, s ⊨ NewAU$_1$ Ù NewAU$_2$
- We already argued that:
  - If **K, s ⊨ AU(y_1, y_2)** then

    K, s ⊨ NewAU$_1$ Ù NewAU$_2$

From CTL to NCTL

$\mathbf{AU}(y_1, y_2) = \neg\mathbf{EU}(\neg y_2, (\neg y_1 \wedge \neg y_2)) \wedge \neg\mathbf{EG}(\neg y_2)$

$\Leftarrow$ We need to argue that:

– If $\mathbf{K}, s \models \mathbf{NewAU}_1 \wedge \mathbf{NewAU}_2$ then

$\mathbf{K}, s \models \mathbf{AU}(y_1, y_2)$

- So assume $\mathbf{K}, s \models \mathbf{NewAU}_1 \wedge \mathbf{NewAU}_2$.
- $\mathbf{NewAU}_1 = \neg\mathbf{EU}(\neg y_2, (\neg y_1 \wedge \neg y_2))$.
- $\mathbf{NewAU}_2 = \neg\mathbf{EG}\neg y_2 = \mathbf{AF}\, y_2$

From CTL to NCTL

- Let p be some path from **s**.
- We need to show that there exists **k ³ 0** such that:
  - **K,** p**(k) ⊨** y_2
  - **K,** p**(j) ⊨** y_1 if **0 £ j < k**.
- But **K, s ⊨ AF** y_2 implies there along any path (and also along p) there exists **k ³ 0** such that:
  - **K,** p**(k) ⊨** y_2
- Assume **k** is the ***least*** such number along p.

From CTL to NCTL

Now consider an arbitrary **m** with **0 £ m < k**.

CLAIM: **K, s(m) ⊨ y_1**

- If the **CLAIM** is true then we are done.
- Suppose instead that **K, s(m) ⊨ Øy_1**.
  - Then **K, s(m) ⊨ Øy_1 Ù Øy_2 (m < k)** ***WHY???***
  - and **K, s(j) ⊨ Øy_2** if **0 £ j < m,** since **j < m < k**
  - Hence **K, s(0) ⊨ EU(Øy_2, Øy_1 Ù Øy_2)**
  - Therefore, **not K, s ⊨ NewAU$_1$** which is a ***contradiction***!

CTL Model Checking

- $\mathbf{K} \models \gamma$ *iff*

  $\mathbf{K}, \mathbf{s}_0 \models \gamma$ for every $\mathbf{s}_0 \in \mathbf{S}_0$.

- The ***CTL model checking problem***.
  - $\mathbf{K} = (\mathbf{S}, \mathbf{S}_0, \mathbf{R}, \mathbf{AP}, \mathbf{L})$ (system model)
  - γ a ***CTL*** formula (spec. of the property)

- Given $\mathbf{K}$ and γ ***determine whether or not*** $\mathbf{K} \models \gamma$

CTL Model Checking

- The actual model checking problem:
  - Given **$K = (S, S_0, R, AP, L)$**
  - Given **$s \in S$**
  - Given y, **an NCTL formula**.
  - Determine whether:

$K, s \models y$

The Sub-formulas of ψ

- **SF(y)** is the ***least set of formulas*** satisfying:
  - y Î **SF(y)**
  - If Øa Î **SF(y)** then a Î **SF(y)**.
  - If a Ú b Î **SF(y)** then a, b Î **SF(y)**
  - If **EX**a Î **SF(y)** then a Î **SF(y)**.
  - If **EU**(a, b) Î **SF(y)** then a, b Î **SF(y)**
  - If **EG**a Î **SF(y)** then a Î **SF(y)**.
- **SF(y)** ---- The ***set of sub-formulas*** of y.

The Labeling Procedure.

- **K = (S, S_0, R, AP, L)**
  - **s Î S**
  - y a ***NCTL*** formula (built out of **AP**).
- **Strategy**:
  - Construct **Labels: S ⟶ $2^{SF(y)}$**
  - **$2^{SF(y)}$**, the set of subsets of **SF(y)**.
  - Each state of **K** is assigned a subset of a **SF(y)** by the Labels function.
- **K, s ⊨ y *iff* y Î Labels(s)**.

The Labels function

- **Stage 1**:
  - For every **t Î S**:
  - **Labels(t) = L(t)** (**K =(S, S_0, R, AP, L)**)
- ….

Assume we have done up to stage **i**.

- **Stage i +1**:
  - For every **t Î S**:
  - If a = Øb then
    a î **Labels(t)** ***iff*** b Ï **Labels(t)**.

The Labels function

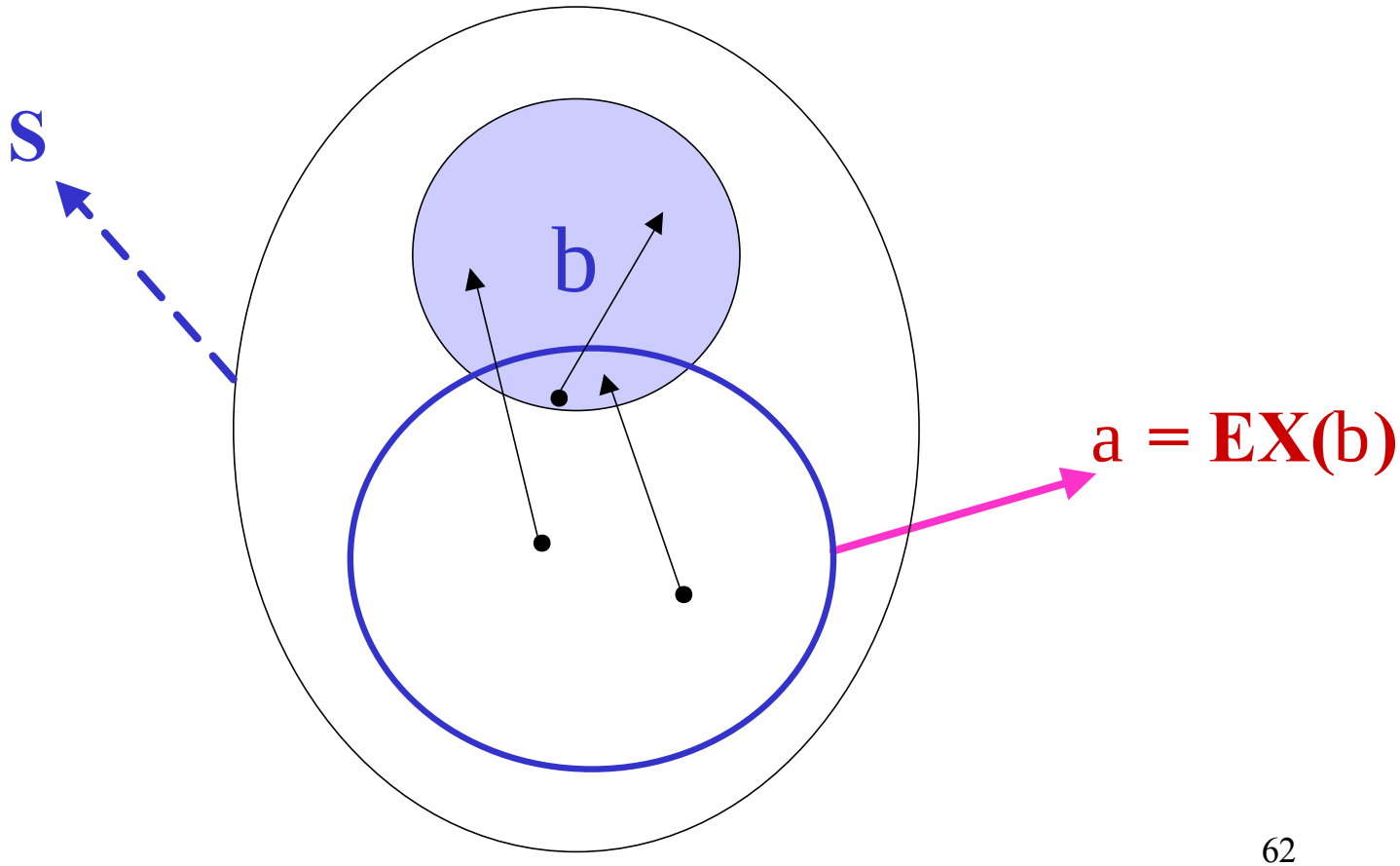
- **Stage i +1**:
  - For every **t** Î **S**:
  - If a = b_1 Ú b_2 then

    a Î **Labels(t)** ***iff*** b_1 Î **Labels(t)** **or** b_2 Î **Labels(t)**
  - If a = **EX**b then

    a Î **Labels(t)** ***iff*** **there exists s Î S such that**

    b Î **Labels(s)** and **R(t, s)**

The Labels Function



Computing the labeling for EX(β)

Complexity: O(|M|)

Algorithm Check_EX(β)

T := {**s** | b Î **Labels(s)**};

while **T** ¹ Æ do

choose **s** Î **T**;

T := **T** \{**s**};

forall **t** Î **S** such that (**t**,**s**) Î **R** do

Labels(t) := **Labels(t)** È {**EX** β};

The Labels Function

- **Stage i +1**:
  - For every **t** Î **S**:
  - If a = **EU**(b_1,b_2) then

    a Î **Labels(t)** *iff*

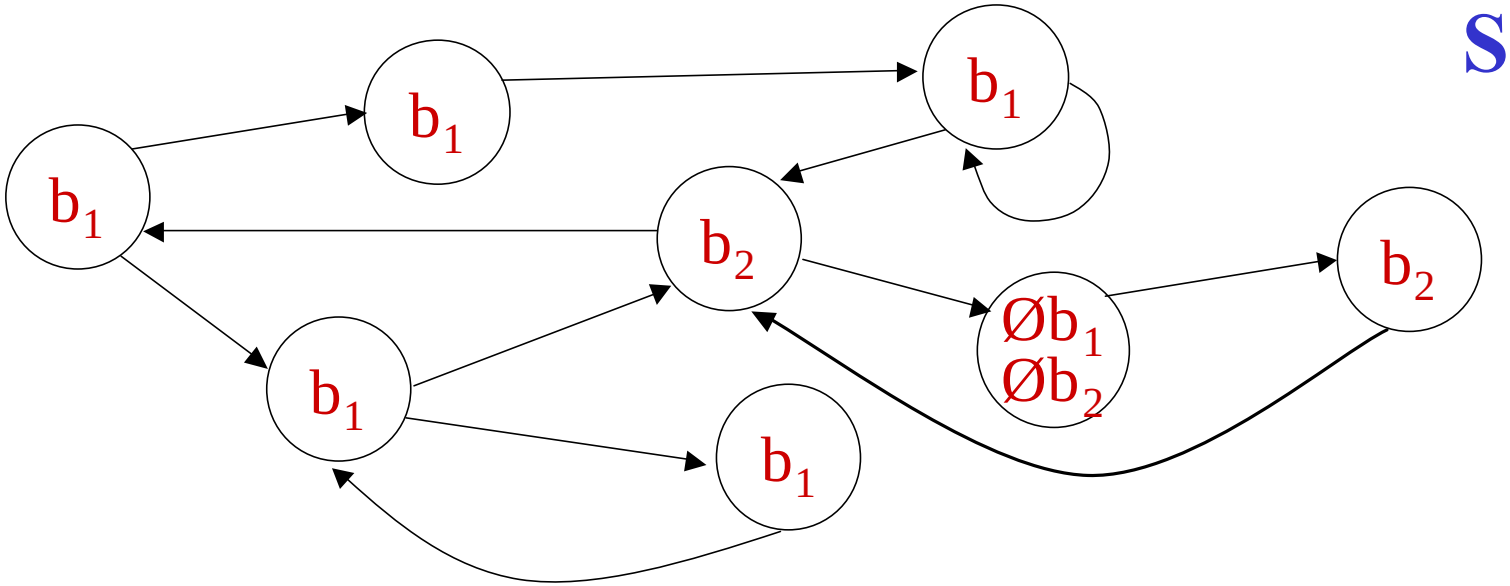
    - b_2 Î **Labels(t)** **or**
    - b_1 Î **Labels(t)** **and** **EU**(b_1,b_2) Î **Labels(s)** **for some s with R(t,s).**

The Labels Function

- Collect in **T** all the states satisfying b_2
  - **all these states do also satisfy EU(b_1,b_2)**.
- Traverse backward **R** from states in **T** and label with **EU(b_1,b_2)** all the states **t** satisfying b_1 and reaching at least a state **s** labeled with **EU(b_1,b_2)**.

If s Î T, t with R(t,s) and b_1 Î Labels(t)

then EU(b_1,b_2) Î Labels(t)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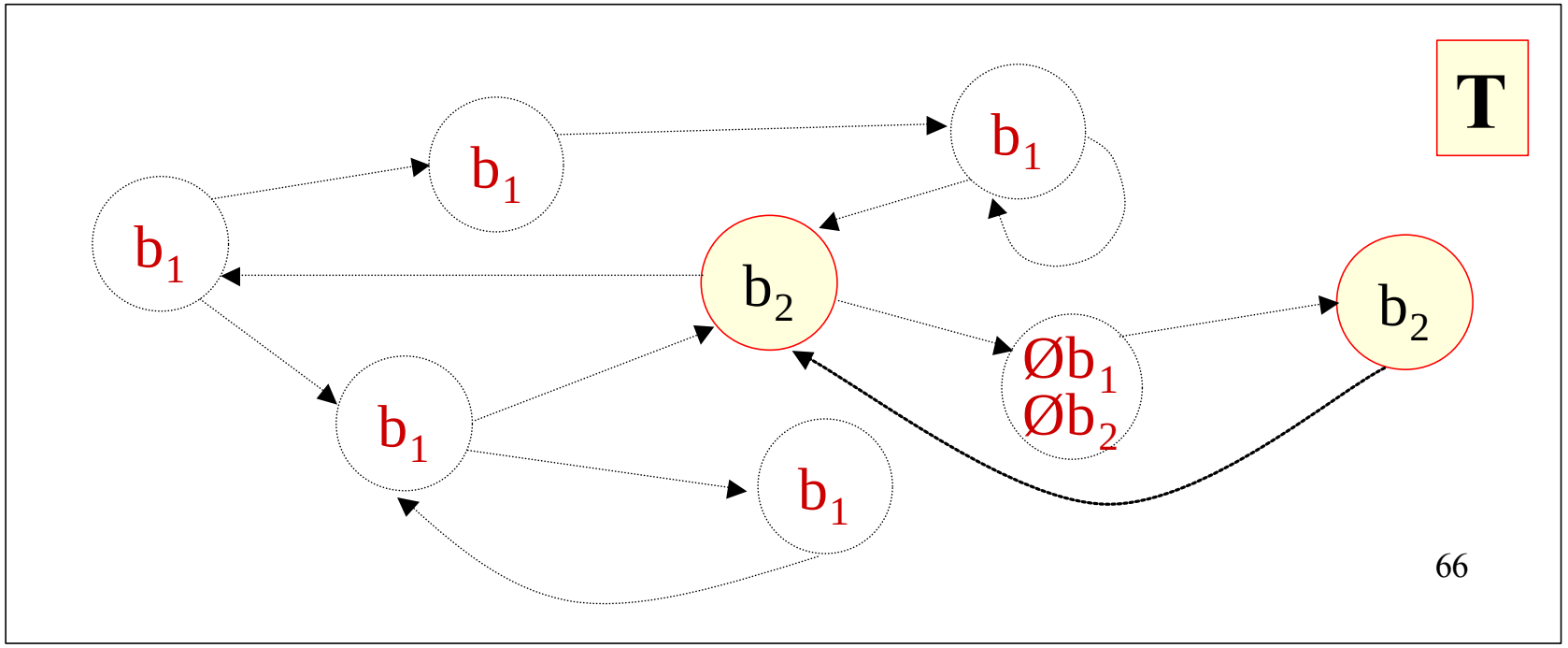
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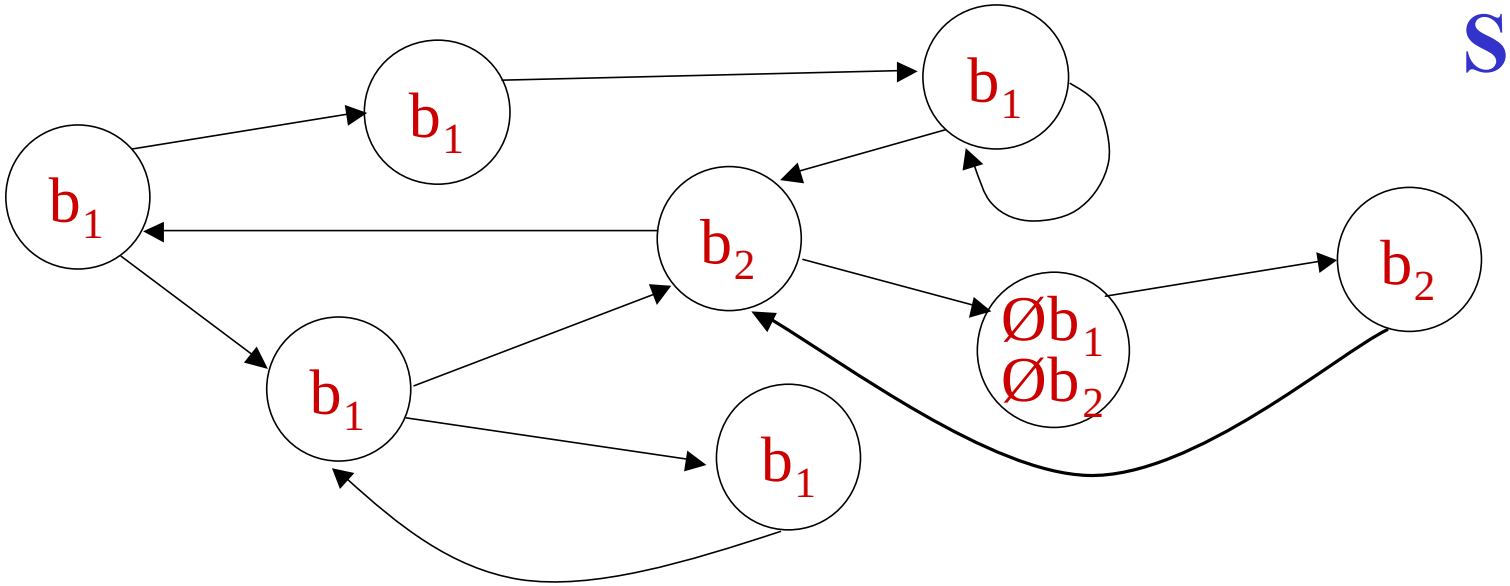
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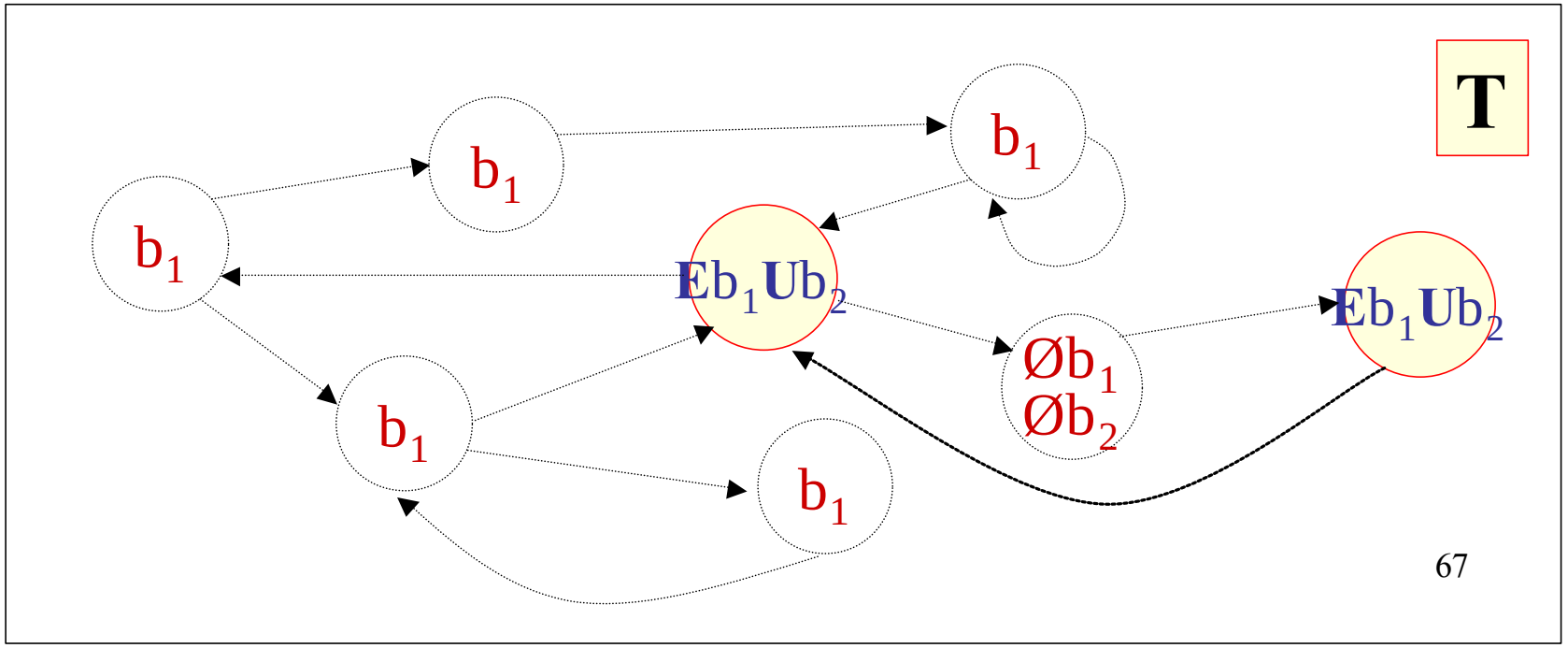
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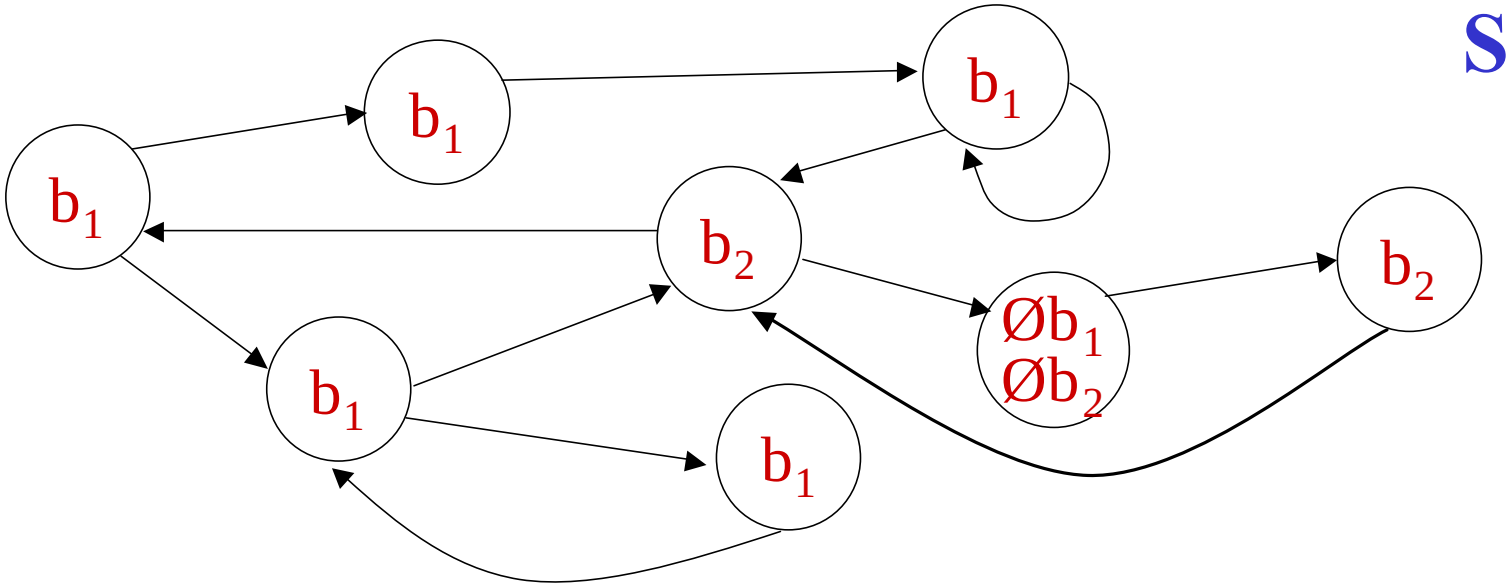
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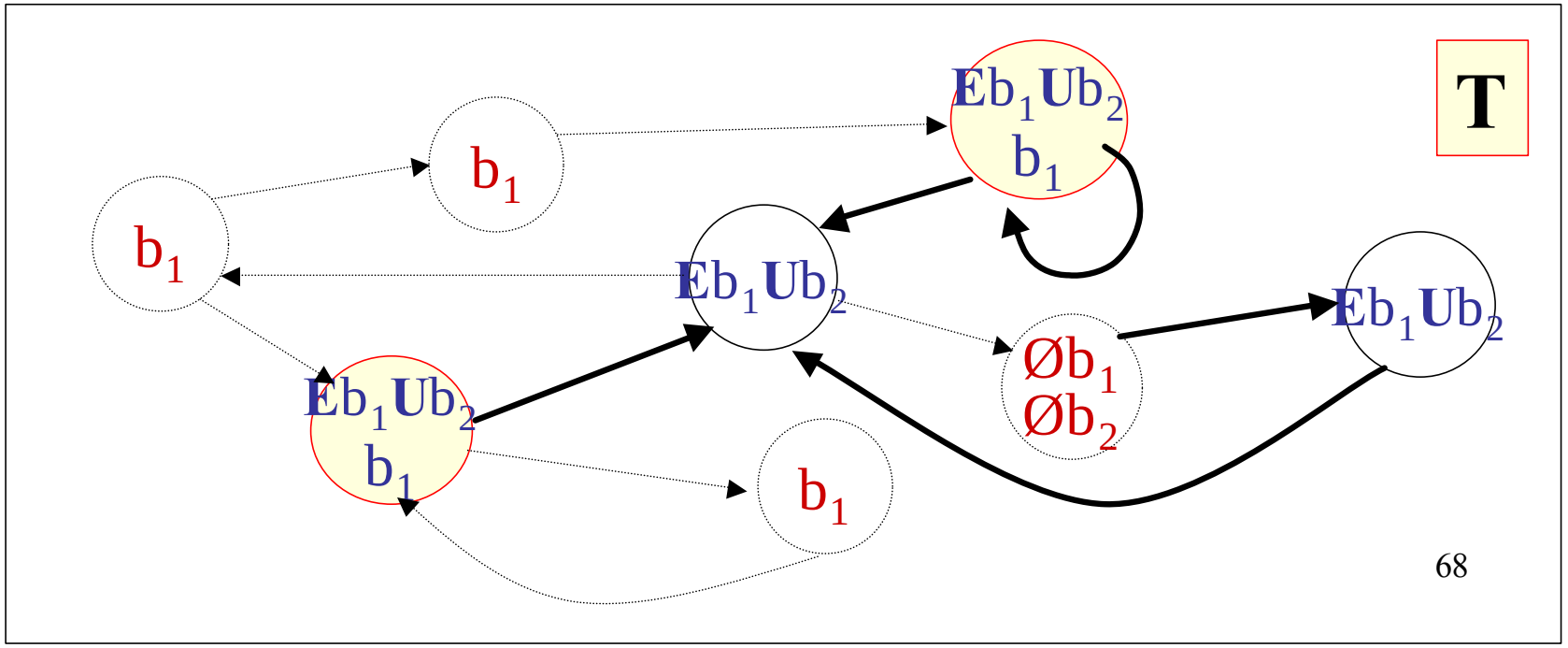
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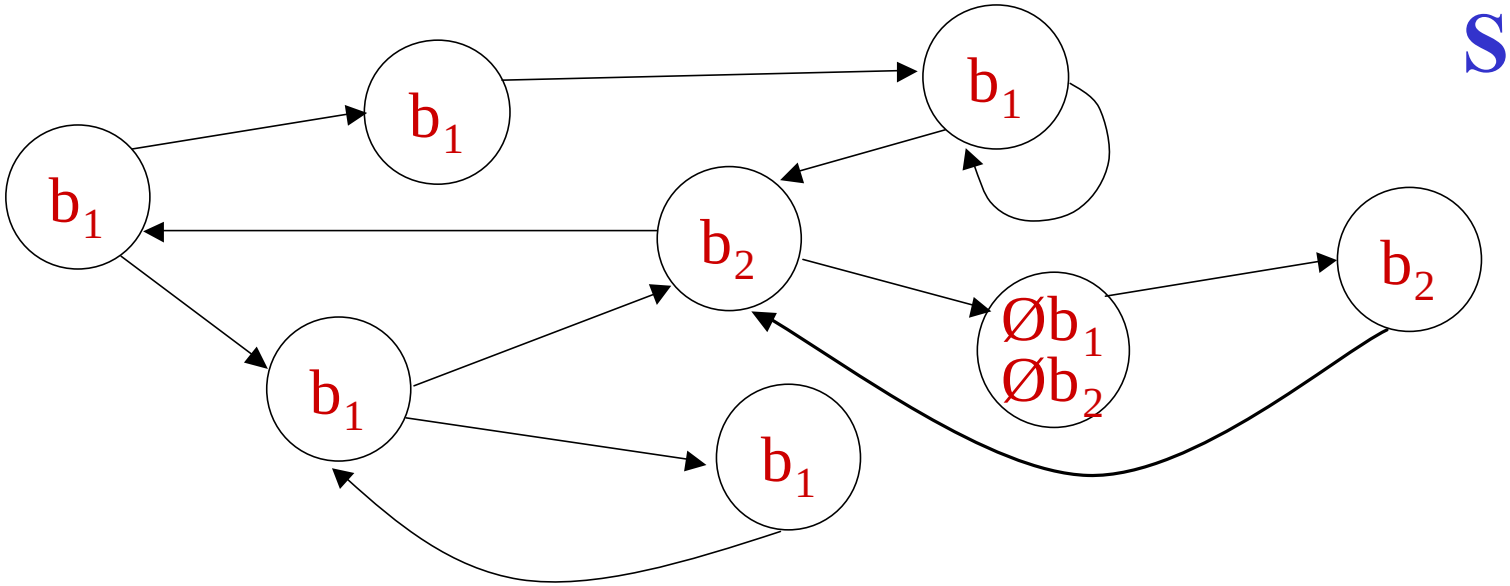
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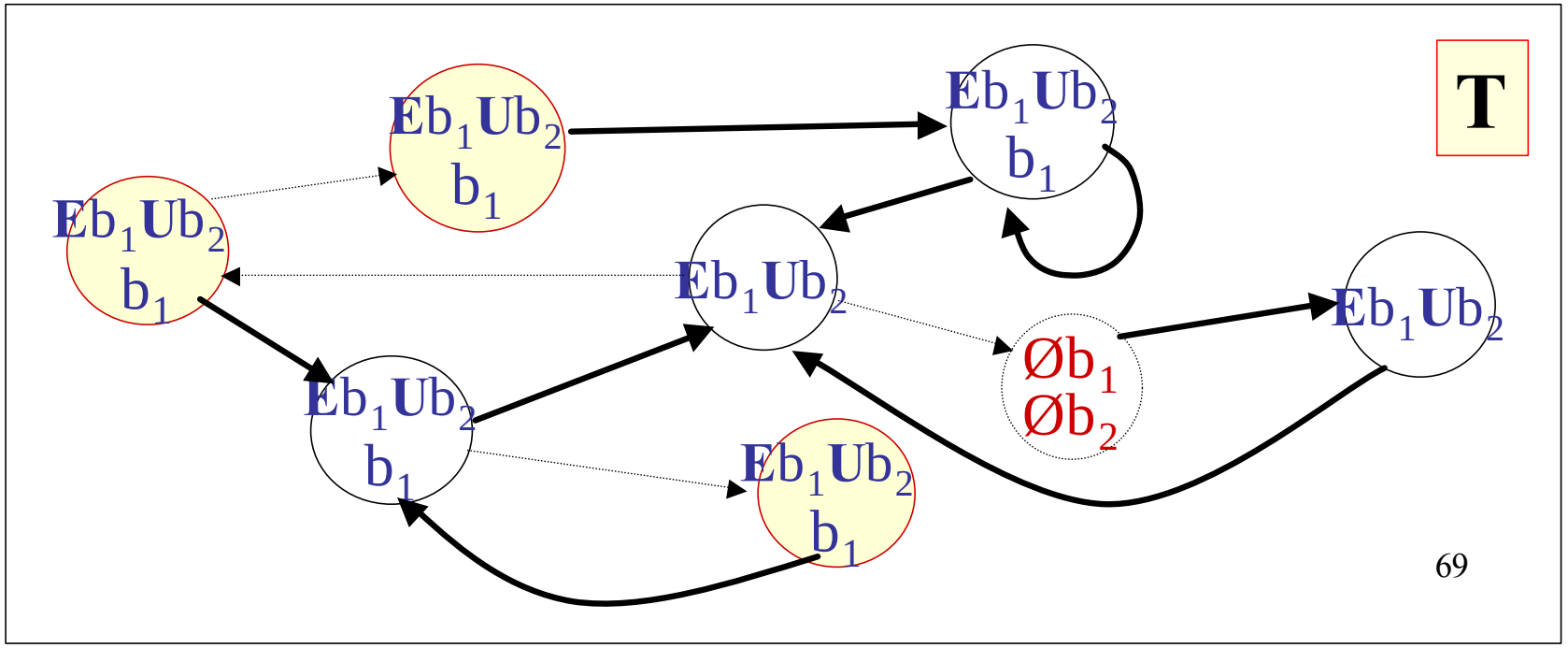
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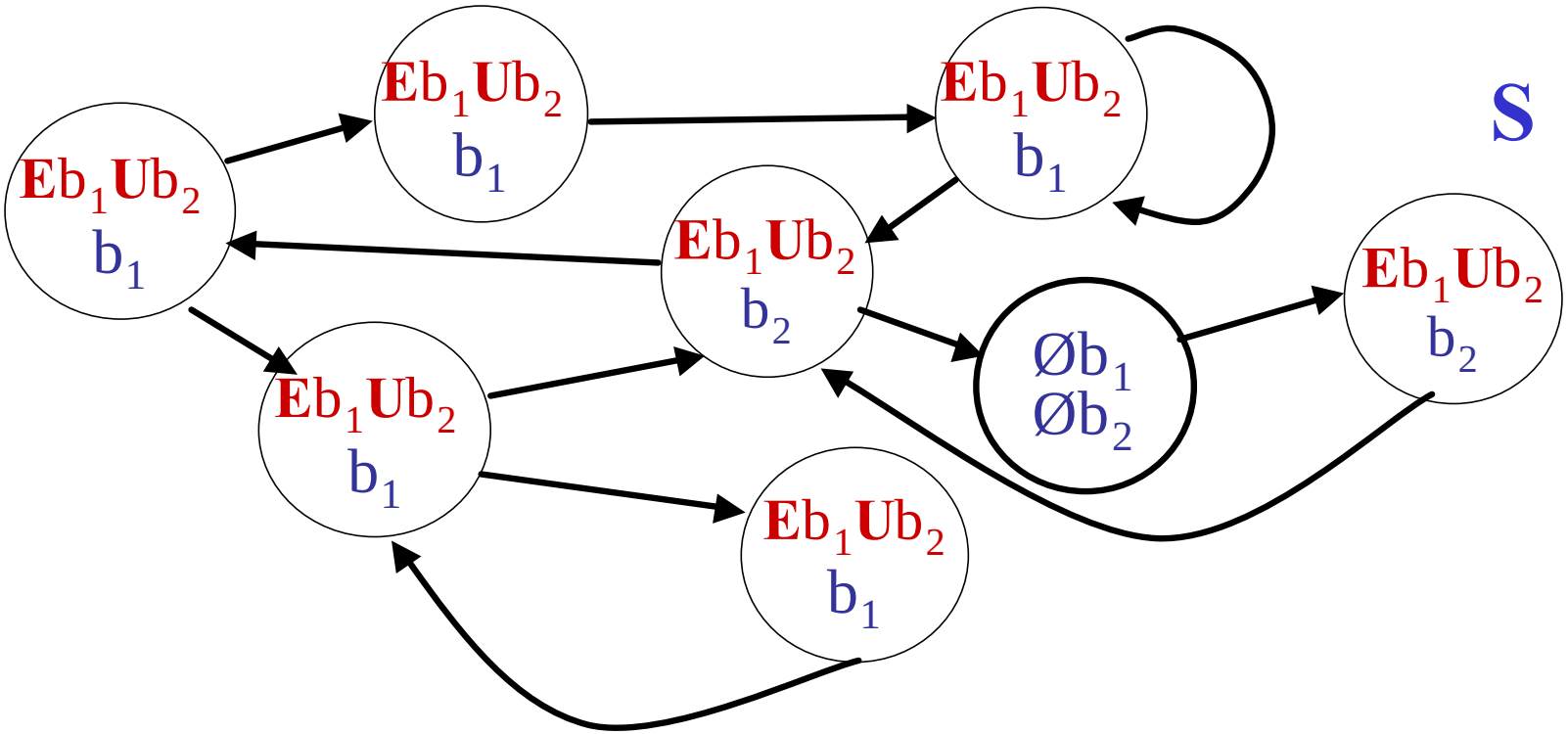
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Computing the labeling for $EU(\beta_1,\beta_2)$

Algorithm Check_$EU(\beta_1,\beta_2)$

T := {**s** | β_2 Î Labels(**s**)};

forall **s** Î **T** do

Labels(**s**) := Labels(**s**) È {**EU**(β_1,β_2)};

while **T** ¹ Æ do

chose **s** Î **T**;

T := **T** \{**s**};

forall **t** Î **S** with (**t**,**s**) Î **R** do

if **EU**(b_1,b_2) Ï Labels(**t**) and β_1 Î Labels(**t**) then

Labels(**t**) := Labels(**t**) È {**EU**(β_1,β_2)};

T := **T** È {**t**};

Complexity: O(|M|)

The Labels Function

- **Stage i +1**:
  - For every **t** Î **S**:
  - If a = **EG**(b) then

    a Î **Labels(t)** ***iff***

    - b Î **Labels(t)** **and** **EG**(b) Î **Labels(s)** **for some s with R(t,s).**

Property of EG(b)

Let **M' = (S',R',L')** be the sub-graph of **M** where

- **S' = { s | M ,s ⊨ b }**
- **R' = $R|_{S' \times S'}$** (the restriction of **R** to **S'**)
- **L' = $L|_{S'}$** (the restriction of **L** to **S'**)

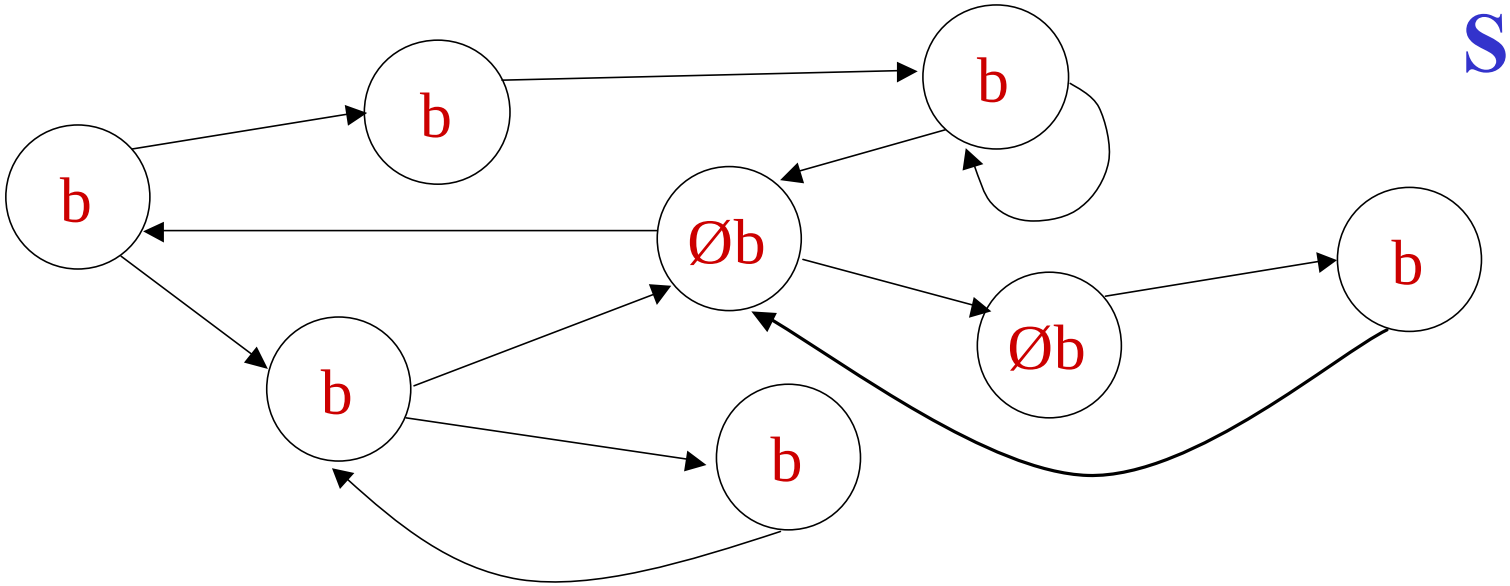
Lemma: M,s ⊨ EG(b) *iff*

1. **s Î S' and**
2. ***there exists a path*** in **M'** leading from **s** to a ***non-trivial strongly connected component*** **C** of the graph **(S',R').**

The Labels Function

- Compute the ***non-trivial strongly connected components*** of the subgraph **S'** whose states all satisfy b
  - **all the states in these components do satisfy EG(b).**
- Traverse backward **R** and label with **EG(b)** the ***states t reaching at least a state s*** labeled with **EG(b)** (note that both **t** and **s** must belong to **S'**).

If t Î S' and R(t,s) then EG(b) Î Labels(t)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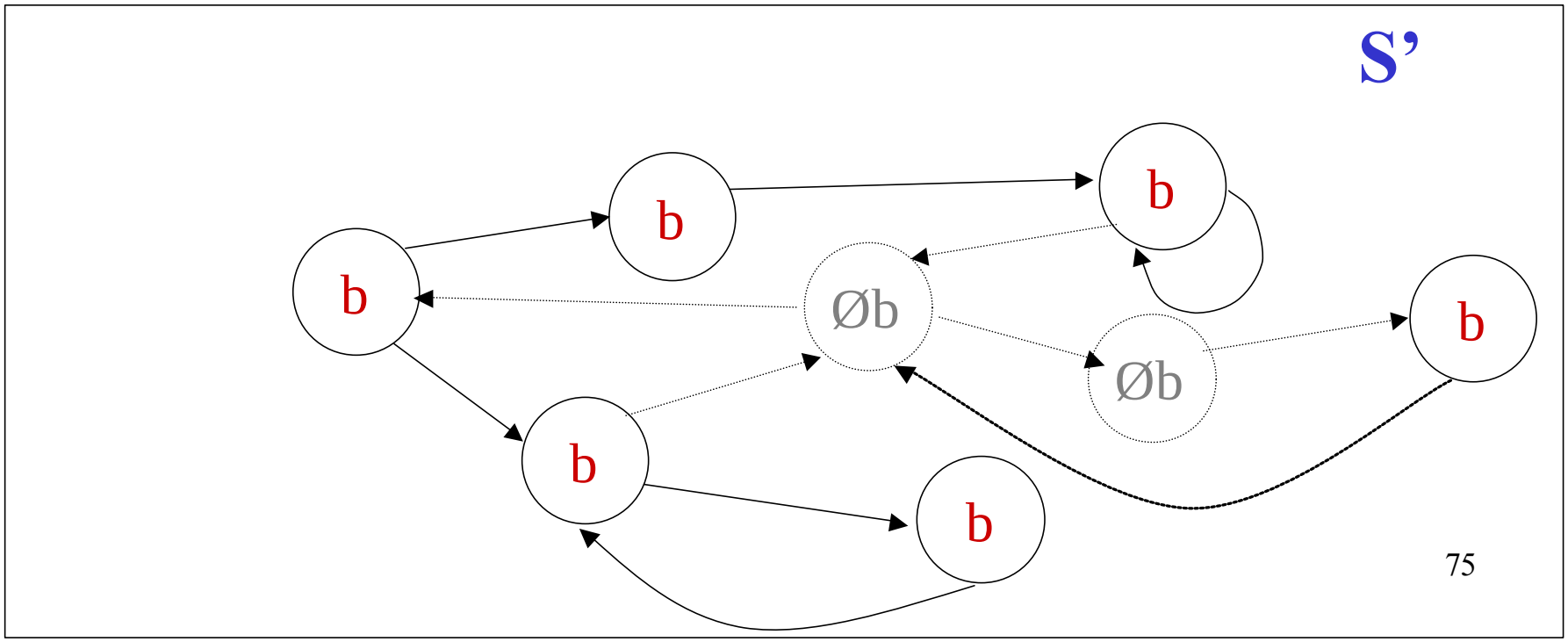
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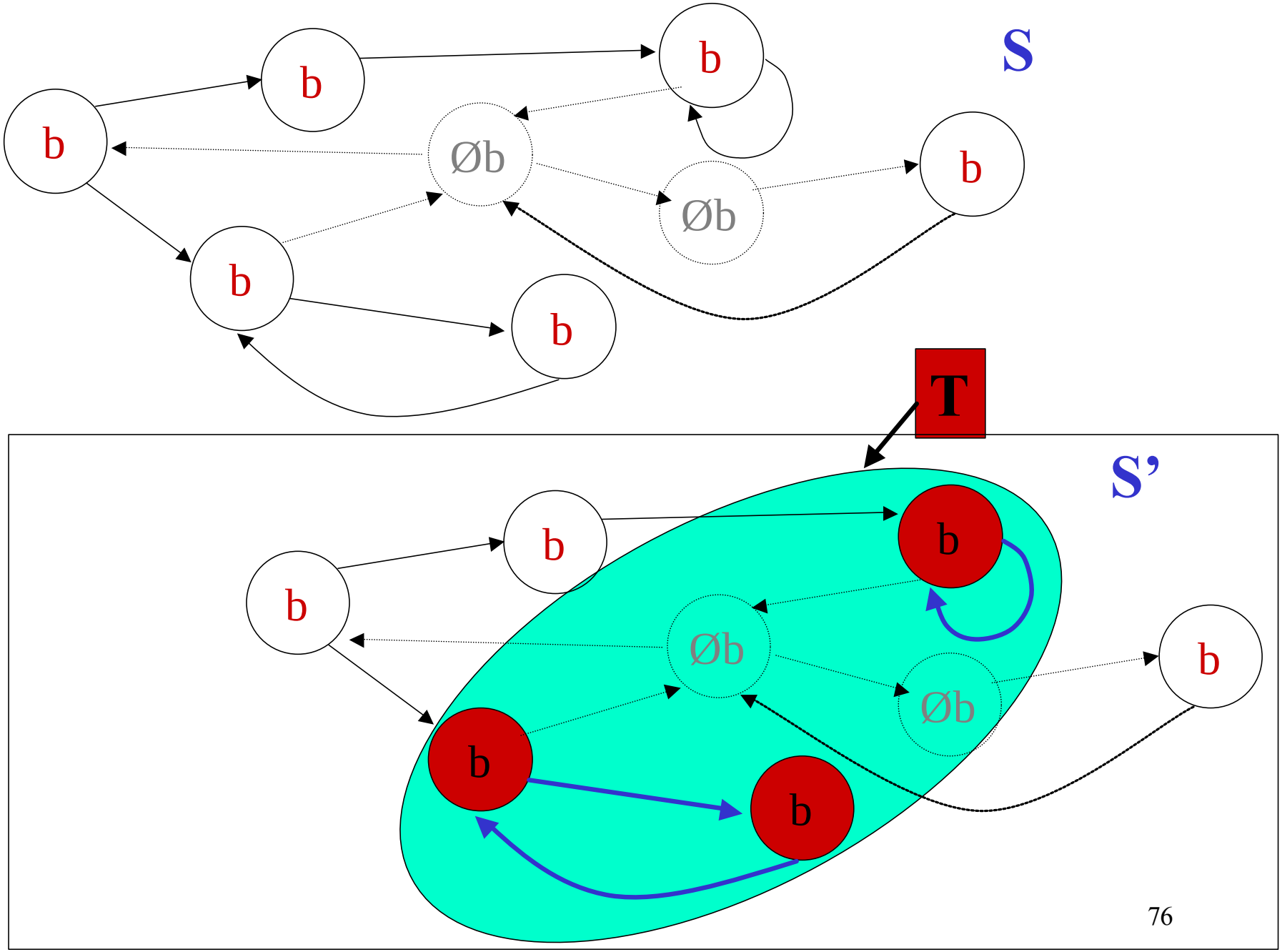
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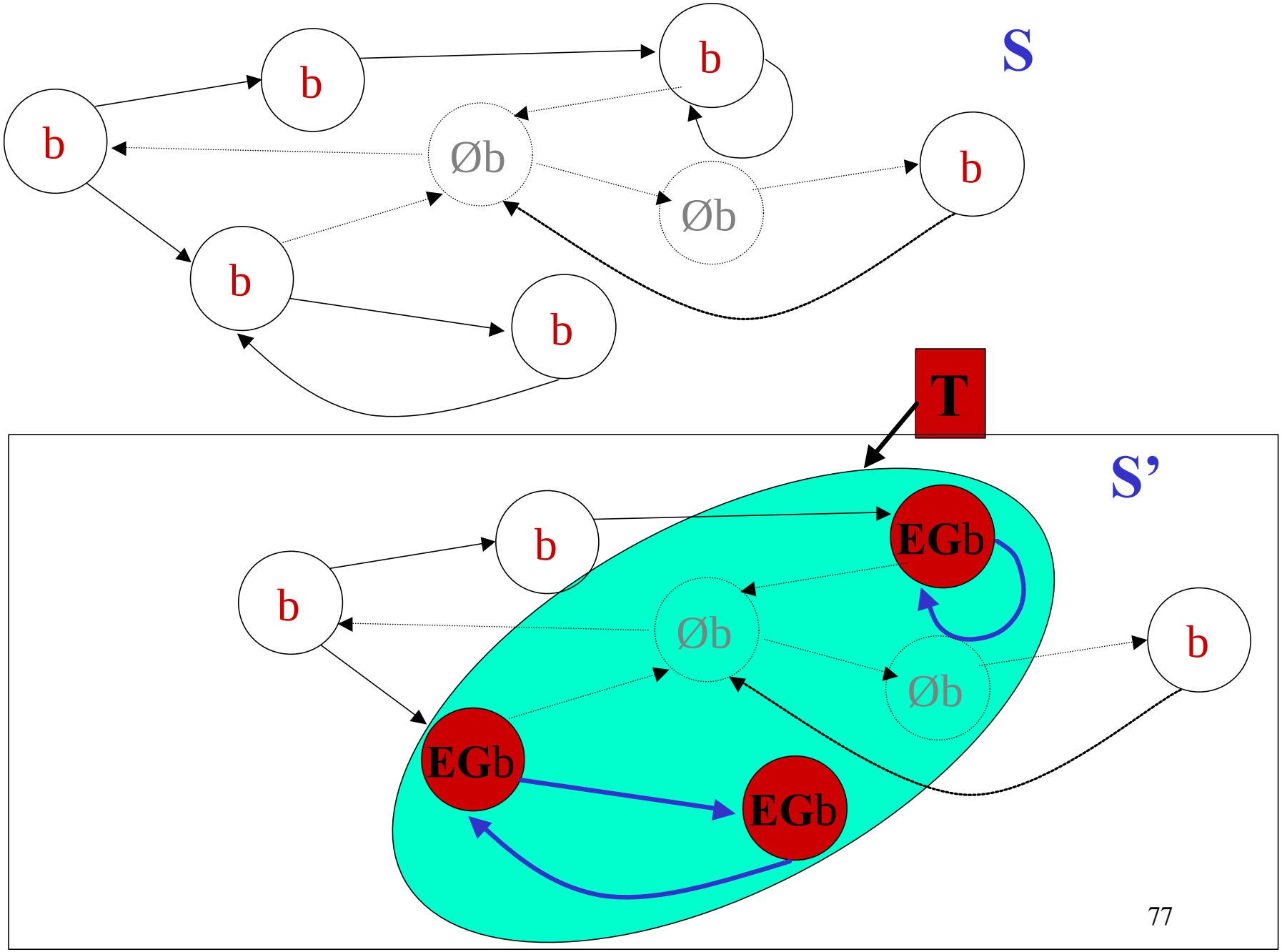
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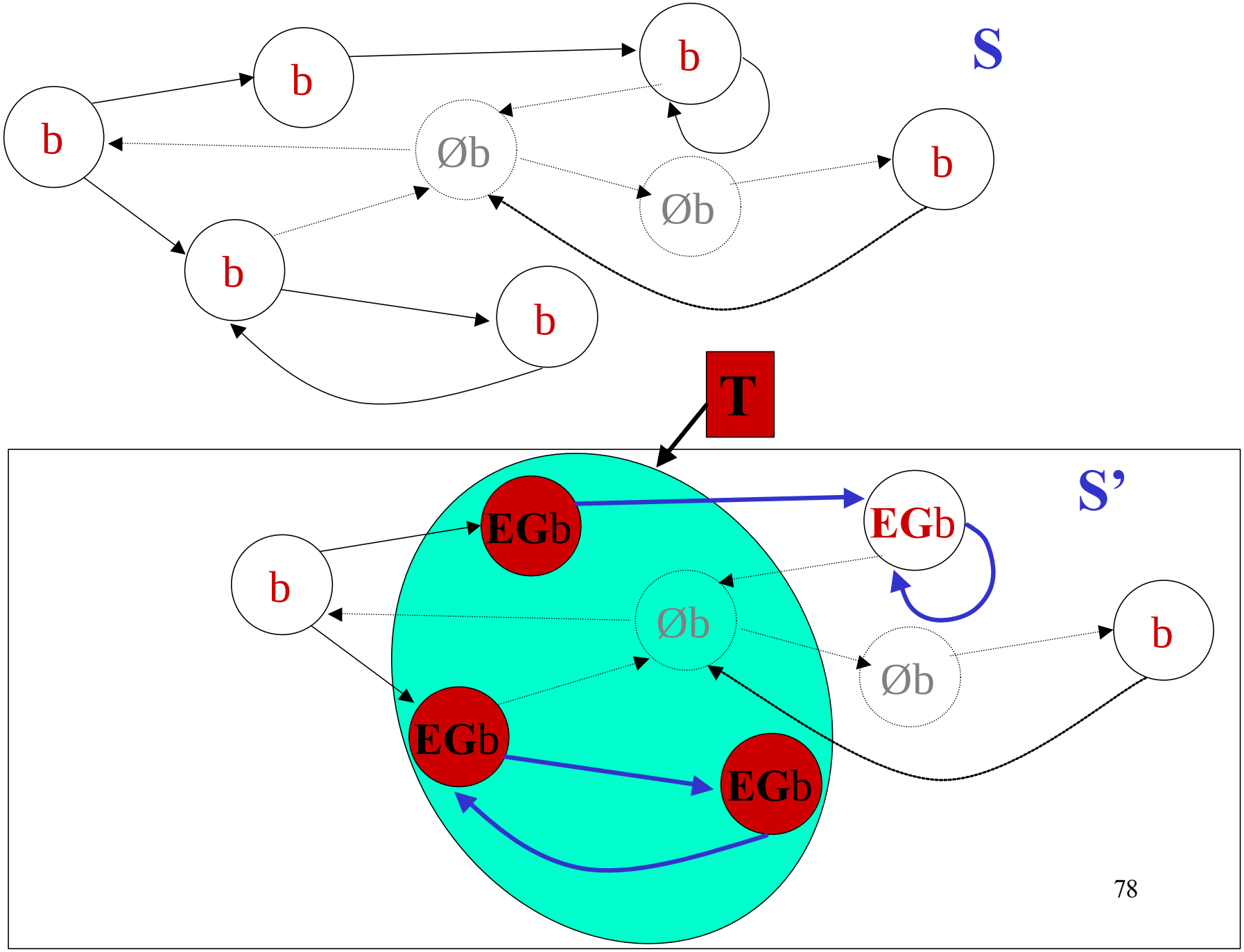
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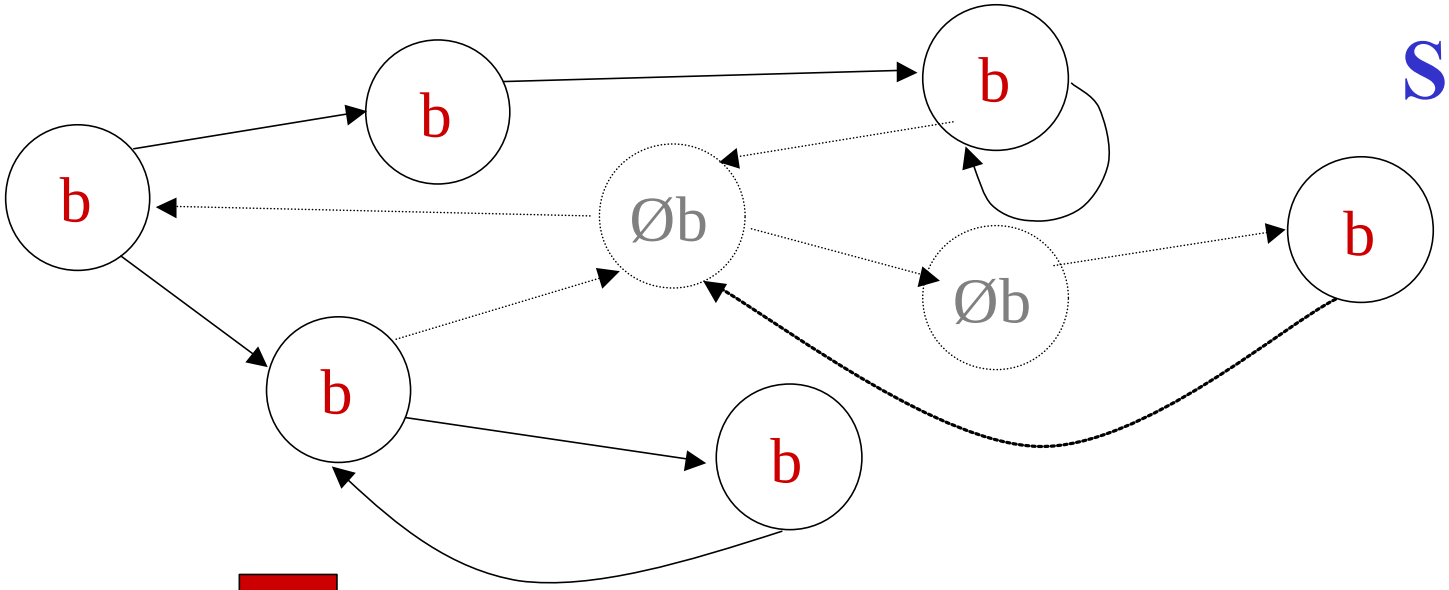
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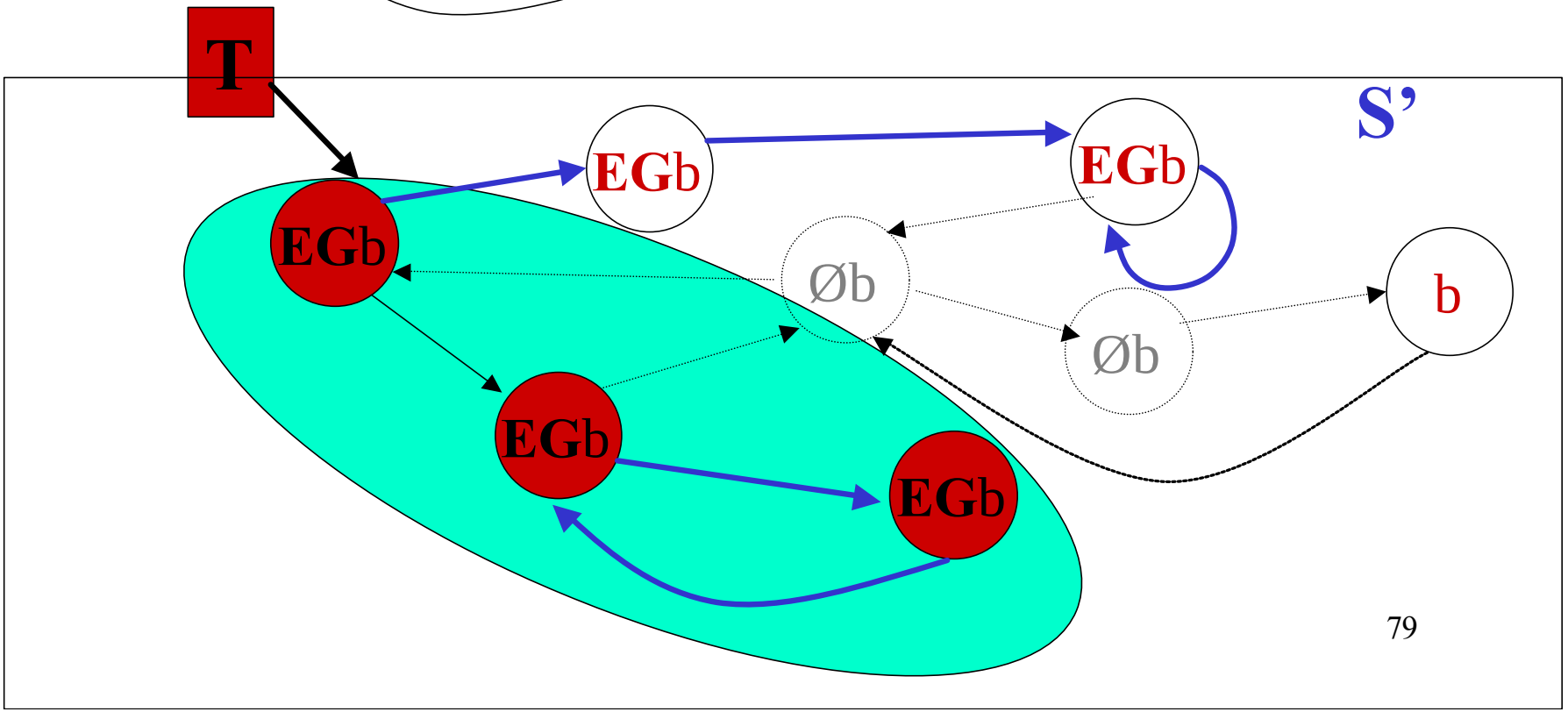
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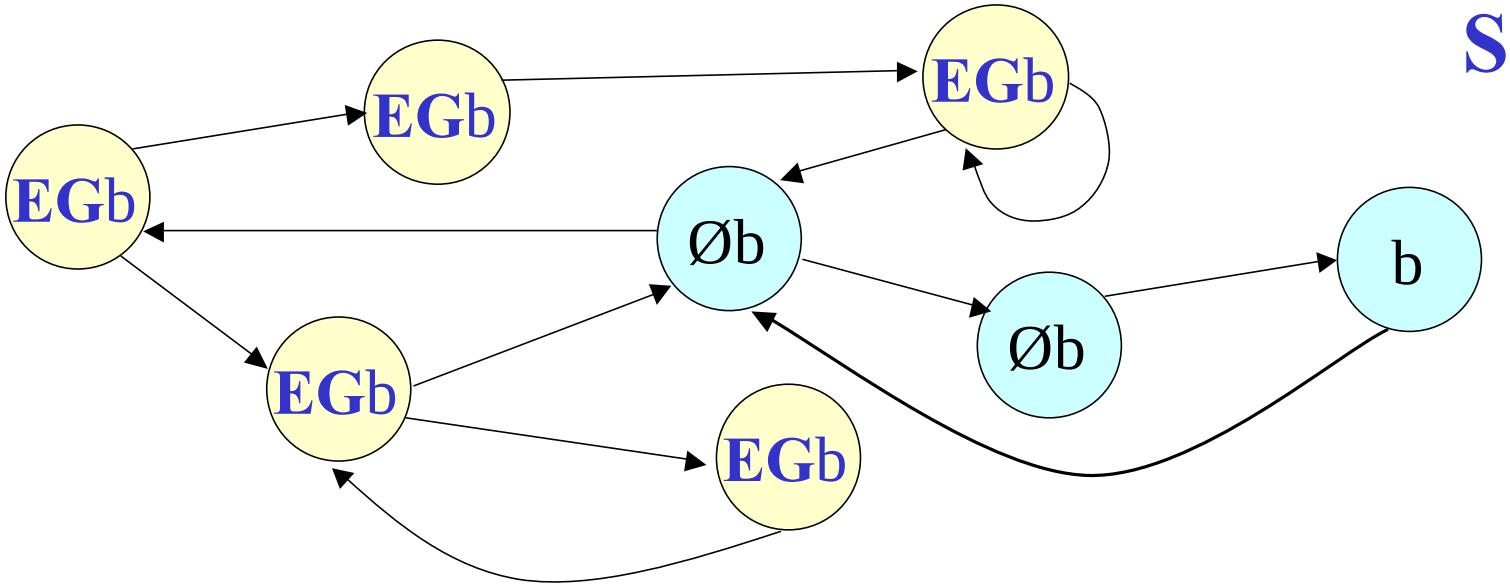
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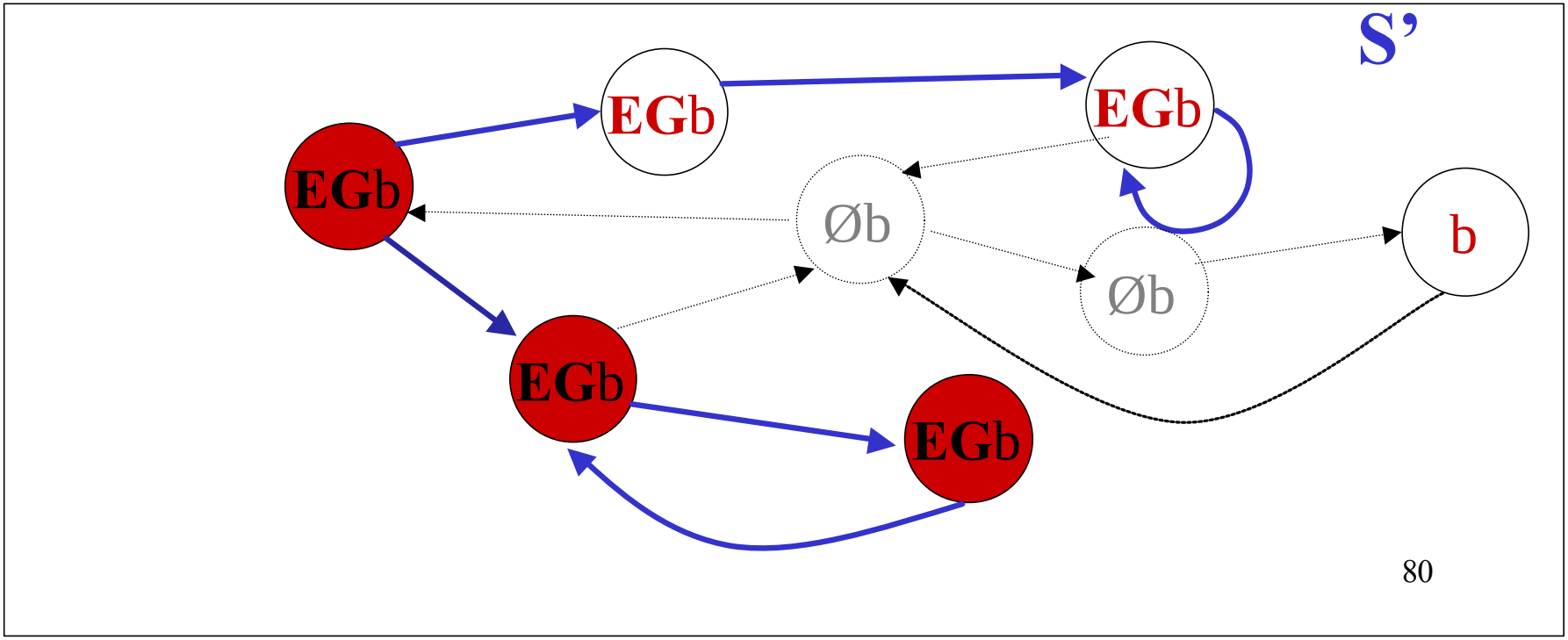
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Computing the labeling for EG(β)

Algorithm Check_EG(β)

Complexity: O(\|M\|)

```
S' := {s | β Î Labels(s)};
SCC := {C | C is a non trivial SCC of S'};
T := È_{C Î SCC} {s | s Î C};
forall s Î T do Labels(s) := Labels(s) È {EG(β)};
while T ¹ Æ do
        chose s Î T;
        T := T \{s};
        forall t Î S' with (t,s) Î R do
            if EG(β) Ï Lables(t) then
                Labels(t) := Labels(t) È {EG(β)};
                T := T È {t};
```

CTL model checking

- The algorithms just presented show that the ***model checking problem*** for ***CTL*** can be solved in ***time linear*** in the size of System **M** and the size of the Property f, namely:

in time **O(|M|×|f|)**

where **|M|** is the ***size of the graph*** underlying **M** and **|f|** is the ***number of subformulae*** of f.

Fixed point characterization

- We will redefine the labeling function in terms of ***fixed point computation***.
- This is a ***nice*** and ***elegant*** algorithmic account.
- It will be used when ***efficient symbolic approach*** will be introduced.

Partial Orders

- A binary relation $\sqsubseteq$ on a set **A** is a ***partial order*** iff $\sqsubseteq$ is ***reflexive***, ***anti-symmetric*** and ***transitive***.
- The pair $\langle A, \sqsubseteq \rangle$ is called a ***partially ordered set*** (or ***poset***).
- **Example:** If **S** is any set and $\subseteq$ is the ordinary subset relation, then $\langle 2^S, \subseteq \rangle$ is a *partially ordered set*.

Upper Bounds

Given **<A, ⊑>** and **A' Í A**

- **a Î A** is an ***upper bound*** of **A'** iff **" a' Î A', a' ⊑ a**
- **a Î A** is a ***least upper bound*** (***lub***) of **A'**, written **⊔A'**, iff
  - **a** is an ***upper bound*** of **A'** and
  - **" a' Î A,** if **a'** is an ***upper bound*** of **A'**, then **a ⊑ a'**

Lower Bounds

Given **<A, ⊑>** and **A' Í A**

- **a Î A** is a ***lower bound*** of **A'** iff **" a' Î A', a ⊑ a'**
- **a Î A** is a ***greatest lower bound*** (***glb***) of **A'**, written **⊓A'**, iff
  - **a** is a ***lower bound*** of **A'** and
  - **" a' Î A,** if **a'** is a ***lower bound*** of **A'**, then **a' ⊑ a**

Complete Lattice

A ***poset*** **<A, ⊑>** is a ***complete lattice*** if, for each **A'** Í **A**, the ***greatest lower bound*** **⊓A'** and the ***least upper bound*** **⊔A'** do exist.

A ***complete lattice*** **<A, ⊑>** has a unique ***greatest element*** **⊔A=⊤** and also a unique ***least element*** **⊓A = ^**.

Complete Lattice

The ***poset*** $<2^S, \subseteq>$ is a ***complete lattice*** where ***intersection*** $\cap$ and ***union*** $\cup$ correspond to $\sqcap$ and $\sqcup$, respectively.

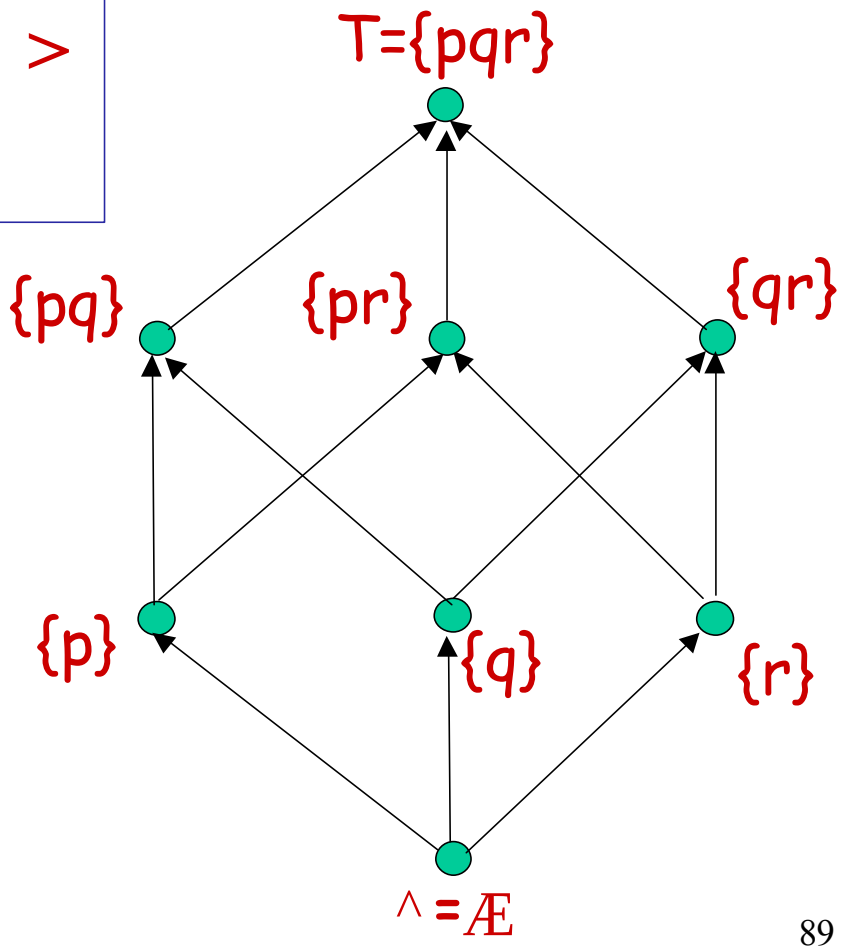
Any two subset of **S** have a ***least upper*** and a ***greatest lower bound***.

Example: **S={a,b,c,d}**. For **{a,c}** and **{b,c}** the ***lub*** is **{c}**, while the ***glb*** is **{a,b,c}**.

There is a unique ***greatest element*** $\cup 2^S = S$ and a unique ***least element*** $\cap 2^S = \emptyset$.

Example of a complete lattice

The complete lattice $\langle 2^S, \subseteq \rangle$ when S is the set {p,q,r}.



Monotonic functions

- A function **F: A ⟶ A** is ***monotonic*** if for each **a,b Î A**, **a ⊑ b** implies **F(a) ⊑ F(b)**.
- In other words, a function **F** is monotonic if it ***preserves the ordering*** ⊑.

Fixed points

- Given a function **F: A ⟶ A**, an element **a Î A** is a ***fixed point*** of **F** if **F(a) = a**.
- **a Î A** is called the ***least fixed point*** of **F** (**mx.F(x)**), if for all **a' Î A** such that **F(a') = a'**, then **a ⊑ a'**.
- **a Î A** is called the ***greatest fixed point*** of **F** (**nx.F(x)**), if for all **a' Î A** such that **F(a') = a'**, then **a' ⊑ a**.

Tarski's Fixed Point theorem

THEOREM: Let **<A, ⊑>** be a ***complete lattice***, and **F: A ⟶ A** a monotonic function. Then **F** has a ***least*** and a ***greatest fixed point*** given, respectively, by:

- **μx.F(x) = ⊓{x Î A | F(x) ⊑ x}**
- **νx.F(x) = ⊔{x Î A| x ⊑ F(x)}**

Fixed point in finite lattices

Let $\langle A, \sqsubseteq \rangle$ be a ***finite complete lattice***, and $F: A \to A$ be a monotonic function.

The ***least element*** of **A**

Then the ***least fixed point*** for **F** is obtained as

$$\mu x.F(x) = F^m(\bot)$$

for some **m**, where $F^0(\bot) = \bot$, and $F^{n+1}(\bot) = F(F^n(\bot))$.

Moreover, the ***greatest fixed point*** for **F** is obtained as

$$\nu x.F(x) = F^k(\top)$$

for some **k**, where $F^0(\top) = \top$, and $F^{n+1}(\top) = F(F^n(\top))$.

The ***greatest element*** of **A**

Generic fixed point algorithm

Algorithm Compute_lfp(**F**:function)

$\mathbf{X_0}$:= ^;

$\mathbf{X_1}$:= $\mathbf{F(X_0)}$;

j=1;

while $\mathbf{X_j}$ ^{1} $\mathbf{X_{j-1}}$

$\mathbf{j := j+1}$;

$\mathbf{X_j := F(X_{j-1})}$;

return $\mathbf{X_j}$

CTL and complete lattices

- Given a Kripke structure $\mathbf{M=<S,S_0,R,L,AP>}$. We will then consider the ***poset*** $\mathbf{<2^S, \sqsubseteq>}$.
- $\mathbf{<2^S, \sqsubseteq>}$ is clearly a ***complete lattice*** (with respect to intersection and union).
- We will identify a ***CTL formula*** with the ***set of states*** which ***satisfy it***.
- In this way we can define ***temporal operators*** as ***functions*** on the ***complete lattice*** $\mathbf{<2^S, \sqsubseteq>}$.

Denotation of a CTL formula

- Given a formula f, let us define its ***denotation*** (in **M**), in symbols |[f]|, as the set of states satisfying the formula:

$$|[f]| = \{ s \mid M,s \vDash f \}$$

- We could then define the cpo **<*CTL*, ⊑ >** by:

$$f \sqsubseteq y \quad \textit{iff} \quad |[f]| \subseteq |[y]|$$

Denotation of a CTL formula

- Given the ***denotation*** of a formula

$$|[f]| = \{ s \mid M,s \models f \}$$

- We could then define the cpo $\langle CTL, \sqsubseteq \rangle$ by:

$$f \sqsubseteq y \quad \textit{iff} \quad |[f]| \subseteq |[y]|$$

- Then $|[\wedge]| = \mathcal{P}$; $|[\top]| = S$;
- $|[p]| = \{ s \mid p \in L(s) \}$;
- $|[\neg f]| = S \setminus |[f]|$;
- $|[f \vee y]| = |[f]| \cup |[y]|$;
- $|[f \wedge y]| = |[f]| \cap |[y]|$;

CTL is closed under ***conjunction*** and ***disjunction***, therefore for any pair of formulae the ***upper*** and ***lower bound*** do exist.

Denotation of a CTL formula

- Given a formula f, let us define its ***denotation*** (in **M**), in symbols **|[f]|**, as the set of states satisfying the formula:

$$|[f]| = \{ s \mid M,s \vDash f \}$$

- ….
- **|[EX f]| = { s | $t. (t Î |[f]| Ç R(s)) }**
- for the other **temporal operators** we would need to use **fixed points….**

Fixed point characterization of $\mathbf{EU(b_1,b_2)}$

- $\mathbf{EU(b_1,b_2)} \circ b_2 \acute{U} (b_1 \grave{U} \mathbf{EX\ EU(b_1, b_2)})$
- $|[\mathbf{EU(b_1,b_2)}]| = \mu\mathbf{Z.}(|[b_2]| \grave{E} (|[b_1]| \c{C} |[\mathbf{EX\ Z}]|))$
- $|[\mathbf{EU(b_1,b_2)}]| =$

  $\mu\mathbf{Z.}(|[b_2]| \grave{E} (|[b_1]| \c{C} \{ \mathbf{s} \mid \$\mathbf{t}\ \hat{I}\ \mathbf{Z} \c{C} \mathbf{R(s)} \}))$

Fixed point characterization of $\mathbf{EU(b_1,b_2)}$

Lemma: Let

$$\mathbf{F(Z) = (|[b_2]| \cup (|[b_1]| \cap \{ s \mid \$t \in Z \cap R(s) \}))}$$

then **F** is a ***monotonic function***, i.e.

$$\mathbf{Z_1 \subseteq Z_2 \text{ implies } F(Z_1) \subseteq F(Z_2)}$$

Fixed point characterization of $\mathbf{EU(b_1,b_2)}$

Theorem:

$$|[\mathbf{EU}(b_1,b_2)]| = \mu Z.(|[b_2]| \cup (|[b_1]| \cap \{ s \mid \exists t \in Z \cap R(s) \}))$$

in other words:

$$\mu Z.(|[b_2]| \cup (|[b_1]| \cap \{ s \mid \exists t \in Z \cap R(s)\})) \subseteq |[\mathbf{EU}(b_1,b_2)]|$$

and

$$|[\mathbf{EU}(b_1,b_2)]| \subseteq \mu Z.(|[b_2]| \cup (|[b_1]| \cap \{ s \mid \exists t \in Z \cap R(s)\}))$$

Computing fixed point for $EU(\beta_1,\beta_2)$

Algorithm Compute_EU(β_1,β_2)

X_0 := |[^]|; /* i.e. X_0 := Æ */

X_1 := |[β_2]| È (|[β_1]| Ç X_0); /* i.e. X_1 := |[β_2]| */

j=1;

while X_j ¹ X_{j-1}

$j := j+1$; T := X_{j-1}; X := Æ

while T ¹ Æ do

chose s Î T;

T := T \{s};

forall t such that s Î R(t) do

X := X È {t};

X_j := |[β_2]| È (|[β_1]| Ç X)

This computes
$X = EX\ X_{j-1}$

Computing fixed point for $\mathbf{EU(b_1,b_2)}$

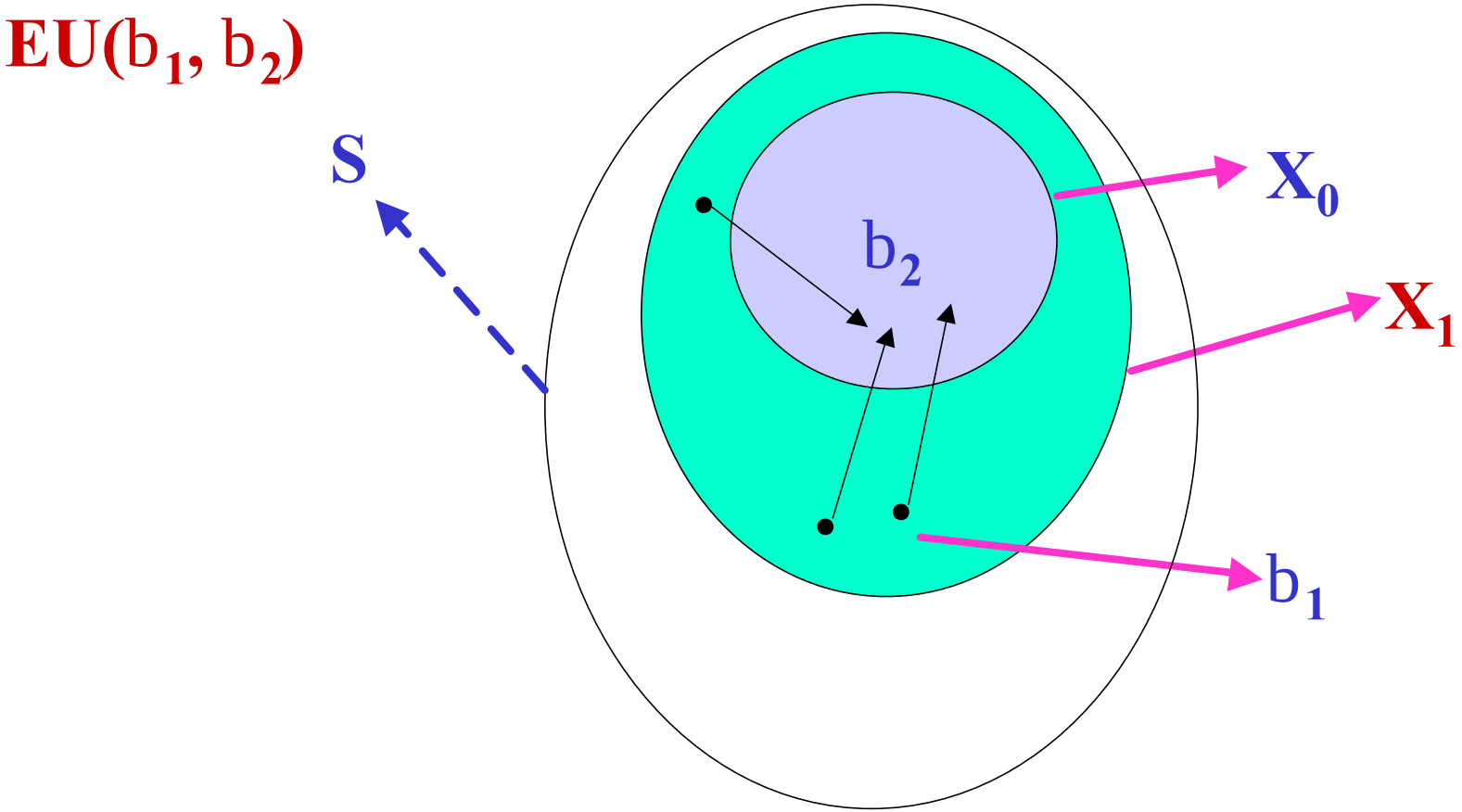
To compute $\mathbf{|[EU(b_1,b_2)]|}$ we can ***construct inductively the set*** of states $\mathbf{X_j}$ as follows:

- $\mathbf{X_1 = \{s \mid s \in |[b_2]|\}}$.
- $\mathbf{X_{j+1} = X_j \cup \{ s \mid s \in |[b_1]|}$ and $\mathbf{R(s,t)}$ for some $\mathbf{t \in X_j\}}$

$\mathbf{|[EU(b_1,b_2)]|}$ is then the set $\mathbf{X}$ such that $\mathbf{X = X_n}$ for $\mathbf{n}$ such that $\mathbf{X_{n+1} = X_n}$.

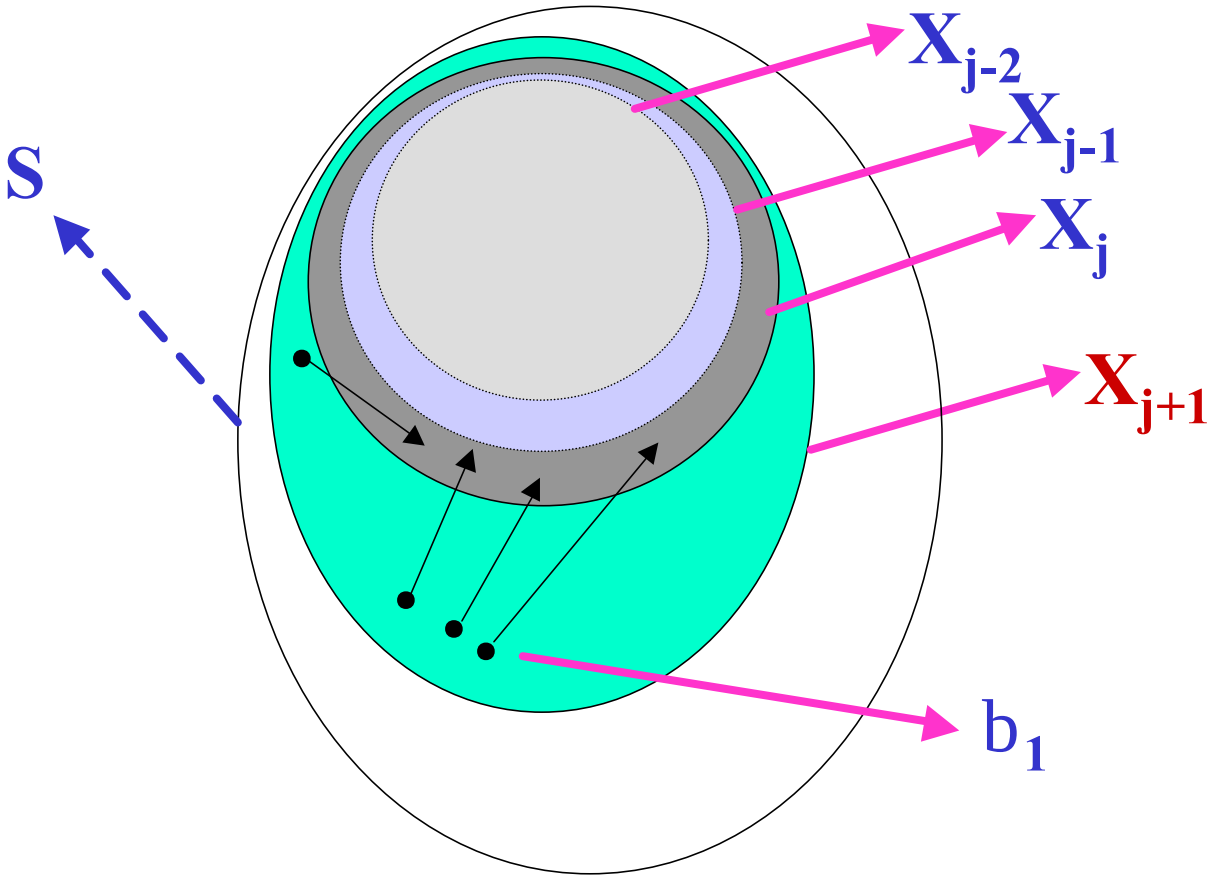
Notice that $\mathbf{n}$ ***must exist*** by ***Tarski's Theorem*** since $\mathbf{X_j \subseteq X_{j+1} \subseteq S}$ (and $\mathbf{S}$ is finite!)

From X_0 to X_1



From X_j to X_{j+1}

EU(b_1, b_2)



Computing fixed point for $EU(\beta_1,\beta_2)$

```
Algorithm Compute_EU(β1,β2)
    X1 := |[β2]|;
    j=1;
    repeat
        j := j+1; T := X := Xj-1;
        while T ¹ Æ do
            chose s Î T;
            T := T \{s};
            forall t such that s Î R(t) do
                if t Î |[β1]| then      /* t Î |[β1]| Ç EX Xj-1 */
                    X := X È {t};
        Xj = X ;
    until Xj-1 = Xj
```

Fixed point characterization of EG(β)

- **EG**(b) º b Ù **EX EG**(b)
- |[**EG**(b)]| = n**Z.**(|[b]| Ç |[**EX Z**]|))
- |[**EG**(b)]| =

  n**Z.**(|[b]| Ç { **s** | $**t** Î **Z** Ç **R**(**s**) })

Computing the fixed point for EG(β)

Algorithm Compute_EG(β)

X_0 := |[⊤]|; /* i.e. X_0 := S */

X_1 := |[β]| Ç X_0; /* i.e. X_1 := |[β]| */

j=1;

while X_j ¹ X_{j-1}

j := j+1; T := X_{j-1}; X := Æ;

while T ¹ Æ do

chose s Î T;

T := T \{s};

forall t such that s Î R(t)

X_j := X_j È {t};

X_j := |[β]| Ç X_j

X = EX X_{j-1}

The Labels function

- To compute $|[\mathbf{EGb}]|$ we can ***construct inductively the set*** of states $\mathbf{X_j}$ as follows:

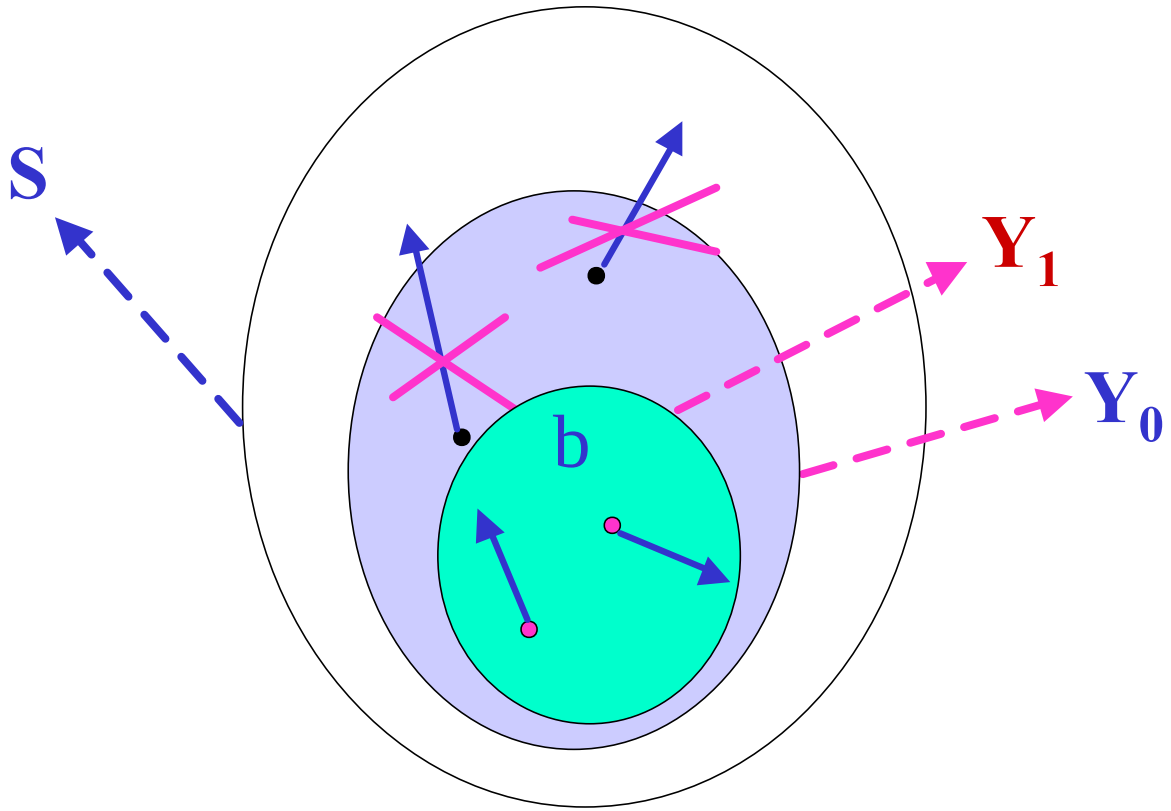
  - $\mathbf{X_1} = |[b]|$.
  - $\mathbf{X_{j+1}} = \mathbf{X_j} - \{ s \mid s \in \mathbf{X_j}$ and
    there does not exist $t \in \mathbf{X_j}$
    such that $\mathbf{R(s, t)}\}$

$|[\mathbf{EGb}]|$ is then the set $\mathbf{X}$ such that $\mathbf{X} = \mathbf{X_n}$ for $\mathbf{m}$ such that $\mathbf{X_{n+1}} = \mathbf{X_n}$.

- Notice that $\mathbf{m}$ ***must exists*** by ***Tarski's Theorem*** since $\emptyset \subseteq \mathbf{X_{j+1}} \subseteq \mathbf{X_j}$

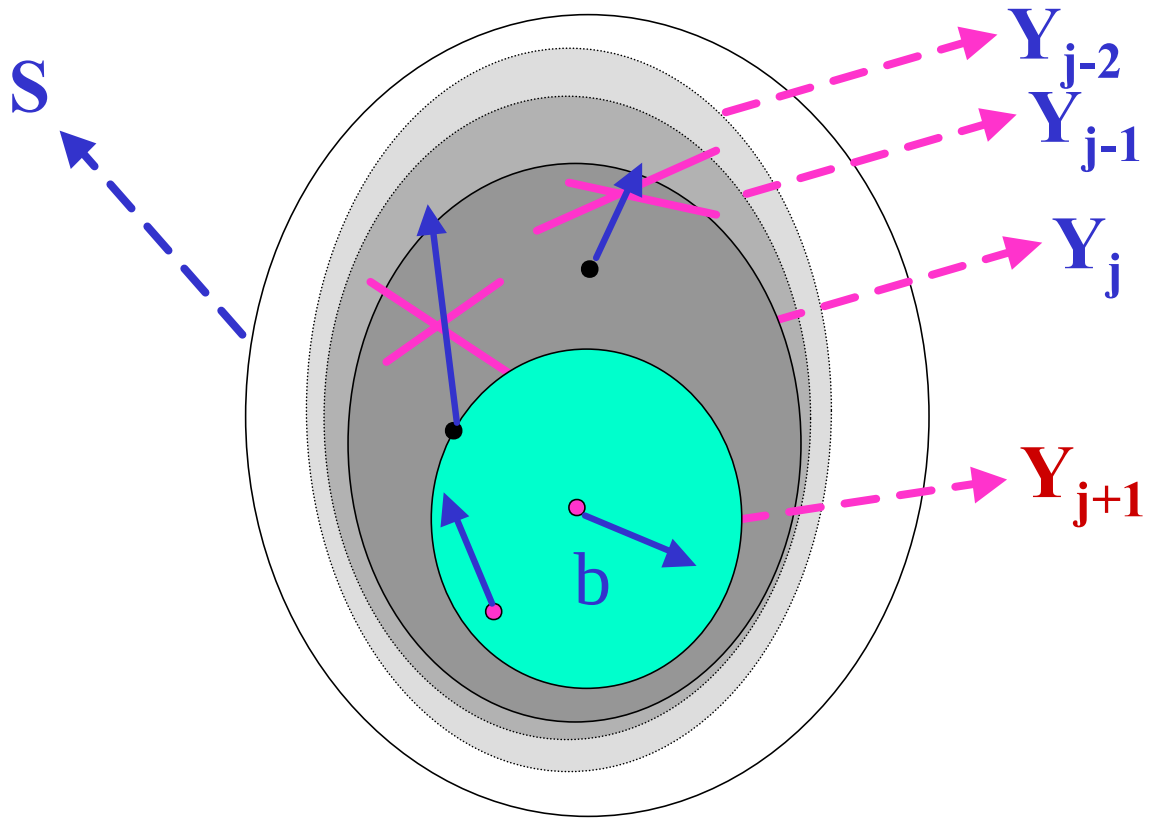
From Y_0 to Y_1

EGb



From Y_j to Y_{j+1}

EGb



Computing the fixed point for EG(β)

Algorithm Compute_EG(β)

X_1 := |[β]|;

j=1;

repeat

j := j+1; T := X_j := X_{j-1};

while T ¹ Æ do

chose s Î T;

T := T \{s};

if for no t Î R(s), t Î X_{j-1} then

X_j := X_j - {s};

until X_j = X_{j-1};