Tecniche di Specifica e di Verifica

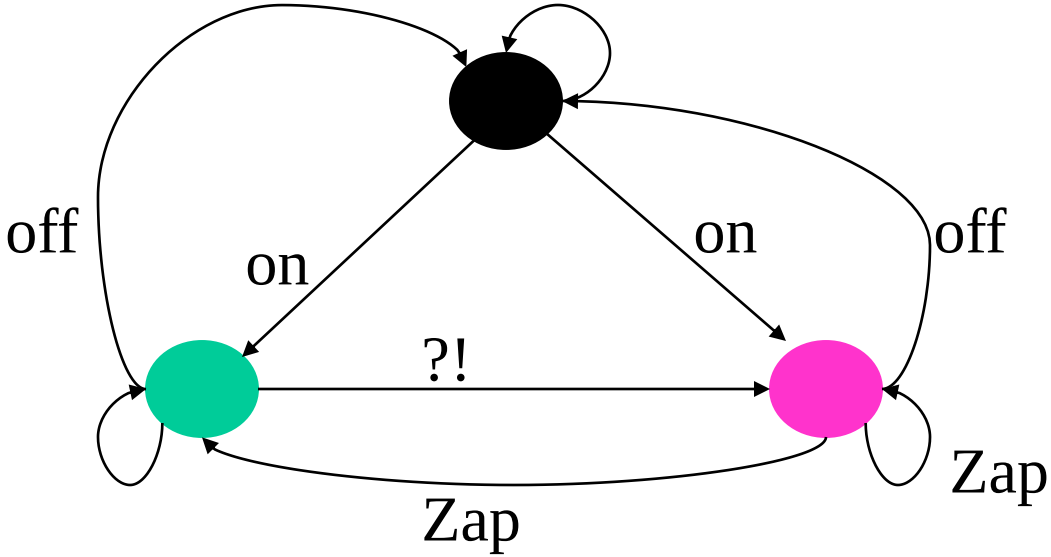
Branching Time Temporal Logics I

Outline

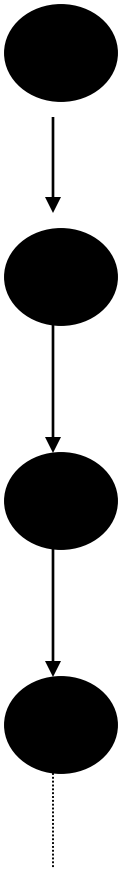
- ***CTL*** (**C**omputation **T**ree **L**ogic)
  - **Branching Time**
  - Unwindings --- computation trees
  - Syntax and semantics of CTL.

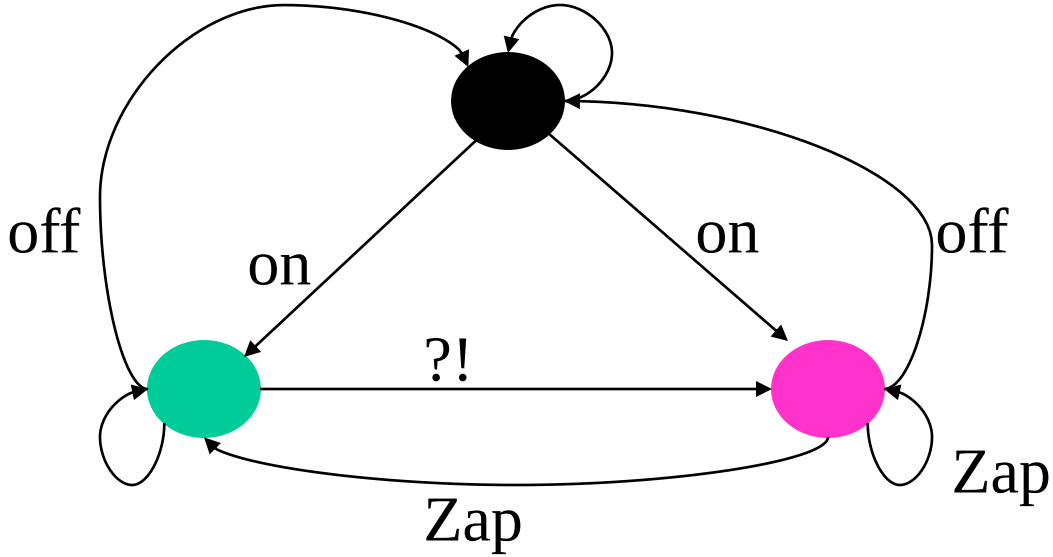
Branching Time Structures

- **Linear Time**:
  - A ***computation*** at its first state satisfies a property.
  - Property ---- **LTL** formula
- **Branching Time**
  - The ***computation tree*** at its root satisfies a property.
  - Property: **CTL** (**CTL***, **μ-calculus**) formula.
  - **Computation Tree**
    - **All *computations*** starting from a state ***glued together*** (to form a tree structure).
- In branching time, ***the decisions*** taken during a run are taken into account.



The TV Example

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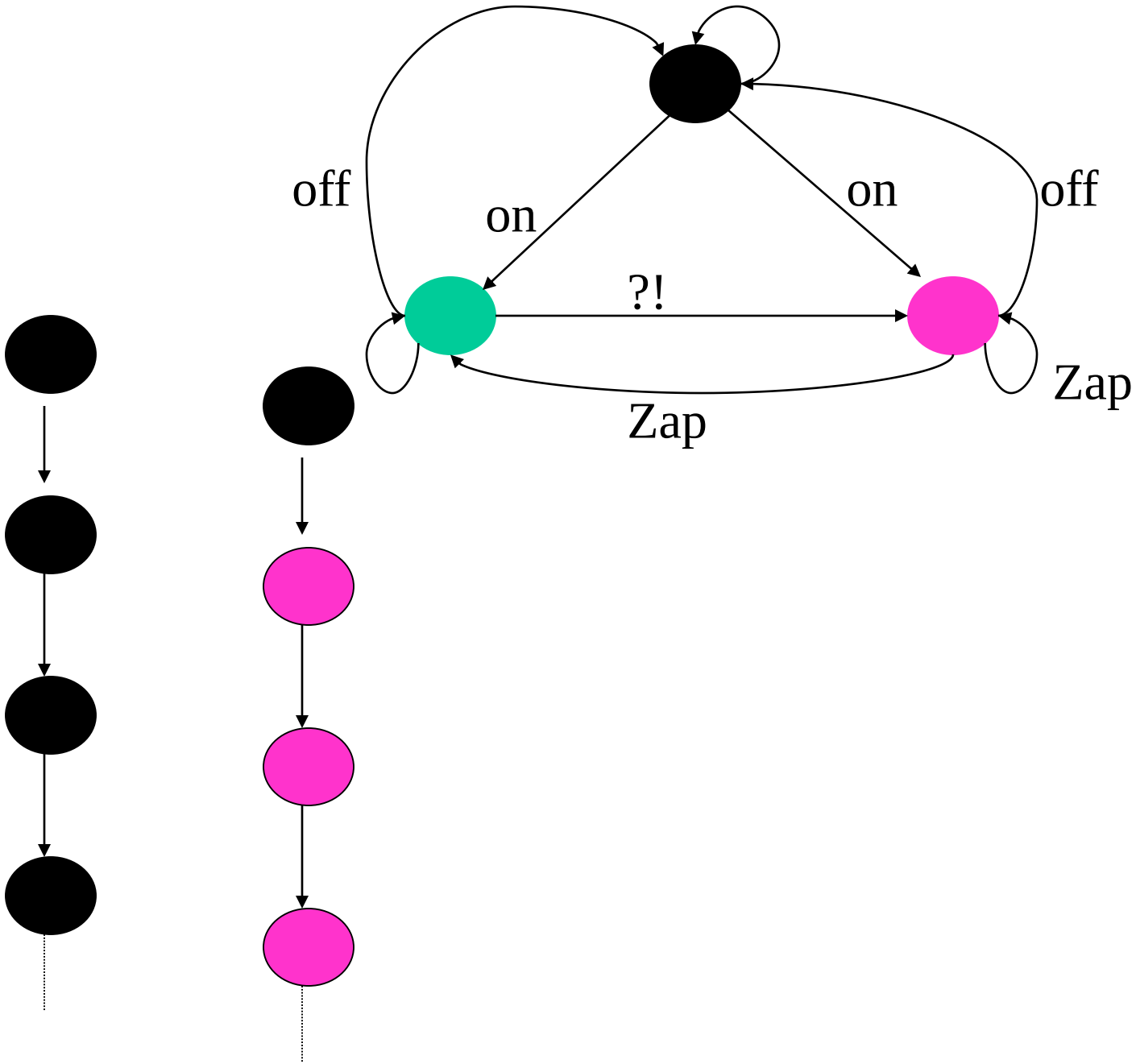
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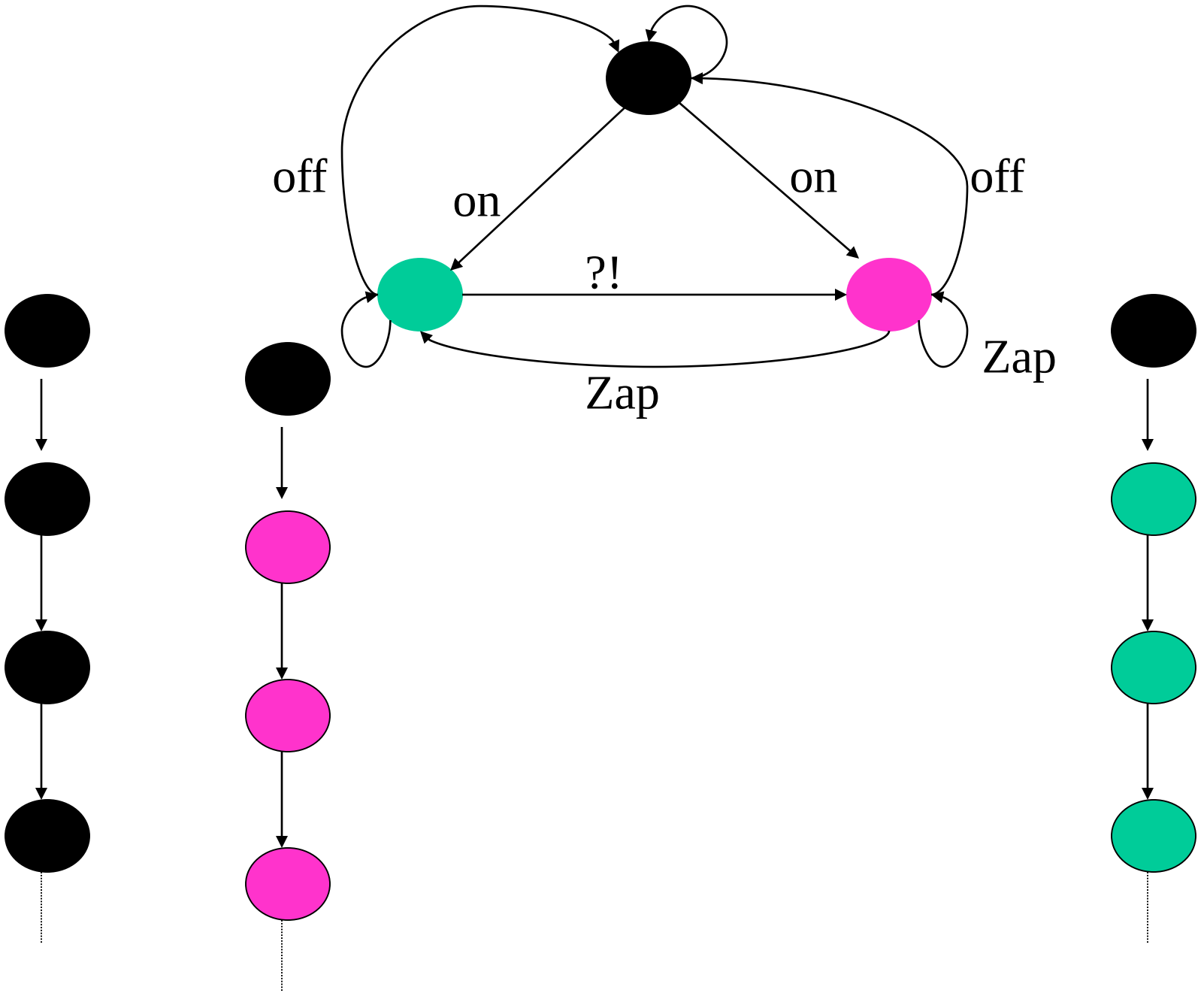
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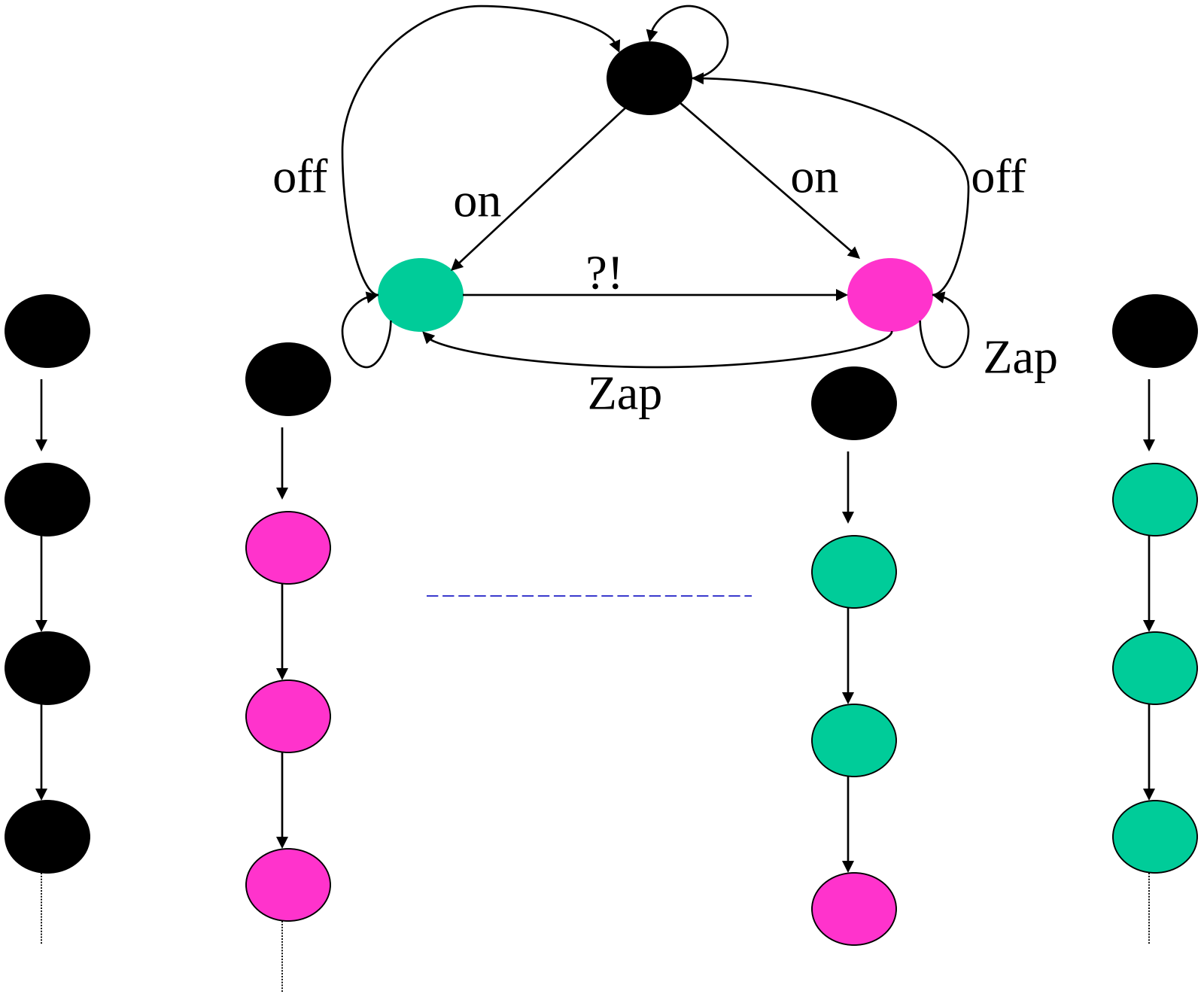
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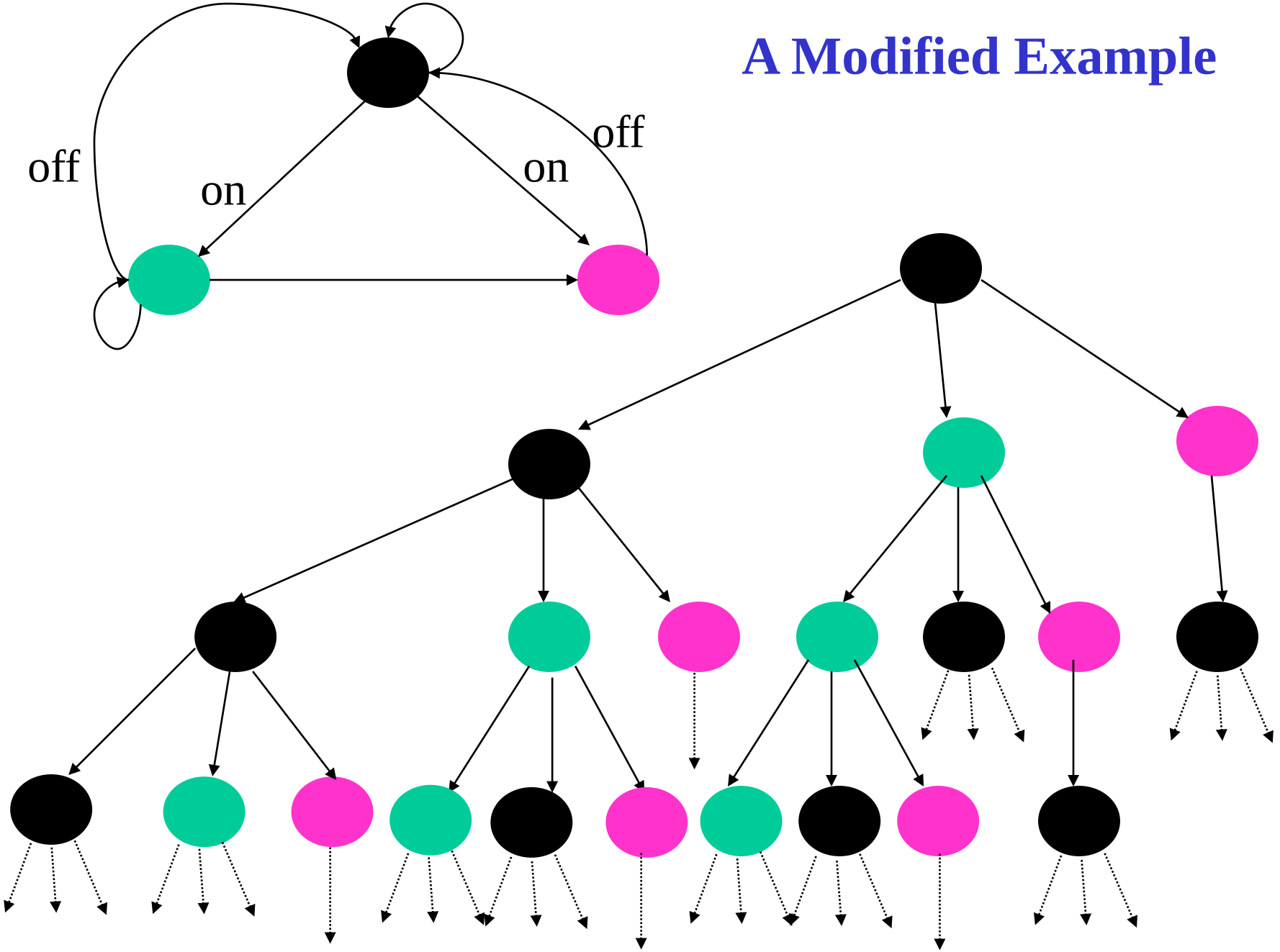
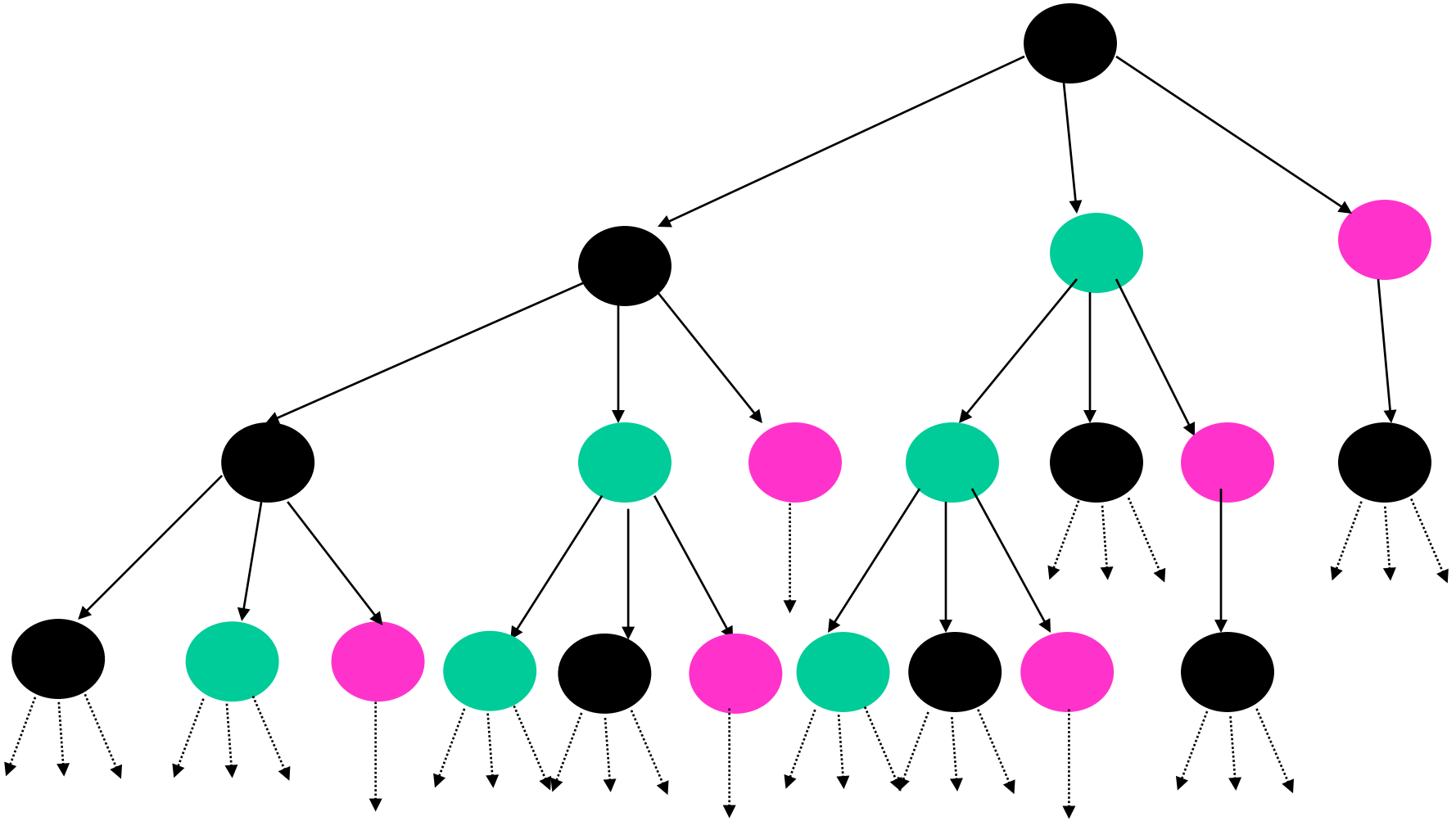
A Modified Example



For every path **π** and every state **s** on that path, there is a path **π'** starting from **s** and a state **s'** on **π'** which is **green**.



Branching Time Temporal Logic

- **$K = (S, S_0, R, AP, L)$**
- **$K, s \models \psi$** -- the computation tree rooted at **s** satisfies **ψ**.
- **$K \models \psi$** iff **$K, s_0 \models \psi$** for every **$s_0 \in S_0$**.
- **Branching Time Temporal Logics**:
  - **CTL**
  - **CTL***
  - (The modal) **μ-calculus**

Unwinding

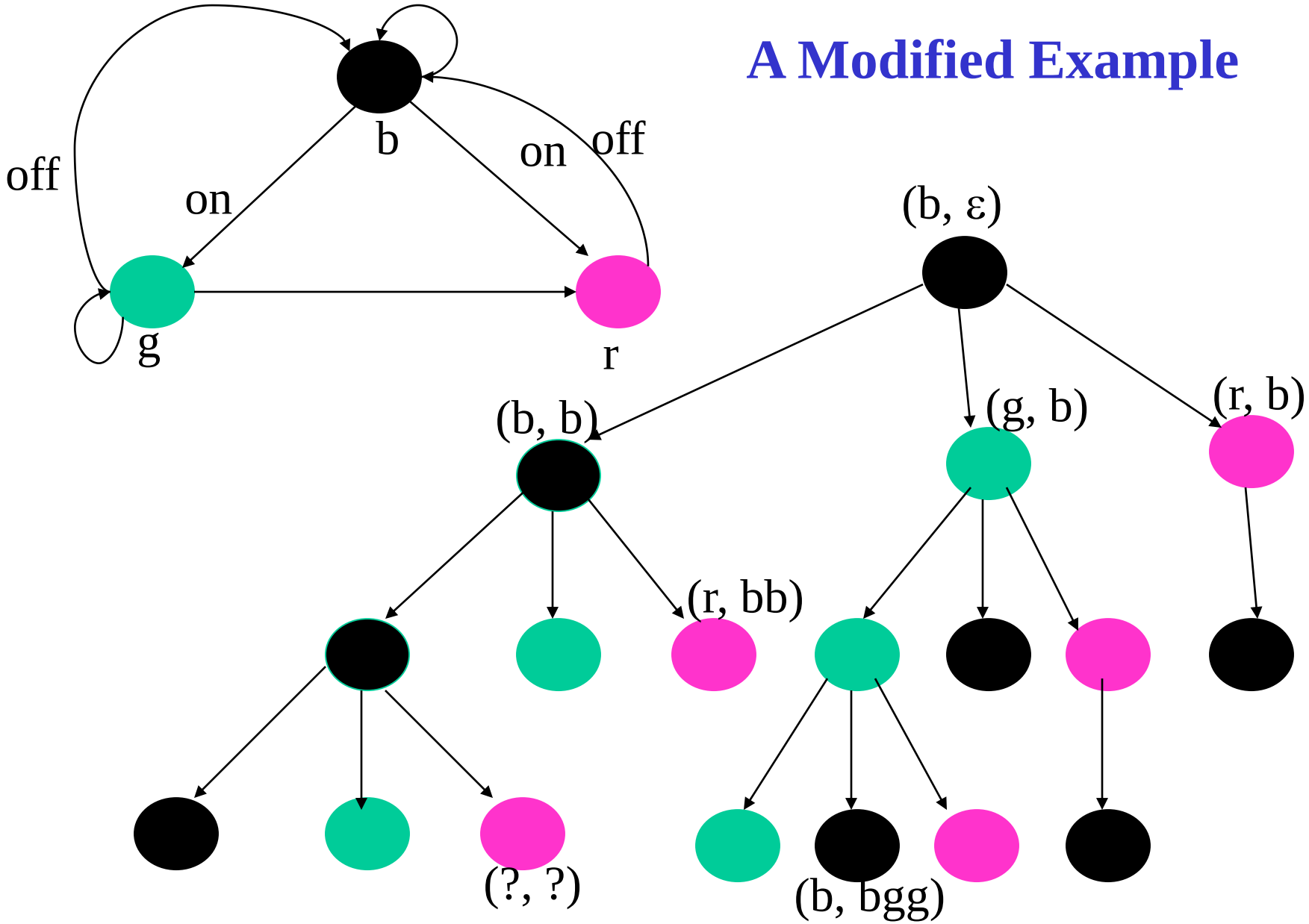
- **$K = (S, S_0, R, AP, L)$** $s_0 \in S$
- **$TR(K, s_0)$** --- The computation tree rooted at s_0.
- **$TR(K, s_0) = (\mathcal{S}_{s0}, (s_0,\varepsilon), \mathcal{R}_{s0}, AP, \mathcal{L}_{s0})$**
  - $(s_0, \varepsilon) \in \mathcal{S}_{s0}$;
  - If $(s_1, \sigma) \in \mathcal{S}_{s0}$ and $R(s_1, s_2)$ then
    - $(s_2, \sigma.s_1) \in \mathcal{S}_{s0}$ and
    - $\mathcal{R}_{s0}((s_1, \sigma), (s_2, \sigma.s_1))$;
    - $\mathcal{L}((s_1, \sigma)) = L(s_1)$.

Therefore, for all $(s, \sigma) \in \mathcal{S}_{s0}$, $s \in S$ and $\sigma = s_0\, s_1 \ldots s_n$ is a path in **K** from s_0 to s_n and $R(s_n, s)$ (hence, $\sigma.s$ is a path in **K** from s_0 to s)

Unwinding

- **TR(K, s)** is almost a Kripke structure.
  - $\mathcal{S}_s$ will tipically be infinite.
  - But $\mathcal{R}_s$ is ***tree-like***.
  - The "***graph***" of **TR(K, s)** is a tree rooted at **(s, ε)**.
- **TR(K, s)** is the ***computation tree rooted*** of **K** at **s**.

A Modified Example



Linear time Vs Branching time

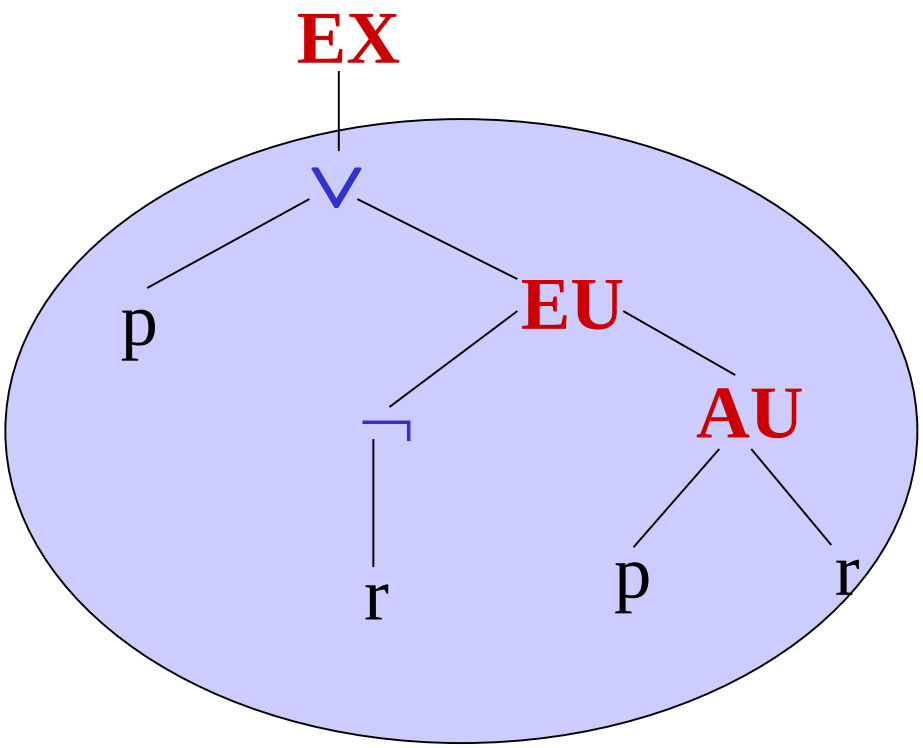
- There are ***properties*** that can be ***expressed in* LTL** but which ***can not be expressed in* CTL**.
- There are ***properties*** that can be ***expressed in* CTL** but ***not in* LTL**.
- The **LTL *model checking problem*** can be ***converted*** into a ***restricted*** kind of a **CTL*** ***model checking problem***.

CTL

- **Syntax**
  - **AP** – a finite set of ***atomic propositions***.
  - **$p \in AP$** is a formula.
  - If ψ and ψ' are formulas then so are $\neg\psi$ and $\psi \vee \psi'$.
  - If ψ is a formula then so is **EX**ψ
  - If ψ_1 and ψ_2 are formulas then so are **EU**(ψ_1, ψ_2) and **AU**(ψ_1, ψ_2).

Formulas

- **EX(p ∨ EU(¬r, AU(p, r)))**



Semantics

- **$K = (S, S_0, R, AP, L)$**
  - **$L : S \longrightarrow 2^{AP}$**
- **ψ** a **CTL** formula and **$s \in S$**
- **$K, s \models \psi$**
- **ψ** (holds) ***is satisfied*** at **s**.
- **FACT**:

$K, s \models \psi$ iff **$TR(K, s), (s, \varepsilon) \models \psi$.**

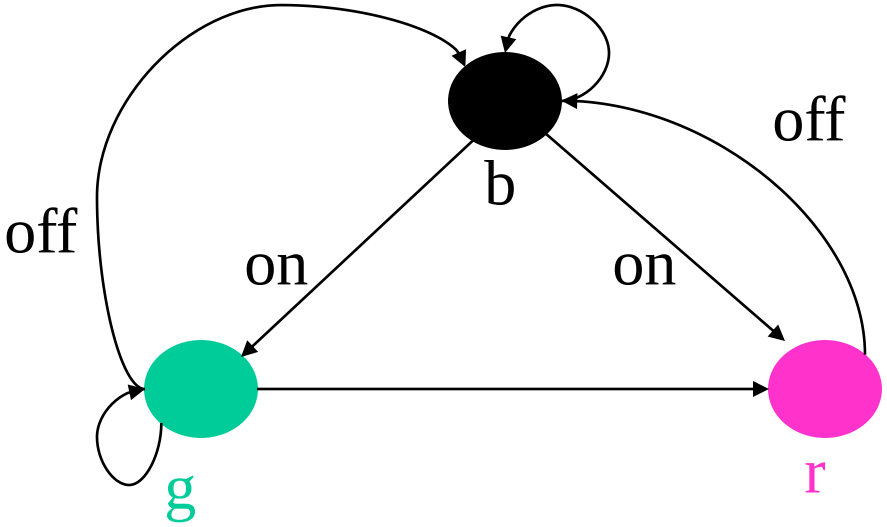
Semantics

- **CTL ::= p | $\neg\psi$ | $\psi_1 \vee \psi_2$ | EX(ψ) |**
  | EU(ψ_1, ψ_2) | AU(ψ_1, ψ_2)
- **K = (S, S_0, R, AP, L) ; L: S $\longrightarrow$ 2^{AP} ; s $\in$ S**
- **K, s $\models$ p** iff **p $\in$ L(s).**
- **K, s $\models$ $\neg\psi$** iff **not K, s $\models$ ψ**
- **K, s $\models$ $\psi_1 \vee \psi_2$** iff
  K, s $\models$ ψ_1 or **K, s $\models$ ψ_2.**

Semantics

- **CTL ::= p | $\neg\psi$ | $\psi_1 \vee \psi_2$ | EX(ψ) |**
  | EU(ψ_1, ψ_2) | AU(ψ_1, ψ_2)
- **K = (S, S_0, R, AP, L) ; L: S $\longrightarrow$ 2^{AP} ; s $\in$ S**
- **K, s $\models$ EX(ψ)** iff there exists **s'** such that:
  - **s $\longrightarrow$ s'** (i.e. **R(s, s')**) and **K, s' $\models$ ψ**

  s has a successor state **s'** at which ψ holds.



AP = {n, h, uh}

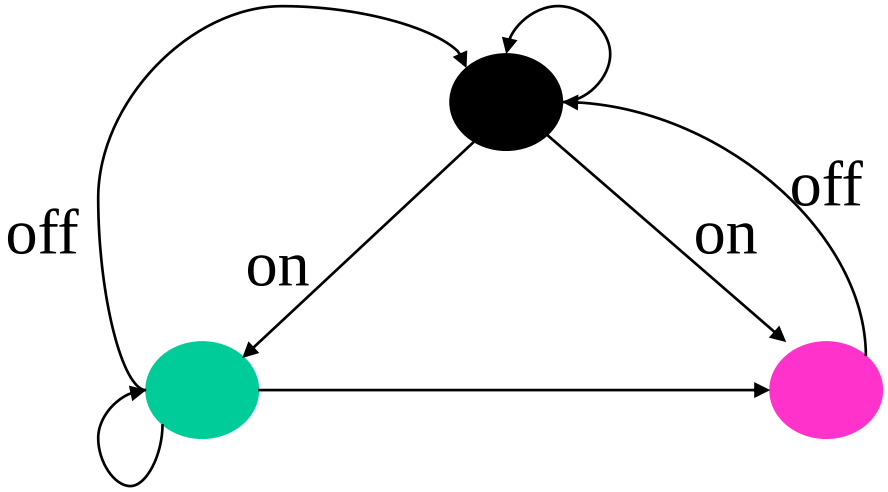
K, b ⊨ EX(uh) ? K, b ⊨ EX(¬uh) ?

K, g ⊨ EX(uh) ?

K, r ⊨ EX(h) ?

Semantics

- $K = (S, S_0, R, AP, L)$; $L: S \longrightarrow 2^{AP}$; $s \in S$
- ***A path from s*** is a (infinite) sequence of states
  $\pi = s_0, s_1, s_2, \ldots, s_i, s_{i+1}, \ldots$ s.t:
  - $s = s_0$
  - $s_i \longrightarrow s_{i+1}$ (i.e. $R(s_i, s_{i+1})$) for every i.
- $\pi(i) = s_i$ the i-th element of π.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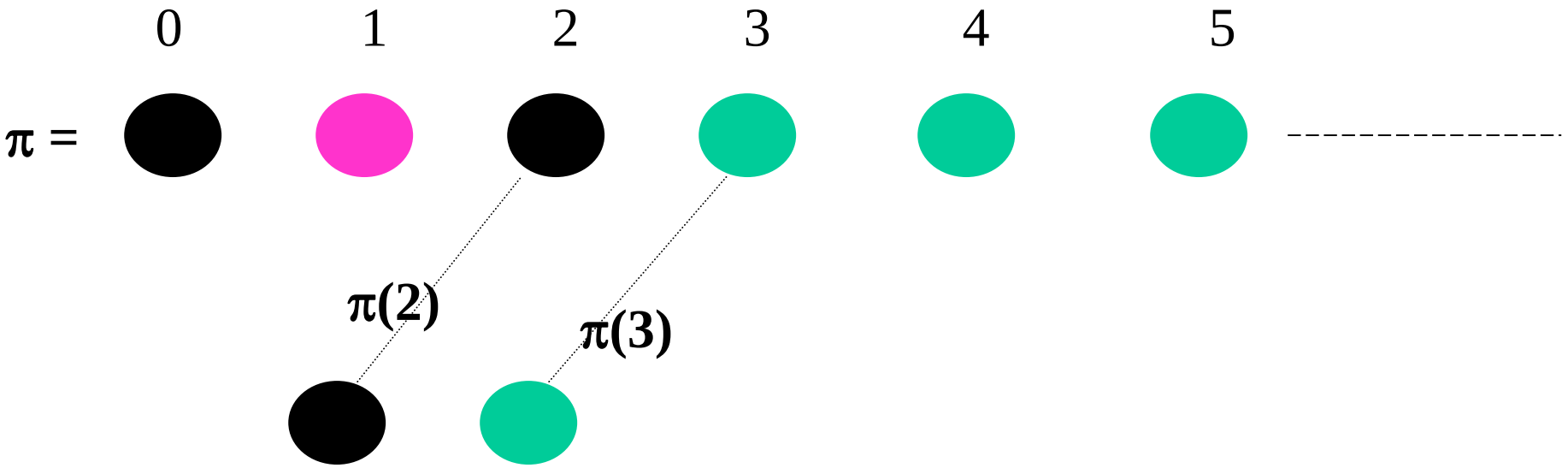
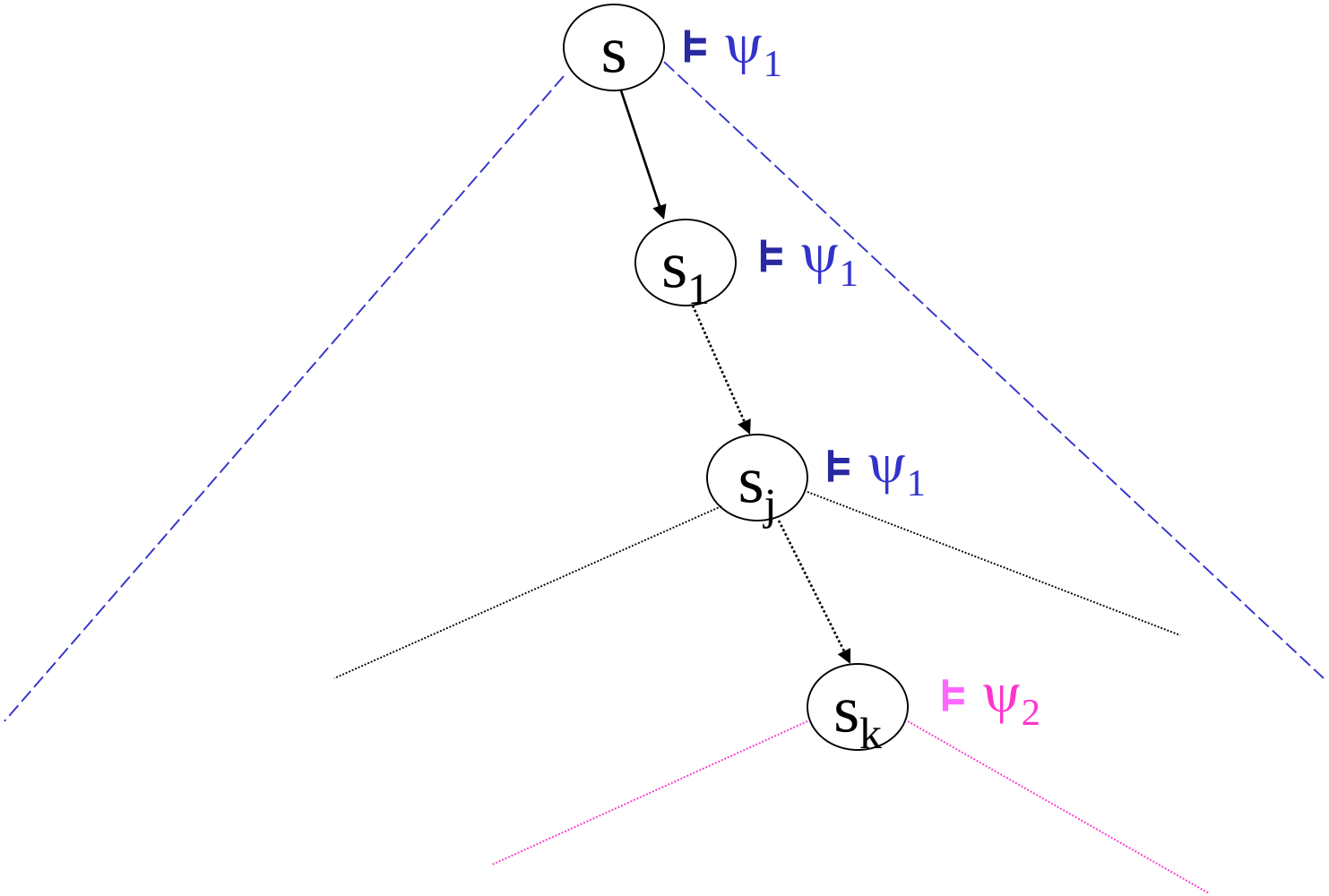

0
1
2
3
4
5
π =
π(2)
π(3)

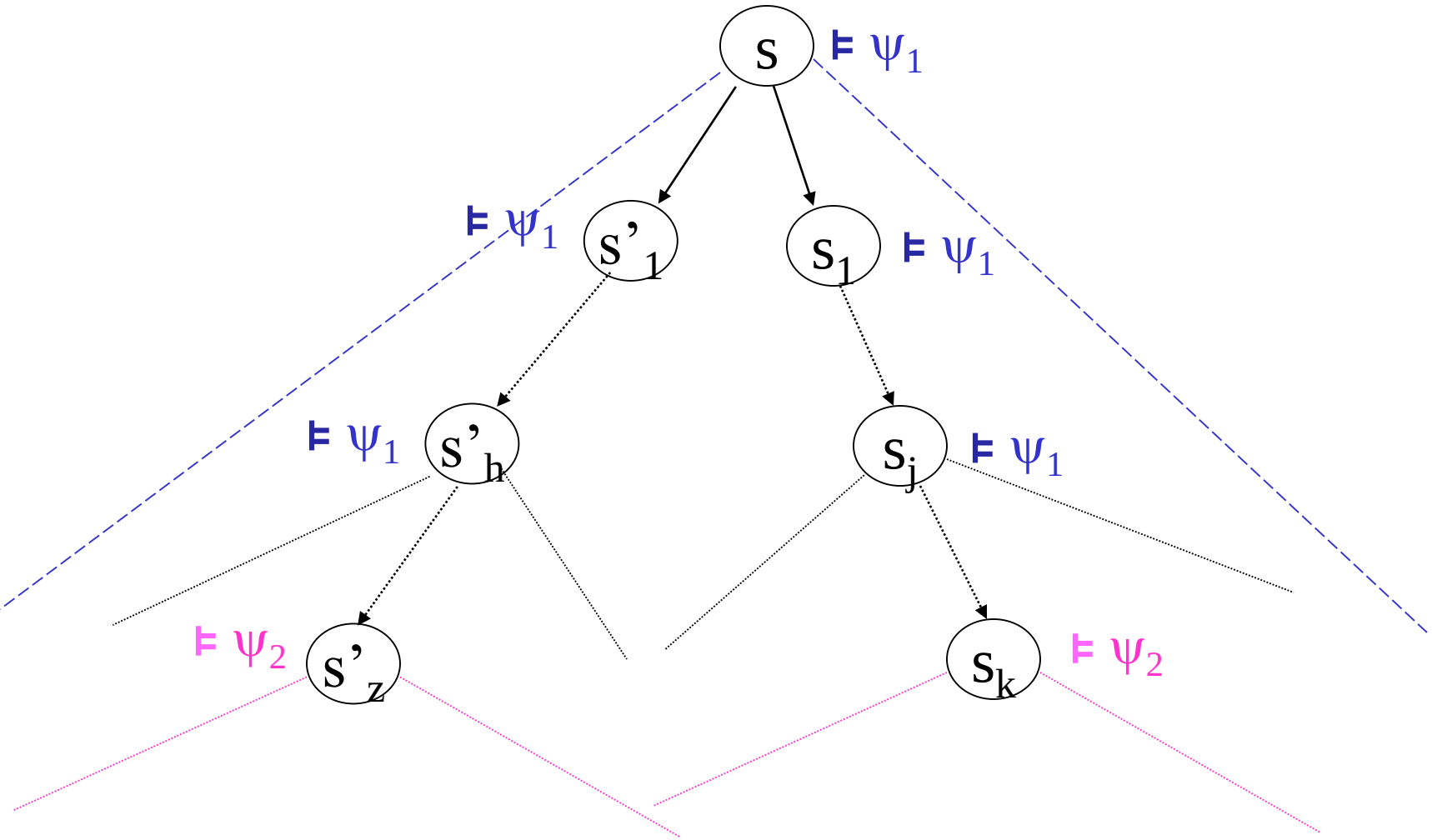
Semantics

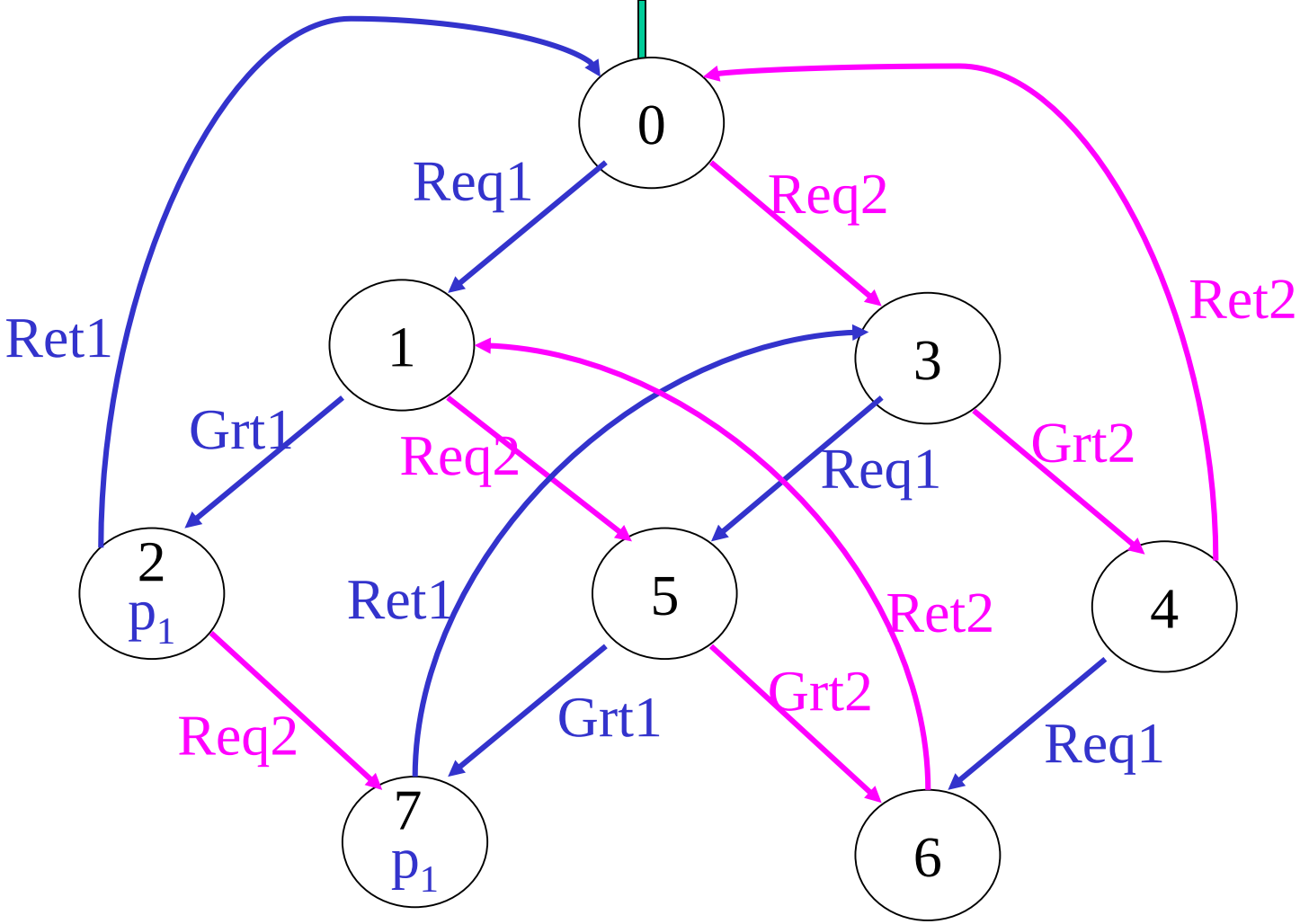
- **CTL ::= $p \mid \neg\psi \mid \psi_1 \vee \psi_2 \mid EX(\psi) \mid$**
  $\mid EU(\psi_1, \psi_2) \mid AU(\psi_1, \psi_2)$
- **$K = (S, S_0, R, AP, L)$; $L: S \longrightarrow 2^{AP}$; $s \in S$**
- **$K, s \vDash EU(\psi_1, \psi_2)$** iff ***there exists a path*** **$\pi = s_0, s_1, \ldots$** from **s** (i.e. **s_0=s**) and **$k \geq 0$** such that:
  - **$K, \pi(k) \vDash \psi_2$**
  - **$K, \pi(j) \vDash \psi_1$** , for all **$0 \leq j < k$**.


s
⊨ ψ1
s1
⊨ ψ1
sj
⊨ ψ1
sk
⊨ ψ2

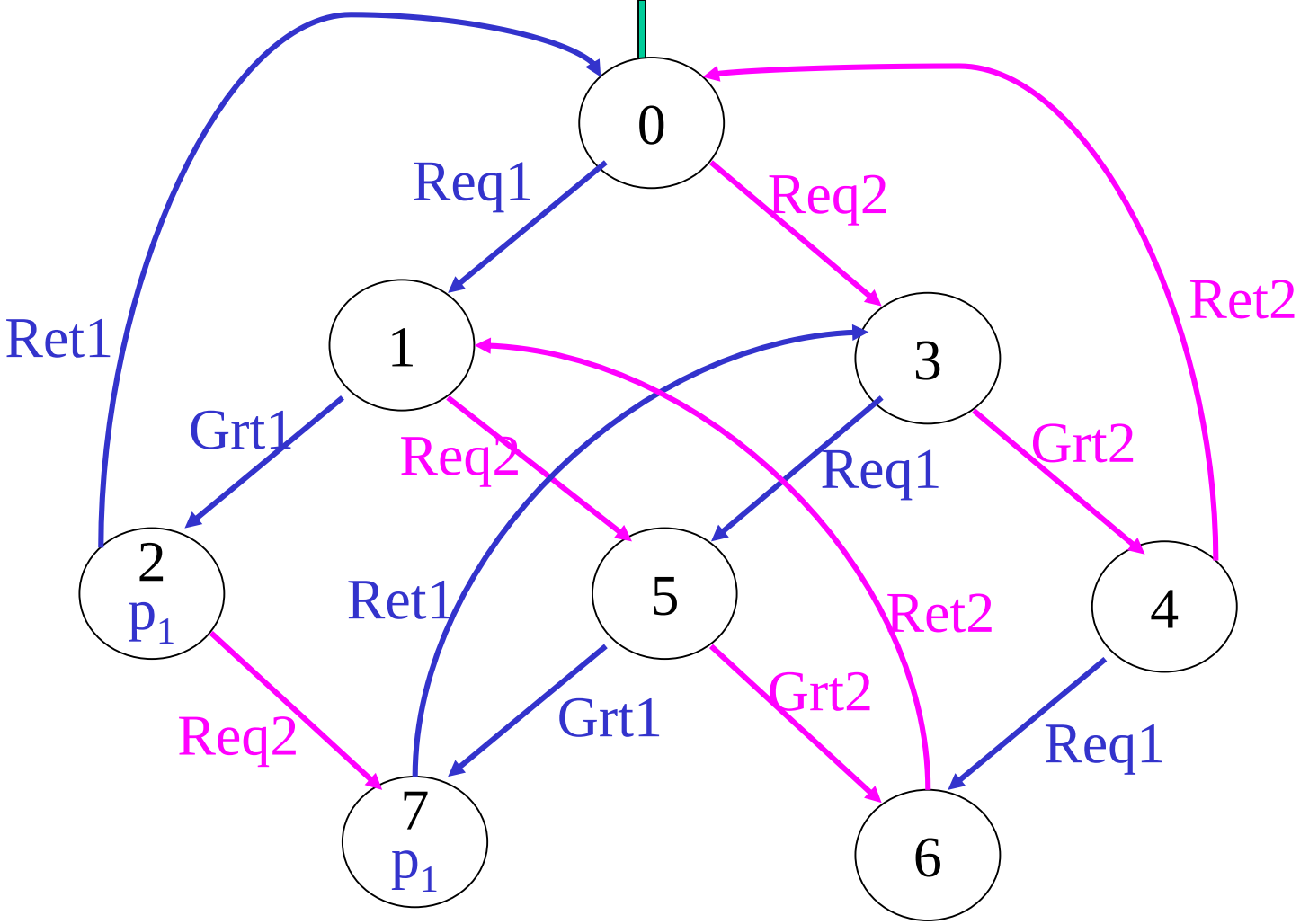
Semantics

- **CTL ::= $p \mid \neg\psi \mid \psi_1 \vee \psi_2 \mid EX(\psi) \mid$**
  $\mid EU(\psi_1, \psi_2) \mid AU(\psi_1, \psi_2)$
- **$K = (S, S_0, R, AP, L)$; $L: S \longrightarrow 2^{AP}$; $s \in S$**
- **$K, s \vDash AU(\psi_1, \psi_2)$** iff ***for every path***
  $\pi = s_0, s_1, \ldots$ from **s** there exists **$k \geq 0$** such that:
  - **$K, \pi(k) \vDash \psi_2$**
  - **$K, \pi(j) \vDash \psi_1$**, for all **$0 \leq j < k$**.

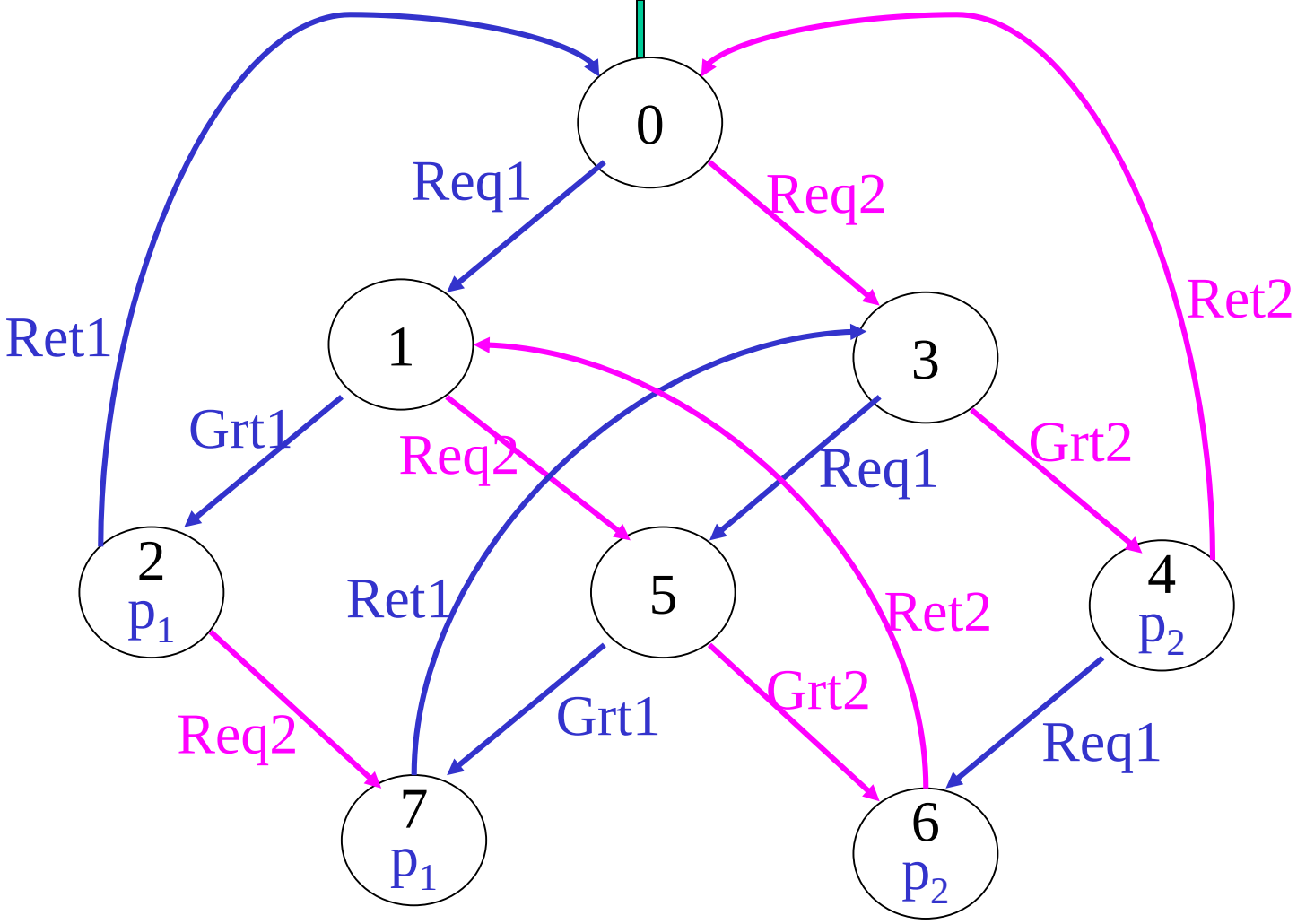

s
⊨ ψ1
⊨ ψ1
s'1
s1
⊨ ψ1
⊨ ψ1
s'h
sj
⊨ ψ1
⊨ ψ2
s'z
sk
⊨ ψ2


0
Req1
Req2
Ret1
Ret2
1
3
Grt1
Req2
Req1
Grt2
2
p1
Ret1
5
Ret2
4
Req2
Grt1
Grt2
Req1
7
p1
6

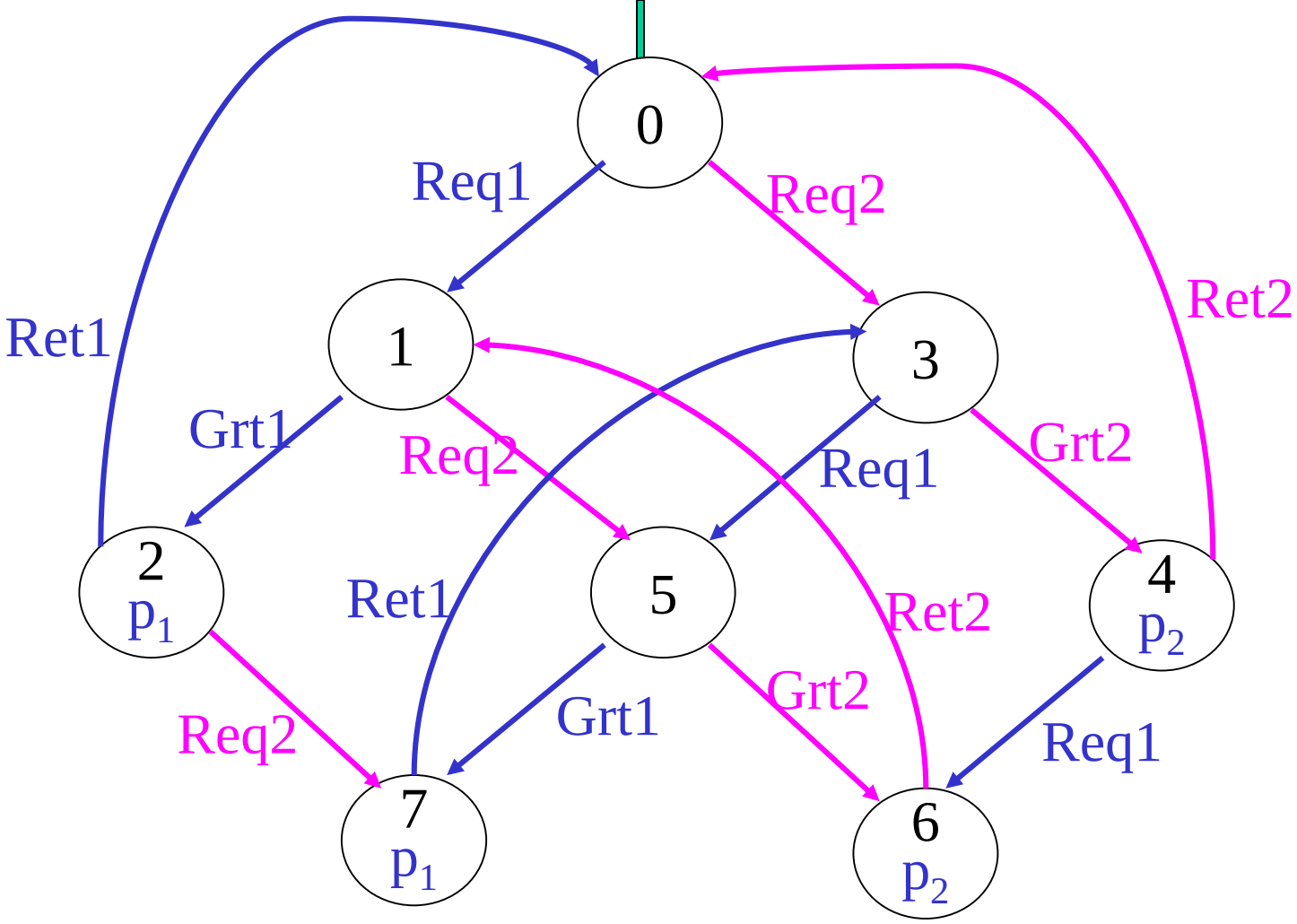
M, 0 ⊨ EU(⊤, p_1) ?


0
Req1
Req2
Ret1
Ret2
1
3
Grt1
Req2
Req1
Grt2
2
p_1
Ret1
5
Ret2
4
Req2
Grt1
Grt2
Req1
7
p_1
6

M, 0 ⊨ AU(⊤, p_1) ?

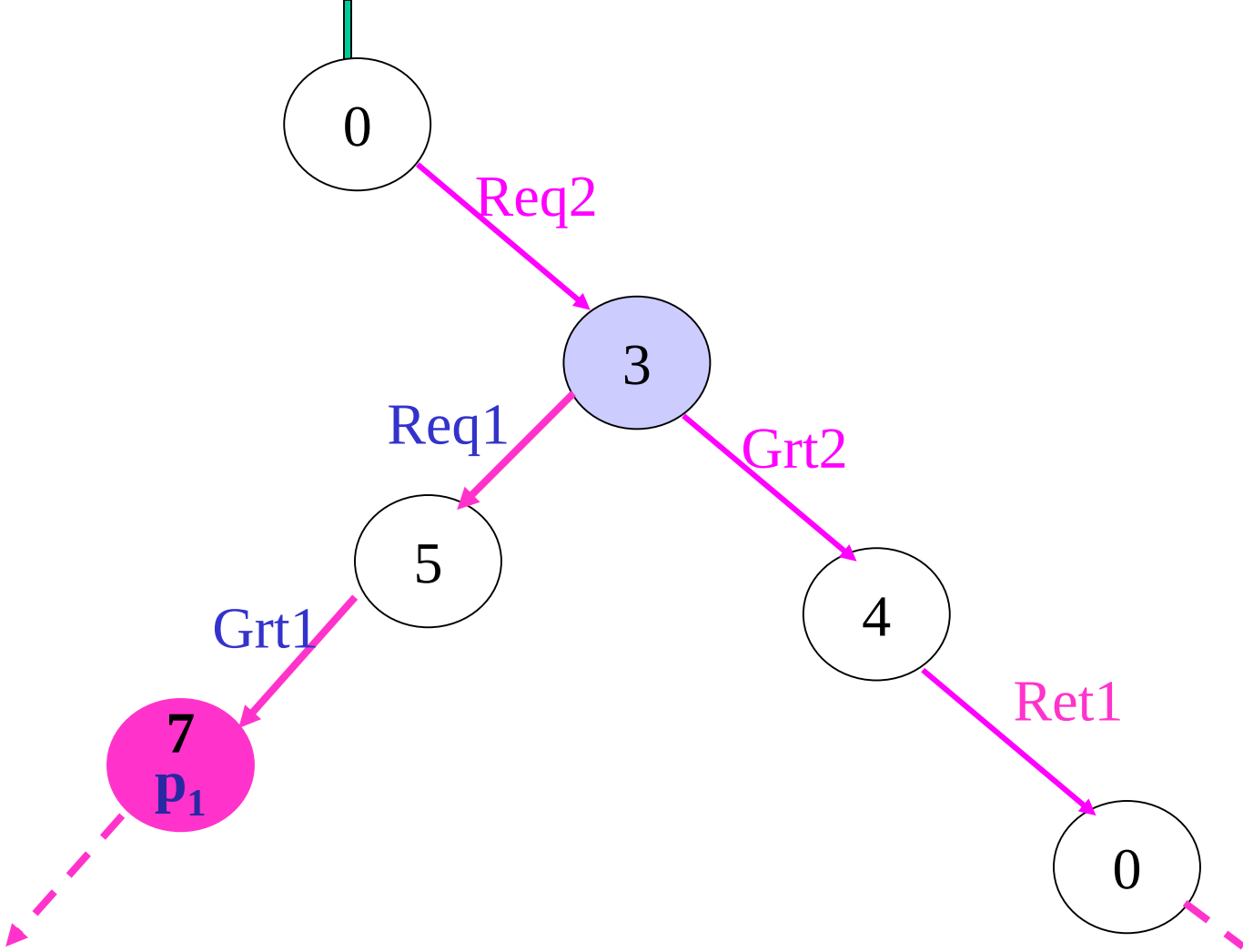


M, 0 ⊨ AU(⊤, $p_1 \vee p_2$) ?



M, 0 ⊨ AU(⊤, EU(⊤, p_1)) ?

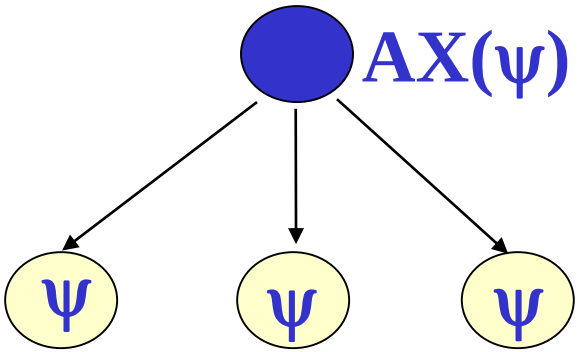
From s_0, ***all*** the computations will reach a point, where it is ***possible*** for 1 to print ***eventually.***



M, 0 ⊨ AU(⊤, EU(⊤, p_1)) ?

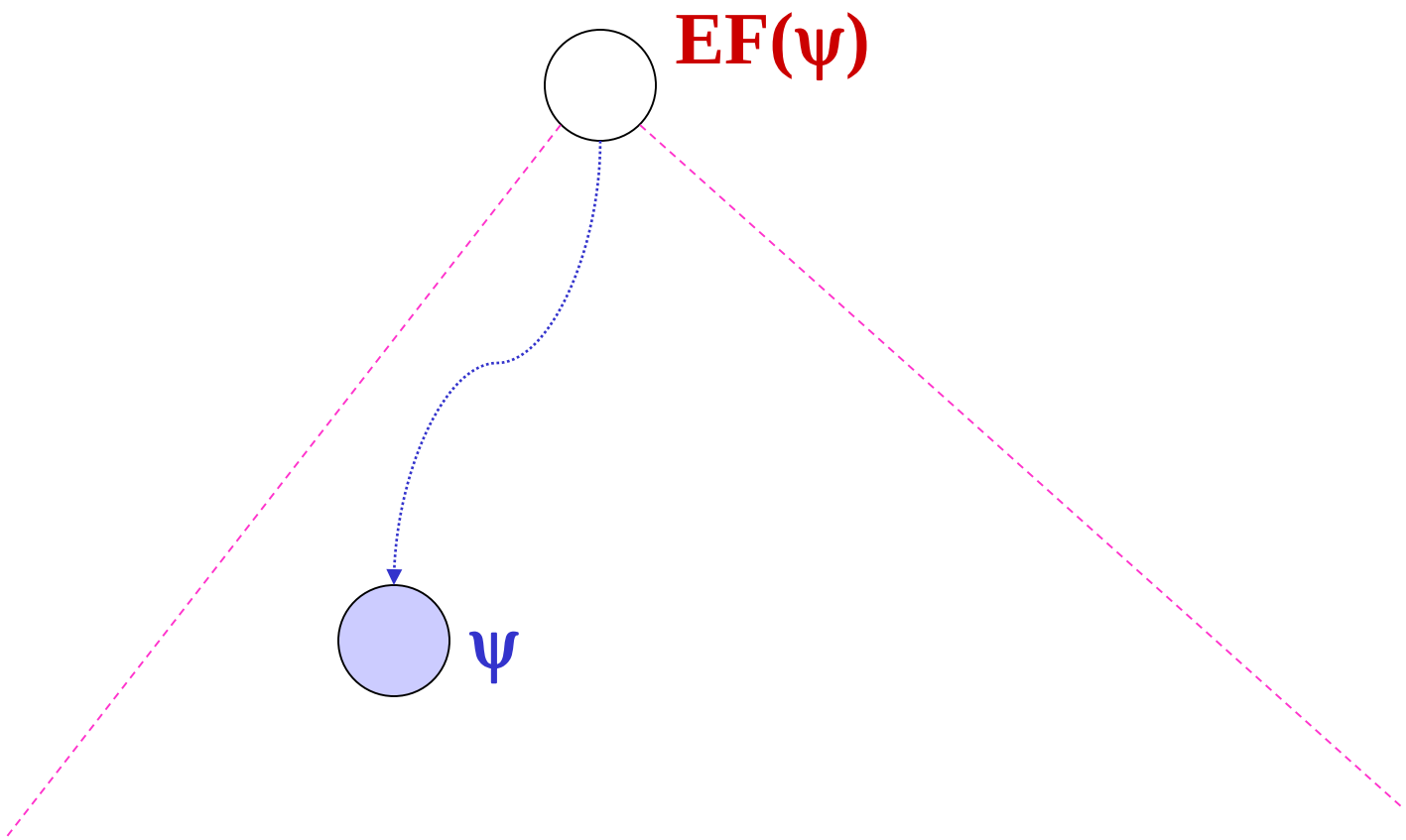
Derived Operators

- **AX(ψ) = ¬EX(¬ψ)**
  - It is not the case there exists a next state at which ψ does not hold, equivalent to
  - ***For every next state*** ψ holds.



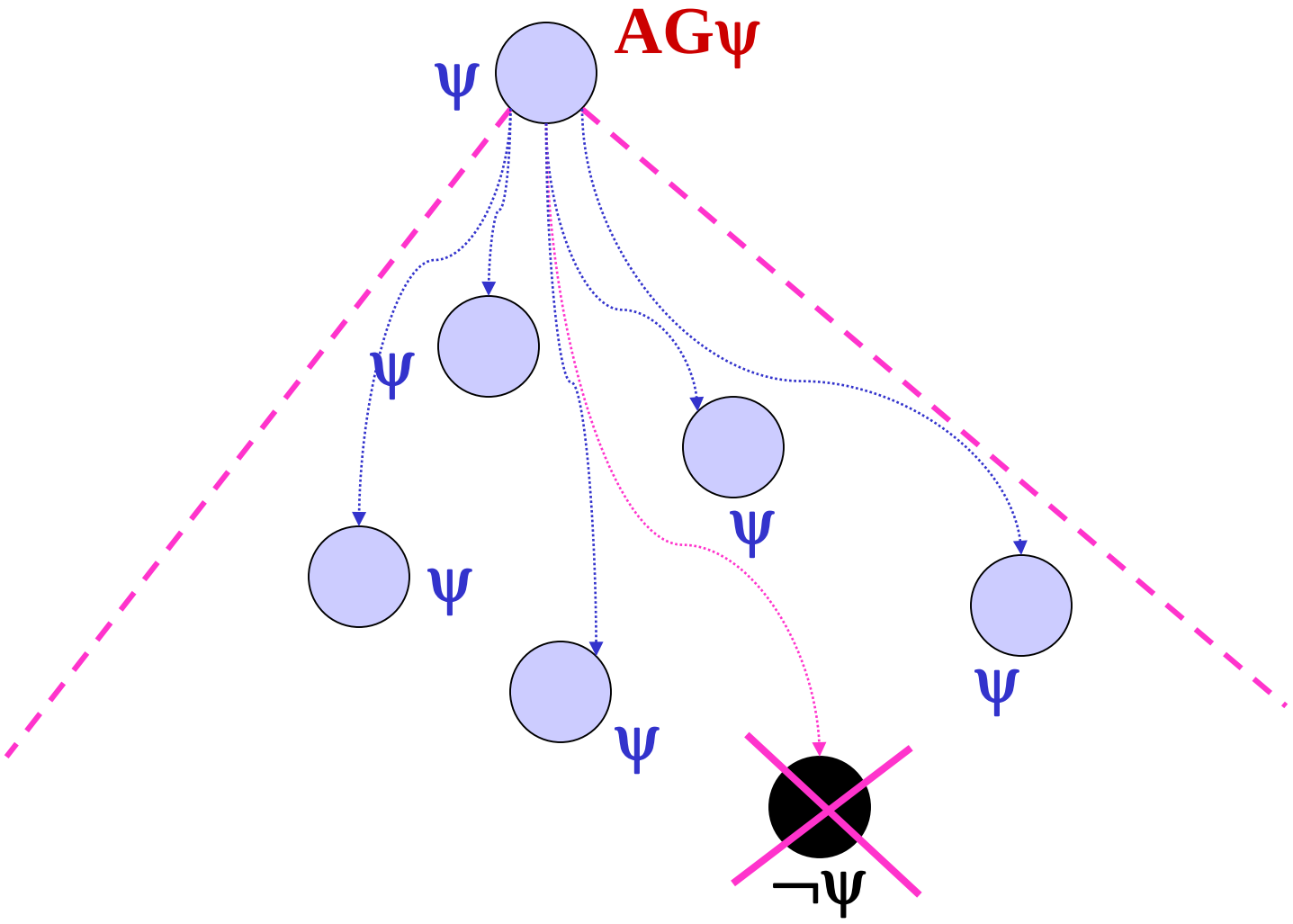
Derived Operators

- **K, s ⊨ EF(ψ)**
- **EF(ψ) = EU(⊤, ψ)**
  - There exists a path **π** (from **s**) and **k ≥ 0** such that:
    - **K, π(k) ⊨ ψ.**


EF(ψ)
ψ

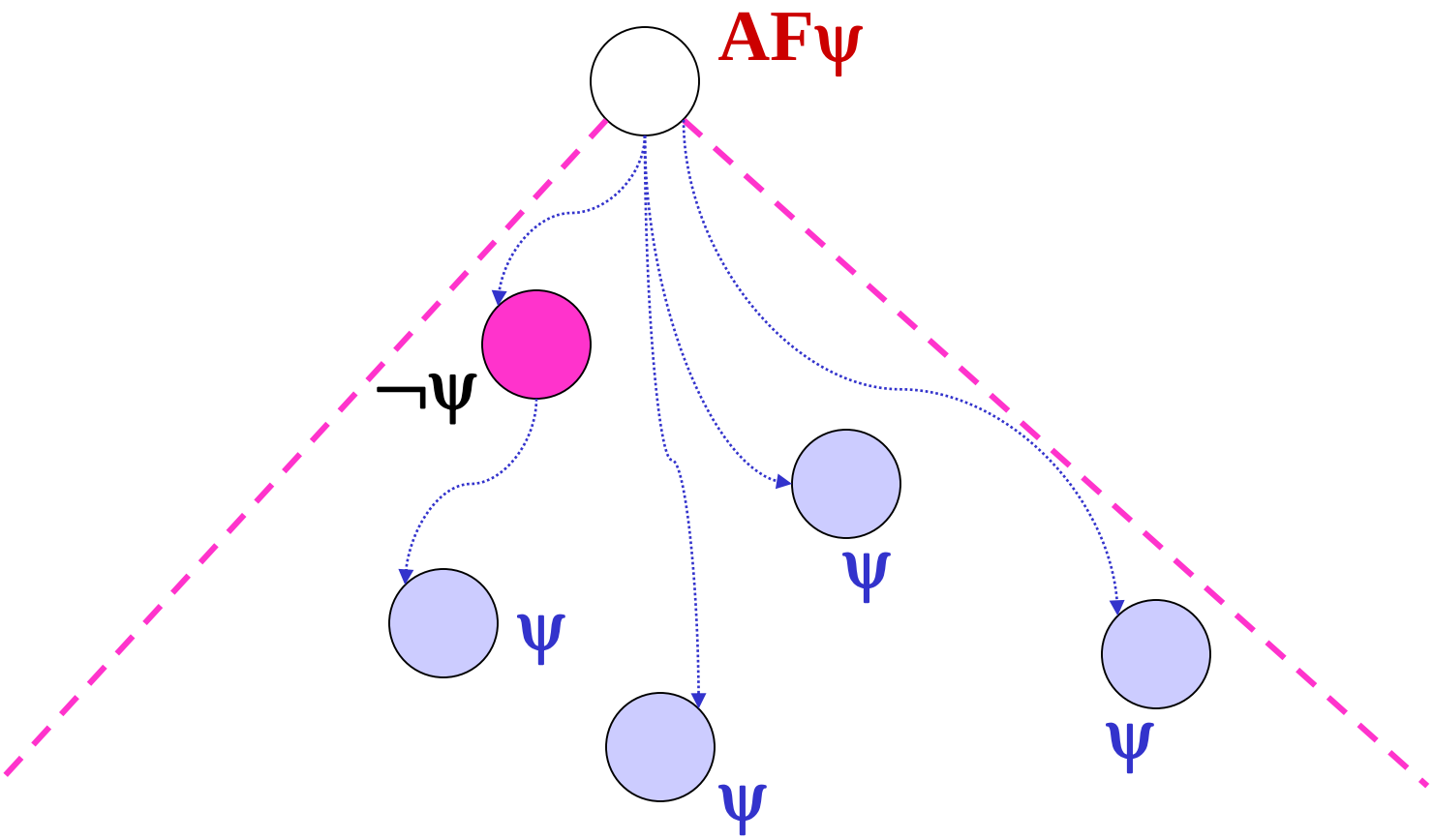
Derived Operators

- **K, s ⊨ AG(ψ)**
- **AG(ψ) = ¬EF(¬ψ)**
  - It is ***not*** the case ***there exists a path π*** (from **s**) and **k ≥ 0** such that:
    - **K, π(k) ⊨ ¬ ψ**
  - ***For every path π*** (from **s**) and ***every*** **k ≥ 0**:
    - **K, π(k) ⊨ ψ**


AGψ
ψ
ψ
ψ
ψ
ψ
ψ
¬ψ

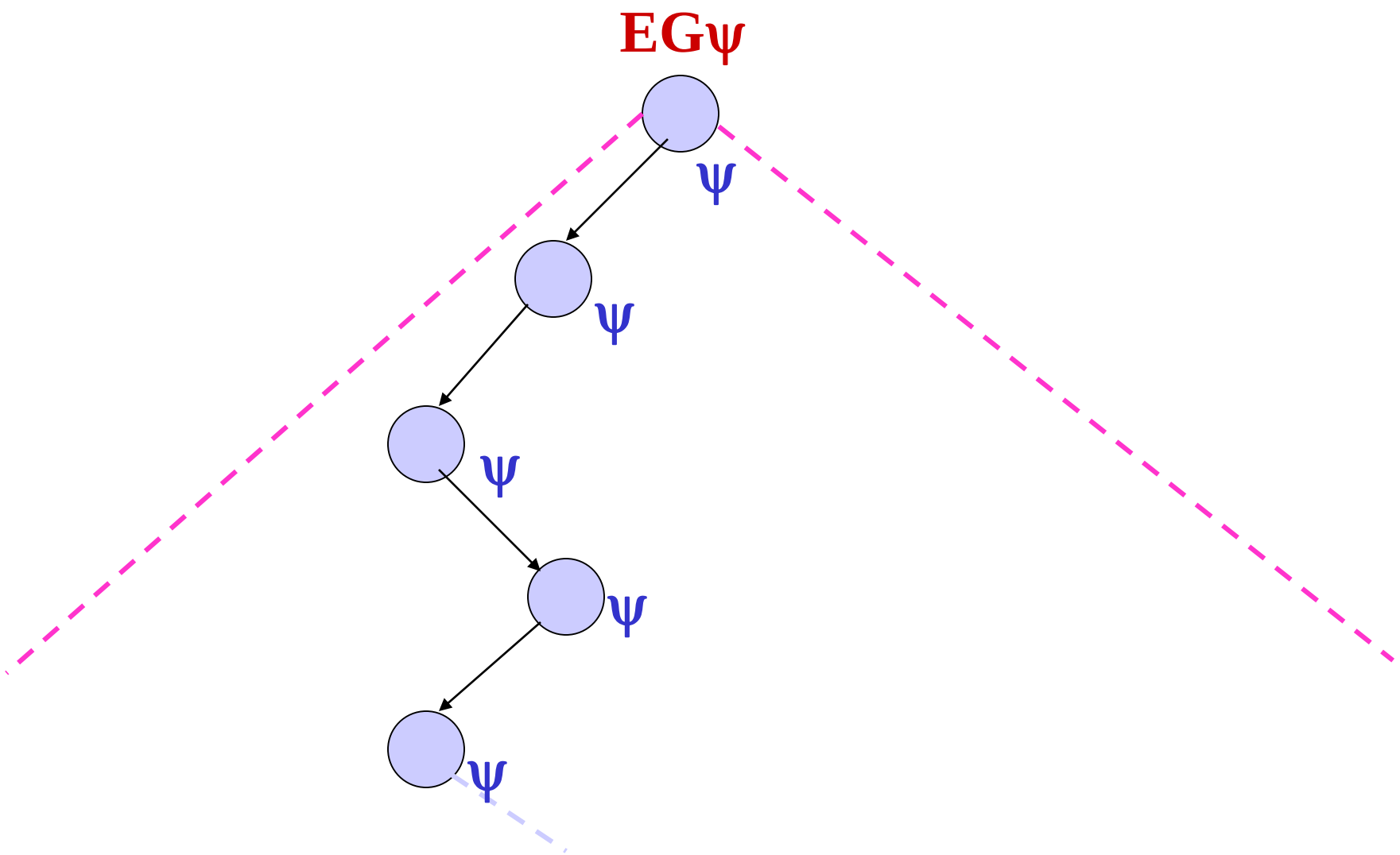
Derived Operators

- **K, s ⊨ AF(ψ)**
- **AF(ψ) = AU(⊤, ψ)**
  - ***For every path π*** from **s**, there ***exists*** **k ≥ 0** such that:
    - **K, π(k) ⊨ ψ.**


AFψ
¬ψ
ψ
ψ
ψ
ψ

Derived Oparators

- **K, s ⊨ EG(ψ)**
- **EG(ψ) = ¬AF(¬ψ)**
  - It is **not** the case that ***for every path π*** from **s** there is a **k ≥ 0** such that **K, π(k) ⊨ ¬ ψ**.
  - ***There exists a path π*** from **s** such that, for every **k ≥ 0**:
    - **K, π(k) ⊨ ψ**.


EGψ
ψ
ψ
ψ
ψ
ψ

A more convenient CTL

- **NCTL ::= p | $\neg\psi$ | $\psi_1 \vee \psi_2$ | EX(ψ) |**
  | EU(ψ_1, ψ_2) | EG(ψ)
- **CTL ::= p | $\neg\psi$ | $\psi_1 \vee \psi_2$ | EX(ψ) |**
  | EU(ψ_1, ψ_2) | AU(ψ_1, ψ_2)
- **NCTL** is more convenient for model checking!
- Clearly **NCTL** can be defined in terms of **CTL**.

A more convenient CTL

- **NCTL ::= p | $\neg\psi$ | $\psi_1 \vee \psi_2$ | EX(ψ) |**
  | EU(ψ_1, ψ_2) | EG(ψ)
- **CTL ::= p | $\neg\psi$ | $\psi_1 \vee \psi_2$ | EX(ψ) |**
  | EU(ψ_1, ψ_2) | AU(ψ_1, ψ_2)
- **CTL** can be defined in terms of **NCTL**!
- The semantics of **NCTL** is given in the obvious way.

A more convenient CTL

- **NCTL ::= $p \mid \neg\psi \mid \psi_1 \vee \psi_2 \mid EX(\psi) \mid$**

  $\mid EU(\psi_1, \psi_2) \mid EG(\psi)$

- **$K, s \models EG(\psi)$** iff ***there exists a path π*** from **s** such that for every **$k \geq 0$**:
  - **$K, \pi(k) \models \psi$**

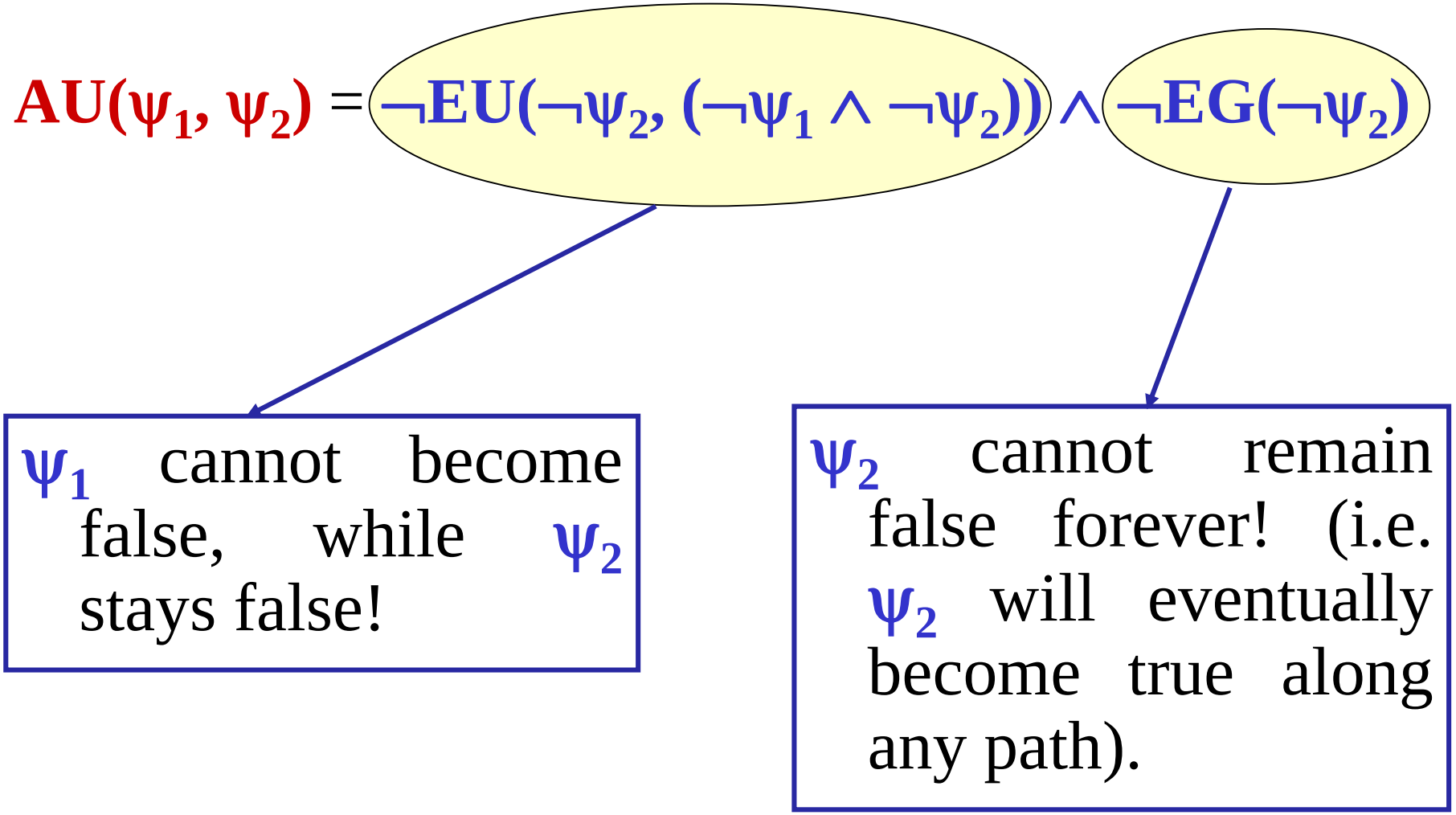
A more convenient CTL

- **NCTL ::= $p \mid \neg\psi \mid \psi_1 \vee \psi_2 \mid EX(\psi) \mid$**
  $\mid EU(\psi_1, \psi_2) \mid EG(\psi)$
- **CTL ::= $p \mid \neg\psi \mid \psi_1 \vee \psi_2 \mid EX(\psi) \mid$**
  $\mid EU(\psi_1, \psi_2) \mid AU(\psi_1, \psi_2)$
- **$AU(\psi_1, \psi_2) = \neg EU(\neg\psi_2, (\neg\psi_1 \wedge \neg\psi_2)) \wedge \neg EG(\neg\psi_2)$**

i.e., along any path: ψ_2 must hold eventually and $(\neg\psi_1 \wedge \neg\psi_2)$ can only happen after ψ_2 (recall the *before operator* of LTL)

A more convenient CTL

$$\mathbf{AU}(\psi_1, \psi_2) = \neg\mathbf{EU}(\neg\psi_2, (\neg\psi_1 \wedge \neg\psi_2)) \wedge \neg\mathbf{EG}(\neg\psi_2)$$



A more convenient CTL

$\mathbf{AU(\psi_1, \psi_2) = \neg EU(\neg\psi_2, (\neg\psi_1 \wedge \neg\psi_2)) \wedge \neg EG(\neg\psi_2)}$

$\Rightarrow$ Assume $\mathbf{K, s \vDash AU(\psi_1, \psi_2)}$

- Let π be a path from **s**. Then there exists $\mathbf{k \geq 0}$ with:
  - $\mathbf{K, s \vDash \psi_2}$
- Hence, **not** $\mathbf{K, s \vDash EG(\neg\psi_2)}$
- Equivalent to $\mathbf{K, s \vDash \neg EG(\neg\psi_2)}$

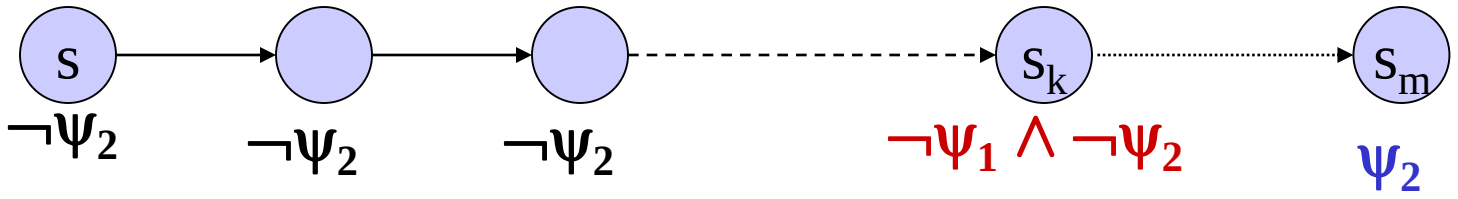
A more convenient CTL

- **$AU(\psi_1, \psi_2)$ = NewAU(ψ_1, ψ_2) =**

  $\neg EU(\neg\psi_2, (\neg\psi_1 \wedge \neg\psi_2)) \wedge \neg EG(\neg\psi_2)$

- Clearly **$K, s \models AU(\psi_1, \psi_2)$** implies **$K, s \models \neg EG(\neg\psi_2)$**

- Let **$K, s \models AU(\psi_1, \psi_2)$**
  - Suppose now **$K, s \models EU(\neg\psi_2, \neg\psi_1 \wedge \neg\psi_2)$**
  - Let **π** be any path from **s** witnessing the above:
  - Let now **k *be the least integer*** such that:
    - **$K, \pi(k) \models \neg\psi_1 \wedge \neg\psi_2$**
    - **$K, \pi(j) \models \neg\psi_2$** for **$0 \leq j < k$.**



- Suppose **K, π(m) ⊨ ψ_2,** required by **K, s ⊨ AU(ψ_1,ψ_2)**
- Take **m** to be the least such number.
- Then **k < m**, since **K, s ⊨ EU($\neg\psi_2$, $\neg\psi_1 \wedge \neg\psi_2$)**
- This implies that **K, π(k) ⊨ $\neg\psi_1$**, for some **0 ≤ k < m**
- Hence **not K, s ⊨ AU(ψ_1, ψ_2)**. ***Contradiction!***
- Thus **K, s ⊨ AU(ψ_1, ψ_2)** also implies:
  - **K, s ⊨ $\neg$EU($\neg\psi_2$, $\neg\psi_1 \wedge \neg\psi_2$)**
- So **K, s ⊨ AU(ψ_1, ψ_2)** implies **K, s ⊨ NewAU(ψ_1, ψ_2)**

From CTL to NCTL

- In a similar way we can argue that:

> ***if*** **K, s ⊨ newAU(ψ_1, ψ_2)**
>
> ***then*** **K, s ⊨ AU(ψ_1, ψ_2).**

- Hence ***CTL*** can be expressed in terms of ***NCTL***.

A more convenient CTL

- **NCTL ::= p | $\neg\psi$ | $\psi_1 \vee \psi_2$ | EX(ψ) |**
  | EU(ψ_1, ψ_2) | EG(ψ)
- **CTL ::= p | $\neg\psi$ | $\psi_1 \vee \psi_2$ | EX(ψ) |**
  | EU(ψ_1, ψ_2) | AU(ψ_1, ψ_2)
- **AU(ψ_1, ψ_2) = NewAU(ψ_1, ψ_2) =**

  $\neg$(EU($\neg\psi_2$, ($\neg\psi_1 \wedge \neg\psi_2$)) $\wedge$ AF(ψ_2)
- **NewAU$_1$ = $\neg$ EU($\neg\psi_2$, ($\neg\psi_1 \wedge \neg\psi_2$)**
- **NewAU$_2$ = AFψ_2**

$\neg$EG$\neg\psi_2$ = AFψ_2

From CTL to NCTL

- Let $\mathbf{K = (S, S_0, R, AP, L)}$ and $\mathbf{s \in S}$.
- We need to argue:
  - $\mathbf{K, s \vDash AU(\psi_1, \psi_2)}$ iff

    $\mathbf{K, s \vDash NewAU_1 \wedge NewAU_2}$
- We already argued that:
  - If $\mathbf{K, s \vDash AU(\psi_1, \psi_2)}$ then

    $\mathbf{K, s \vDash NewAU_1 \wedge NewAU_2}$

From CTL to NCTL

$\mathbf{AU(\psi_1, \psi_2) = \neg EU(\neg\psi_2, (\neg\psi_1 \wedge \neg\psi_2)) \wedge \neg EG(\neg\psi_2)}$

$\Leftarrow$ We need to argue that:

– If $\mathbf{K, s \vDash NewAU_1 \wedge NewAU_2}$ then

$\mathbf{K, s \vDash AU(\psi_1, \psi_2)}$

- So assume $\mathbf{K, s \vDash NewAU_1 \wedge NewAU_2}$.

- $\mathbf{NewAU_1 = \neg EU(\neg\psi_2, (\neg\psi_1 \wedge \neg\psi_2)).}$
- $\mathbf{NewAU_2 = \neg EG\neg\psi_2 = AF\psi_2}$

From CTL to NCTL

- Let π be some path from **s**.
- We need to show that there exists $k \geq 0$ such that:
  - $K, \pi(k) \models \psi_2$
  - $K, \pi(j) \models \psi_1$ if $0 \leq j < k$.
- But $K, s \models AF\ \psi_2$ implies that along any path (and also along π) there exists $k \geq 0$ such that:
  - $K, \pi(k) \models \psi_2$
- Assume **k** is the ***least*** such number along π.

From CTL to NCTL

Now consider an arbitrary **m** with **$0 \leq m < k$**.

CLAIM: **$K, \sigma(m) \models \psi_1$**

- If the **CLAIM** is true then we are done.
- Suppose instead that **$K, \sigma(m) \models \neg\psi_1$**.
  - Then **$K, \sigma(m) \models \neg\psi_1 \wedge \neg\psi_2$** (**$m < k$**) ***WHY???***
  - and **$K, \sigma(j) \models \neg\psi_2$** if **$0 \leq j < m$**, since **$j < m < k$**
  - Hence **$K, \sigma(0) \models EU(\neg\psi_2, \neg\psi_1 \wedge \neg\psi_2)$**
  - Therefore, **not** **$K, s \models NewAU_1$** which is a ***contradiction***!

CTL Model Checking

- **$K \models \psi$** ***iff***

  $K, s_0 \models \psi$ for every **$s_0 \in S_0$**.

- The ***CTL model checking problem***.
  - **$K = (S, S_0, R, AP, L)$** (system model)
  - **ψ** a ***CTL*** formula (spec. of the property)

- Given **K** and **ψ** ***determine whether or not*** **$K \models \psi$**

CTL Model Checking

- The actual model checking problem:
  - Given **$K = (S, S_0, R, AP, L)$**
  - Given **$s \in S$**
  - Given **ψ**, **an NCTL formula**.
  - Determine whether:

$K, s \models \psi$

The Sub-formulas of ψ

- **SF(ψ)** is the ***least set of formulas*** satisfying:
  - $\psi \in SF(\psi)$
  - If $\neg\alpha \in SF(\psi)$ then $\alpha \in SF(\psi)$.
  - If $\alpha \vee \beta \in SF(\psi)$ then $\alpha, \beta \in SF(\psi)$
  - If $EX\alpha \in SF(\psi)$ then $\alpha \in SF(\psi)$.
  - If $EU(\alpha, \beta) \in SF(\psi)$ then $\alpha, \beta \in SF(\psi)$
  - If $EG\alpha \in SF(\psi)$ then $\alpha \in SF(\psi)$.
- **SF(ψ)** ---- The ***set of sub-formulas*** of ψ.

The Labeling Procedure.

- **K = (S, S_0, R, AP, L)**
  - **s ∈ S**
  - **ψ** a ***NCTL*** formula (built out of **AP**).
- **Strategy**:
  - Construct **Labels: S ⟶ $2^{SF(\psi)}$**
  - **$2^{SF(\psi)}$**, the set of subsets of **SF(ψ)**.
  - Each state of **K** is assigned a subset of a **SF(ψ)** by the Labels function.
- **K, s ⊨ ψ** ***iff*** **ψ ∈ Labels(s)**.

The Labels function

- **Stage 0**: consider the atomic propositions only
  - For every **$t \in S$**:
  - **Labels(t) = L(t)** (**$K = (S, S_0, R, AP, L)$**)
- ….

Assume we have done up to stage **i** (all subfomrulae of length **i** already processed)

- **Stage i +1**: consider subfomrulae **α** of length **i+1**
  - For every **$t \in S$**:
  - If **$\alpha = \neg\beta$** then

    $\alpha \in$ Labels(t) ***iff*** **$\beta \notin$ Labels(t)**.

The Labels function

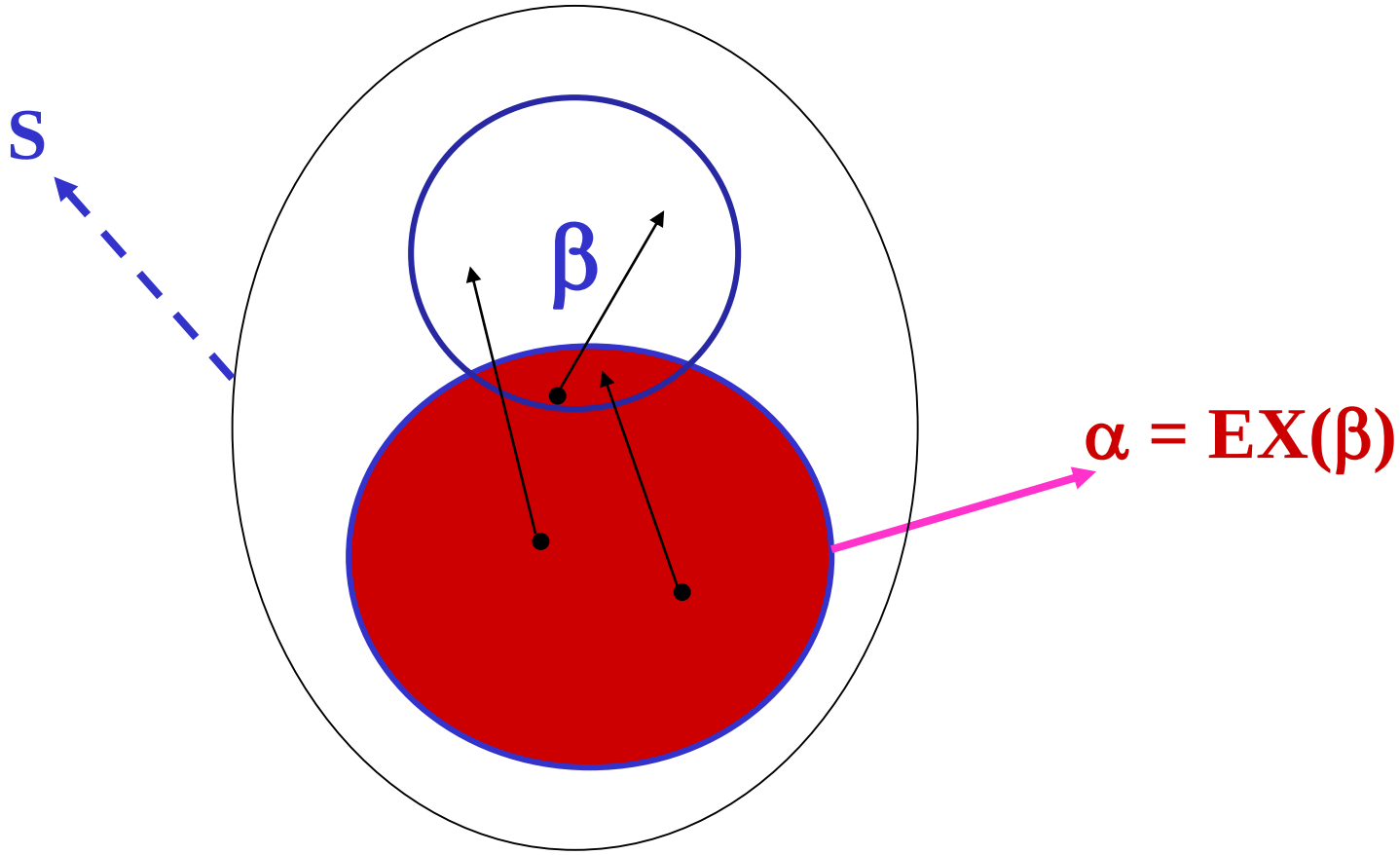
- **Stage i +1**: consider subfomrulae α of length **i+1**
  - For every $t \in S$:
  - If $\alpha = \beta_1 \vee \beta_2$ then

    $\alpha \in$ **Labels(t)** ***iff*** $\beta_1 \in$ **Labels(t) or** $\beta_2 \in$ **Labels(t)**
  - If $\alpha = EX\beta$ then

    $\alpha \in$ **Labels(t)** ***iff*** **there exists** $s \in S$ **such that**

    $\beta \in$ **Labels(s)** and **R(t, s) [i.e.** $t \rightarrow s$ **]**

The Labels Function



Computing the labeling for EX(β)

Complexity: O(\|M\|)

Algorithm Check_EX(β)

T := {**s** | β ∈ **Labels(s)**};

while **T** ≠ ∅ do

choose **s** ∈ **T**;

T := **T** \{**s**};

for each **t** ∈ **S** such that **t** → **s** do

Labels(**t**) := **Labels**(**t**) ∪ {**EX** β};

The Labels Function

- **Stage i +1**: consider subfomrulae α of length **i+1**
  - For every $t \in S$:
  - If $\alpha = EU(\beta_1,\beta_2)$ then

    $\alpha \in Labels(t)$ ***iff***
    - $\beta_2 \in Labels(t)$ **or**
    - $\beta_1 \in Labels(t)$ **and** $EU(\beta_1,\beta_2) \in Labels(s)$ **for some s with** $t \rightarrow s$.

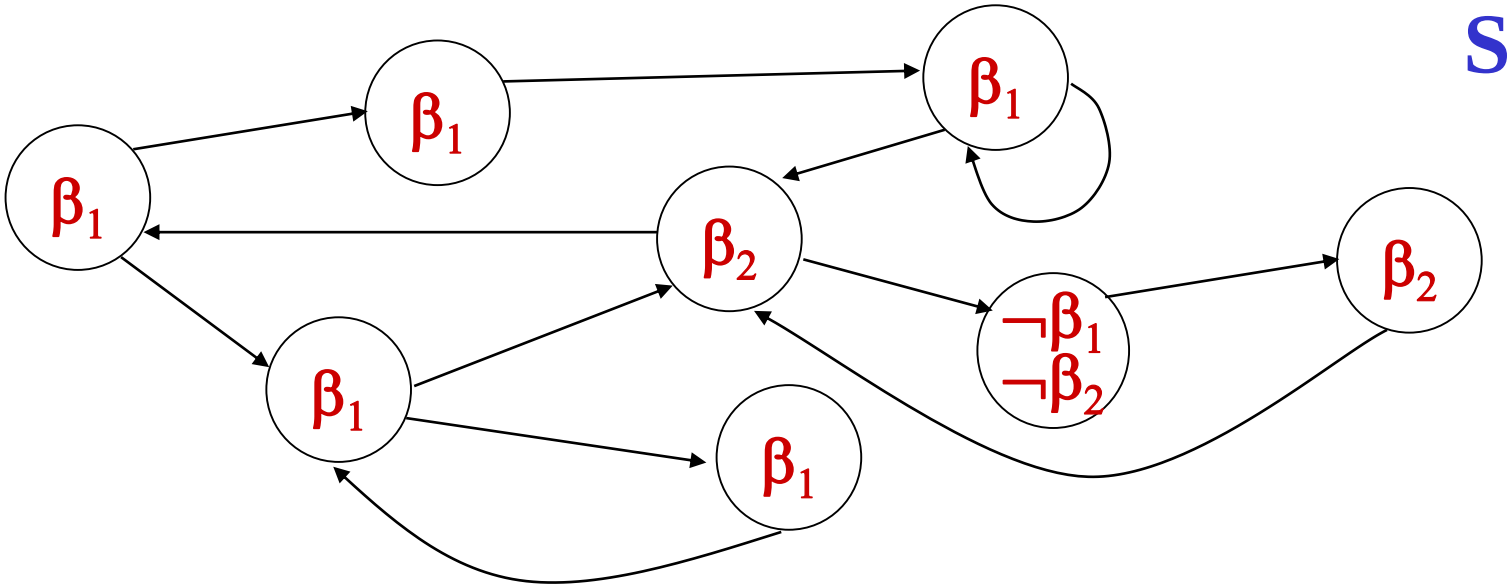
The Labels Function

- Collect in **T** all the states satisfying β_2
  - **all these states do also satisfy $EU(\beta_1,\beta_2)$.**
- Traverse backward $\rightarrow$ from states in **T** and label with $EU(\beta_1,\beta_2)$ all the states **t** satisfying β_1 and reaching at least a state **s** labeled with $EU(\beta_1,\beta_2)$.

If $s \in T$, t with $t \rightarrow s$ and $\beta_1 \in Labels(t)$ then

$EU(\beta_1,\beta_2) \in Labels(t)$

Recall that: $EU(\beta_1,\beta_2) = (\beta_2 \vee (\beta_1 \wedge EXEU(\beta_1,\beta_2)))$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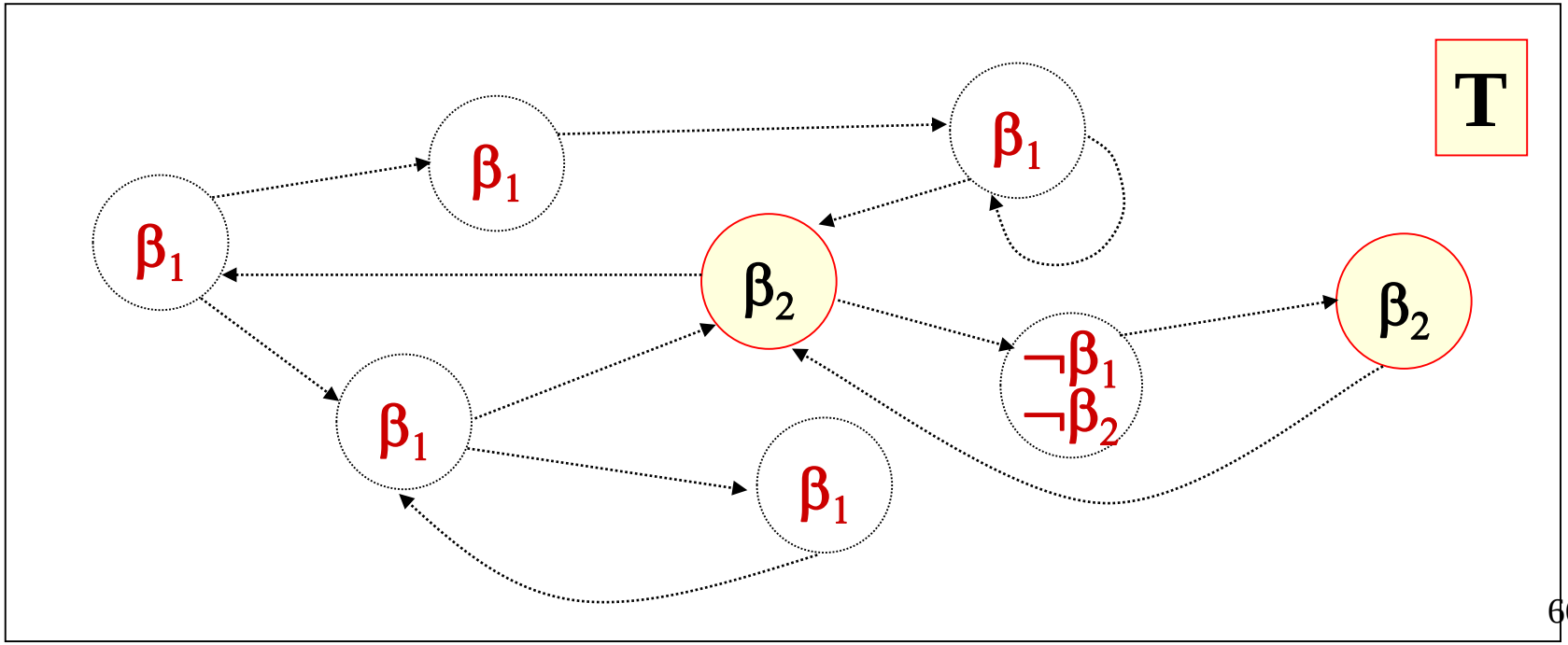
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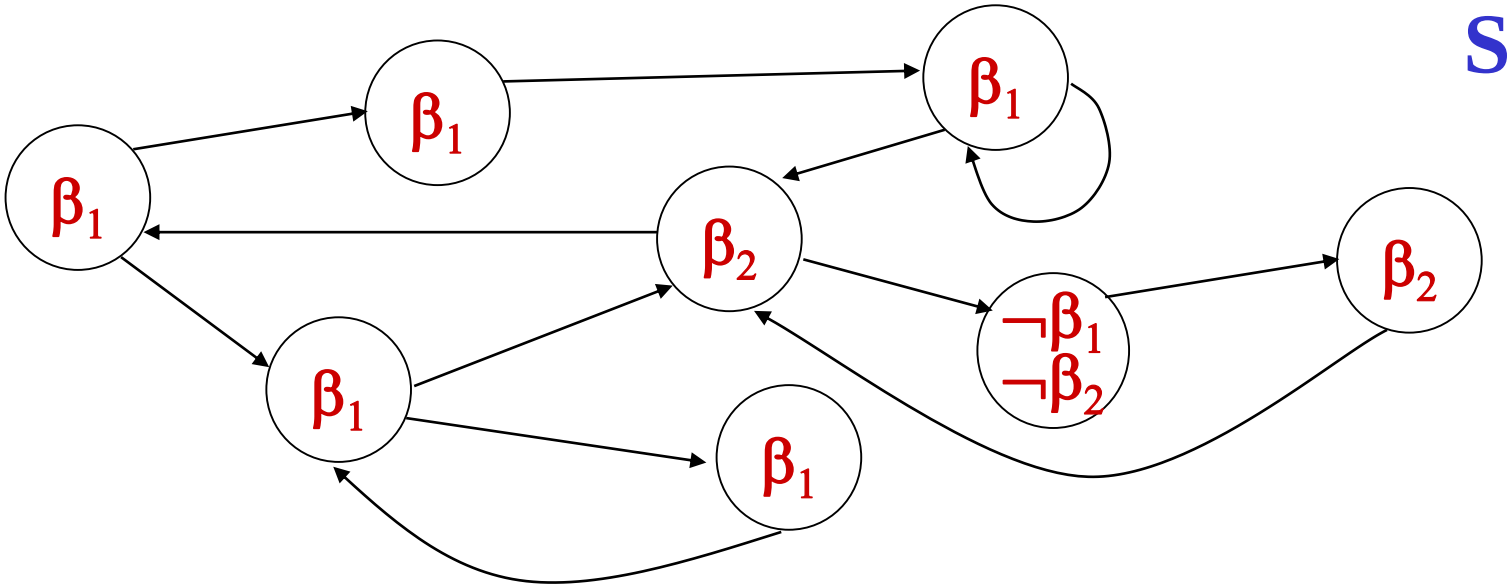
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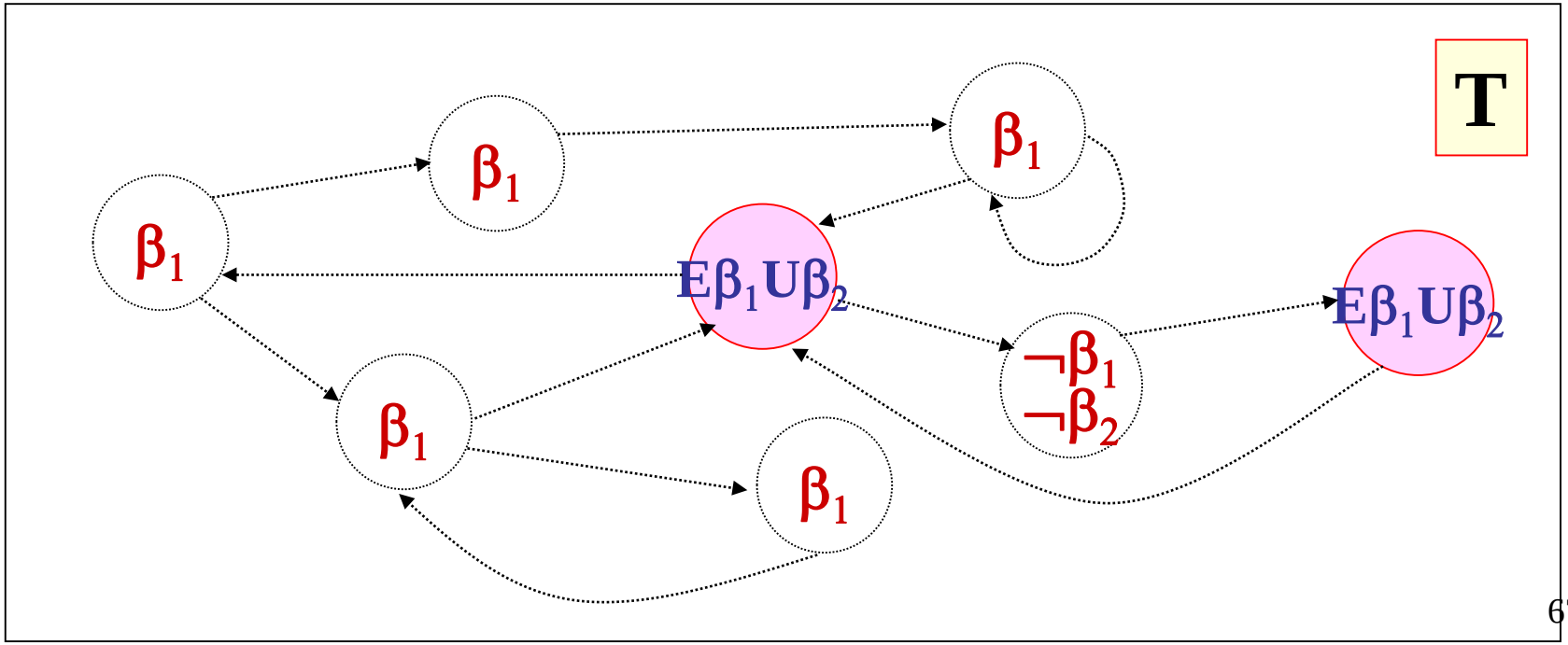
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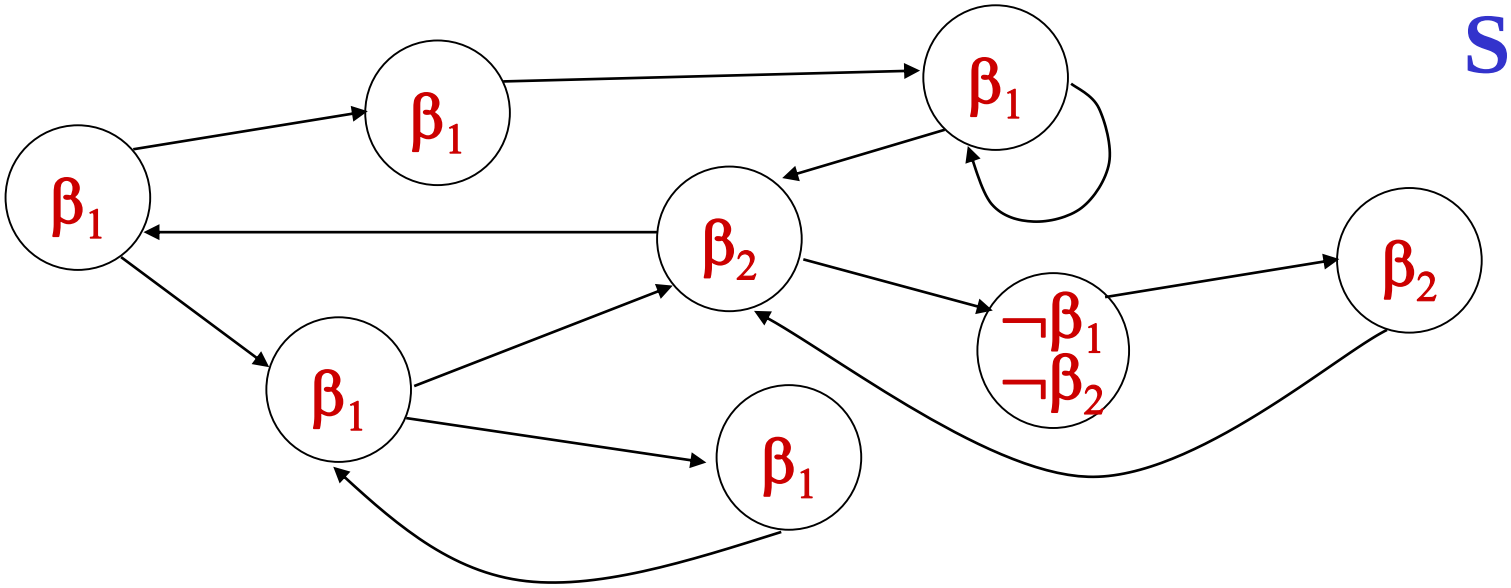
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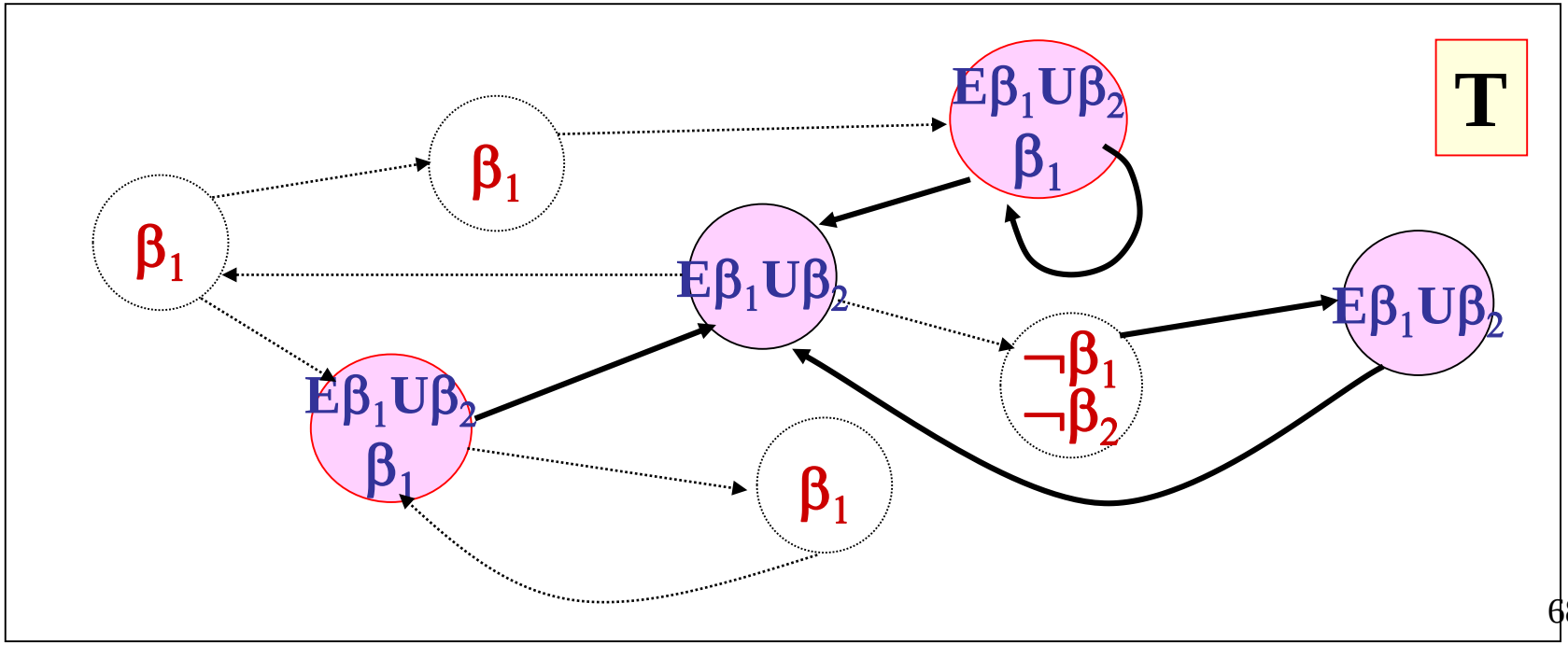
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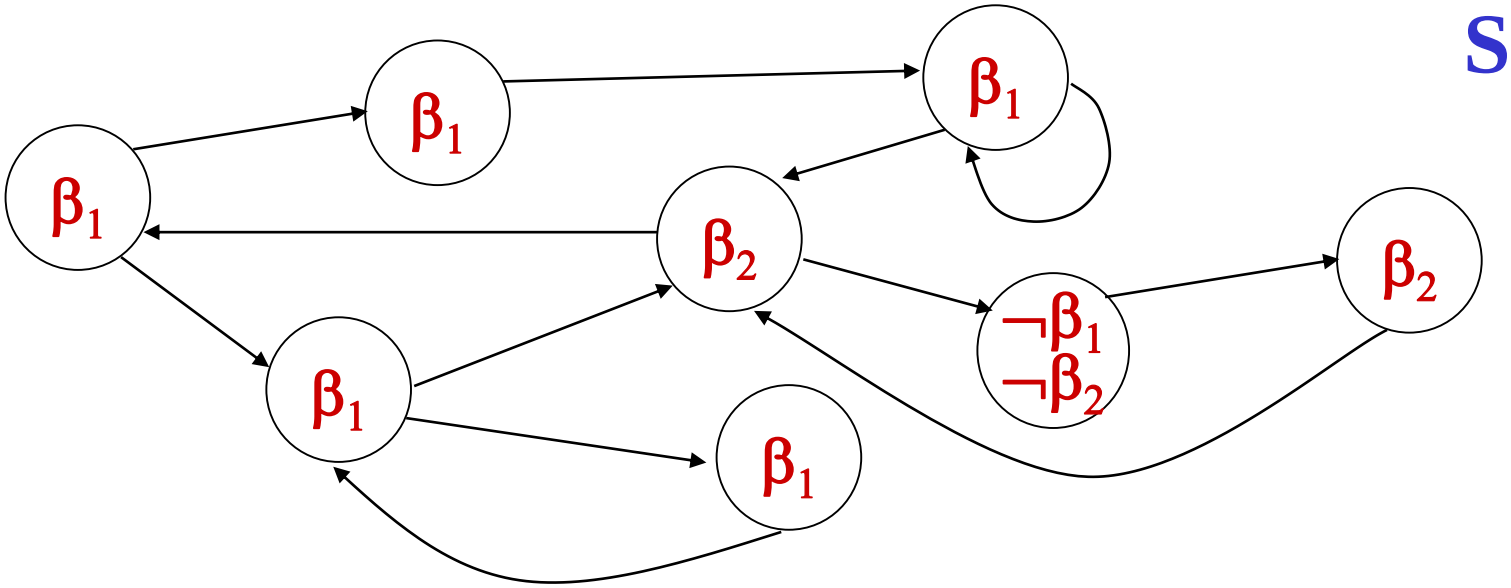
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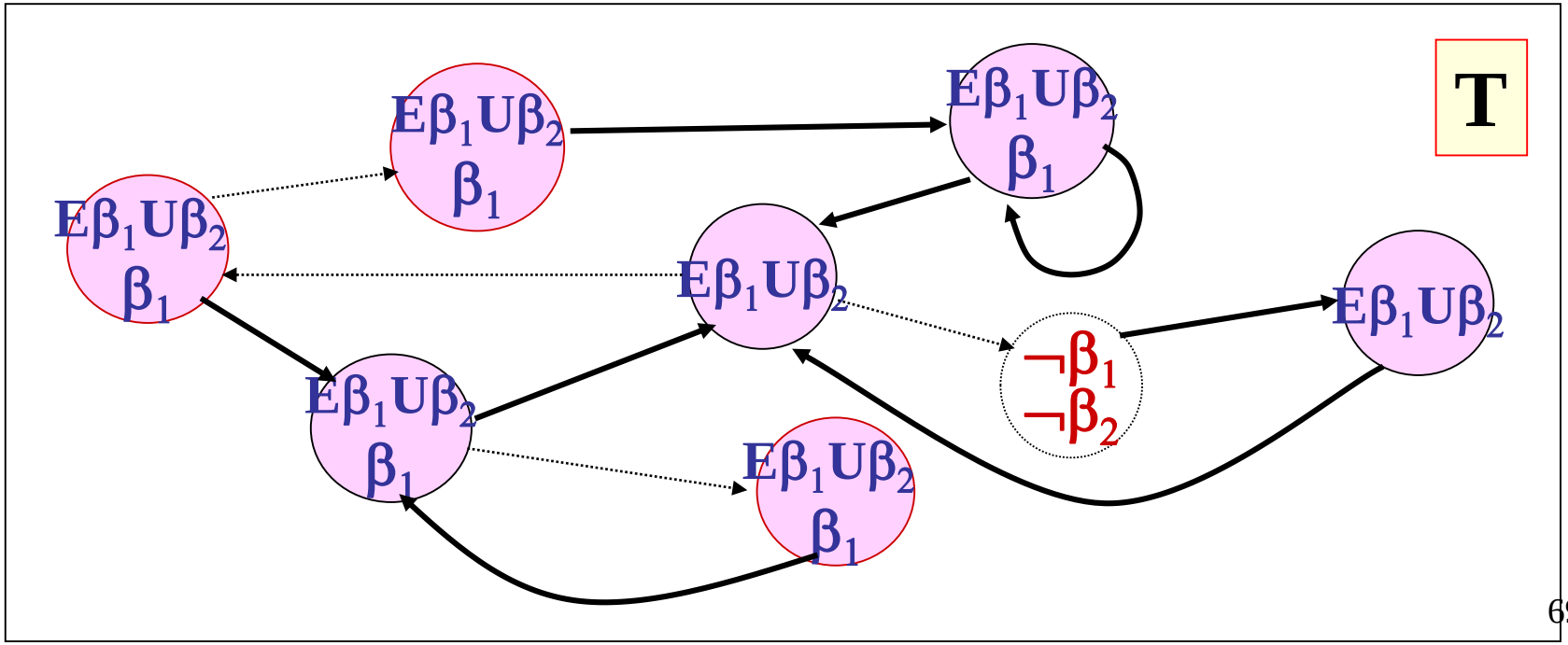
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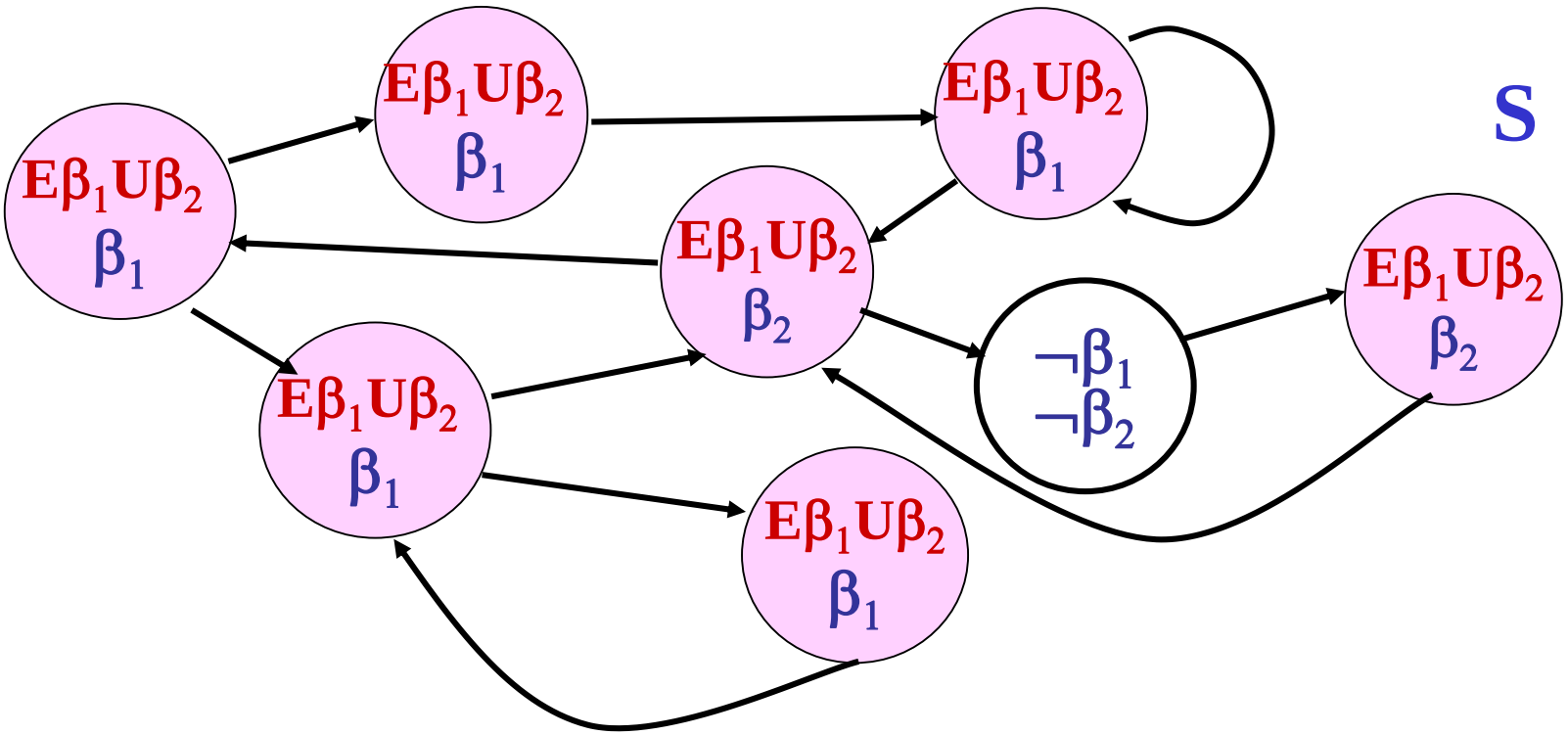
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Computing the labeling for $EU(\beta_1,\beta_2)$

Algorithm Check_$EU(\beta_1,\beta_2)$

T := {**s** | $\beta_2 \in$ Labels(**s**)};

for each **s** ∈ **T** do

Labels(**s**) := Labels(**s**) ∪ {**EU**(β_1,β_2)};

while **T** ≠ ∅ do

chose **s** ∈ **T**;

T := **T** \{**s**};

for each **t** ∈ **S** with **t** → **s** do

if **EU**(β_1,β_2) ∉ Labels(**t**) and $\beta_1 \in$ Labels(**t**) then

Labels(**t**) := Labels(**t**) ∪ {**EU**(β_1,β_2)};

T := **T** ∪ {**t**};

Complexity: O(|M|)

The Labels Function

- **Stage i +1**: consider subfomrulae **α** of length **i+1**
  - For every **t ∈ S**:
  - If **α = EG(β)** then

    α ∈ Labels(t) ***iff***

    - **β ∈ Labels(t) and EG(β) ∈ Labels(s) for some s with t → s.**

Property of EG(β)

Let **M' = (S',R',L')** be the sub-graph of **M** where

- **S' = { s | M ,s ⊨ β }**
- **R' = $R|_{S' \times S'}$** (the restriction of **R** to **S'**)
- **L' = $L|_{S'}$** (the restriction of **L** to **S'**)

Lemma: M,s ⊨ EG(β) *iff*

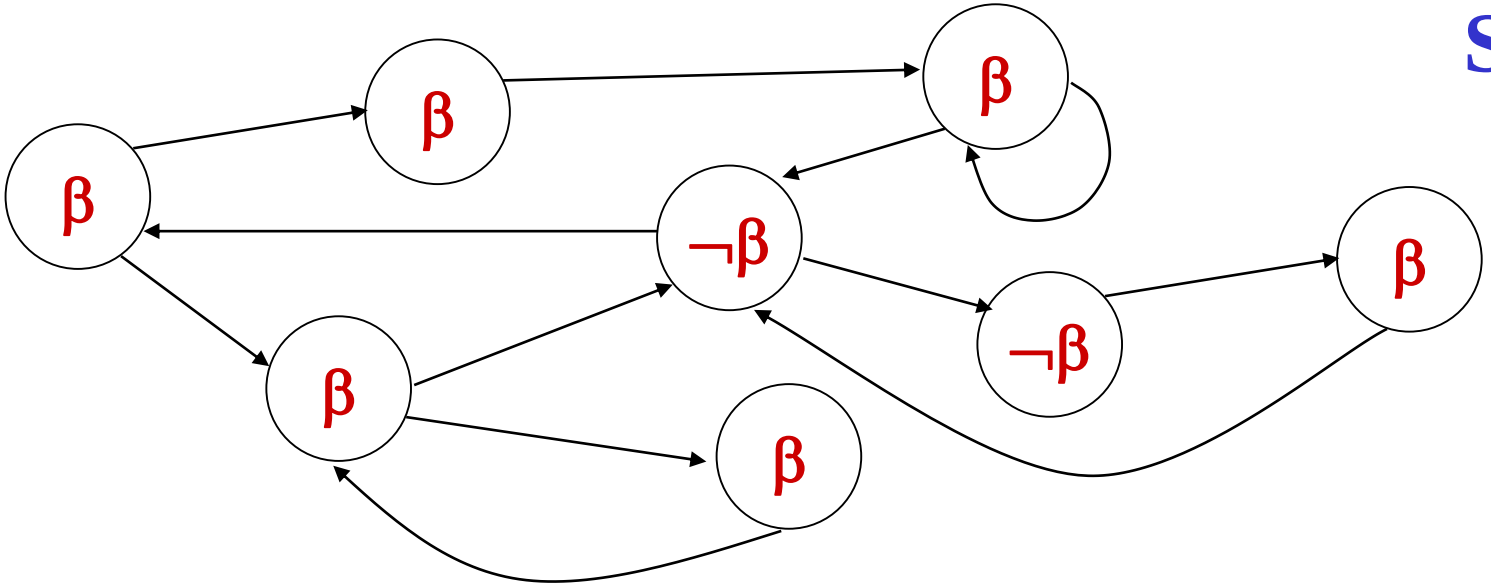
1. **s ∈ S' and**
2. ***there exists a path*** in **M'** leading from **s** to a ***non-trivial strongly connected component*** **C** of the graph **(S',R').**

The Labels Function

- Compute the subgraph **S'** whose states satisfy **β**
- Take ***non-trivial strongly connected components*** of **S'**
  - **all the states in these components do satisfy EG(β).**
- Traverse backward **→'** and label with **EG(β)** the ***states t reaching at least a state s*** labeled with **EG(β)** (note that both **t** and **s** belong to **S'**).

If t ∈ S' and R(t,s) then EG(β) ∈ Labels(t)

Recall that: **EG β = β ∧ EXEG β**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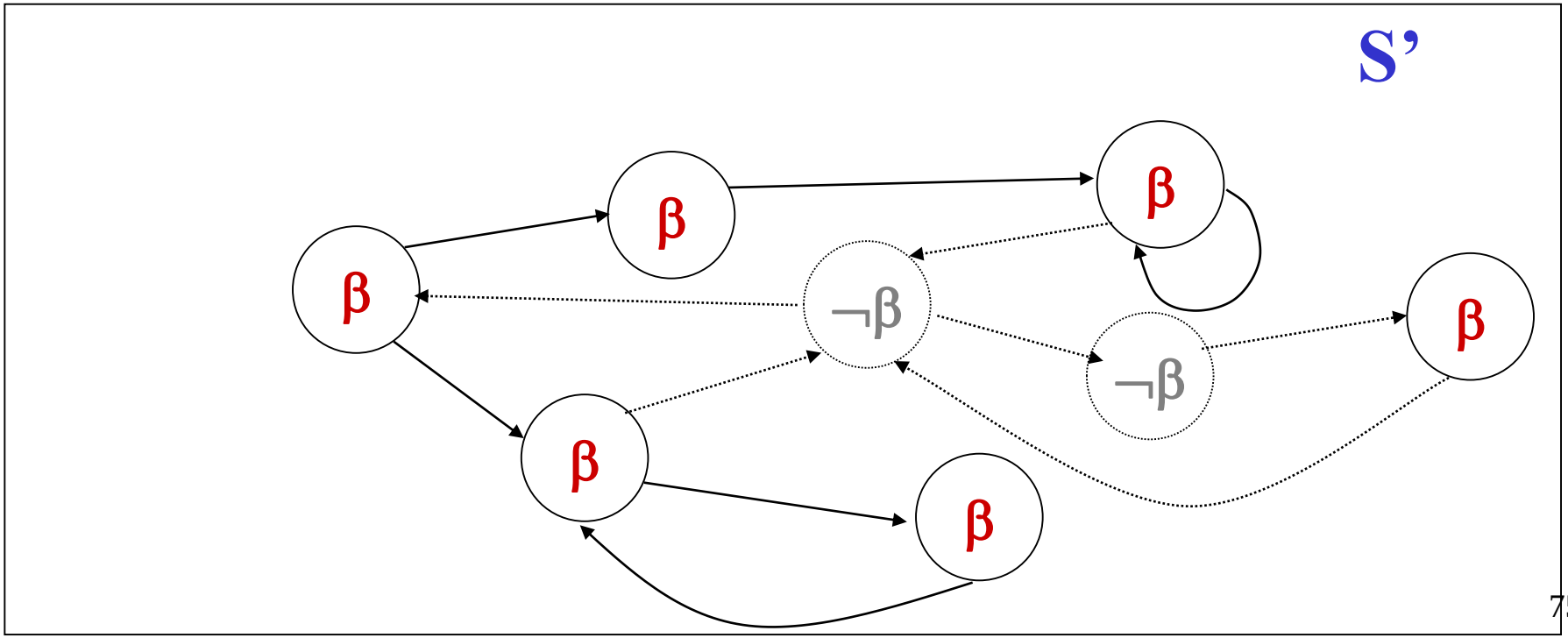
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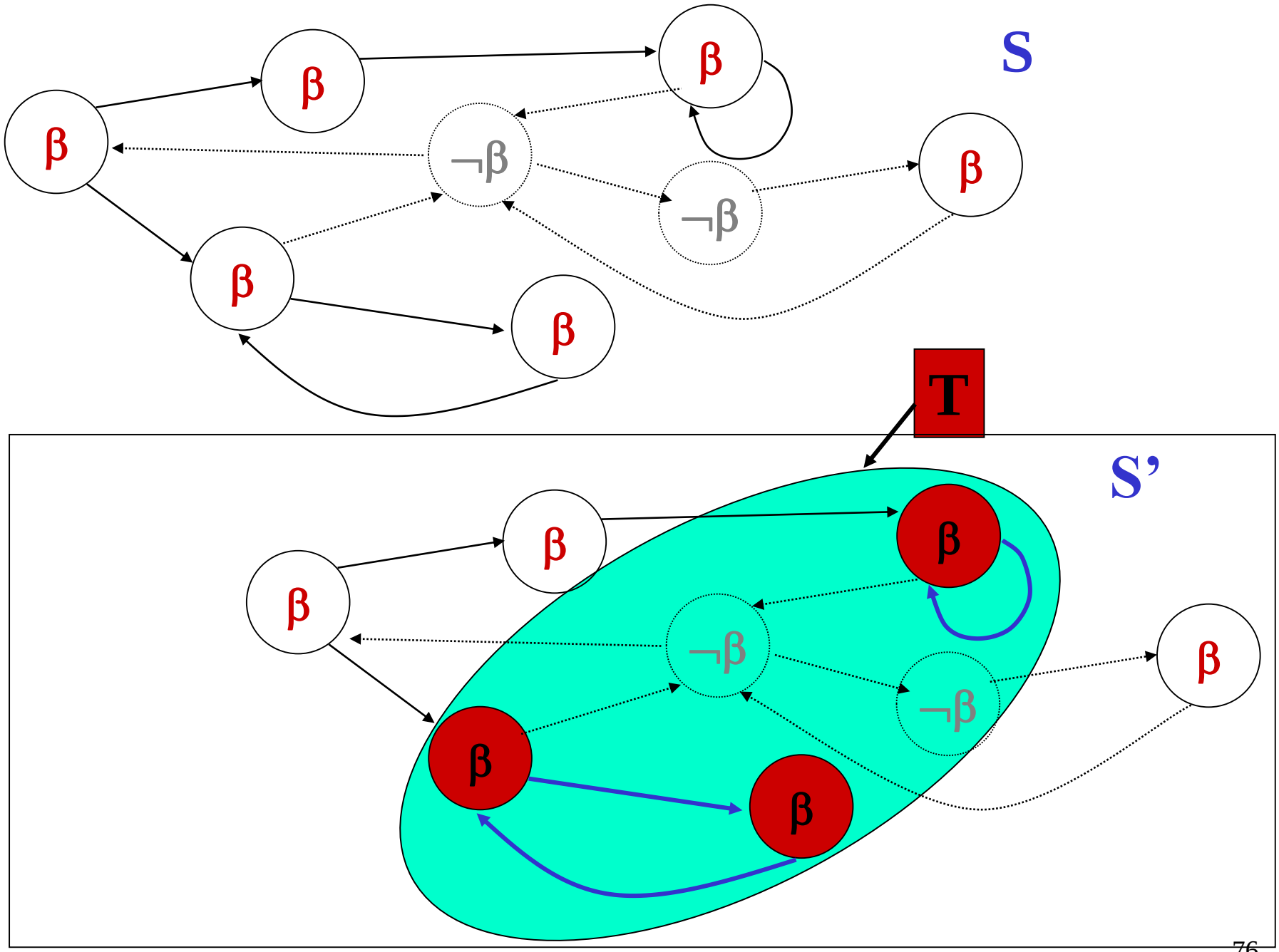
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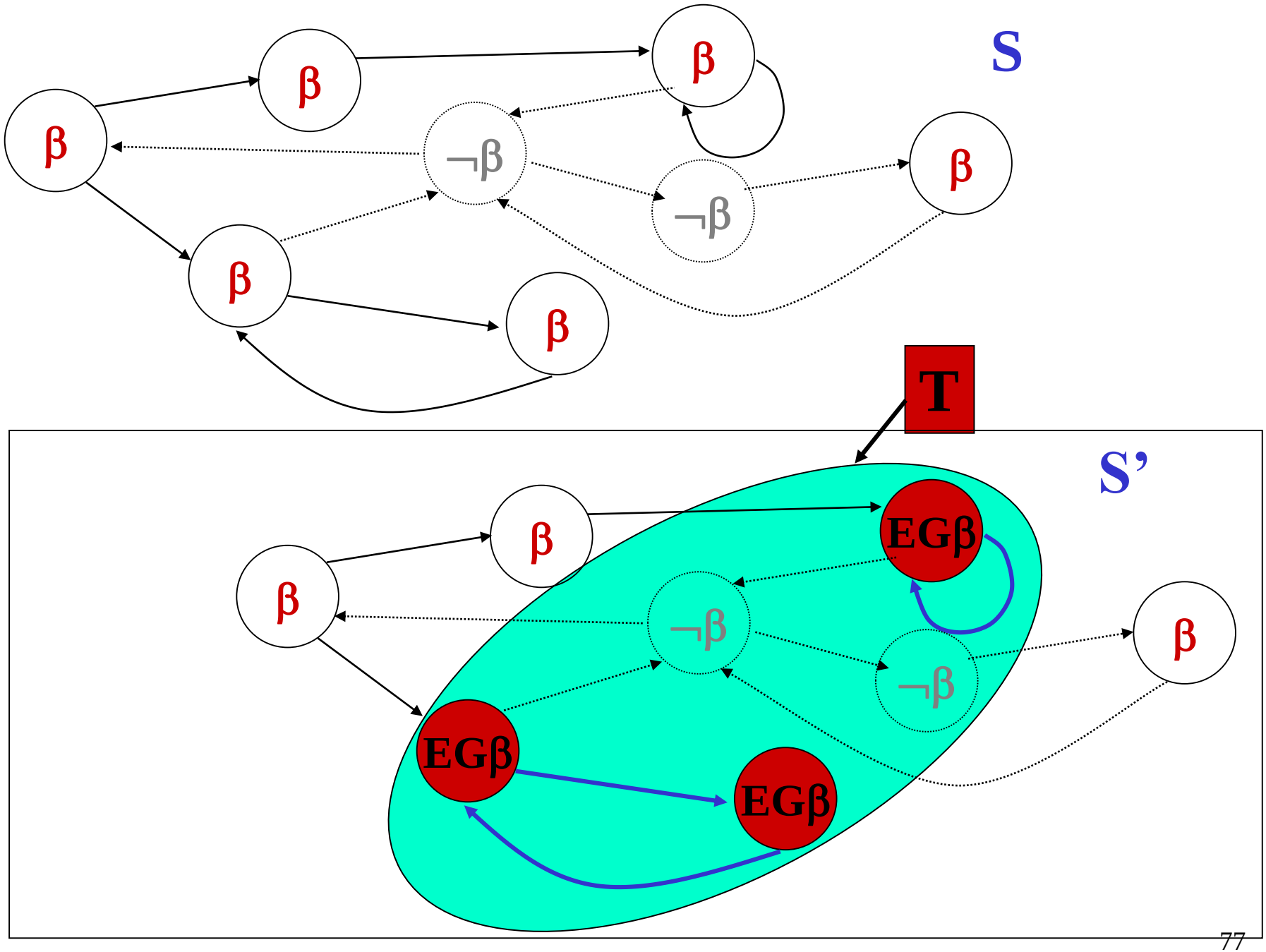
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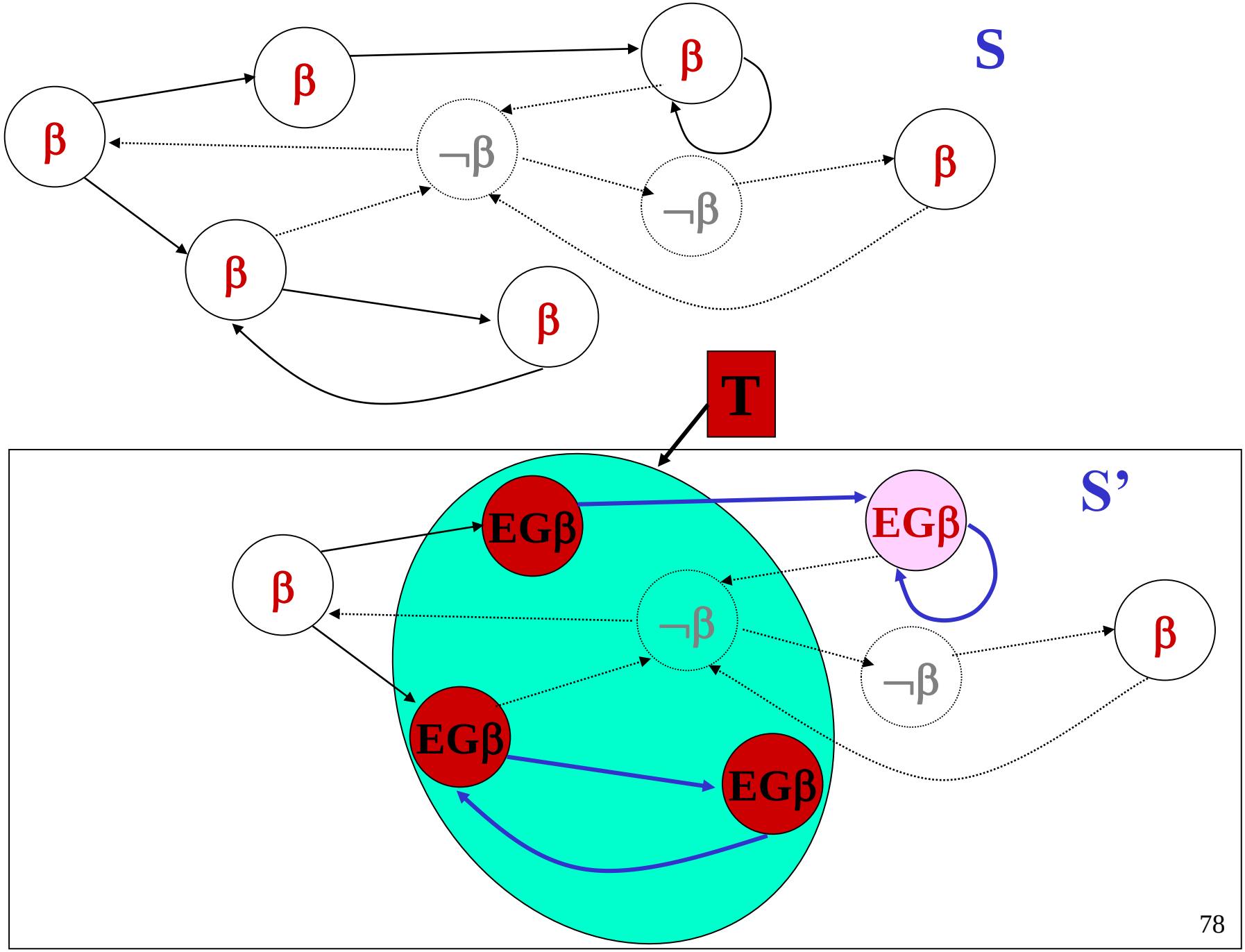
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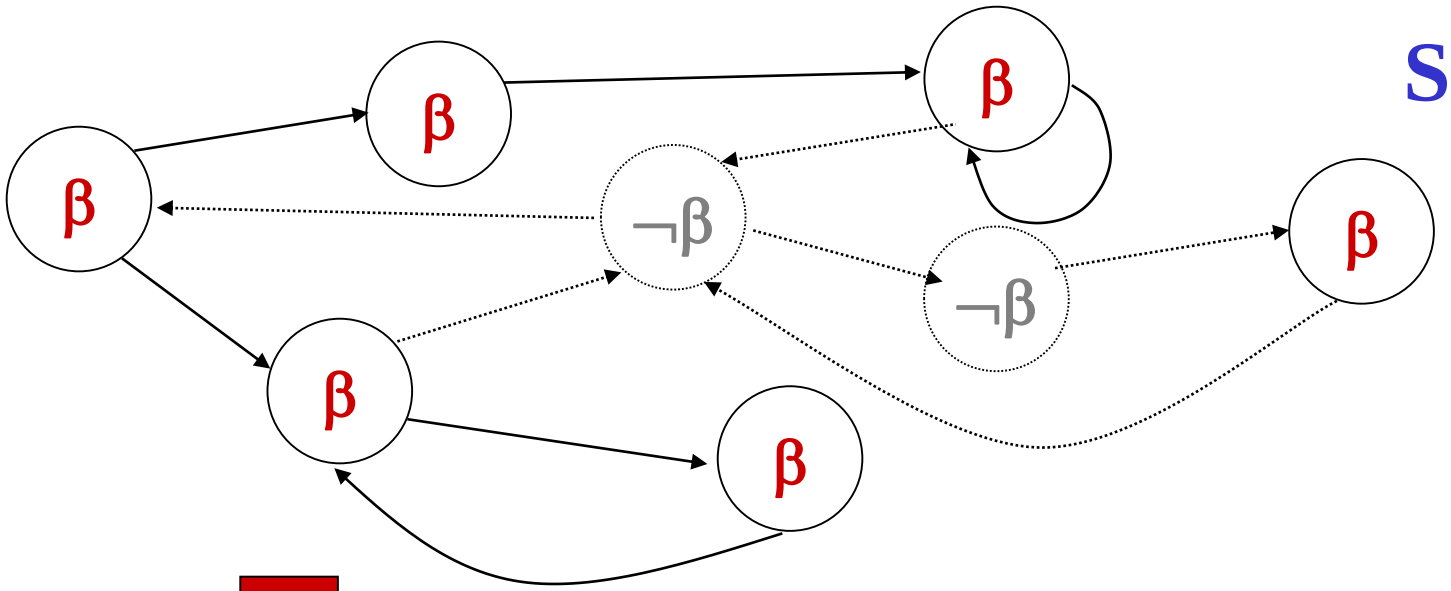
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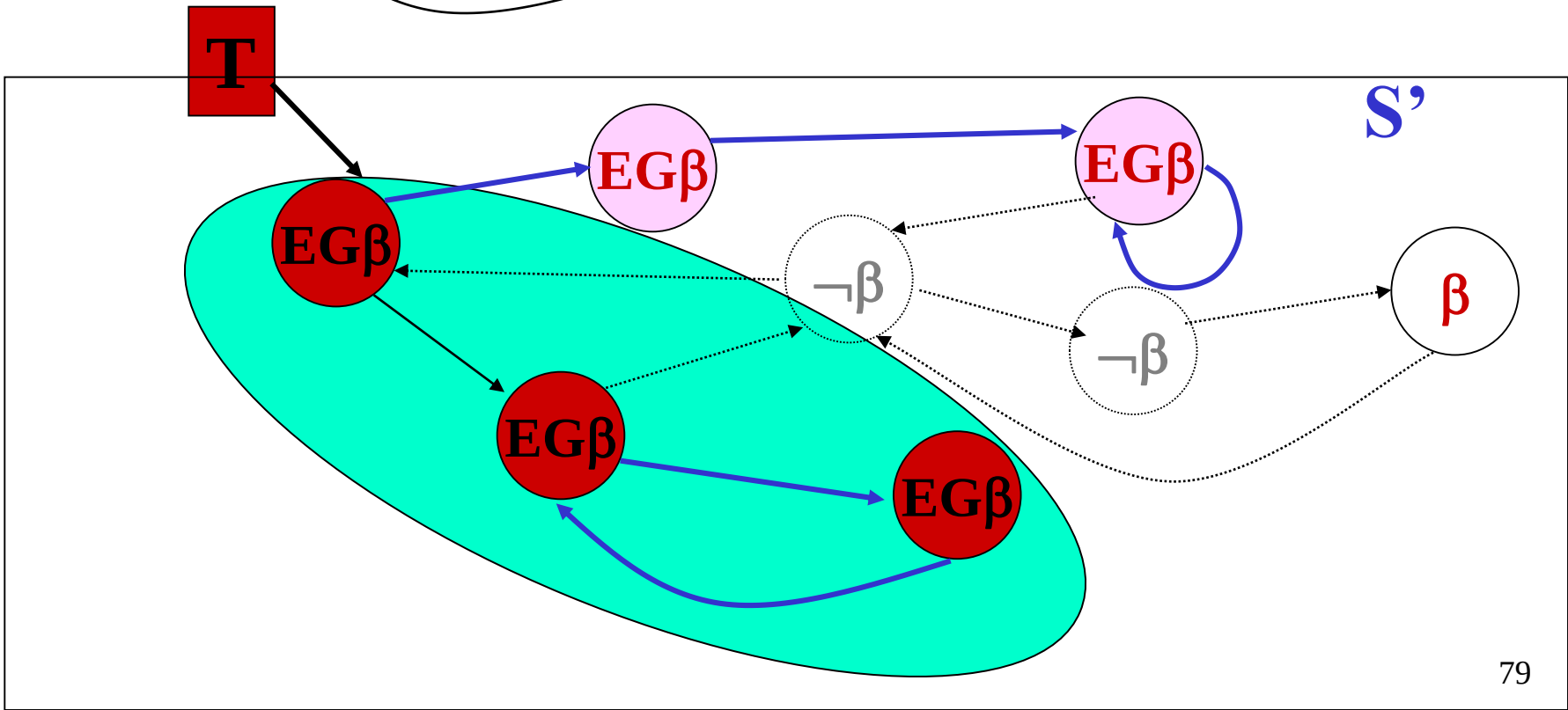
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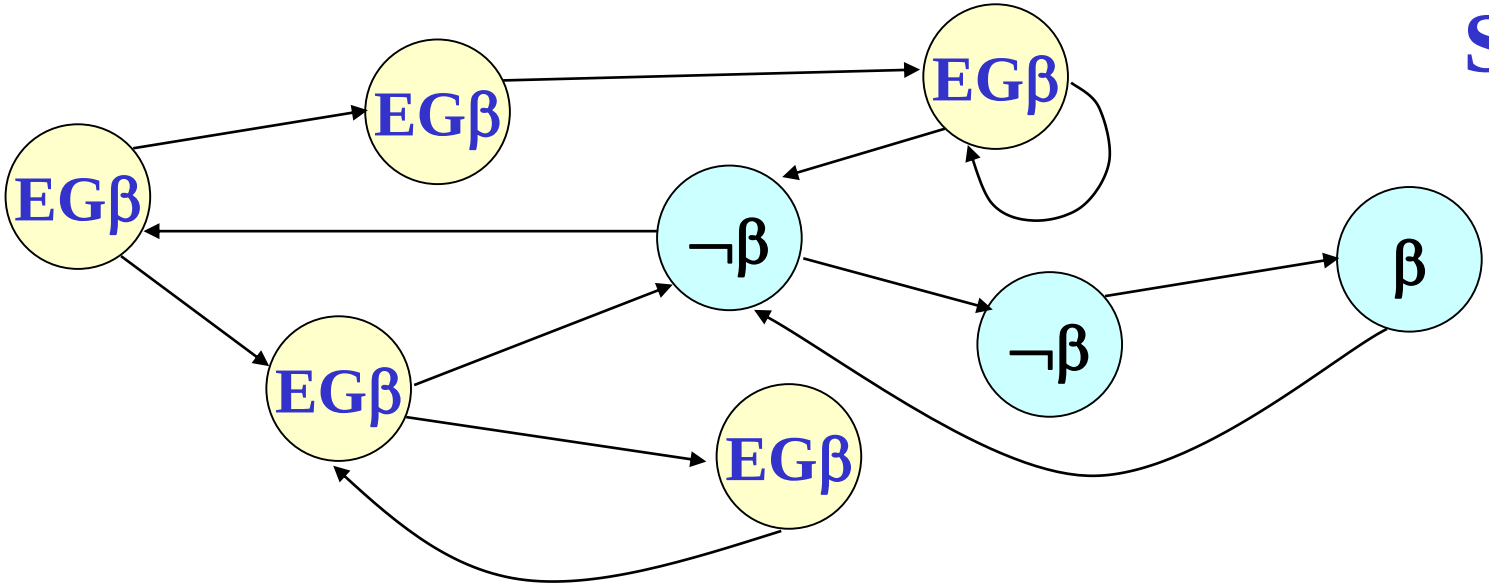
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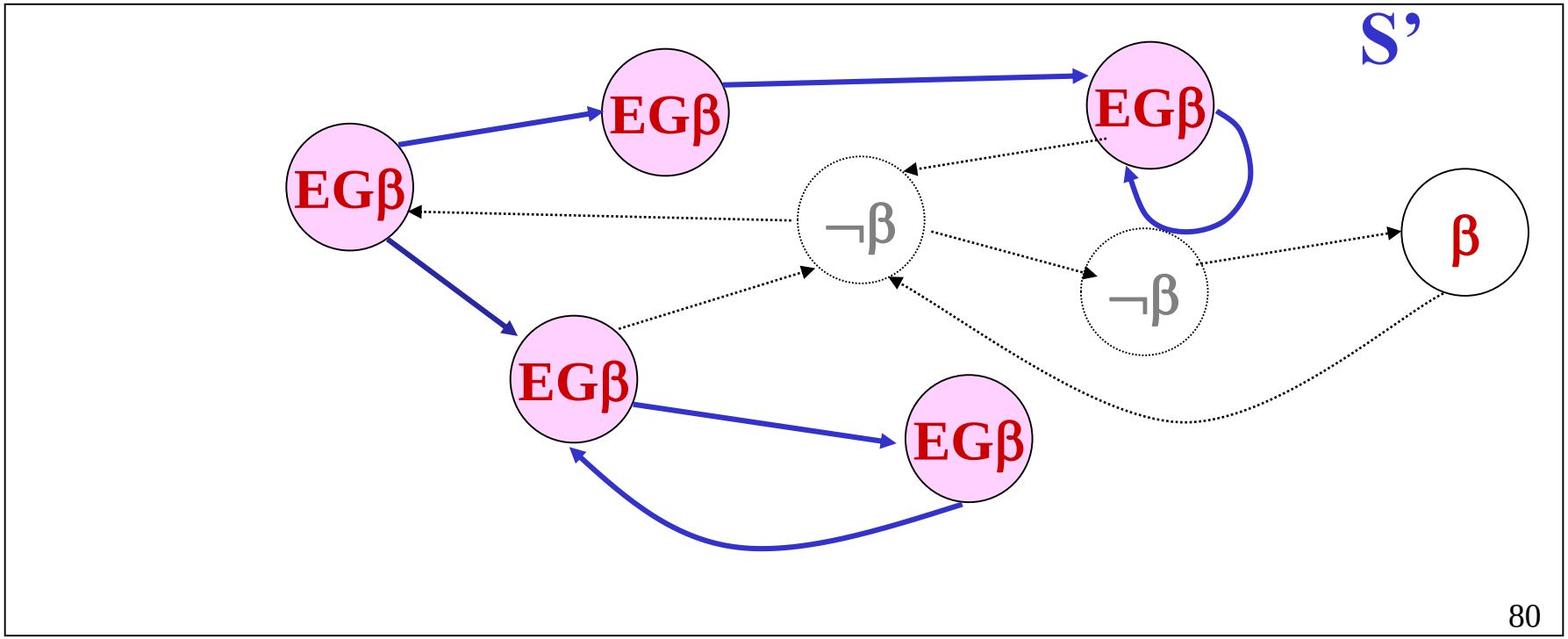
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Computing the labeling for EG(β)

Algorithm Check_EG(β)

Complexity: O(|M|)

S' := {**s** | β ∈ Labels(**s**)};

SCC := {**C** | **C** is a non trivial **SCC** of **S'**};

T := $\bigcup_{C \in SCC}$ {**s** | **s** ∈ **C**};

for each **s** ∈ **T** do Labels(**s**) := Labels(**s**) ∪ {**EG(**β**)**};

while **T** ≠ ∅ do

chose **s** ∈ **T**;

T := **T** \{**s**};

for each **t** ∈ **S'** with **t** → **s** do

if **EG(**β**)** ∉ Lables(**t**) then

Labels(**t**) := Labels(**t**) ∪ {**EG(**β**)**};

T := **T** ∪ {**t**};

CTL model checking

- The algorithms just presented show that the ***model checking problem*** for ***CTL*** can be solved in ***time linear*** in the size of System **M** and the size of the Property ϕ, namely:

in time $\mathbf{O(|M| \cdot |\phi|)}$

where $|\mathbf{M}|$ is the ***size of the graph*** underlying **M** and $|\phi|$ is the ***number of subformulae*** of ϕ.

Fixed point characterization

- We will redefine the labeling function in terms of ***fixed point computation***.
- This is a ***nice*** and ***elegant*** algorithmic account.
- It will be used when ***efficient symbolic approach*** will be introduced.

Partial Orders

- A binary relation $\sqsubseteq$ on a set **A** is a ***partial order*** iff $\sqsubseteq$ is ***reflexive***, ***anti-symmetric*** and ***transitive***.
- The pair $\langle A, \sqsubseteq \rangle$ is called a ***partially ordered set*** (or ***poset***).
- **Example:** If **S** is any set and $\subseteq$ is the ordinary subset relation, then $\langle 2^S, \subseteq \rangle$ is a *partially ordered set*.

Upper Bounds

Given $\langle A, \sqsubseteq \rangle$ and $A' \subseteq A$

- $a \in A$ is an ***upper bound*** of A' iff $\forall a' \in A', a' \sqsubseteq a$
- $a \in A$ is a ***least upper bound*** (***lub***) of A', written $\sqcup A'$, iff
  - a is an ***upper bound*** of A' and
  - $\forall a' \in A$, if a' is an ***upper bound*** of A', then $a \sqsubseteq a'$

Lower Bounds

Given $\langle A, \sqsubseteq \rangle$ and $A' \subseteq A$

- $a \in A$ is a ***lower bound*** of A' iff $\forall a' \in A', a \sqsubseteq a'$
- $a \in A$ is a ***greatest lower bound*** (***glb***) of A', written $\sqcap A'$, iff
  - a is a ***lower bound*** of A' and
  - $\forall a' \in A$, if a' is a ***lower bound*** of A', then $a' \sqsubseteq a$

Complete Lattice

A ***poset*** **<A, ⊑>** is a ***complete lattice*** if, for each **A' ⊆ A**, the ***greatest lower bound*** **⊓A'** and the ***least upper bound*** **⊔A'** do exist.

A ***complete lattice*** **<A, ⊑>** has a unique ***greatest element*** **⊔A=⊤** and also a unique ***least element*** **⊓A = ⊥**.

Complete Lattice

The ***poset*** $\langle 2^S, \subseteq \rangle$ is a ***complete lattice*** where ***intersection*** $\cap$ and ***union*** $\cup$ correspond to $\sqcap$ and $\sqcup$, respectively.

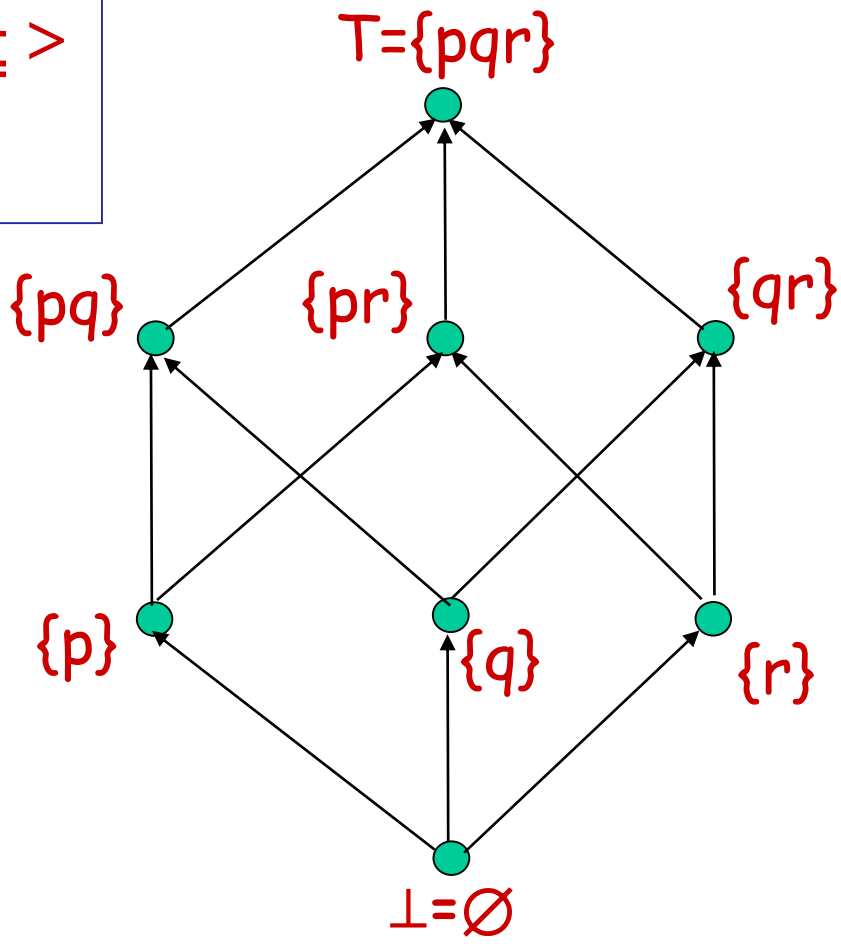
Any two subset of **S** have a ***least upper*** and a ***greatest lower bound***.

Example: **S={a,b,c,d}**. For **{a,c}** and **{b,c}** the ***glb*** is **{c}**, while the ***lub*** is **{a,b,c}**.

There is a unique ***greatest element*** $\cup 2^S = S$ and a unique ***least element*** $\cap 2^S = \varnothing$.

Example of a complete lattice

The complete lattice $<2^S, \subseteq>$ when S is the set {p,q,r}.



Monotonic functions

- A function **F: A ⟶ A** is ***monotonic*** if for each **a,b ∈ A**, **a ⊑ b** implies **F(a) ⊑ F(b)**.
- In other words, a function **F** is monotonic if it ***preserves the ordering*** ⊑.

Fixed points

- Given a function **F: A ⟶ A**, an element **a ∈ A** is a ***fixed point*** of **F** if **F(a) = a**.
- **a ∈ A** is called the ***least fixed point*** of **F (μx.F(x))**, if for all **a' ∈ A** such that **F(a') = a'**, then **a ⊑ a'**.
- **a ∈ A** is called the ***greatest fixed point*** of **F (νx.F(x))**, if for all **a' ∈ A** such that **F(a') = a'**, then **a' ⊑ a**.

Tarski's Fixed Point theorem

THEOREM: Let **<A, ⊑>** be a ***complete lattice***, and **F: A ⟶ A** a monotonic function. Then **F** has a ***least*** and a ***greatest fixed point*** given, respectively, by:

- **μx.F(x) = ⊓{x ∈ A | F(x) ⊑ x}**
- **νx.F(x) = ⊔{x ∈ A| x ⊑ F(x)}**

Fixed point in finite lattices

Let $\langle A, \sqsubseteq \rangle$ be a ***finite complete lattice***, and $F: A \rightarrow A$ be a monotonic function.

The ***least element*** of **A**

Then the ***least fixed point*** for **F** is obtained as

$$\mu x.F(x) = F^m(\bot)$$

for some **m**, where $F^0(\bot) = \bot$, and $F^{n+1}(\bot) = F(F^n(\bot))$.

Moreover, the ***greatest fixed point*** for **F** is obtained as

$$\nu x.F(x) = F^k(\top)$$

for some **k**, where $F^0(\top) = \top$, and $F^{n+1}(\top) = F(F^n(\top))$.

The ***greatest element*** of **A**

Generic fixed point algorithm

Algorithm Compute_lfp(**F**:function)

$\mathbf{X_0} := \bot$;

$\mathbf{X_1} := \mathbf{F}(\mathbf{X_0})$;

j=1;

while $\mathbf{X_j} \neq \mathbf{X_{j-1}}$

$\mathbf{j} := \mathbf{j+1}$;

$\mathbf{X_j} := \mathbf{F}(\mathbf{X_{j-1}})$;

return $\mathbf{X_j}$

CTL and complete lattices

- Given a Kripke structure $\mathbf{M=<S,S_0,R,L,AP>}$. We will then consider the ***poset*** $\mathbf{<2^S,\subseteq>}$.
- $\mathbf{<2^S,\subseteq>}$ is clearly a ***complete lattice*** (with respect to intersection and union).
- We will identify a ***CTL formula*** with the ***set of states*** which ***satisfy it***.
- In this way we can define ***temporal operators*** as ***functions*** on the ***complete lattice*** $\mathbf{<2^S,\subseteq>}$.

Denotation of a CTL formula

- Given a formula ϕ, let us define its ***denotation*** (in **M**), in symbols $|[\phi]|$, as the set of states satisfying the formula:

$$|[\phi]| = \{ s \mid M,s \vDash \phi \}$$

- We could then define the cpo $<CTL, \sqsubseteq>$ by:

$$\phi \sqsubseteq \psi \quad \textit{iff} \quad |[\phi]| \subseteq |[\psi]|$$

Denotation of a CTL formula

- Given the ***denotation*** of a formula

$$|[\phi]| = \{ s \mid M,s \models \phi \}$$

- We could then define the cpo $\langle CTL, \sqsubseteq \rangle$ by:

$$\phi \sqsubseteq \psi \quad \textit{iff} \quad |[\phi]| \subseteq |[\psi]|$$

- Then $|[\bot]| = \varnothing$; $|[\top]| = S$;
- $|[p]| = \{ s \mid p \in L(s) \}$;
- $|[\neg\phi]| = S \setminus |[\phi]|$;
- $|[\phi \vee \psi]| = |[\phi]| \cup |[\psi]|$;
- $|[\phi \wedge \psi]| = |[\phi]| \cap |[\psi]|$;

CTL is closed under ***conjunction*** and ***disjunction***, therefore for any pair of formulae the ***upper*** and ***lower bound*** do exist.

Denotation of a CTL formula

- Given a formula ϕ, let us define its ***denotation*** (in **M**), in symbols $|[\phi]|$, as the set of states satisfying the formula:

$$|[\phi]| = \{ s \mid M,s \vDash \phi \}$$

- ….
- $|[EX\phi]| = \{ s \mid \exists t. (t \in |[\phi]| \cap R(s)) \}$
- for the other **temporal operators** we would need to use **fixed points….**

Fixed point characterization of $\mathbf{EU(\beta_1,\beta_2)}$

- $\mathbf{EU(\beta_1,\beta_2) \equiv \beta_2 \vee (\beta_1 \wedge EX\ EU(\beta_1, \beta_2))}$
- $\mathbf{|[EU(\beta_1,\beta_2)]| = \mu Z.(|[\beta_2]| \cup (|[\beta_1]| \cap |[EX\ Z]|))}$
- $\mathbf{|[EU(\beta_1,\beta_2)]| =}$

$$\mathbf{\mu Z.(|[\beta_2]| \cup (|[\beta_1]| \cap \{ s \mid \exists t \in Z \cap R(s) \}))}$$

Fixed point characterization of $\mathbf{EU(\beta_1,\beta_2)}$

Lemma: Let

$$\mathbf{F(Z) = (|[\beta_2]| \cup (|[\beta_1]| \cap \{ s \mid \exists t \in Z \cap R(s) \}))}$$

then **F** is a ***monotonic function***, i.e.

$$\mathbf{Z_1 \subseteq Z_2 \text{ implies } F(Z_1) \subseteq F(Z_2)}$$

Fixed point characterization of $\mathbf{EU(\beta_1,\beta_2)}$

Theorem:

$$|[\mathbf{EU}(\beta_1,\beta_2)]| = \mu Z.(|[\beta_2]| \cup (|[\beta_1]| \cap \{ s \mid \exists t \in Z \cap R(s) \}))$$

in other words:

$$\mu Z.(|[\beta_2]| \cup (|[\beta_1]| \cap \{ s \mid \exists t \in Z \cap R(s)\})) \subseteq |[\mathbf{EU}(\beta_1,\beta_2)]|$$

and

$$|[\mathbf{EU}(\beta_1,\beta_2)]| \subseteq \mu Z.(|[\beta_2]| \cup (|[\beta_1]| \cap \{ s \mid \exists t \in Z \cap R(s)\}))$$

Computing fixed point for $EU(\beta_1,\beta_2)$

Algorithm Compute_EU(β_1,β_2)

$X_0 := |[\bot]|$; /* i.e. $X_0 := \emptyset$ */

$X_1 := |[\beta_2]| \cup (|[\beta_1]| \cap X_0)$; /* i.e. $X_1 := |[\beta_2]|$ */

j=1;

while $X_j \neq X_{j-1}$

$j := j+1$; $T := X_{j-1}$; $X := \emptyset$

while $T \neq \emptyset$ do

chose $s \in T$;

$T := T \setminus \{s\}$;

forall t such that $s \in R(t)$ do

$X := X \cup \{t\}$;

$X_j := |[\beta_2]| \cup (|[\beta_1]| \cap X)$

This computes $X = EX\ X_{j-1}$

Computing fixed point for $\mathbf{EU(\beta_1,\beta_2)}$

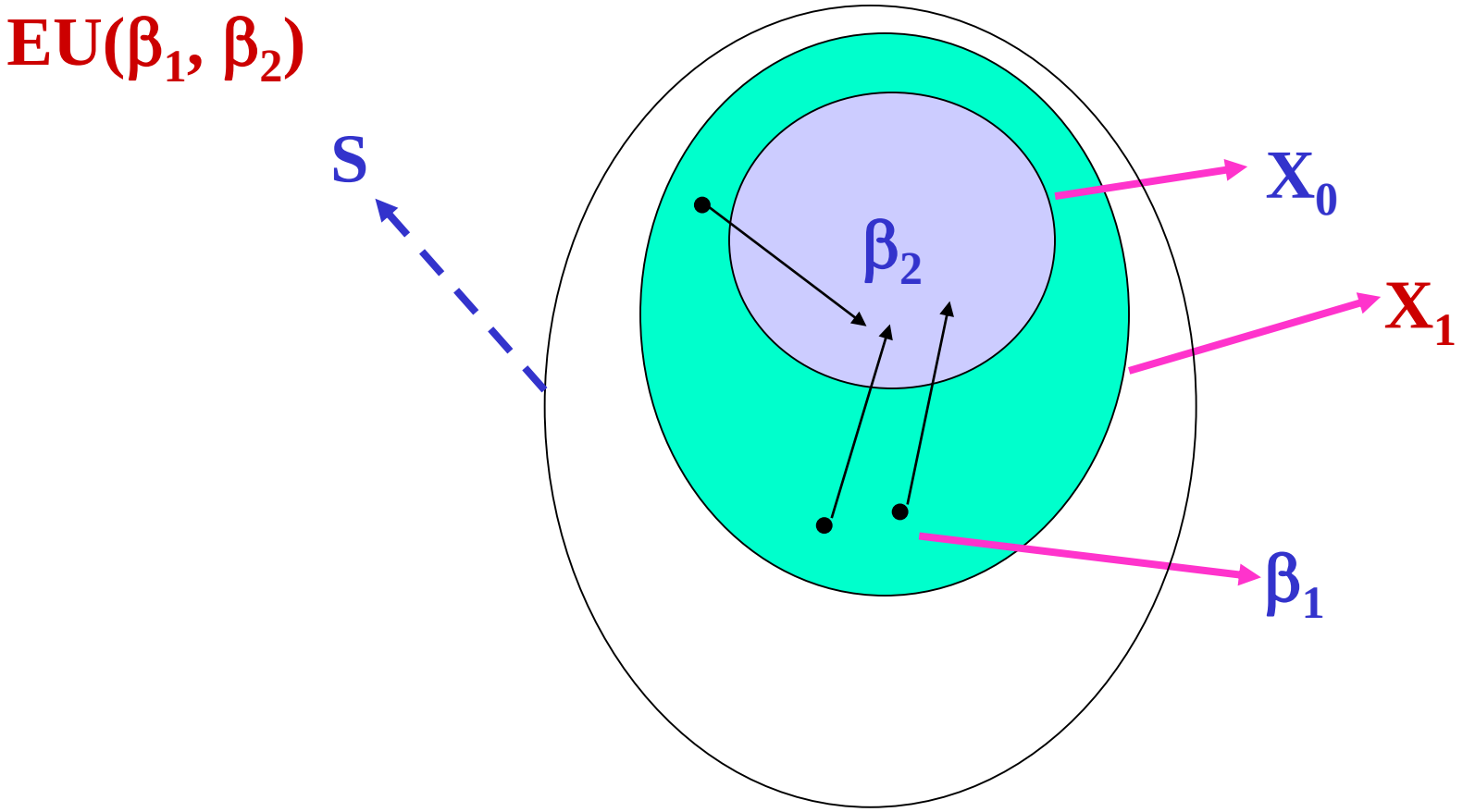
To compute $\mathbf{|[EU(\beta_1,\beta_2)]|}$ we can ***construct inductively the set*** of states $\mathbf{X_j}$ as follows:

- $\mathbf{X_1 = |[\beta_2]|}$.
- $\mathbf{X_{j+1} = X_j \cup \{ s \mid s \in |[\beta_1]|}$ and $\mathbf{R(s,t)}$ for some $\mathbf{t \in X_j\}}$

$\mathbf{|[EU(\beta_1,\beta_2)]|}$ is then the set $\mathbf{X}$ such that $\mathbf{X = X_n}$ for $\mathbf{n}$ such that $\mathbf{X_{n+1} = X_n}$.

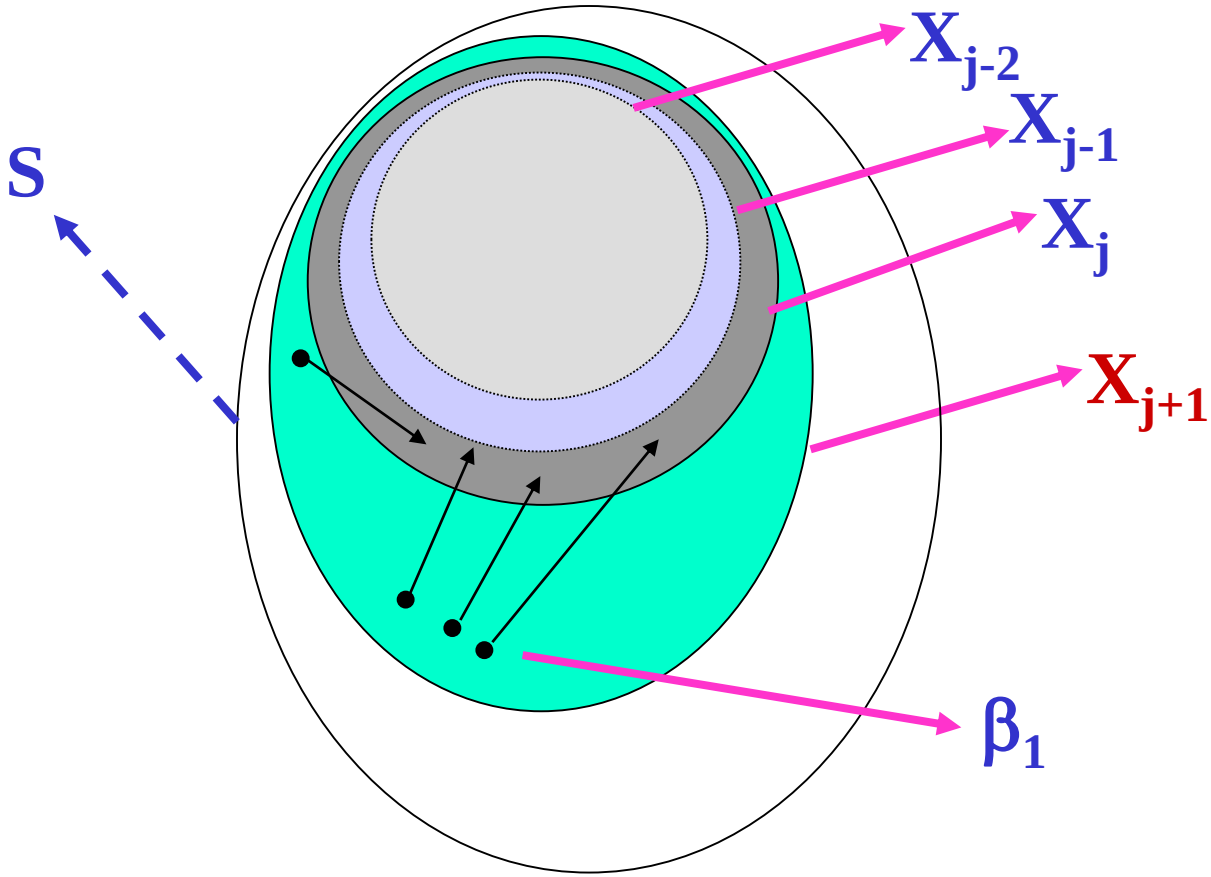
Notice that $\mathbf{n}$ ***must exist*** by ***Tarski's Theorem*** since $\mathbf{X_j \subseteq X_{j+1} \subseteq S}$ (and $\mathbf{S}$ is finite!)

From X_0 to X_1



From X_j to X_{j+1}

$EU(\beta_1, \beta_2)$



Computing fixed point for $EU(\beta_1,\beta_2)$

Algorithm Compute_EU(β_1,β_2)

$X_1 := |[\beta_2]|$;

j=1;

repeat

$j := j+1$; $T := X := X_{j-1}$;

while $T \neq \emptyset$ do

chose $s \in T$;

$T := T \setminus \{s\}$;

forall t such that $s \in R(t)$ do

if $t \in |[\beta_1]|$ then /* $t \in |[\beta_1]| \cap EX\, X_{j-1}$ */

$X := X \cup \{t\}$;

$X_j = X$;

until $X_{j-1} = X_j$

Fixed point characterization of EG(β)

- **$EG(\beta) \equiv \beta \wedge EX\ EG(\beta)$**
- **$|[EG(\beta)]| = \nu Z.((|[\beta]| \cap |[EX\ Z]|)\ )$**
- **$|[EG(\beta)]| =$**

  $\nu Z.((|[\beta]| \cap \{\ s \mid \exists t \in Z \cap R(s)\ \})$

Computing the fixed point for EG(β)

Algorithm Compute_EG(β)

$X_0 := |[\top]|$; /* i.e. $X_0 := S$ */

$X_1 := |[\beta]| \cap X_0$; /* i.e. $X_1 := |[\beta]|$ */

$j=1$;

while $X_j \neq X_{j-1}$

$j := j+1$; $T := X_{j-1}$; $X := \emptyset$;

while $T \neq \emptyset$ do

chose $s \in T$;

$T := T \setminus \{s\}$;

forall t such that $s \in R(t)$

$X_j := X_j \cup \{t\}$;

$X_j := |[\beta]| \cap X_j$

$X = EX\ X_{j-1}$

The Labels function

- To compute $|[EG\beta]|$ we can ***construct inductively the set*** of states X_j as follows:

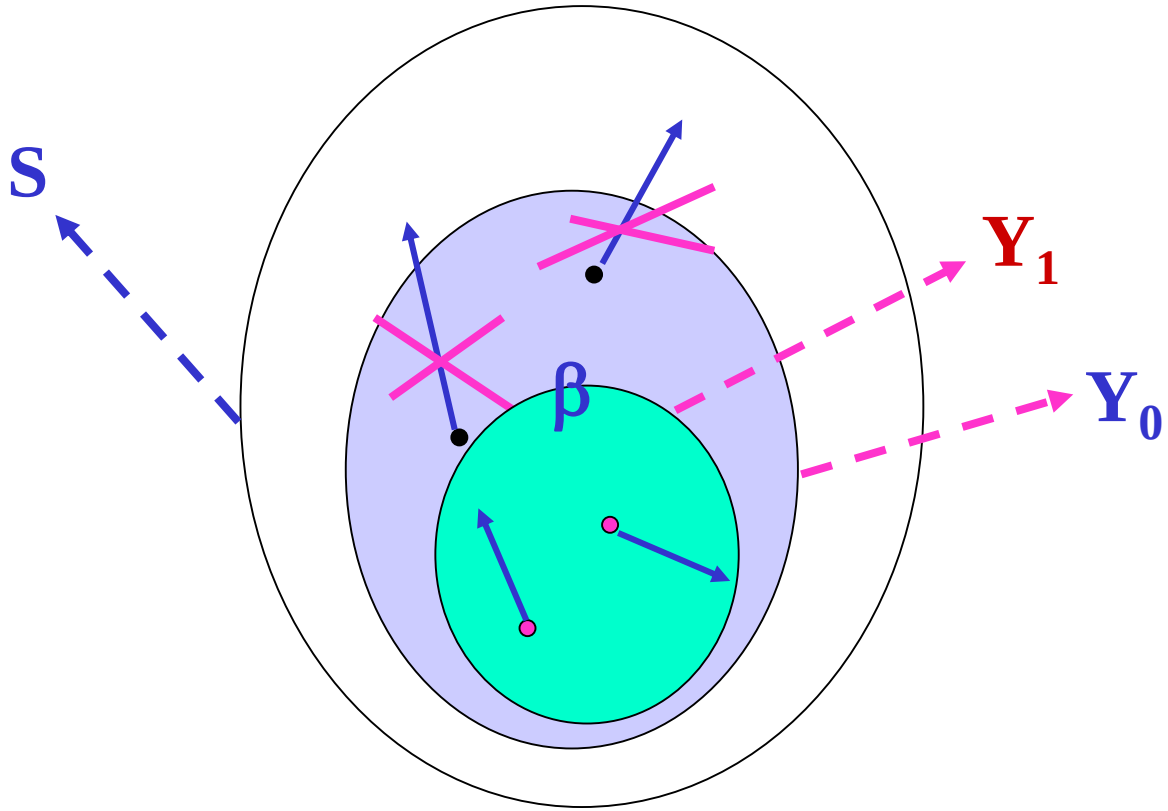
  - $X_1 = |[\beta]|$.
  - $X_{j+1} = X_j - \{ s \mid s \in X_j$ and ***there does not exist*** $t \in X_j$ such that $R(s, t)\}$

$|[EG\beta]|$ is then the set X such that $X = X_n$ for m such that $X_{m+1} = X_m$.

- Notice that m ***must exists*** by ***Tarski's Theorem*** since $\emptyset \subseteq X_{j+1} \subseteq X_j$

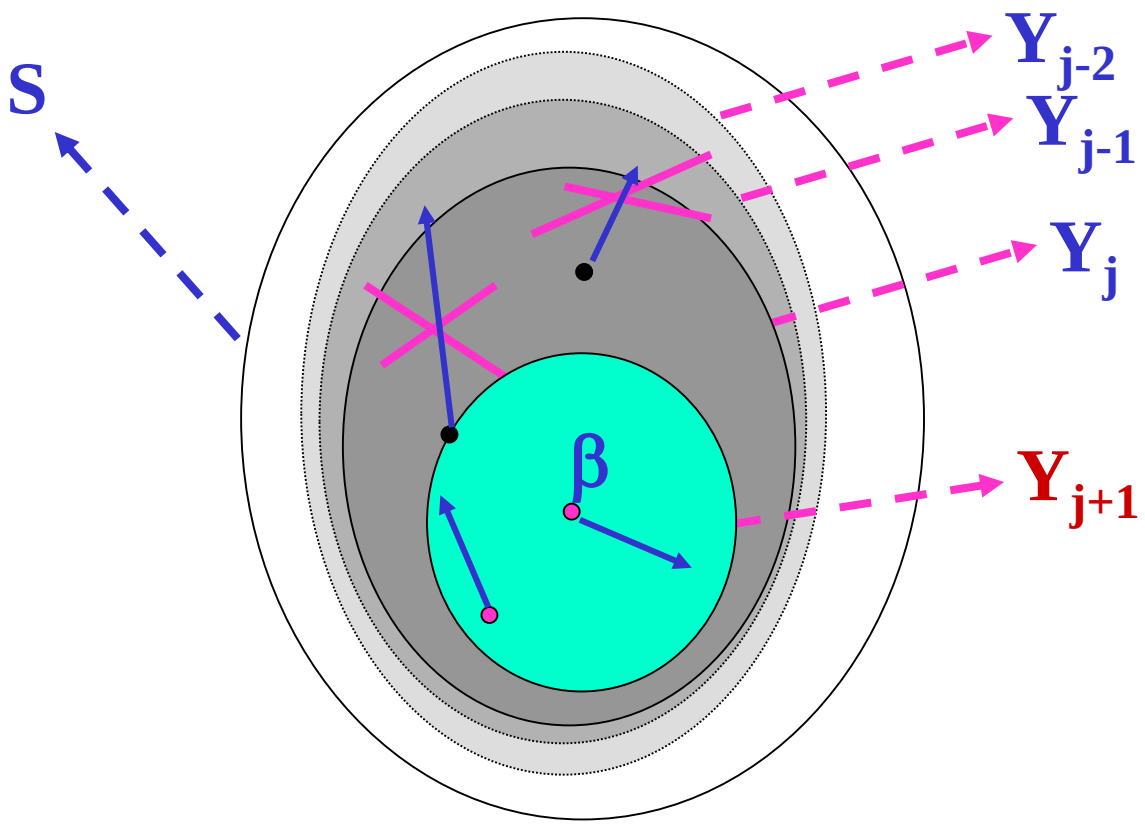
From Y_0 to Y_1

EGβ



From Y_j to Y_{j+1}

EGβ



Computing the fixed point for EG(β)

Algorithm Compute_EG(β)

$X_1 := |[\beta]|$;

$j=1$;

repeat

$j := j+1$; $T := X_j := X_{j-1}$;

while $T \neq \emptyset$ do

chose $s \in T$;

$T := T \setminus \{s\}$;

if for all $t \in R(s)$, $t \notin X_{j-1}$ then

$X_j := X_j - \{s\}$;

until $X_j = X_{j-1}$;