Chapter 30

DEMAND ANALYSIS

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Contents

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0. Introduction

The empirical analysis of consumer behavior has always held a central position in econometrics and many of what are now standard techniques were developed in response to practical problems in interpreting demand data. An equally central position in economic analysis is held by the theory of consumer behavior which has provided a structure and language for model formulation and data analysis. Demand analysis is thus in the rare position in econometrics of possessing long interrelated pedigrees on both theoretical and empirical sides. And although the construction of models which are both theoretically and empirically satisfactory is never straightforward, no one who reads the modern literature on labor supply, on discrete choice, on asset demands, on transport, on housing, on the consumption function, on taxation or on social choice, can doubt the current vigor and power of utility analysis as a tool of applied economic reasoning. There have been enormous advances towards integration since the days when utility theory was taught as a central element in microeconomic courses but then left unused by applied economists and econometricians.

Narrowly defined, demand analysis is a small subset of the areas listed above, referring largely to the study of commodity demands by consumers, most usually based on aggregate data but occasionally, and more so recently, on cross-sections or even panels of households. In this chapter, I shall attempt to take a somewhat broader view and discuss, if only briefly, the links between conventional demand analysis and such topics as labor supply, the consumption function, rationing, index numbers, equivalence scales and consumer surplus. Some of the most impressive recent econometric applications of utility theory are in the areas of labor supply and discrete choice, and these are covered in other chapters. Even so, a very considerable menu is left for the current meal. Inevitably, the choice of material is my own, is partial (in both senses), and does not pretend to be a complete survey of recent developments. Nor have I attempted to separate the economic from the statistical aspects of the subject. The strength of consumer demand analysis has been its close articulation of theory and evidence and the theoretical advances which have been important (particularly those concerned with duality) have been so precisely because they have permitted a more intimate contact between the theory and the interpretation of the evidence. It is not possible to study applied demand analysis without keeping statistics and economic theory simultaneously in view.

The layout of the chapter is as follows. Section 1 is concerned with utility and the specification of demand functions and attempts to review the theory from the point of view of applied econometrics. Duality aspects are particularly emphasized. Section 2 covers what I shall call 'naive' demand analysis, the estimation and testing, largely on aggregate time series data, of 'complete' systems of demand equations linking quantities demanded to total expenditure and prices. The label "naive" implies simplicity neither in theory nor in econometric technique. Instead, the adjective refers to the belief that, by itself, the simple, static, neoclassical model of the individual consumer could (or should) yield an adequate description of aggregate time-series data. Section 3 is concerned with microeconomic or cross-section analysis including the estimation of Engel curves, the treatment of demographic variables, and the particular econometric problems which arise in such contexts. There is also a brief discussion of the econometric issues that arise when consumers face non-linear budget constraints. Sections 4 and 5 discuss two theoretical topics of considerable empirical importance, separability and aggregation. The former provides the analysis underpinning econometric analysis of subsystems on the one hand and of aggregates, or supersystems, on the other. The latter provides what justification there is for grouping over different consumers. Econometric analysis of demand under conditions of rationing or quantity constraints is discussed in Section 6. Section 7 provides a brief overview of three important topics which, for reasons of space, cannot be covered in depth, namely, intertemporal demand analysis, including the analysis of the consumption function and of durable goods, the choice over qualities, and the links between demand analysis and welfare economics, particularly as concerns the measurement of consumer surplus, cost-of-living index numbers and the costs of children. Many other topics are inevitably omitted or dealt with less fully than is desirable; some of these are covered in earlier surveys by Goldberger (1967), Brown and Deaton (1972) and Barten (1977).

1. Utility and the specification of demand

1.1. Assumptions for empirical analysis

As is conventional, I begin with the specification of preferences. The relationship "is at least as good as", written $\succsim$, is assumed to be reflexive, complete, transitive and continuous. If so, it may be represented by a utility function, $v(q)$ say, defined over commodity vector q with the property that the statement $q^A \succsim q^B$ for vectors q^A and q^B is equivalent to the statement $v(q^A) \geq v(q^B)$. Clearly, for most purposes, it is more convenient to work with a utility function than with a preference ordering. There seem few prior empirical grounds for objecting to reflexivity, completeness, transitivity or continuity, nor indeed to the assumption that $v(q)$ is monotone increasing in q. Again, for empirical work, there is little objection to the assumption that preferences are *convex*, i.e. that for $q^A \gtrsim q^B$, and for $0 \leq \lambda \leq 1$, $\lambda q^A + (1-\lambda) q^B \gtrsim q^B$. This translates immediately into quasi-concavity of the utility function $v(q)$, i.e. for q^A, q^B, $0 \leq \lambda \leq 1$,

$$v(q^A) \geq v(q^B) \quad \text{implies} \quad v(\lambda q^A + (1-\lambda) q^B) \geq v(q^B). \tag{1}$$

Henceforth, I shall assume that the consumer acts so as to maximise the monotone, continuous and quasi-concave utility function $v(q)$.

It is common, in preparation for empirical work, to assume, in addition to the above properties, that the utility function is *strictly* quasi-concave (so that for $0 < \lambda < 1$ the second inequality in (1) is strict), *differentiable*, and that all goods are *essential*, i.e. that in all circumstances all goods are bought. All these assumptions are convenient in particular situations. But they are all restrictive and all rule out phenomena that are likely to be important in some empirical situations. Figure 1 illustrates in two dimensions. All of the illustrated indifference curves are associated with quasi-concave utility functions, but only A is either differentiable or strictly quasi-concave. The flat segments on B and C would be ruled out by strict quasi-concavity; hence, strictness ensures single-val-

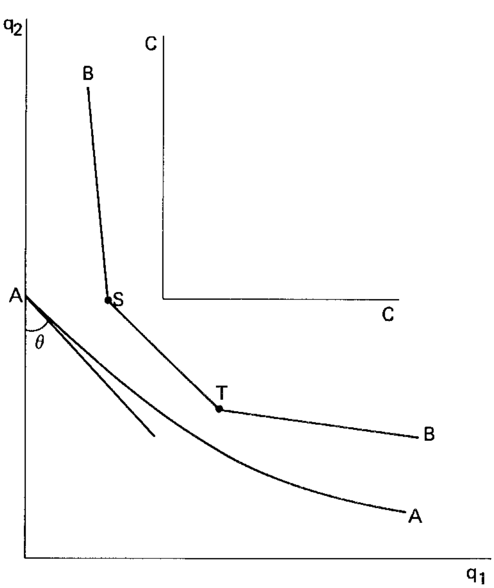

Figure 1 Indifference curves illustrating quasi-concavity, differentiability and essential goods.

ued demand functions. Empirically, flats are important because they represent perfect substitutes; for example, between S and T on B, the precise combination of q_1 and q_2 makes no difference and this situation is likely to be relevant, say, for two varieties of the same good. Non-differentiabilities occur at the kink points on the curves B and C. With a linear budget constraint, kinks imply that for relative prices within a certain range, two or more goods are bought in fixed proportions. Once again, this may be practically important and fixed relationships between complementary goods are often a convenient and sensible modelling strategy. The n-dimensional analogue of the utility function corresponding to C is the fixed coefficient or Leontief utility function

$$v(q) = \min\{\alpha_1 q_1, \alpha_2 q_2, \ldots, \alpha_n q_n\}. \tag{2}$$

For positive parameters $\alpha_1, \ldots, \alpha_n$. Finally curve A illustrates the situation where q_2 is essential but q_1 is not. As q_2 tends to zero, its marginal value relative to that of q_1 tends to infinity along any given indifference curve. Many commonly used utility functions impose this condition which implies that q_2 is always purchased in positive amounts. But for many goods, the behavior with respect to q_1 is a better guide; if $p_1 > p_2\theta$, the consumer on indifference curve A buys none of q_1. Data on individual households always show that, even for quite broad commodity groups, many households do not buy all goods. It is therefore necessary to have models that can deal with this fact.

1.2. *Lagrangians and matrix methods*

If $v(q)$ is strictly quasi-concave and differentiable, the maximization of utility subject to the budget constraint can be handled by Lagrangian techniques. Writing the constraint $p \cdot q = x$ for price vector p and *total expenditure* x, the first-order conditions are

$$\frac{\partial v(q)}{\partial q_i} = \lambda p_i, \tag{3}$$

which, under the given assumptions, solve for the demand functions

$$q_i = g_i(x, p). \tag{4}$$

For example, the *linear expenditure system* has utility function

$$u = \Pi(q_i - \gamma_i)^{\beta_i}, \tag{5}$$

for parameters γ and β, the first-order conditions of which are readily solved to give the demand functions

$$p_i q_i = p_i \gamma_i + \beta_i (x - p \cdot \gamma). \tag{6}$$

In practice, the first-order conditions are rarely analytically soluble even for quite simple formulations (e.g. Houthakker's (1960) "direct addilog" $u = \sum \alpha_i q_i^{\beta_i}$), nor is it at all straightforward to pass back from given demand functions to a closed form expression for the utility function underlying them, should it indeed exist.

The generic properties of demands are frequently derived from (3) by total differentiation and matrix inversion to express $\mathrm{d}q$ as a function of $\mathrm{d}x$ and $\mathrm{d}p$, the so-called "fundamental matrix equation" of consumer demand analysis, see Barten (1966) originally and its frequent later exposition by Theil, e.g. (1975b, pp. 14ff), also Phlips (1974, 1983, p. 47), Brown and Deaton (1972, pp. 1160–2). However, such an analysis requires that $v(q)$ be twice-differentiable, and it is usually assumed in addition that utility has been monotonically transformed so that the Hessian is non-singular and negative definite. Neither of these last assumptions follows in any natural way from reasonable axioms; note in particular that is is *not* always possible to transform a quasi-concave function by means of a monotone increasing function into a concave one, see Kannai (1977), Afriat (1980). Hence, the methodology of working through first-order conditions involves an expansive and complex web of restrictive and unnatural assumptions, many of which preclude consideration of phenomena requiring analysis. Even in the hands of experts, e.g. the survey by Barten and Bohm (1980), the analytical apparatus becomes very complex. At the same time, the difficulty of solving the conditions in general prevents a close connection between preferences and demand, between the a priori and the empirical.

1.3. Duality, cost functions and demands

There are many different ways of representing preferences and great convenience can be obtained by picking that which is most appropriate for the problem at hand. For the purposes of generating empirically useable models in which quantities are a function of prices and total expenditure, dual representations are typically most convenient. In this context, duality refers to a switch of variables, from quantities to prices, and to the respecification of preferences in terms of the latter. Define the *cost function*, sometimes *expenditure function*, by

$$c(u, p) = \left\{ \min_q p \cdot q;\ v(q) \geq u \right\}. \tag{7}$$

If x is the total budget to be allocated, then x will be the cheapest way of reaching whatever u can be reached at p and x, so that

$$c(u, p) = x. \tag{8}$$

The function $c(u, p)$ can be shown to be continuous in both its arguments, monotone increasing in u and monotone non-decreasing in p. It is linearly homogeneous and *concave* in prices, and first and second differentiable almost everywhere. It is *strictly* quasi-concave if $v(q)$ is *differentiable* and *everywhere* differentiable if $v(q)$ is *strictly* quasi-concave. For proofs and further discussions see McFadden (1978), Diewert (1974a), (1980b) or, less rigorously, Deaton and Muellbauer (1980a, Chapter 2).

The empirical importance of the cost function lies in two features. The first is the 'derivative property', often known as Shephard's Lemma, Shephard (1953). By this, whenever the derivative exists

$$\frac{\partial c(u, p)}{\partial p_i} \equiv h_i(u, p) = q_i. \tag{9}$$

The functions $h_i(u, p)$ are known as *Hicksian* demands, in contrast to the *Marshallian* demands $g_i(x, p)$. The second feature is the Shephard–Uzawa duality theorem [again see McFadden (1978) or Diewert (1974a), (1980b)] which given convex preferences, allows a constructive recovery of the utility function from the cost function. Hence, all the information in $v(q)$ which is relevent to behavior and empirical analysis is encoded in the function $c(u, p)$. Or put another way, any function $c(u, p)$ with the correct properties can serve as an alternative to $v(q)$ as a basis for empirical analysis. The direct utility function need never be explicitly evaluated or derived; if the cost function is correctly specified, corresponding preferences always exist. The following procedure is thus suggested in empirical work. Starting from some linearly homogeneous concave cost function $c(u, p)$, derive the Hicksian demand functions $h_i(u, p)$ by differentiation. These can be converted into Marshallian demands by substituting for u from the inverted form of (8); this is written

$$u = \psi(x, p), \tag{10}$$

and is known as the *indirect* utility function. (The original function $v(q)$ is the *direct* utility function and the two are linked by the identity $\psi(x, p) \equiv v\{g(x, p)\}$ for utility maximizing demands $g(x, p)$). Substituting (10) into (9) yields

$$q_i = h_i(u, p) = h_i\{\psi(x, p), p\} = g_i(x, p), \tag{11}$$

which can then be estimated. Of course, the demands corresponding to the original cost function may not fit the data or may have other undesirable properties for the purpose at hand. To build this back into preferences, we must be able to go from $g_i(x, p)$ back to $c(u, p)$. But, from Shephard's Lemma, $q_i = g_i(x, p)$ may be rewritten as

$$\frac{\partial c(u, p)}{\partial p_i} = g_i\{c(u, p), p\}, \tag{12}$$

which may be solved for $c(u, p)$ provided the mathematical integrability conditions are satisfied. These turn out to be equivalent to Slutsky symmetry, so that demand functions displaying symmetry always imply some cost function, see, for example, Hurwicz and Uzawa (1971) for further details. If the Slutsky matrix is also negative semi-definite (together with symmetry, the 'economic' integrability condition), the cost function will be appropriately concave which it must be to represent preferences. This possibility, of moving relatively easily between preferences and demands, is of vital importance if empirical knowledge is to be linked to economic theory.

An alternative and almost equally straightforward procedure is to start from the indirect utility function $\psi(x, p)$. This must be zero degree homogeneous in x and p and quasi-convex in p and Shephard's Lemma takes the form

$$q_i = g_i(x, p) = \frac{-\partial\psi(x, p)/\partial p_i}{\partial\psi(x, p)/\partial x}, \tag{13}$$

a formula known as *Roy's identity*, Roy (1942). This is sometimes done in "normalized" form. Clearly, $\psi(x, p) = \psi(1, p/x) = \psi^*(r)$ where $r = p/x$ is the vector of normalized prices. Hence, using ψ^* instead of ψ, Roy's identity can be written in the convenient form

$$w_i = \frac{p_i q_i}{x} = \frac{\partial\psi^*/\partial \log r_i}{\sum_k \partial\psi^*/\partial \log r_k} = \frac{\partial \log c(u, p)}{\partial \log p_i}, \tag{14}$$

where the last equality follows from rewriting (9).

One of the earliest and best practical examples of the use of these techniques is Samuelson's (1947–8) derivation of the utility function (5) from the specification of the linear expenditure system suggested earlier by Klein and Rubin (1947–8). A more recent example is provided by the following. In 1943, Holbrook Working suggested that a useful form of Engel curve was given by expressing the budget share of good i, w_i, as a linear function of the logarithm of total expenditure.

Hence,

$$w_i = \alpha_i + \beta_i \ln x, \tag{15}$$

for parameters α and β, generally functions of prices, and this form was supported in later comparative tests by Leser (1963). From (14), the budget shares are the logarithmic derivatives of the cost function, so that (15) corresponds to differential equations of the form

$$\frac{\partial \ln c(u, p)}{\partial \ln p_i} = \alpha_i(p) + \beta_i(p) \ln c(u, p), \tag{16}$$

which give a solution of the general form

$$\ln c(u, p) = u \ln b(p) + (1-u) \ln a(p), \tag{17}$$

where $\alpha_i(p) = (a_i \ln b - b_i \ln a)/(\ln b - \ln a)$ and $\beta_i(p) = b_i/(\ln b - \ln a)$ for $a_i = \partial \ln a/\partial \ln p_i$ and $b_i = \partial \ln b/\partial \ln p_i$. The form (17) gives the cost function as a utility-weighted geometric mean of the linear homogeneous functions $a(p)$ and $b(p)$ representing the cost functions of the very poor $(u=0)$ and the very rich $(u=1)$ respectively. Such preferences have been called the PIGLOG class by Muellbauer (1975b), (1976a), (1976b). A full system of demand equations within the Working–Leser class can be generated by suitable choice of the functions $b(p)$ and $a(p)$. For example, if

$$\begin{aligned} \ln a(p) &= a_o + \sum \alpha_k \ln p_k + \tfrac{1}{2} \sum_k \sum_m \gamma^*_{km} \ln p_k \ln p_m, \\ \ln b(p) &= \ln a(p) + \beta_0 \Pi p_k^{\beta_k}, \end{aligned} \tag{18}$$

we reach the "almost ideal demand system" (AIDS) of Deaton and Muellbauer (1980b) viz

$$\begin{aligned} w_i &= \alpha_i + \beta_i \ln(x/P) + \sum_j \gamma_{ij} \ln p_j, \\ \ln P &= \alpha_0 + \sum \alpha_k \ln p_k + \tfrac{1}{2} \sum_k \sum_m \gamma_{km} \ln p_k \ln p_m, \end{aligned} \tag{19}$$

and $\gamma_{ij} = \frac{1}{2}(\gamma^*_{ij} + \gamma^*_{ji})$. A variation on the same theme is to replace the geometric mean (17) by a mean of order ε

$$c(u, p) = \left\{ u b(p)^{\varepsilon} + (1-u) a(p)^{\varepsilon} \right\}^{1/\varepsilon}, \tag{20}$$

with Engel curves

$$w_i = \alpha_i + \beta_i x^{-\varepsilon}. \tag{21}$$

This is Muellbauer's PIGL class; equation (21), in an equivalent Box–Cox form, has recently appeared in the literature as the "generalized Working model", see Tran van Hoa, Ironmonger, and Manning (1983) and Tran van Hoa (1983).

I shall return to these and similar models below, but for the moment note how the construction of these models allows empirical knowledge of demands to be built into the specification of preferences. This works at a less formal level too. For example, prior information may relate to the shape of indifference curves, say that two goods are poor substitutes or very good substitutes as the case may be. This translates directly into curvature properties of the cost function; 'kinks' in quantity space turn into 'flats' in price space and vice versa so that the specification can be set accordingly. For further details, see the elegant diagrams in McFadden (1978).

The duality approach also provides a simple demonstration of the generic properties of demand functions which have played such a large part in the testing of consumer rationality, see Section 2 below. The budget constraint implies immediately that the demand functions *add-up* (trivially) and that they are zero-degree *homogeneous* in prices and total expenditure together (since the budget constraint is unaffected by proportional changes in p and x). Shephard's Lemma (9) together with the mild regularity conditions required for Young's Theorem implies that

$$\frac{\partial h_i}{\partial p_j} = \frac{\partial^2 c}{\partial p_j \partial p_i} = \frac{\partial^2 c}{\partial p_i \partial p_j} = \frac{\partial h_j}{\partial p_i}, \tag{22}$$

so that, if s_{ij}, the Slutsky substitution term is $\partial h_i / \partial p_j$, the matrix of such terms, S, is *symmetric*. Furthermore, since $c(u, p)$ is a concave function of p, S must be *negative semi-definite*. (Note that the homogeneity of $c(u, p)$ implies that p lies in the nullspace of S). Of course, S is not directly observed, but it can be evaluated using (12); differentiating with respect to p_j gives the *Slutsky equation*.

$$s_{ij} = \frac{\partial g_i}{\partial p_j} + \frac{\partial g_i}{\partial x} q_j. \tag{23}$$

Hence to the extent that $\partial g_i / \partial p_j$ and $\partial g_i / \partial x$ can be estimated econometrically, symmetry and negative semi-definiteness can be checked. I shall come to practical attempts to do so in the next section.

1.4. Inverse demand functions

In practical applications, it is occasionally necessary to estimate prices as a function of quantities rather than the other way round. An approach to specification exists for this case which is precisely analogous to that suggested above. From the direct utility function and the first-order conditions (10), apply the budget constraint $p \cdot q = x$ to give

$$\frac{p_i q_i}{x} = \frac{\partial v / \partial \ln q_i}{\sum_k \partial v / \partial \ln q_k}, \tag{24}$$

which is the dual analogue of (14), though now determination goes from the quantities q to the normalized prices p/x. Alternatively, define the distance function $d(u, q)$, dual to the cost function, by

$$d(u,q) = \min_p \{\, p \cdot q;\ \psi(1,p) \le u \}. \tag{25}$$

The distance function has properties analogous to the cost function and, in particular,

$$p_i / x = \partial d(u,q) / \partial q_i = a_i(u,q), \tag{26}$$

are the inverse compensated demand functions relating an indifference curve u and a quantity ray q to the price to income ratios at the intersection of q and u. See McFadden (1978), Deaton (1979) or Deaton and Muellbauer (1980a, Chapter 2.7) for fuller discussions.

Compensated and uncompensated inverse demand functions can be used in exactly the same way as direct demand functions and are appropriate for the analysis of situations when quantities are predetermined and prices adjust to clear the market. Hybrid situations can also be analysed with some prices fixed and some quantities fixed; again see McFadden (1978) for discussion of "restricted" preference representation functions. Note one final point, however. The Hessian matrix of the distance function $d(u,q)$ is the *Antonelli* matrix A with elements

$$a_{ij} = \frac{\partial^2 d}{\partial q_i \partial q_j} = a_{ji} = \frac{\partial a_i(u,q)}{\partial q_j}, \tag{27}$$

which can be used to define q-substitutes and q-complements just as the Slutsky matrix defines p-substitutes and p-complements, see Hicks (1956) for the original discussion and derivations. Unsurprisingly the Antonelli and Slutsky matrices are intimately related and given the close parallel been duality and matrix inversion, it is appropriate that they should be generalised inverses of one another. For example, using ∇ to denote the vector of price or quantity partial derivatives, (9) and (26) combine to yield

$$q = \nabla c\{u, \nabla d\{u, \nabla c(u, p)\}\}. \tag{28}$$

Hence, differentiating with respect to p/x and repeatedly applying the chain rule, we obtain at once

$$S^* = S^* A S^*. \tag{29}$$

Similarly,

$$A = A S^* A, \tag{30}$$

where $S^* = xS$. Note that the homogeneity restrictions imply $Aq = S^*p = 0$ which together with (29) and (30) complete the characterization as generalized inverses. These relationships also allow passage from one type of demand function to another so that the Slutsky matrix can be calculated from estimates of indirect demand functions while the Antonelli matrix may be calculated from the usual demands. The explicit formula for the latter is easily shown to be

$$A = (xS + qq')^{-1} - x^{-2}pp', \tag{31}$$

with primes denoting transposition, see Deaton (1981a). The Antonelli matrix has important applications in measuring quantity index numbers, see, e.g. Diewert (1981, 1983) and in optimal tax theory, see Deaton (1981a). Formula (31) allows its calculation from an estimate of the Slutsky matrix.

This brief review of the theory is sufficient to permit discussion of a good deal of the empirical work in the literature. Logically, questions of aggregation and separability ought to be treated first, but since they are not required for an understanding of what follows, I shall postpone their discussion to Section 4.

2. Naive demand analysis

Following Stone's first empirical application of the linear expenditure system in 1954, a good deal of attention was given in the subsequent literature to the problems involved in estimating complete, and generally nonlinear, systems of demand equations. Although the issues are now reasonably well understood, they deserve brief review. I shall use the linear expenditure system as representative of the class

$$p_{it}q_{it} = f_i(p_t, x_t;\ b) + u_{it}, \tag{32}$$

for commodity i on observation t, parameter vector b, and error u_{it}. For the linear expenditure system the function takes the form

$$f_i(p_t, x_t;\ b) = \gamma_i p_{it} + \beta_i(x_t - p_t \cdot \gamma). \tag{33}$$

2.1. *Simultaneity*

The first problem of application is to give a sensible interpretation to the quantity x_t. In loose discussion of the theory x_t is taken as "income" and is assumed to be imposed on the consumer from outside. But, if q_t is the vector of commodity purchases in period t, then (a) only exceptionally is any real consumer given a predetermined and inflexible limit for total commodity expenditure and (b) the only thing which expenditures add up to is total expenditure *defined* as the sum of expenditures. Clearly then, x_t is in general jointly endogenous with the expenditures and ought to be treated as such, a point argued, for example, by Summers (1959), Cramer (1969) and more recently by Lluch (1973), Lluch and Williams (1974). The most straightforward solution is to instrument x_t and there are no shortages of theories of the consumption function to suggest exogenous variables. However, in the spirit of demand analysis this can be formalized rather neatly using any intertemporally separable utility function. For example, loosely following Lluch, an intertemporal or extended linear expenditure system can be proposed of the form

$$p_{it}q_{it} = p_{it}\gamma_{it} + \beta_{it}\left\{W - \sum_{\tau=t}^{L}\sum_{k} p^*_{\tau k}\gamma_{\tau k}\right\} + v_{it}, \tag{34}$$

where the γ_{it} and β_{it} parameters are now specific to periods (needs vary over the life-cycle), W is the current present discounted value of present and future income and current financial assets, and $p^*_{\tau k}$ is the current discounted price of good k in future period $\tau (p^*_{tk} = p_{tk}$ since t is the present). As with any such system based on intertemporally separable preferences, see Section 4 below, (34) can be solved for x_t by summing the left-hand side over i and the result, i.e. the consumption function, used to substitute for W. Hence (34) *implies* the familiar static linear expenditure system, i.e.

$$p_{it}q_{it} = p_{it}\gamma_{it} + \frac{\beta_{it}}{\beta_t}\left\{x_t - \sum_k p_{kt}\gamma_{kt}\right\} + \left\{v_{it} - \frac{\beta_{it}}{\beta_t}v_t\right\}, \tag{35}$$

where $v_t = \sum v_{it}$, $\beta_t = \sum \beta_{it}$ and it is assumed, as is reasonable, that $\beta_t \neq 0$. This not only relates the parameters in the static version (33) to their intertemporal counterparts, but it also gives valuable information about the structure of the error term in (32). Given this, the bias introduced by ignoring the simultaneity between x_t and $p_{it}q_{it}$ can be studied. For the usual reasons, it will be small if the equations fit well, as Prais (1959) argued in his reply to Summers (1959). But there is a rather more interesting possibility. It is easily shown, on the basis of (35), that

$$\mathrm{Cov}(x_t, u_{it}) = \sum_k \sigma_{ik} - \frac{\beta_{it}}{\beta_t}\sum_k \sum_m \sigma_{km}, \tag{36}$$

where σ_{ij} is the (assumed constant) covariance between v_{it} and v_{jt}, i.e.

$$\mathrm{Cov}(v_{it}, v_{js}) = \delta_{ts}\sigma_{ij}. \tag{37}$$

where σ_{ts} is the Kronecker delta. Clearly, the covariance in (36) is zero if $\sum_k \sigma_{ik}/\sum \sigma_{km} = \beta_{it}/\beta_t$. One specialized theory which produces exactly this relationship is Theil's (1971b, 1974, 1975a, 1975b, pp. 56–90, 1979) "rational random behaviour" under which the variance, covariance matrix of the errors v_{it} is rendered proportional to the Slutsky matrix by consumers' trading-off the costs of exact maximization against the utility losses of not doing so. If this model is correct, there is no simultaneity bias, see Deaton (1975a, pp. 161–8) and Theil (1976, pp. 4–6, 80–82) for applications. However, most econometricians would tend to view the error terms as reflecting, at least in part, those elements *not* allowed for by the theory, i.e. misspecifications, omitted variables and the like. Even so, it is not implausible that (36) should be close to zero since the requirement is that error covariances between each category and total expenditure should be proportional to the marginal propensity to spend for that good. This is a type of "error separability" whereby omitted variables influence demands in much the same way as does total outlay.

In general, simultaneity will exist and the issue deserves to be taken seriously; it is likely to be particularly important in cross-section work, where occasional large purchases affect both sides of the Engel curve. Ignoring it may also bias the other tests discussed below, see Altfield (1985).

2.2. Singularity of the variance–covariance matrix

The second problem arises from the fact that with x_t *defined* as the sum of expenditures, expenditures automatically add-up to total expenditure identically, i.e. without error. Hence, provided f_i in (32) is properly chosen, we must have

$$\sum_i p_{it}q_{it} = x_t; \qquad \sum f_i(p_t, x_t; b) = x_t; \qquad \sum_i u_{it} = 0. \tag{38}$$

Writing Ω as the $n \times n$ contemporaneous variance–covariance matrix of the u_{it}'s with typical element ω_{ij}, i.e.

$$E(u_{it}, u_{js}) = \delta_{ts}\omega_{ij}, \tag{39}$$

then the last part of (38) clearly implies

$$\sum_j \omega_{ij} = \sum_i \omega_{ij} = 0, \tag{40}$$

so that the variance–covariance matrix is singular. If (32) is stacked in the usual way as an nT observation regression, its covariance matrix is $\Omega \otimes I$ which cannot have rank higher than $(n-1)T$. Hence, the usual generalized least squares estimator or its non-linear analogue is not defined since it would require the non-existent inverse $\Omega^{-1} \otimes I$.

This non-existence is, however, a superficial problem. For a set of equations such as (32) satisfying (38), one equation is essentially redundant and all of its parameters can be inferred from knowledge of those in the other equations. Hence, attempting to estimate all the parameters in all equations is equivalent to including some parameters more than once and leads to exactly the same problems as would arise if, for example, some independent variables were included more than once on the right hand side of an ordinary single-variable regression. The solution is obviously to drop one of the equations and estimate the resulting $(n-1)$ equations by GLS, Zellner's (1962) seemingly unrelated regressions estimator (SURE), or similar technique. Papers by McGuire, Farley, Lucas and Winston (1968) and by Powell (1969) show that the estimates are invariant to the particular equation which is selected for omission. Barten (1969) also considered the maximum-likelihood estimation of such systems when the errors follow the multivariate normal assumption. If Ω_n is the variance–covariance matrix of the system (32) excluding the nth equation, a sample of T observations has a log-likelihood conditional on normality of

$$\ln L = -\frac{T}{2}(n-1)\ln 2\pi - \frac{T}{2}\ln\det\Omega_n - \frac{1}{2}\sum_{t=1}^{T} u'_{(n)t}\Omega_n^{-1}u_{(n)t}, \tag{41}$$

where $u_{(n)}$ is the $(n-1)$-vector of u_{it} excluding element n. Barten defines a new non-singular matrix V by

$$V = \Omega + \kappa ii, \tag{42}$$

where i is the normalized vector of units, i.e. $i_i = 1/n$, and $0 < \kappa < \infty$. Then (41) may be shown to be equal to

$$\ln L = \frac{T}{2}\{\ln \kappa + \ln n - (n-1)\ln 2\pi - \ln \det V\} - \frac{1}{2}\sum_{t=1}^{T} u_t' V^{-1} u_t. \tag{43}$$

This formulation establishes that the likelihood is independent of the equation deleted (and incidentally of κ since (41) does not depend on it) and also returns the original symmetry to the problem. However, in practice, the technique of dropping one equation is usually to be preferred since it reduces the dimension of the parameter vector to be estimated which tends to make computation easier.

Note two further issues associated with singularity. First, if the system to be estimated is a "subsystem" of commodities that does not exhaust the budget, the variance covariance matrix of the residuals need not, and usually will not be singular. In consequence, SURE or FIML (see below) can be carried out directly on the subsystem. However, it is still necessary to assume a non-diagonal variance–covariance matrix; overall singularity precludes *all* goods from having orthogonal errors and there is usually no good reason to implicitly confine all the off-diagonal covariances to the omitted goods. Second, there are additional complications if the residuals are assumed to be serially correlated. For example, in (32), it might be tempting to write

$$u_{it} = \rho_i u_{it-1} + \varepsilon_{it}, \tag{44}$$

for serially uncorrelated errors ε_{it}. If R is the diagonal matrix of ρ_i's, (44) implies that

$$\Omega = R\Omega R + \Sigma, \tag{45}$$

where Σ is the contemporaneous variance–covariance matrix of the ε's. Since $\Omega i = \Sigma i = 0$, we must have $\Omega\rho = 0$, which, since i spans the null space of Ω, implies that $\rho \propto i$, i.e. that all the ρ_i's are the same, a result first established by Berndt and Savin (1975). Note that this does *not* mean that (44) with $\rho_i = \rho$ for all i is a sensible specification for autocorrelation in singular systems. It would seem better to allow for autocorrelation at an earlier stage in the modeling, for example by letting v_{it} be autocorrelated in (34) and following through the consequences for the compound errors in (35). In general, this will imply vector autoregressive structures, as, for example, in Guilkey and Schmidt (1973) and Anderson and Blundell (1982). But provided autocorrelation is handled in a way that respects the singularity (as it should be), so that the omitted equation is not implicitly treated differently from the others, then it will always be correct to estimate by dropping one equation since all the relevant information is contained in the other $(n-1)$.

2.3. *Estimation*

For estimation purposes, rewrite (32) in the form

$$y_{ti} = f_{ti}(\beta) + u_{ti}, \tag{46}$$

with $t = 1,\dots,T$ indexing observations and $i = 1,\dots,(n-1)$ indexing goods. I shall discuss only the case where u_{ti} are independently and identically distributed as multivariate normal with zero mean and nonsingular covariance matrix Ω. [For other specifications, see, e.g. Woodland (1979)]. Since Ω is not indexed on t, homoskedasticity is being assumed; this is always more likely to hold if the y_{ti}'s are the budget *shares* of the goods, not quantities or expenditures. Using budget shares as dependent variables also ensures that the R^2 statistics mean something. Predicting better than $w_{it} = \alpha_i$ is an achievement (albeit a mild one), while with quantities or expenditures, R^2 tend to be extremely high no matter how poor the model.

Given the variance–covariance matrix Ω, typical element ω_{ij}, the MLE's of β, $\tilde{\beta}$ say, satisfy the first-order conditions, for all i,

$$\sum_t \sum_l \sum_k \frac{\partial f_{tk}}{\partial \beta_i} \omega^{kl} \{ y_{tl} - f_{tl}(\tilde{\beta}) \} = 0, \tag{47}$$

where ω^{kl} is the (k,l)th element of Ω^{-1}. These equations also define the linear or non-linear GLS estimator. Since Ω is usually unknown, it can be replaced by its maximum likelihood estimator,

$$\tilde{\omega}_{ij} = \frac{1}{T} \sum_t \{ y_{ti} - f_{ti}(\tilde{\beta}) \} \{ y_{tj} - f_{tj}(\tilde{\beta}) \}. \tag{48}$$

If $\tilde{\omega}_{ij}$ replaces ω_{ij} in (47) and (47) and (48) are solved simultaneously, $\tilde{\beta}$ and $\tilde{\Omega}$ are the full-information maximum likelihood estimators (FIML). Alternatively, some consistent estimator of β can be used in place of $\tilde{\beta}$ in (48) and the resulting $\tilde{\Omega}$ used in (47); the resulting estimates of β will be asymptotically equivalent to FIML. Zellner's (1962) seemingly unrelated regression technique falls in this class, see also Gallant (1975) and the survey by Srivastava and Dwivedi (1979) for variants. *Consistency* of estimation of $\tilde{\beta}$ in (47) is unaffected by the choice of Ω; the MLE's of β and Ω are asymptotically independent, as calculation of the information matrix will show. All this is standard enough, except possibly for computation, but the use of standard algorithms such as those of Marquardt (1963), scoring, Berndt, Hall, Hall and Hausman (1974), Newton–Raphson, Gauss–Newton all work well for these models, see Quandt (1984) in this Handbook for a survey. Note also Byron's (1982) technique for estimating very large symmetric systems.

Nevertheless, there are a number of problems, particularly concerned with the estimation of the covariance matrix Ω, and these may be severe enough to make the foregoing estimators undesirable, or even infeasible. Taking feasibility first, note that the estimated covariance matrix $\tilde{\Omega}$ given by (48) is the mean of T matrices each of rank 1 so that its rank cannot be greater than T. In consequence, systems for which $(n-1) > T$ cannot be estimated by FIML or SURE if the inverse of the estimated $\tilde{\Omega}$ is required. Even this underestimates the problem. In the linear case (e.g. the Rotterdam system considered below) the demand system becomes the classical multivariate regression model

$$Y = XB + U, \tag{49}$$

with Y a $(T \times (n-1))$ matrix, X a $(T \times K)$ matrix, $B(k \times (n-1))$ and $U(T \times (n-1))$. (The nth equation has been dropped). The estimated variance–covariance matrix from (48) is then

$$\tilde{\Omega} = \frac{1}{T} Y'\left(I - X(X'X)^{-1}X'\right)Y. \tag{50}$$

Now the idempotent matrix in backets has rank $(T-k)$ so that the inverse will not exist if $n-1 > T-k$. Since X is likely to contain at least $n+2$ variables (prices, the budget and a constant), an eight commodity system would require *at least* 19 observations. Non-linearities and cross-section restrictions can improve matters, but they need not. Consider the following problem, first pointed out to me by Teun Kloek. The AIDS system (19) illustrates most simply, though the problem is clearly a general one. Combine the two parts of (19) into a single set of equations,

$$\begin{aligned} w_{it} = {} & (\alpha_i - \beta_i\alpha_0) + \beta_i \ln x_t + \sum_j (\gamma_{ij} - \beta_i\alpha_j)\ln p_{jt} \\ & - \tfrac{1}{2}\beta_i \sum_k \sum_m \gamma_{km} \ln p_{kt} \ln p_{mt} + u_{it}. \end{aligned} \tag{51}$$

Not counting α_0, which is unidentified, the system (without restrictions) has a total of $(2+n)(n-1)$ parameters–$(n-1)$ α's and β's, and $n(n-1)$ γ's–or $(n+2)$ per equation as in the previous example. But now, each equation has $2+(n-1)n$ parameters since all γ's always appear. In consequence, if the constant, $\ln x$, $\ln p$, and the cross-terms are linearly independent in the sample, and if $T<2+(n-1)n$, it is possible to choose parameters such that the calculated residuals for any one (arbitrarily chosen) equation will be exactly zero for all sample points. For these parameters, one row and one column of the estimated $\tilde{\Omega}$ will also be zero, its determinant will be zero and the log likelihood (41) or (43) will be infinite. Hence full information MLE's do not exist. In such a case, at least 56 observations would be necessary to estimate an 8 commodity disaggregation. All these cases are variants of the familiar "undersized sample" problem in FIML estimation of simultaneous equation systems and they set upper limits to the amount of commodity disaggregation that can be countenanced on any given time-series data.

Given a singular variance–covariance matrix, for whatever reason, the log likelihood (41) which contains the term $-T/2\ \log\det\tilde{\Omega}$, will be infinitely large and FIML estimates do not exist. Nor, in general, can (47) be used to calculate GLS or SURE estimators if a singular estimate of Ω is employed. However, there are a number of important special cases in which (47) has solutions that can be evaluated even when Ω is singular (though it is less than clear what is the status of these estimators). For example, in the classical multivariate regression model (49), the solution to (47) is the OLS matrix estimator $\hat{B}=(X'X)^{-1}X'Y$ which does not involve Ω, see e.g. Goldberger (1964, pp. 207–12). Imposing identical *within equation* restrictions on (49), e.g. homogeneity, produces another (restricted) classical model with the same property. With *cross-equation* restrictions of the form $R\beta=r$, e.g. symmetry, for stacked β, $\tilde{\beta}$, the solution to (47) is

$$\tilde{\beta}=\hat{\beta}+\left\{\Omega\otimes(X'X)^{-1}\right\}R'\left[R\left\{\Omega\otimes(X'X)^{-1}\right\}R'\right]^{-1}(r-R\hat{\beta}),\tag{52}$$

which, though involving Ω, can still be calculated with Ω singular provided the matrix in square brackets is non-singular. I have not been able to find the general conditions on (47) that allow solutions of this form, nor is it clear that it is important to do so. General non-linear systems will not be estimable on undersized samples, and except in the cases given where closed-form solutions exist, attempts to solve (47) and (48) numerically will obviously fail.

The important issue, of course, is the small sample performance of estimators based on near-singular or singular estimates of Ω. In most time series applications with more than a very few commodities, $\tilde{\Omega}$ is likely to be a poor estimator of Ω and the introduction of very poor estimates of Ω into the procedure for parameter estimation is likely to give rise to extremely inefficient estimates of the latter. Paradoxically, the search for (asymptotic) efficiency is likely to lead, in this case, to much greater (small-sample) inefficiency than is actually obtainable. Indeed it may well be that estimation techniques which do not depend on estimating Ω will give better estimates in such situations. One possibility is the minimization of the *trace* of the matrix on the right-hand side of (48) rather than its *determinant* as required by FIML. This is equivalent to (non-linear) least squares applied to the sum of the residual sums of squares over each equation and can be shown to be ML if (the true) $\Omega = \sigma^2(I - ii')$ for some σ^2, see Deaton (1975a, p. 39). There is some general evidence that such methods can dominate SURE and FIML in small samples, see again Srivastava and Dwivedi (1979). Fiebig and Theil (1983) and Theil and Rosalsky (1984) have carried out Monte Carlo simulations of symmetry constrained linear systems, i.e. with estimators of the form (52). The system used has 8 commodities, 15 observations and 9 explanatory variables so that their estimate of $\tilde{\Omega}$ from (50) based on the unconstrained regressions is singular. Fiebig and Theil find that replacing Ω by $\tilde{\Omega}$ yielded "estimates with greatly reduced efficiency and standard errors which considerably underestimate the true variability of these estimates". A number of alternative specifications for were examined and Theil and Rosalsky found good performance in terms of MSE for Deaton's (1975a) specification $\Omega = \sigma^2(\hat{v} - vv')$ where v is the sample mean of the vector of budget shares and $\hat{v}$ is the diagonal matrix of v's. Their results also give useful information on procedures for evaluating standard errors. Define the matrix $A(\Sigma)$, element a_{ij} by

$$a_{ij}(\Sigma) = \sum_t \sum_l \sum_k \frac{\partial f_{tk}}{\partial \beta_i} \sigma^{kl} \frac{\partial f_t}{\partial \beta_j}, \tag{53}$$

where σ^{kl} is the (k, l)th element of Σ^{-1}, so that $\{A(\tilde{\Omega})\}^{-1}$ is the conventionally used (asymptotic) variance–covariance matrix of the FIML estimates $\tilde{\beta}$ from (47). Define also $B(\Sigma, \Omega)$ by

$$b_{ij}(\Sigma, \Omega) = \sum_t \sum_k \sum_l \sum_m \sum_n \frac{\partial f_{tk}}{\partial \beta_i} \sigma^{kl} \omega_{lm} \sigma^{mn} \frac{\partial f_{tn}}{\partial \beta_j}. \tag{54}$$

Hence, if β^* is estimated from (47) using some assumed variance–covariance matrix $\bar{\Omega}$ say (as in the experiments reported above), then the variance–covariance matrix V^* is given by

$$V^* = A(\bar{\Omega}) B(\bar{\Omega}, \Omega) A(\bar{\Omega}). \tag{55}$$

Fiebig and Theil's experiments suggest good performance if Ω in $B(\bar{\Omega}, \Omega)$ is replaced by $\tilde{\Omega}$ from (48).

2.4. Interpretation of results

It is perhaps not surprising that authors who finally surmounted the obstacles in the way of estimating systems of demand equations should have professed themselves satisfied with their hard won results. Mountaineers are not known for criticising the view from the summit. And certainly, models such as the linear expenditure system, or which embody comparably strong assumptions, yield very high R^2 statistics for expenditures or quantities with t-values that are usually closer to 10 than to unity. Although there are an almost infinite number of studies using the linear expenditure system from which to illustrate, almost certainly the most comprehensive is that by Lluch, Powell and Williams (1977) who fit the model (or a variant) to data from 17 developed and developing countries using an eightfold disaggregation of commodities. Of the 134 R^2 statistics reported (for 2 countries 2 of the groups were combined) 40 are greater than 0.99, 104 are greater than 0.95 and only 14 are below 0.90. (Table 3.9 p. 49). The parameter estimates nearly all "look sensible" and conform to theoretical restrictions, i.e. marginal propensities to consume are positive yielding, in the case of the linear expenditure system, a symmetric negative semi-definite Slutsky matrix. However, as is almost invariably the case with the linear expenditure system, the estimated residuals display substantial positive autocorrelation. Table 3.10 in Lluch, Powell and Williams displays Durbin–Watson statistics for all countries and commodities: of the 134 ratios, 60 are less than 1.0 and only 15 are greater than 2.0. Very similar results were found in my own, Deaton (1975a), application of the linear expenditure system to disaggregated expenditures in post-war Britain. Such results suggest that the explanatory power of the model reflects merely the common upward time trends in individual and total expenditures. The estimated β parameters in (33), the marginal propensities to consume, will nevertheless be sensible, since the model can hardly fail to reflect the way in which individual expenditures evolve relative to their sum over the sample as a whole. Obtaining sensible estimates of marginal propensities to spend on time-series data is not an onerous task. Nevertheless, the model singularly fails to account for variations around trend, the high R^2 statistics could be similarly obtained by replacing total expenditure by virtually any trending variable, and the t-values are likely to be grossly overestimated in the presence of the very severe autocorrelation, see, e.g. Malinvaud (1970, pp. 521–2) and Granger and Newbold (1974). In such circumstances, the model is almost certainly a very poor approximation to whatever process actually generated the data and should be abandoned in favor of more appropriate alternatives. It makes little sense to "treat" the autocorrelation by transforming the residuals by a Cochrane–Orcutt type technique, either based on (44) with a common parameter, or using a full vector autoregressive specification. [See Hendry (1980) for some of the consequences of trying to do so in similar situations.]

In spite of its clear misspecifications, there may nevertheless be cases where the linear expenditure system or a similar model may be the best that can be done. Because of its very few parameters, $(2n-1)$ for an n commodity system, it can be estimated in situations (such as the LDC's in Lluch, Powell and Williams book) where data are scarce and less parsimonious models cannot be used. In such situations, it will at the least give a theoretically consistent interpretation of the data, albeit one that is probably wrong. But in the absence of alternatives, this may be better than nothing. Even so, it is important that such applications be seen for what they are, i.e. untested theory with "sensible" parameters, and not as fully-tested data-consistent models.

2.5. *Flexible functional forms*

The immediately obvious problem with the linear expenditure system is that it has too few parameters to give it a reasonable chance of fitting the data. Referring back to (33) and dividing through by p_i, it can be seen that the γ_i parameters are essentially intercepts and that, apart from them, there is only one free parameter per equation. Essentially, the linear expenditure system does little more than fit bivariate regressions between individual expenditures and their total. Of course, the prices also enter the model but all own- and cross-price effects must also be allowed for within the two parameters per equation, one of which is an intercept. Clearly then, in interpreting the results from such a model, for example, total expenditure elasticities, own and cross-price elasticities, substitution matrices, and so on, there is no way to sort out which numbers are determined by measurement and which by assumption. Certainly, econometric analysis requires the application of prior reasoning and theorizing. But it is not helped if the separate influences of measurement and assumption cannot be practically distinguished.

Such difficulties can be avoided by the use of what are known as "flexible functional forms," Diewert (1971). The basic idea is that the choice of functional form should be such as to allow at least one free parameter for the measurement of each effect of interest. For example, the basic linear regression with intercept is a flexible functional form. Even if the true data generation process is not linear, the linear model without parameter restrictions can offer a first-order Taylor approximation around at least one point. For a system of $(n-1)$ independent demand functions, $(n-1)$ intercepts are required, $(n-1)$ parameters for the total expenditure effects and $n(n-1)$ for the effects of the n prices. Barnett (1983b) offers a useful discussion of how Diewert's definition relates to the standard mathematical notions of approximation.

Flexible functional form techniques can be applied either to demand functions or to preferences. For the former, take the differential of (9) around some convenient point, i.e.

$$dq_i = h_{i0} + h_{iu}\,du + \sum_j s_{ij}\,dp_j. \tag{56}$$

But from (10) and (14)

$$d\ln u = \left(d\ln x - \sum_k w_k d\ln p_k\right)\cdot(\partial \ln c/\partial \ln u)^{-1}, \tag{57}$$

so that writing $dq_i = q_i\,d\ln q_i$ and multiplying (56) by p_i/x, the approximation becomes

$$w_i\,d\ln q_i = a_i + b_i(d\ln x - w\cdot d\ln p) + \sum_j c_{ij}\,d\ln p_j, \tag{58}$$

where

$$\begin{aligned} a_i &= p_i h_{i0}/x \\ b_i &= \frac{u p_i h_{iu}}{x}\left(\frac{\partial \ln c}{\partial \ln u}\right)^{-1} = p_i\frac{\partial q_i}{\partial x} \\ c_{ij} &= p_i s_{ij} p_j/x. \end{aligned} \tag{59}$$

Eq. (58), with a_i, b_i and c_{ij} parametrized, is the *Rotterdam* system of Barten (1966), (1967), (1969) and Theil (1965), (1975b), (1976). It clearly offers a local first-order approximation to the underlying relationship between q, x and p.

There is, of course, no guarantee that a function $h_i(u, p)$ exists which has a_i, b_i and c_{ij} constant. Indeed, if it did, Young's theorem gives $h_{iuj} = h_{iju}$ which, from (59), is easily seen to hold only if $c_{ij} = -(\delta_{ij}b_i - b_i b_j)$. If imposed, this restriction would remove the system's ability to act as a flexible functional form. (In fact, the restriction implies unitary total expenditure and own-price elasticities). Contrary to assertions by Phlips (1974, 1983), Yoshihara (1969), Jorgenson and Lau (1976) and others, this only implies that it is not sensible to impose the restriction; it does not affect the usefulness of (58) for approximation and study of the true demands via the approximation, see also Barten (1977) and Barnett (1979b).

Flexible functional forms can also be constructed by approximating *preferences* rather than demands. By Shephard's Lemma, an order of approximation in prices (or quantities) – but *not* in utility – is lost by passing from preferences to demands, so that in order to guarantee a first-order linear approximation in the latter, *second-order* approximation must be guaranteed in preferences. Beyond that, one can freely choose to approximate the direct utility function, the indirect utility function, the cost-function or the distance function provided only that the appropriate quasi-concavity, quasi-convexity, concavity and homogeneity restrictions are observed. The best known of these approximations is the *translog*, Sargan (1971), Christensen, Jorgenson and Lau (1975) and many subsequent applications. See in particular Jorgenson, Lau and Stoker (1982) for a comprehensive treatment. The *indirect translog* gives a quadratic approximation to the indirect function $\psi^*(r)$ for normalized prices, and then uses (14) to derive the system of share equations. The forms are

$$\psi^*(r) = \alpha_0 + \sum \alpha_k \ln r_k + \tfrac{1}{2} \sum_k \sum_j \beta^*_{kj} \ln r_k \ln r_j \tag{60}$$

$$w_i = \frac{\alpha_i + \sum_j \beta_{ij} \ln r_j}{\sum \alpha_k + \sum_k \sum_j \beta_{kj} \ln r_j}, \tag{61}$$

where $\beta_{ij} = \frac{1}{2}(\beta^*_{ij} + \beta^*_{ji})$. In estimating (61), some normalization is required, e.g. that $\sum \alpha_k = 1$. The *direct translog* approximates the direct utility function as a quadratic in the vector q and it yields an equation of the same form as (61) with w_i on the left-hand side but with q_i replacing r_i on the right. Hence, while (61) views the budget share as being determined by quantity adjustment to exogenous price to outlay ratios, the direct translog views the share as adapting by prices adjusting to exogenous quantities. Each could be appropriate under its own assumptions, although presumably not on the same set of data. Yet another flexible functional form with close affinities to the translog is the second-order approximation to the cost function offered by the AIDS, eqs. (17), (18) and (19) above. Although the translog considerably predates the AIDS, the latter is a good deal simpler to estimate, at least if the price index $\ln P$ can be adequately approximated by some fixed pre-selected index.

The AIDS and translog models yield demand functions that are first-order flexible subject to the theory, i.e. they automatically possess symmetric substitution matrices, are homogeneous, and add up. However, trivial cases apart, the AIDS cost function will not be *globally* concave nor the translog indirect utility function globally convex, though they can be so over a restricted range of r (see below). The functional forms for both systems are such that, by relaxing certain restrictions, they can be made first-order flexible without theoretical restrictions, as is the Rotterdam system. For example, in the AIDS, eq. (19), the restrictions $\gamma_{ij} = \gamma_{ji}$ and $\sum_j \gamma_{ij} = 0$ can be relaxed while, in the indirect translog, eq. (61), $\beta_{ij} = \beta_{ji}$ can be relaxed and $\ln x$ included as a separate variable without necessarily assuming that its coefficient equals $-\sum \beta_{ij}$. Now, if the theory is correct, and the flexible functional form is an adequate representation of it over the data, the restrictions should be satisfied, or at least not significantly violated. Similarly, for the Rotterdam system, if the underlying theory is correct, it might be expected that its approximation by (58) would estimate derivatives conforming to the theoretical restrictions. From (59), homogeneity requires $\sum c_{ij} = 0$ and symmetry $c_{ij} = c_{ji}$. Negative semi-definiteness of the Slutsky matrix can also be imposed (globally for the Rotterdam model and at a point for the other models) following the work of Lau (1978) and Barten and Geyskens (1975).

The AIDS, translog, and Rotterdam models far from exhaust the possibilities and many other flexible functional forms have been proposed. Quadratic logarithmic approximations can be made to distance and cost functions as well as to utility functions. The direct quadratic utility function $u = (q-a)'A(q-a)$ is clearly flexible, though it suffers from other problems such as the existence of "bliss" points, see Goldberger (1967). Diewert (1973b) suggested that $\psi^*(r)$ be approximated by a "Generalized Leontief" model

$$\psi^*(r) = \left\{\delta_0 + 2\sum_i \delta_i r_i^{1/2} + \sum_i \sum_j \gamma_{ij} r_i^{1/2} r_j^{1/2}\right\}^{-1}. \tag{62}$$

This has the nice property that it is globally quasi-convex if $\delta_i \geq 0$ and $\gamma_{ij} \geq 0$ for all i, j; it also generalizes Leontief since with $\delta_0 = \delta_i = 0$ and $\gamma_{ij} = 0$ for $i \neq j$, $\psi^*(r)$ is the indirect utility function corresponding to the Leontief preferences (2). Berndt and Khaled (1979) have, in the production context, proposed a further generalization of (62) where the $\frac{1}{2}$ is replaced by a parameter, the "generalized Box–Cox" system.

There is now a considerable body of literature on testing the symmetry and homogeneity restrictions using the Rotterdam model, the translog, or these other approximations, see, e.g. Barten (1967), (1969), Byron (1970a), (1970b), Lluch (1971), Parks (1969), Deaton (1974a), (1978), Deaton and Muellbauer (1980b), Theil (1971a), (1975b), Christensen, Jorgensen and Lau (1975), Christensen and Manser (1977), Berndt, Darrough and Diewert (1977), Jorgenson and Lau (1976), and Conrad and Jorgenson (1979). Although there is some variation in results through different data sets, different approximating functions, different estimation and testing strategies, and different commodity disaggregations, there is a good deal of accumulated evidence rejecting the restrictions. The evidence is strongest for homogeneity, with less (or perhaps no) evidence against symmetry over and above the restrictions embodied in homogeneity. Clearly, for any one model, it is impossible to separate failure of the model from failure of the underlying theory, but the results have now been replicated frequently using many different functional forms, so that it seems implausible that an inappropriate specification is at the root of the difficulty. There are many possible substantive reasons why the theory as presented might fail, and I shall discuss several of them in subsequent sections. However, there are a number of arguments questioning this sort of procedure for testing. One is a statistical issue, and questions have been raised about the appropriateness of standard statistical tests in this context; I deal with these matters in the next subsection. The other arguments concern the nature of flexible functional forms themselves.

Empirical work by Wales (1977), Thursby and Lovell (1978), Griffin (1978), Berndt and Khaled (1979), and Guilkey and Lovell (1980) cast doubt on the ability of flexible functional forms both to mimic the properties of actual preferences and technologies, and to behave "regularly" at points in price-outlay space other than the point of local approximation (i.e. to generate non-negative, downward sloping demands). Caves and Christensen (1980) investigated theoretically the global properties of the (indirect) translog and the generalized Leontief forms. For a number of two and three commodity homothetic and non-homothetic systems, they set the parameters of the two systems to give the same pattern of budget shares and substitution elasticities at a point in price space, and then mapped out the region for which the models remained regular. Note that regularity is a mild requirement; it is a minimal condition and does not by itself suggest that the system is a good approximation to true preferences or behavior. It is not possible here to reproduce Caves and Christensen's diagrams, nor do the authors give any easily reproducible summary statistics. Nevertheless, although both systems *can* do well (e.g. when substitutability is low so that preferences are close to Leontief, the GL is close to globally regular, and similarly for the translog when preferences are close to Cobb–Douglas), there are also many cases where the regular regions are worringly small. Of course, these results apply only to the translog and the GL systems, but I see no reason to suppose that similar problems would not occur for the other flexible functional forms discussed above.

These results raise questions as to whether Taylor series approximations, upon which most of these functional forms are based, are the best type of approximations to work with, and there has been a good deal of recent activity in exploring alternatives. Barnett (1983a) has suggested that Laurent series expansions are a useful avenue to explore. The Laurent expansion of a function $f(x)$ around the point x_0 takes the form

$$f(x)=\sum_{n=-\infty}^{+\infty} a_n(x-x_0)^n, \tag{63}$$

and Barnett has suggested generalizing the GL form (62) to

$$\{\psi^*(r)\}^{-1}=a_0+2a'v+v'Av-2b'\bar{v}-\bar{v}'B\bar{v}, \tag{64}$$

where $v_i=r_i^{1/2}$ and $\bar{v}_i=r_i^{-1/2}$. The resulting demand system has too many parameters to be estimated in most applications, and has more than it needs to be a second-order flexible functional form. To overcome this, Barnett suggests setting $b = 0$, the diagonal elements of B to zero, and forcing the off-diagonal elements of both A and B to be non-negative (the Laurent model (64) like the GL model (62) is globally regular if all the parameters are non-negative). The resulting budget equations are

$$w_i = \left(a_i v_i + a_{ii} v_i + \sum_{j \neq i} a_{ij}^2 v_i v_j + \sum_{j \neq i} b_{ij}^2 \bar{v}_j \bar{v}_i \right) / D, \tag{65}$$

where D is the sum over i of the bracketed expression. Barnett calls this the miniflex Laurent model. The squared terms guarantee non-negativity, but are likely to cause problems with multiple optima in estimation. Barnett and Lee (1983) present results comparable to those of Caves and Christensen's which suggest that the miniflex Laurent has a substantially larger regular region than either translog or GL models.

A more radical approach has been pioneered by Gallant, see Gallant (1981), and Gallant and Golub (1983), who has shown how to approximate indirect utility functions using Fourier series. Interestingly, Gallant replicates the Christensen, Jorgenson and Lau (1975) rejection of the symmetry restriction, suggesting that their rejection is not caused by the approximation problems of the translog. Fourier approximations are superior to Taylor approximations in a number of ways, not least in their ability to keep their approximating qualities in the face of the separability restrictions discussed in Section 4 below. However, they are also heavily parametrized and superior approximation may be being purchased at the expense of low precision of estimation of key quantities. Finally, many econometricians are likely to be troubled by the sinusoidal behavior of fitted demands when projected outside the region of approximation. There is something to be said for using approximating functions that are themselves plausible for preferences and demands.

The whole area of flexible functional forms is one that has seen enormous expansion in the last five years and perhaps the best results are still to come. In particular, other bases for spanning function space are likely to be actively explored, see, e.g. Barnett and Jones (1983).

2.6. *Statistical testing procedures*

The principles involved are most simply discussed within a single model and for convenience I shall use the Rotterdam system written in the form, $i = 1, \ldots, (n-1)$

$$w_i \,\mathrm{d}\ln q_i = a_i + b_i \,\mathrm{d}\ln \bar{x}_t + \sum_j \gamma_{ij} \,\mathrm{d}\ln p_j + u_{it}, \tag{66}$$

where $\mathrm{d}\ln \bar{x}_t$ is an abbreviated form of the term in (58) and, in practice, the differentials would be replaced by finite approximations, see Theil (1975b, Chapter 2) for details. I shall omit the nth equation as a matter of course so that Ω stands for the $(n-1)\times(n-1)$ variance–covariance matrix of the u's.

The u_t vectors are assumed to be identically and independently distributed as $N(0,\Omega)$. I shall discuss the testing of two restrictions: *homogeneity* $\sum_j \gamma_{ij} = 0$, and symmetry, $\gamma_{ij} = \gamma_{ji}$.

Equation (66) is in the classical multivariate regression form (49), so equation by equation OLS yields SURE and FIML estimates. Let $\hat{\beta}$ be the stacked vector of OLS estimates and $\hat{\Omega}$ for the unrestricted estimate of the variance–covariance matrix (50). If the matrix of unrestricted residuals $Y - X\hat{B}$ is denoted by $\hat{E}$, (50) takes the form

$$\hat{\Omega} = T^{-1}\hat{E}'\hat{E}. \tag{67}$$

Testing homogeneity is relatively straightforward since the restrictions are *within* equation restrictions. A simple way to proceed is to substitute $\gamma_{in} = -\sum_1^{n-1}\gamma_{ij}$ into (66) to obtain the restricted model

$$w_i \mathrm{d}\ln q_i = a_i + b_i \mathrm{d}\ln \bar{x}_t + \sum_{j=1}^{n-1} \gamma_{ij}\left(\mathrm{d}\ln p_j - \mathrm{d}\ln p_n\right), \tag{68}$$

and re-estimate. Once again OLS is SURE is FIML and the restriction can be tested equation by equation using standard text-book F-tests. These are *exact* tests and no problems of asymptotic approximation arise. For examples, see Deaton and Muellbauer's (1980b) rejections of homogeneity using AIDS. If an overall test is desired, a Hotelling T^2 test can be constructed for the system as a whole, see Anderson (1958 pp. 207–10) and Laitinen (1978). Laitinen also documents the divergence between Hotelling's T^2 and its limiting χ^2 distribution when the sample size is small relative to the number of goods, see also Evans and Savin (1982). In consequence, homogeneity should *always* be tested using exact F or T^2 statistics and *never* using asymptotic test statistics such as uncorrected Wald, likelihood ratio, or Lagrange multiplier tests. However, my reading of the literature is that the rejection of homogeneity in practice tends to be confirmed using exact tests and is not a statistical illusion based on the use of inappropriate asymptotics.

Testing *symmetry* poses much more severe problems since the presence of the cross-equation restrictions makes estimation more difficult, separates SUR from FIML estimators and precludes exact tests. Almost certainly the simplest testing procedure is to use a Wald test based on the unrestricted (or homogeneous) estimates. Define R as the $\frac{1}{2}n(n-1)\times(n-1)(n+2)$ matrix representing the symmetry (and homogeneity) restrictions on β, so that

$$(R\beta)' = (\gamma_{12} - \gamma_{21}, \gamma_{13} - \gamma_{31}, \dots, \gamma_{(n-1)n} - \gamma_{n(n-1)}). \tag{69}$$

Then, under the null hypothesis of homogeneity and symmetry combined,

$$W_1 = \hat{\beta}'R'\left[R\{\hat{\Omega}\otimes(X'X)^{-1}\}R'\right]^{-1}R\hat{\beta}, \tag{70}$$

is the Wald test statistic which is asymptotically distributed as $\chi^2_{1/2n(n-1)}$. Apart from the calculation of W_1 itself, computation requires no more than OLS estimation. Alternatively, the symmetry constrained estimator $\tilde{\beta}$ given by (52) with $r = 0$, can be calculated. From this, restricted residuals E can be derived, and a new (restricted) estimate of Ω, $\tilde{\Omega}$, i.e.

$$\tilde{\Omega} = T^{-1}\tilde{E}'\tilde{E}. \tag{71}$$

The new estimate of $\tilde{\Omega}$ can be substituted into (52) and iterations continued to convergence yielding the FIML estimators of β and Ω. Assume that this process has been carried out and that (at the risk of some notational confusion) $\tilde{\beta}$ and $\tilde{\Omega}$ are the final estimates. A likelihood ratio test can then be computed according to

$$W_2 = T\ln\{\det\tilde{\Omega}/\det\hat{\Omega}\}, \tag{72}$$

and W_2 is also asymptotically distributed as $\chi^2_{1/2n(n-1)}$. Finally, there is the Lagrange multiplier, or score test, which is derived by replacing $\hat{\Omega}$ in (70) by $\tilde{\Omega}$, so that

$$W_3 = \hat{\beta}'R'\left[R\{\tilde{\Omega}\otimes(X'X)^{-1}\}R'\right]^{-1}R\hat{\beta}, \tag{73}$$

with again the same limiting distribution.

From the general results of Berndt and Savin (1977), it is known that $W_1 \geq W_2 \geq W_3$; these are mechanical inequalities that always hold, no matter what the configuration of data, parameters, and sample size. In finite samples, with inaccurate and inefficient estimates of Ω, the asymptotic theory may be a poor approximation and the difference between the three statistics may be very large. In my own experience I have encountered a case with 8 commodities and 23 observations where W_1 was more than a hundred times greater than W_3. Meisner (1979) reports experiments with the Rotterdam system in which the null hypothesis was correct. With a system of 14 equations and 31 observations, W_1 rejected symmetry at 5% 96 times out of 100 and at 1% 91 times out of 100. For 11 equations the corresponding figures were 50 and 37. Bera, Byron and Jarque (1981) carried out similar experiments for W_2 and W_3. From the inequalities, we know that rejections will be less frequent, but it was still found that, with n large relative to $(T-k)$ both W_2 and W_3 grossly over-rejected.

These problems for testing symmetry are basically the same as those discussed for estimation in (2.3) above; typical time series are not long enough to give reliable estimates of the variance–covariance matrix, particularly for large systems. For estimation, and for the testing of within equation restrictions, the difficulties can be circumvented. But for testing cross-equation restrictions, such as symmetry, the problem remains. For the present, it is probably best to suspend judgment on the existing tests of symmetry (positive or negative) and to await theoretical or empirical developments in the relevant test statistics. [See Byron and Rosalsky (1984) for a suggested ad hoc size correction that appears to work well in at least some situations.]

2.7. *Non-parametric tests*

All the techniques of demand analysis so far discussed share a common approach of attempting to fit demand functions to the observed data and then enquiring as to the compatibility of these fitted functions with utility theory. If unlimited experimentation were a real possibility in economics, demand functions could be accurately determined. As it is, however, what is observed is a *finite* collection of pairs of quantity and price vectors. It is thus natural to argue that the basic question is whether or not these observed pairs are consistent with any preference ordering whatever, bypassing the need to specify particular demands or preferences. It may well be true that a given set of data is perfectly consistent with utility maximization and yet be very poorly approximated by AIDS, the translog, the Rotterdam system or any other functional form which the limited imagination of econometricians is capable of inventing.

Non-parametric demand analysis takes a direct approach by searching over the price-quantity vectors in the data for evidence of inconsistent choices. If these do exist, a utility function exists and algorithms exist for constructing it (or at least one out of the many possible). The origins of this type of analysis go back to Samuelson's (1938) introduction of revealed preference analysis. However, the recent important work on developing test criteria is due to Hanoch and Rothschild (1972) and especially to Afriat (1967), (1973), (1976), (1977) and (1981). Unfortunately, some of Afriat's best work has remained unpublished and the published work has often been difficult for many economists to understand and assimilate. However, as the techniques involved have become more widespread in economics, other workers have taken up the topic, see the interpretative essays by Diewert (1973a) and Diewert and Parkan (1978) – the latter contains actual test results – and also the recent important work by Varian (1982, 1983).

Afriat proposes that a finite set of data be described as cyclically consistent if, for any "cycle", $a, b, c, \ldots, r, a$ of indices, $p^a \cdot q^a \geq p^a \cdot q^b$, $p^b \cdot q^b \geq p^b \cdot q^c$,

$\ldots, p^r q^r \geq p^r q^a$, then it must be true that $p^a \cdot q^a = p^a \cdot q^b$, $p^b q^b = p^b q^c, \ldots, p^r q^r = p^r q^a$. He then shows that cyclical consistency is necessary and sufficient for the finite set of points to be consistent with the existence of a continuous, non-satiated, concave and monotonic utility function. Afriat also provides a constructive method of evaluating such a utility function. Varian (1982) shows that cyclical consistency is equivalent to a "generalized axiom of revealed preference" (GARP) that is formulated as follows. Varian defines q^i as *strictly directly revealed preferred* to q, written $q^i P^0 q$ if $p^i q^i > p^i q$, i.e. q^i was bought at p^i even though q cost less. Secondly q^i is *revealed preferred* to q, written $q^i R q$, if $p^i q^i \geq p^i q^j$, $p^j q^j \geq p^j q^k, \ldots, p^m q^m \geq p^m q$, for some sequence of observations $(q^i, q^j, \ldots, q^m)$, i.e. q^i is indirectly or directly (weakly) revealed preferred to q. GARP then states that $q^i R q^j$ implies not $q^j P^0 q^i$, and all the nice consequences follow. Varian has also supplied an efficient and easily used algorithm for checking GARP, and his methods have been widely applied. Perhaps not surprisingly, the results show few conflicts with the theory, since on aggregate time series data, most quantities consumed increase over time so that contradictions with revealed preference theory are not possible; each new bundle was unobtainable at the prices and incomes of all previous periods.

Since these methods actually allow the construction of a well-behaved utility function that accounts exactly for most aggregate time-series data, the rejections of the theory based on parametric models (and on semi-parametric models like Gallant's Fourier system) must result from rejection of functional form and not from rejection of the theory per se. Of course, one could regard the non-parametric utility function as being a very profligately parametrized parametric utility function, so that if the object of research is to find a reasonably parsimonious theory-consistent formulation, the non-parametric results are not very helpful.

Afriat's and Varian's work, in particular see Afriat (1981) and Varian (1983), also allows testing of restricted forms of preferences corresponding to the various kinds of separability discussed in Section 4. Varian has also shown how to handle goods that are rationed or not freely chosen, as in Section 6 below. Perhaps most interesting are the tests for homotheticity, a condition that requires the utility function to be a monotone increasing transform of a linearly homogeneous function and which implies that all total expenditure elasticities are unity. Afriat (1977) showed that for two periods, 0 and 1, the necessary and sufficient condition for consistency with a homothetic utility function is that the Laspeyres price index be no less than the Paasche price index, i.e. that

$$\frac{p^1 \cdot q^0}{p^0 \cdot q^0} \geq \frac{p^1 \cdot q^1}{p^0 \cdot q^1}. \tag{74}$$

For many periods simultaneously, Afriat (1981) shows that the Laspeyres index between any two periods i and j, say, should be no less than the chain-linked Paasche index obtained by moving from i to j in any number of steps. Given that no one using *any* parametric form has ever suggested that all total expenditure elasticities are unity, it comes as something of a surprise that the Afriat condition appears to be acceptable for an 111 commodity disaggregation of post-war U.S. data, see Manser and McDonald (1984).

Clearly, more work needs to be done on reconciling parametric and non-parametric approaches. The non-parametric methodology has not yet been successfully applied to cross-section data because it provides no obvious way of dealing with non-price determinants of demand. There are also difficulties in allowing for "disturbance terms" so that failures of, e.g. GARP, can be deemed significant or insignificant, but see the recent attempts by Varian (1984) and by Epstein and Yatchew (1985).

3. Cross-section demand analysis

Although the estimation of complete sets of demand functions on time-series data has certainly been the dominant concern in demand analysis in recent years, a much older literature is concerned with the analysis of "family budgets" using sample-survey data on cross-sections of households. Until after the Second World War, such data were almost the only sources of information on consumer behavior. In the last few years, interest in the topic has once again become intense as more and more such data sets are being released in their individual microeconomic form, and as computing power and econometric technique develop to deal with them. In the United Kingdom, a regular Family Expenditure Survey with a sample size of 7000 households has been carried out annually since 1954 and the more recent tapes are now available to researchers. The United States has been somewhat less forward in the area and until recently, has conducted a Consumer Expenditure Survey only once every decade. However, a large rotating panel survey has recently been begun by the B.L.S. which promises one of the richest sets of data on consumer behavior ever available and it should help resolve many of the long-standing puzzles over differences between cross-section and time-series results. For example, most very long-run time-series data sets which are available show a rough constancy of the food share, see Kuznets (1962), (1966), Deaton (1975c). Conversion to farm-gate prices, so as to exclude the increasing component of transport and distribution costs and built in services, gives a food share which declines, but does so at a rate which is insignificant in comparison to its rate of decline with income in cross-sections [for a survey of cross-section results, see Houthakker (1957)]. Similar problems exist with other categories of expenditure as well as with the relationship between total expenditure and income.

There are also excellent cross-section data for many less developed countries, in particular from the National Sample Survey in India, but also for many other South–East Asian countries and for Latin America. These contain a great wealth of largely unexploited data, although the pace of work has recently been increasing, see, for example, the survey paper on India by Bhattacharrya (1978), the work on Latin America by Musgrove (1978), Howe and Musgrove (1977), on Korea by Lluch, Powell and Williams (1977, Chapter 5) and on Sri Lanka by Deaton (1981c).

In this section, I deal with four issues. The first is the specification and choice of functional form for Engel curves. The second is the specification of how expenditures vary with household size and composition. Third, I discuss a group of econometric issues arising particularly in the analysis of micro data with particular reference to the treatment of zero expenditures, including a brief assessment of the Tobit procedure. Finally, I give an example of demand analysis with a non-linear budget constraint.

3.1. Forms of Engel curves

This is very much a traditional topic to which relatively little has been added recently. Perhaps the classic treatment is that of Prais and Houthakker (1955) who provide a list of functional forms, the comparison of which has occupied many manhours on many data sets throughout the world. The Prais–Houthakker methodology is unashamedly pragmatic, choosing functional forms on grounds of fit, with an attempt to classify particular forms as typically suitable for particular types of goods, see also Tornqvist (1941), Aitchison and Brown (1954–5), and the survey by Brown and Deaton (1972) for similar attempts. Much of this work is not very edifying by modern standards. The functional forms are rarely chosen with any theoretical model in mind, indeed all but one of Prais and Houthakker's Engel curves are incapable of satisfying the adding-up requirement, while, on the econometric side, satisfactory methods for comparing different (non-nested) functional forms are very much in their infancy. Even the apparently straightforward comparison between a double-log and a linear specification leads to considerable difficulties, see the simple statistic proposed by Sargan (1964) and the theoretically more satisfactory (but extremely complicated) solution in Aneuryn–Evans and Deaton (1980).

More recent work on Engel curves has reflected the concern in the rest of the literature with the theoretical plausibility of the specification. Perhaps the most general results are those obtained in a paper by Gorman (1981), see also Russell (1983) for alternative proofs. Gorman considers Engel curves of the general form

$$w_i = \sum_{r \in R} a_{ir}(p)\phi_r(\ln x), \tag{75}$$

where R is some finite set and $\phi_r(\ )$ are a series of functions. If such equations are to be theory consistent, there must exist a cost function $c(u, p)$ such that

$$\frac{\partial \ln c(u, p)}{\partial \ln p_i} = \sum_{r \in R} a_{ir}(p)\phi_r\{\ln c(u, p)\}. \tag{76}$$

Gorman shows that for these partial differential equations to have a solution, (a) the rank of the matrix formed from the coefficients $a_{ir}(p)$ can be no larger than 3 and (b), the functions $\phi_n(\ )$ must take specific restricted forms. There are three generic forms for (75), two of which are reproduced below

$$w_i = a_i(p) + b_i(p)\ln x + d_i(p) \sum_{m=1}^{M} \gamma_m(p)(\ln x)^m \tag{77}$$

$$w_i = a_i(p) + b_i(p) \sum_{\sigma_m \in S_-} \mu_m(p)x^{\sigma_m} + d_i(p) \sum_{\sigma_m \in S_+} \theta_m(p)x^{\sigma_m}, \tag{78}$$

where S is a finite set of elements σ_i, S_- its negative elements and S_+ its positive elements. A third form allows combinations of trigonometrical functions of x capable of approximating a quite general function of x. However, note that the γ_m, μ_m and θ_m functions in (77) and (78) are *not* indexed on the commodity subscript i, otherwise the rank condition on a_{ir} could not hold.

Equations (77) and (78) provide a rich source of Engel curve specifications and contain as special cases anumber of important forms. From (77), with $m = 1$, the form proposed by Working and Leser and discussed above, see (15), is obtained. In econometric specifications, $a_i(p)$ adds to unity and $b_i(p)$ to zero, as will their estimates if OLS is applied to each equation separately. The log quadratic form

$$w_i = a_i(p) + b_i(p)\ln x + d_i(p)(\ln x)^2, \tag{79}$$

was applied in Deaton (1981c) to Sri Lankan micro household data for the food share where the quadratic term was highly significant and a very satisfactory fit was obtained (an R^2 of 0.502 on more than 3,000 observations.) Note that, while for a single commodity, higher powers of $\ln x$ could be added, doing so in a complete system would require cross-equation restrictions since, according to (77), the ratios of coefficients on powers beyond unity should be the same for all commodities. Testing such restrictions (and Wald tests offer a very simple method – see Section 4(a) below) provides yet another possible way of testing the theory.

Equation (78) together with $S = \{-1, 1, 2, \ldots, r, \ldots\}$ gives general polynomial Engel curves. Because of the rank condition, the quadratic with $S = \{-1, 1\}$ is as general as any, i.e.

$$p_i q_i = b_i^*(p) + a_i(p)x + \mathrm{d}_i^*(p)x^2, \tag{80}$$

where $b_i^*(p) = b_i(p)\mu_m(p)$ and $\mathrm{d}_i^*(p) = \mathrm{d}_i(p)\theta_m(p)$. This is the "quadratic expenditure system" independently derived by Howe, Pollak and Wales (1979), Pollak and Wales (1978) and (1980). The cost function underlying (80) may be shown to be

$$c(u, p) = \alpha(p) - \frac{\beta(p)}{u + \gamma(p)}, \tag{81}$$

where the links between the a_i, b_i^* and d_i^* on the one hand and the α, β and γ on the other are left to the interested reader. (With $\ln c(u, p)$ on the left hand side, (81) also generates the form (79)). This specification, like (79), is also of considerable interest for time-series analysis since, in most such data, the range of variation in x is much larger than that in relative prices and it is to be expected that a higher order of approximation in x than in p would be appropriate. Indeed, evidence of failure of linearity in time-series has been found in several studies, e.g. Carlevaro (1976). Nevertheless, in Howe, Pollak and Wales' (1979) study using U.S. data from 1929–1975 for four categories of expenditure, tests against the restricted version represented by the linear expenditure system yielded largely insignificant results. On grouped British cross-section data pooled for two separate years and employing a threefold categorization of expenditures, Pollak and Wales (1978) obtain a χ^2 values of 8.2 (without demographics) and 17.7 (with demographics) in likelihood ratio tests against the linear expenditure system. These tests have 3 degrees of freedom and are notionally significant at the 5% level (the 5% critical value of a χ_3^2 variate is 7.8) but the study is based on only 32 observations and involves estimation of a 3×3 unknown covariance matrix. Hence, given the discussion in Section 2.6 above, a sceptic could reasonably remain unconvinced of the importance of the quadratic terms for this particular data set.

Another source of functional forms for Engel curves is the study of conditions under which it is possible to aggregate over consumers and I shall discuss the topic in Section 5 below.

3.2. Modelling demographic effects

In cross-section studies, households typically vary in much more than total expenditure; age and sex composition varies from household to household, as do the numbers and ages of children. These demographic characteristics have been the object of most attention and I shall concentrate the discussion around them, but other household characteristics can often be dealt with in the same way, (e.g. race, geographical region, religion, occupation, pattern of durable good ownership, and so on). If the vector of these characteristics is a, and superscripts denote individual households, the general model becomes

$$q_i^h = g_i(x^h, p, a^h), \tag{82}$$

with g_i taken as common and, in many studies, with p assumed to be the same across the sample and suppressed as an argument in the function.

The simplest methodology is to estimate a suitable linearization of (82) and one question which has been extensively investigated in this way is whether there are economies of scale to household size in the consumption of some or all goods. A typical approach is to estimate

$$\ln q_i^h = \alpha_i + \beta_i \ln x^h + \gamma_i \ln n^h + u_i, \tag{83}$$

where n^h is the (unweighted) number of individuals in the household. Tests are then conducted for whether $(\gamma_i + \beta_i - 1)$ is negative (economies of scale), zero (no economies or diseconomies) or positive (diseconomies of scale), since this magnitude determines whether, at a given level of per capita outlay, quantity per head decreases, remains constant, or increases. For example, Iyengar, Jain and Srinivasan (1968), using (83) on data from the 17th round of the Indian N.S.S. found economies of scale for cereals and for fuel and light, with roughly constant returns for milk and milk products and for clothing.

A more sophisticated approach attempts to relate the effects of characteristics on demand to their role in preferences, so that the theory of consumer behavior can be used to suggest functional forms for (82) just as it is used to specify relationships in terms of prices and outlay alone. Such models can be used for welfare analysis as well as for the interpretation of demand; I deal with the latter here leaving the welfare applications to Section 7 below. A fairly full account of the various models is contained in Deaton and Muellbauer (1980a, Chapter 8) so that the following is intended to serve as only a brief summary.

Fully satisfactory models of household behavior have to deal both with the specification of needs or preferences at the individual level and with the question of how the competing and complementary needs of different individuals are reconciled within the overall budget constraint. The second question is akin to the usual question of social choice, and Samuelson (1956) suggested that family utility u, might be written as

$$u^h = V\{u^1(q^1), \ldots\ldots, u^{n^h}(q^{n^h})\}, \tag{84}$$

for the n^h individuals in household h. Such a form allows decentralized budgeting over members subject to central (parental) control over members' budgets. Presumably the problems normally inherent in making interpersonal comparisons of welfare are not severe within a family since, typically, such allocations seem to be made in a satisfactory manner. Building on this idea, Muellbauer (1976c) has suggested that utility is equalised within the family (e.g. for a maximin social welfare function), so that if $\gamma^r(u, p)$ is the cost function for individual r, the family cost function is given by

$$c^h(u, p) = \sum_{r=1}^{n^h} \gamma^r(u, p) = x, \tag{85}$$

which, if needs can be linked to, say, age through the γ functions, would yield an applicable specification with strong restrictions on behavior. However, such models are somewhat artificial in that they ignore the 'public' or shared goods in family consumption, though suitable modifications can be made. They also lack empirical sharpness in that the consumption vectors of individual family members are rarely observed. The exception is in the case of family labor supply, see Chapter 32 of this volume.

Rather more progress has been made in the specification of needs under the assumption that the family acts as a homogeneous unit. The simplest possibility is that, for a given welfare level, costs are affected multiplicatively by some index depending on characteristics and welfare, i.e.

$$c^h(u^h, p, a^h) = m(a^h, u^h)c(u^h, p), \tag{86}$$

where $c(u^h, p)$ is the cost function for some reference household type, e.g. one with a single adult. The index $m(a^h, u^h)$ can then be thought of as the number of adult equivalences generated by a^h at the welfare level u^h. Taking logarithms and differentiating (86) with respect to $\ln p_i$ gives

$$w_i^h = \frac{\ln c(u^h, p)}{\partial \ln p_i}, \tag{87}$$

which is independent of a^h. Hence, if households face the same prices, those with the same consumption patterns w_i have the same u^h, so that by comparing their outlays the ratio of their costs is obtained. By (86), this ratio is the equivalence scale $m(a^h, u^h)$. This procedure derives directly from Engel's (1895) pioneering work, see Prais and Houthakker (1955). In practice, a single good, food, is usually used although there is no reason why the model cannot be applied more generally under suitable specification of the m and c functions in (86), see e.g. Muellbauer

(1977). For examples of the usual practice, see Jackson (1968), Orshansky (1965), Seneca and Taussig (1971) and Deaton (1981c).

Although the Engel model is simple to apply, it has the long recognised disadvantage of neglecting any commodity specific dimension to needs. Common observation suggests that changes in demographic composition cause substitution of one good for another as well as the income effects modelled by (86) and (87). In a paper of central importance to the area, Barten (1964) suggested that household utility be written

$$u^h = v(q^*), \tag{88}$$

$$q_i^* = q_i / m_i(a^h). \tag{89}$$

So that, using Pollak and Wales' (1981) later terminology, the demographic variables generate indices which "scale" commodity consumption levels. The Barten model is clearly equivalent to writing the cost function in the form

$$c^h(u^h, p, a^h) = c(u^h\ p^*), \tag{90}$$

$$p_i^* = p_i m_i(a^h), \tag{91}$$

for a cost function $c(u, p)$ for the reference household. Hence, if $g_i(x, p)$ are the Marshallian demands for the household, household h's demands are given by

$$q_i^h = m_i(a^h) g_i(x^h, p^*). \tag{92}$$

Differentiation with respect to a_j gives

$$\frac{\partial \ln q_i}{\partial a_j} = \frac{\partial \ln m_i}{\partial a_j} + \sum_{k=1}^{n} e_{ik} \frac{\partial \ln m_k}{\partial a_j}, \tag{93}$$

where e_{ik} is the cross-price elasticity between i and k. Hence, a change in demographic composition has a direct affect through the change in needs (on m_i) and an indirect effect through the induced change in the "effective" price structure. It is this recognition of the quasi-price substitution effects of demographic change, that "a penny bun costs threepence when you have a wife and child" that is the crucial contribution of the Barten model. The specification itself may well neglect other important aspects of the problem, but this central insight is of undeniable importance.

The main competition to the Barten specification comes from the model originally due to Sydenstricker and King (1921) but rediscovered and popularized by Prais and Houthakker (1955). This begins from the empirical specification, apparently akin to (89)

$$q_i/m_i(a^h) = f_i\left(x^h/m_0(a^h)\right), \tag{94}$$

where $m_i(a^h)$ is the *specific* commodity scale, and $m_0(a^h)$ is some *general* scale. In contrast to (93), we now have the relationship

$$\frac{\partial \ln q_i}{\partial a_j} = \frac{\partial \ln m_i}{\partial a_j} - e_i \frac{\partial \ln m_0}{\partial a_j}, \tag{95}$$

so that the substitution effects embodied in (93) are no longer present. Indeed, if $x^h/m_0(a^h)$ is interpreted as a welfare indicator (which is natural in the context) (94) can only be made consistent with (88) and (89) if indifference curves are Leontief, ruling out all substitution in response to relative price change, see Muellbauer (1980) for details, and Pollak and Wales (1981) for a different interpretation.

On a single cross-section, neither the Barten model nor the Prais–Houthakker model are likely to be identifiable. That there were difficulties with the Prais–Houthakker formulation has been recognized for some time, see Forsyth (1960) and Cramer (1969) and a formal demonstration is given in Muellbauer (1980). In the Barten model, (93) may be rewritten in matrix notation as

$$F = (I + E)M, \tag{96}$$

and we seek to identify M from observable information on F. In the most favorable case, E may be assumed to be known (and suitable assumptions may make this practical even on a cross-section, see Section 4.2 below). The problem lies in the budget constraint, $p \cdot q = x$ which implies $w'[I + E] = 0$ so that the matrix $(I + E)$ has at most rank n–1. Hence, for any given F and E, both of which are observable, there exist an infinite number of M matrices satisfying (96). In practice, with a specific functional form, neither F nor E may be constant over households so that the information matrix of the system could conceivably not be singular. However, such identification, based on choice of functional form and the existence of high nonlinearities, is inherently controversial. A much better solution is the use of several cross-sections between which there is price variation and, in a such a case, several quite general functional forms are fully identified. For the Prais–Houthakker model, (95) may be written as

$$F = M - em', \tag{97}$$

where $m = \partial \ln m^0/\partial a$. From the budget constraint, $w'F = 0$ so that $m' = w'M$

which yields

$$F = (I - ew')M. \tag{98}$$

Once again $(I - ew')$ is singular, and the identification problem recurs. Here price information is likely to be of less help since, with Leontief preferences, prices have only income effects. Even so, it is not difficult to construct Prais–Houthakker models which identified given sufficient variation in prices.

Since Prais and Houthakker, the model has nevertheless been used on a number of occasions, e.g by Singh (1972), (1973), Singh and Nagar (1973), and McClements (1977) and it is unclear how identification was obtained in these studies. The use of a double logarithmic formulation for f_i helps; as is well-known, such a function cannot add up even *locally*, see Willig (1976), Varian (1978), and Deaton and Muellbauer (1980a, pp 19–20) so that the singularity arguments given above cannot be used. Nevertheless, it seems unwise to rely upon a clear misspecification to identify the parameters of the model. Coondoo (1975) has proposed using an assumed independence of m_0 on x as an identifying restriction; this is ingenious but, unfortunately, turns out to be inconsistent with the model. There are a number of other possible means of identification, see Muellbauer (1980), but essentially the only practical method is the obvious one of assuming a priori a value for one of the m_i's. By this means, the model can be estimated and its results compared with those of the Barten model. Some results for British data are given in Muellbauer (1977) (1980) and are summarized in Deaton and Muellbauer (1980a, pp 202–5). In brief, these suggest that each model is rather extreme, the Prais–Houthakker with its complete lack of substitution and the Barten with its synchronous equivalence of demographic and price substitution effects. If both models are normalized to have the same food scale, the Prais–Houthakker model also tends to generate the higher scales for other goods since, unless the income effects are very large, virtually all variations with composition must be ascribed directly to the m_i's. The Barten scales are more plausible but evidence suggests that price effects and demographic effects are not linked as simply as is suggested by (93).

Gorman (1976) has proposed an extension to (90) which appears appropriate in the light of this evidence. In addition to the Barten substitution responses he adds fixed costs of children $\gamma_i(a^h)$ say; hence (90) becomes

$$c^h(u^h, p, a^h) = p\cdot\gamma(a^h) + c(u^h, p^*), \tag{99}$$

with (94) retained as before. Clearly, (99) generates demands of the form

$$q_i^h = \gamma_i(a^h) + g_i(x^h - p\cdot\gamma(a^h), p^*). \tag{100}$$

Pollak and Wales (1981) call the addition of fixed costs "demographic translating" as opposed to "demographic scaling" of the Barten model; the Gorman model (99) thus combines translating and scaling. In their paper, Pollak and Wales test various specifications of translating and scaling. Their results are not decisive but tend to support scaling; with little additional explanatory power from translating once scaling has been allowed for. Note, however, that the translating term in (99) might itself form the starting point for the modelling, just as did the multiplicative term in the Engel model. If the scaling terms in (99) are dropped, so that p replaces p^*, and if it is recognized that the child cost term $p \cdot \gamma(a^h)$ is likely to be zero for certain "adult" goods, then for i an adult good, we have

$$q_i^h = h_i(u^h, p), \tag{101}$$

independent of a^h. For all such goods, additional children exert only income effects, a proposition that can be straightforwardly tested by comparing the ratios of child to income derivatives across goods, while families with the same outlay on adult goods can be identified as having the same welfare level. This is the model first proposed by Rothbarth (1943) and later implemented by Henderson (1949–50a) (1949–50b) and Nicholson (1949), see also Cramer (1969). Deaton and Muellbauer (1983) have recently tried to reestablish it as a simply implemented model that is superior to the Engel formulation for applications where computational complexity is a problem.

3.3. Zero expenditures and other problems

In microeconomic data on consumers expenditure, it is frequently the case that some units do not purchase some of the commodities, alcohol and tobacco being the standard examples. This is of course entirely consistent with the theory of consumer behavior; for example, two goods (varieties) may be very close to being perfect substitutes so that (sub) utility for the two might be

$$u = \alpha_1 q_1 + \alpha_2 q_2, \tag{102}$$

so that, if outlay is x, the demand functions are

$$\begin{aligned} q_i &= x_i/p_i \qquad \text{if } p_i/p_j < \alpha_i/\alpha_j \\ &= 0 \qquad \text{otherwise,} \end{aligned} \tag{103}$$

for $i, j = 1,2$ and for $p_1\alpha_2 \neq p_2\alpha_1$. It is not difficult to design more complex (and more realistic) models along similar lines. For a single commodity, many of these models can be made formally equivalent to the Tobit, Tobin (1958) model

$$y_i^* = x_i'\beta + u_i$$

$$\left.\begin{aligned} y_i &= y_i^* \\ &= 0 \end{aligned}\right\} \quad \begin{aligned} &\text{if } y_i^* \geq 0 \\ &\text{otherwise,} \end{aligned} \tag{104}$$

and the estimation of this is well-understood.

However, there are a number of extremely difficult problems in applying the Tobit model to the analysis of consumer behavior. First, there is typically more than one good and whenever the demand for one commodity switches regime (i.e. becomes positive having been zero, or vice versa), there are, in general, regime changes in all the other demands, if only to satisfy the budget constraint. In fact, the situation is a good deal more complex since, as will be discussed in Section 6 below, non-purchase is formally equivalent to a zero ration and the imposition of such rations changes the functional form for other commodities in such a way as to generate both income and substitution effects. With a n goods in the budget, and assuming at least one good purchased, there are 2^{n-1} possible regimes, each with its own particular set of functional forms for the non-zero demands. Wales and Woodland (1983) have shown how, in principle, such a problem can be tackled and have estimated such a system for a three good system using a quadratic (direct) utility function. Even with these simplifying assumptions, the estimation is close to the limits of feasibility. Lee and Pitt (1983) have demonstrated that a dual approach is as complicated. An alternative approach may be possible if only a small number (one or two) commodities actually take on zero values in the sample. This is to condition on non-zero values, omitting all observations where a zero occurs, and to allow specifically for the resulting sample selection bias in the manner suggested, for example, by Heckman (1979). This technique has been used by Blundell and Walker (1982) to estimate a system of commodity demands simultaneously with an hours worked equation for secondary workers.

The second problem is that it is by no means obvious that the Tobit specification is correct, even for a single commodity. In sample surveys, zeros frequently occur simply because the item was not bought over a relatively short enumeration period (usually one or two weeks, and frequently less in developing countries). Hence, an alternative to (104) might be

$$y_i^* = x_i'\beta + u_i,$$

$$\left.\begin{aligned} y_i &= y_i^*/\pi_i \\ y_i &= 0 \end{aligned}\right\} \quad \begin{aligned} &\text{with probability } \pi_i, \\ &\text{with probability } (1-\pi_i). \end{aligned} \tag{105}$$

Hence, if, $p(u_i)$ is the p.d.f. of u_i the likelihood for the model is

$$L = \prod_0 (1-\pi_i) \prod_+ \pi_i p(\pi_i y_i - x_i'\beta). \tag{106a}$$

This can be maximized directly to estimate β and π_i given some low parameter specification for π_i. But note in particular that for $\pi_i = \pi$ for all i and u_i taken as i.i.d.$N(0, \sigma^2)$ the likelihood is, for n_0 the number of zero y_i's,

$$L = (1-\pi)^{n_0} \prod_+ \phi\left(\frac{x_i'\beta}{\pi}, \frac{\sigma}{\pi}\right). \tag{106b}$$

Hence OLS on the positive y_i's alone is consistent and fully efficient for β/π and σ/π. The MLE of π is simply the ratio of the number of positive y_i's to the sample size, so that, in this case, all parameters are easily estimated. If this is the true model, Tobit will not generally be consistent. However, note that (105) allows y_i to be negative (although this may be very improbable) and ideally the Tobit and the binary model should be combined. A not very successful attempt to do this is reported in Deaton and Irish (1984). See also Kay, Keen and Morris (1984) for discussion of the related problem of measuring total expenditure when there are many zeroes.

In my view, the problem of dealing appropriately with zero expenditures is currently one of the most pressing in applied demand analysis. We do not have a theoretically satisfactory and empirically implementable method for modelling zeroes for more than a few commodities at once. Yet all household surveys show large fractions of households reporting zero purchases for some goods. Since household surveys typically contain several thousands observations, it is important that procedures be developed that are also computationally inexpensive.

There are also a number of other problems which are particularly acute in cross-section analysis and are not specific to the Tobit specification. *Heteroscedasticity* tends to be endemic in work with micro data and, in my own practical experience, is extremely difficult to remove. The test statistics proposed by Breusch and Pagan (1979) and by White (1980) are easily applied, and White has proposed an estimator for the variance-covariance matrix which is consistent under heteroscedasticity and does not require any specification of its exact form. Since an adequate specification seems difficult in practice, and since in micro studies efficiency is rarely a serious problem, White's procedure is an extremely valuable one and should be applied routinely in large cross-section regressions. Note, however, that with Tobit-like models, untreated heteroscedasticity generates inconsistency in the parameter estimates, see Chapter 27, thus presenting a much more serious problem. The heteroscedasticity introduced by *grouping* has become less important as grouped data has given way to the analysis of the original micro observations, but see Haitovsky (1973) for a full discussion.

Finally, there are a number of largely unresolved questions about the way in which survey design should be taken into account (if at all) in econometric analysis. One topic is whether or not to use inverse probability weights in regression analysis, see e.g. DuMouchel and Duncan (1983) for a recent discussion. The other concerns the possible implications for regression analysis of Godambe's (1955) (1966) theorem on the non-existence of uniformly minimum variance or maximum likelihood estimators for means in finite populations, see Cassel, Sarndal and Wretman (1977) for a relatively cool discussion.

3.4. Non-linear budget constraints

Consumer behavior with non-linear budget constraints has been extensively discussed in the labor supply literature, where tax systems typically imply a non-linear relationship between hours worked and income received, see Chapter 32 in this Handbook and especially Hausman (1985). I have little to add to Hausman's excellent treatment, but would nevertheless wish to emphasize the potential for these techniques in demand analysis, particularly in "special"

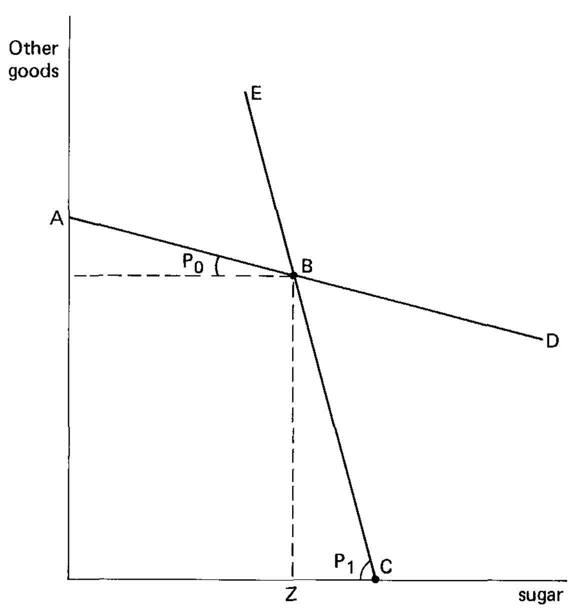

Figure 2. Budget constraint for a fair price shop.

markets. Housing is the obvious example, but here I illustrate with a simple case based on Deaton (1984). In many developing countries, the government operates so-called "fair price" shops in which certain commodities, e.g. sugar or rice, are made available in limited quantities at subsidized prices. Typically, consumers can buy more than the fair price allocation in the free market at a price p_1, with $p_1 > p_0$ the fair price price. Figure 2 illustrates for "sugar" versus a numeraire good with unit price. Z is the amount available in the fair price shop and the budget constraint assumes that resale of surplus at free market prices is impossible.

There are two interrelated issues here for empirical modelling. At the micro level, using cross-section data, we need to know how to use utility theory to generate Engel curves. At the macro-level, it is important to know how the two prices p_0 and p_1 and the quantity Z affect total demand. As usual, we begin with the indirect utility function, though the form of this can be dictated by prior beliefs about demands (e.g. there has been heavy use of the indirect utility function associated with a linear demand function for a single good – for the derivation, see Deaton and Muellbauer (1980a, p. 96) (1981) and Hausman (1980)). Maximum utility along AD is $u_0 = \psi(x, p, p_0)$ with associated demand, by Roy's identity, of $s_0 = g(x, p, p_0)$. Now, by standard revealed preference, if $s_0 < Z$, s_0 is optimal since BC is obtainable by a consumer restricted to being within AD. Similar, maximum utility along EC is $u_1 = \psi(x + (p_1 - p_0)Z, p, p_1)$ with $s = g(x + (p_1 - p_0)Z, p, p_1)$. Again, if $s_1 > Z$, then s_1 is optimal. The remaining case is $s_0 > Z$ *and* $s_1 < Z$ (both of which are infeasible), so that sugar demand is exactly Z (at the kink B). Hence, for individual h with expenditure x^h and quota Z^h, the demand functions are given by

$$s^h = g^h(x^h, p, p_0) \qquad \text{if } g^h(x^h, p, p_0) < Z^h \tag{107}$$

$$s^h = g^h(x^h + (p_1 - p_0)Z^h, p, p_1) \qquad \text{if } g^h(x^h + (p_1 - p_0)Z^h, p, p_1) > Z^h \tag{108}$$

$$s^h = Z^h \qquad \text{if } g^h(x^h + (p_1 - p_0)Z^h, p, p_1) \le Z^h \le g^h(x^h, p, p_0) \tag{109}$$

Figure 3 gives the resulting Engel curve. Estimation on cross-section data is straightforward by an extension of the Tobit method; the demand functions g^h are endowed with taste variation in the form of a normally distributed random term, and a likelihood with three "branches" corresponding to $s^h < Z^h$, $s^h = Z^h$, and $s^h > Z^h$ is constructed. The middle branch corresponds to the zero censoring for Tobit; the outer two are analogous to the non-censored observations in Tobit.

The aggregate free-market demand for sugar can also be analysed using the model. To simplify, assume that households differ only in outlay, x^h. Define x_T by $g\{x_T + (p_1 - p_0)Z, p, p_1\} = Z$, so that consumers with $x > x_T$ enter the free

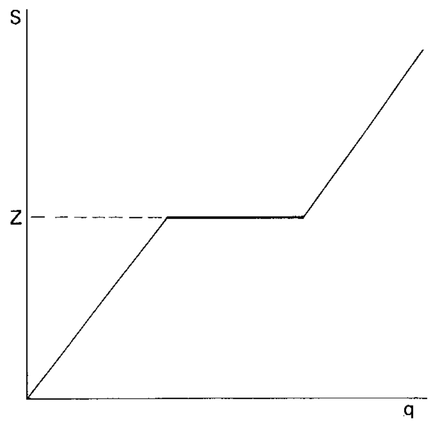

Figure 3. Engel curve with a non-linear budget constraint.

market. Hence per capita free market demand is

$$s = \int_{x_T}^{\infty} \{ g(x + (p_1 - p_0)Z, p, p_1) - Z \} \mathrm{d}F(x) \tag{110}$$

$$\frac{\partial s}{\partial Z} = \int_{x_T}^{\infty} \left\{ \frac{\partial s}{\partial x}(p_1 - p_0) - 1 \right\} \mathrm{d}F(x) - \{ g(x_T + (p_1 - p_0)Z, p, p_1) - Z \} f(x_T) \tag{111}$$

which, from the definition of x_T is simply

$$\frac{\partial s}{\partial Z} = \int_{x_T}^{\infty} \left\{ \frac{\partial s}{\partial x}(p_1 - p_0) - 1 \right\} \mathrm{d}F(x). \tag{112}$$

Since, at the entensive margin, consumers buy nothing in the free market, only the intensive margin is of importance. Note that *all* of these estimations and calculations take a particularly simple form if the Marshallian demand functions are assumed to be linear, so that, even in this non-standard situation, linearity can still greatly simplify.

The foregoing is a very straightforward example but is illustrates the flavor of the analysis. In practice, non-linear budget constraints may have several kink points and the budget set may be non-convex. While such things can be dealt with, e.g. see King (1980), or Hausman and Wise (1980) for housing, and Reece and Zieschang (1984) for charitable giving, the formulation of the likelihood becomes increasingly complex and the computations correspondingly more burdensome. While virtually all likelihood functions can be maximized *in principle*, doing so for real applied examples with several thousand observations can be prohibitively expensive.

4. Separability

In the conventional demand analysis discussed so far, a number of important assumptions have not been justified. First, demand within each period is analysed conditional on total expenditure and prices for that period alone, with no mention of the broader determinants of behavior, wealth, income, other prices and so on. Second, considerations of labor supply were completely ignored. Third, no attention was given to questions of consumption and saving or to the problems arising for goods which are sufficiently durable to last for more than one period. Fourth, the practical analysis has used, not the elementary goods of the theory, but rather aggregates such as food, clothing, etc., each with some associated price index. Separability of one sort or another is behind each of these assumptions and this section gives the basic results required for applied analysis. No attempt is made to give proofs, for more detailed discussion the reader may consult Blackorby, Primont and Russell (1978), Deaton and Muellbauer (1980a Chapter 5) or the original creator of much of the material given here, Gorman (1959) (1968) as well as many unpublished notes.

4.1. *Weak separability*

Weak separability is the central concept for much of the analysis. Let q^A be some subvector of the commodity vector q so that $q=(q^A,q^{\bar{A}})$ without loss of generality. q^A is then said to be (weakly) *separable* if the direct utility function takes the form

$$u=v\left(v_A\left(q^A\right),q^{\bar{A}}\right), \tag{113}$$

$v_A(q^A)$ is the subutility (or felicity) function associated with q^A. This equation is equivalent to the existence of a preference ordering over q^A alone; choices over the q^A bundles are consistent independent of the vector $q^{\bar{A}}$. More symmetrically, preferences as a whole are said to be separable if q can be partitioned into $(q^A,q^B,\dots q^N)$ such that

$$u=v\left(v_A\left(q^A\right),v_B\left(q^B\right),\dots,v_N\left(q^N\right)\right). \tag{114}$$

Since v is increasing in the subutility levels, it is immediately obvious that maximization of overall u implies maximization of the subutilities subject to whatever is optimally spent on the groups. Hence, (113) implies the existence of subgroup demands

$$q_i^A = g_i^A(x^A, p^A), \tag{115}$$

where $x^A = p^A \cdot q^A$, while (115) has the same implication for all groups. Hence, if preferences in a life-cycle model are weakly separable over time periods, commodity demand functions conditional on x and p for each time period are guaranteed to exist. Similarly, if goods are separable from leisure, commodity demand functions of the usual type can be justified.

Tests of these forms of separability can be based on the restrictions on the substitution matrix implied by (115). If i and j are two goods in distinct groups, $i \in G$, $j \in H$, $G \neq H$, then the condition

$$s_{ij} = \mu_{GH} \frac{\partial q_i}{\partial x} \cdot \frac{\partial q_j}{\partial x}, \tag{116}$$

for some quantity μ_{GH} (independent of i and j) is both necessary and sufficient for (114) to hold. If a general enough model of substitution can be estimated, (116) can be used to test separability, and Byron (1968), Jorgenson and Lau (1975) and Pudney (1981b), have used essentially this technique to find separability patterns between goods within a single period. Barnett (1979a) has tested the important separability restriction between goods and leisure using time series American data and decisively rejects it. If widely repeated, this result would suggest considerable misspecification in the traditional studies. It is also possible to use a single cross-section to test separability between goods and leisure. Consider the following cost function proposed by Muellbauer (1981b).

$$c(u, \omega, p) = d(p) + b(p)\omega + \{a(p)\}^{1-\delta}\omega^{\delta}u, \tag{117}$$

where ω is the wage $d(p)$, $b(p)$ and $a(p)$ are functions of p, homogenous of degrees, 1, 0 and 1 respectively. Shephard's Lemma gives immediately

$$\left.\begin{aligned} q_i &= \alpha_i + \beta_i\omega + \gamma_i\mu \\ \omega h &= \alpha_0 + \beta_0\omega + \gamma_0\mu \end{aligned}\right\}, \tag{118}$$

for transfer income μ, hours worked h and parameters α, β, γ all constant in a single cross-section. It may be shown that (117) satisfies (114) for leisure vis-à-vis goods if and only if $b(p)$ is a constant, which for (118) implies that β_i/γ_i be independent of i, $i = 1, \ldots, n$. This can be tested by first estimating (114) as a system by OLS equation by equation and then computing the Wald test for the

$(n-1)$ restrictions, $i=1,\ldots,(n-1)$

$$\beta_i\gamma_n - \gamma_i\beta_n = 0. \tag{119}$$

This does not involve estimating the restricted nonlinear model. My own results on British data, Deaton (1981b), suggest relatively little conflict with separability, however, earlier work by Atkinson and Stern (1981) on the same data but using an ingenious adaptation of Becker's (1965) time allocation model, suggests the opposite. Blundell and Walker (1982), using a variant of (117) reject the hypothesis that wife's leisure is separable from goods. Separability between different time periods is much more difficult to test since it is virtually impossible to provide general unrestricted estimates of the substitution responses between individual commodities across different time periods.

Subgroup demand functions are only a part of what the applied econometrician needs from separability. Just as important is the question of whether it is possible to justify demand functions for commodity composites in terms of total expenditure and composite price indices. The Hicks (1936) composite commodity theorem allows this, but only at the price of assuming that there are no relative price changes within subgroups. Since there is no way of guaranteeing this, nor often even of checking it, more general conditions are clearly desirable. In fact, the separable structure (114) may be sufficient in many circumstances. Write u_A, u_B, etc. for the values of the felicity functions and $c_A(u_A, p^A)$ etc. for the subgroup cost functions corresponding to the $v_A(q^A)$ functions. Then the problem of choosing the group expenditure levels $x_A, x_B,\ldots$ can be written as

$$\begin{aligned}&\max u = v(u_A, u_B,\ldots,u_N),\\ &\text{s.t. } x = \sum_R c_R(u_R, p^R).\end{aligned} \tag{120}$$

Write

$$c_R(u_R, p^R) = c_R(u_R, \bar{p}^R)\cdot\frac{c_R(u_R, p^R)}{c_R(u_R, \bar{p}^R)}, \tag{121}$$

for some fixed prices $\bar{p}^R$. For such a fixed vector, $c_R(u_R, \bar{p}^R)$ is a welfare indicator or quantity index, while the ratio $c_R(u_R, p^R)/c_R(u_R, \bar{p}^R)$ is a true (sub) cost-of-living price index comparing p^R and $\bar{p}^R$ using u_R as reference, see Pollak (1975). Finally, since $u_R = \psi_R(c_R(u_R, \bar{p}^R), \bar{p}^R)$, (120) may be written

$$\begin{aligned}&\max u = v\{\psi_A(c_A(u_A, \bar{p}^A), \bar{p}^A), \psi_B(\quad), \qquad\qquad \},\\ &\text{s.t.} \sum c_R(u_R, \bar{p}_R)\cdot\frac{c_R(u_R, p^R)}{c_R(u_R, \bar{p}^R)} = x,\end{aligned} \tag{122}$$

which is a standard utility maximization problem in which the constant price utility levels $c_R(u_R, \bar{p}^R)$ are the quantities and the indices $c_R(u_R, p^R)/c_R(u_R, \bar{p}^R)$ are the prices. Of course, neither of these quantities is directly observable and the foregoing analysis is useful only to the extent that $c_R(u_R, \bar{p}^R)$ is adequately approximated by the constant price composite $q^R \cdot \bar{p}^R$ and the price index by the implicit price deflator $p^R \cdot q^R/\bar{p}^R \cdot q^R$. The approximations will be exact under the conditions of the composite commodity theorem, but may be very good in many practical situations where prices are highly but not perfectly collinear. If so, the technique has the additional advantage of justifying the price and quantity indices typically available in the national accounts statistics. An ideal solution not relying on approximations requires quantity indices depending only on quantities and price indices depending only on prices. Given weak separability, this is only possible if *either* each subcost function is of the form $c_G(u_G, p^G) = \theta_G(u_G)b_G(p^G)$ so that the subgroup demands (11) display unit elasticity for all goods with respect to group outlay *or* each indirect felicity function takes the "*Gorman generalized polar form*"

$$u_G = F_G[x_G/b_G(p^G)] + a_G(p^G), \tag{123}$$

for suitable functions F_G, b_G and a_G, the first monotone increasing, the latter two linearly homogeneous, *and* the utility function (114) or (120) must be additive in the individual felicity functions. Additivity is restrictive even between groups, and will be further discussed below, but (123) permits fairly general forms of Engel curves, e.g. the Working form, AIDS, PIGL and the translog (61) if $\Sigma_k\Sigma_j\beta_{kj} = 0$. See Blackorby, Boyce and Russell (1978) for an empirical application, and Anderson (1979) for an attempt to study the improvement over standard practice of actually computing the Gorman indices. In spite of this analysis, there seems to be a widespread belief in the profession that homothetic weak separability is *necessary* for the empirical implementation of two-stage budgeting (which is itself almost the only sensible way to deal with very large systems) – see the somewhat bizarre exchanges in the 1983 issue of the *Journal of Business and Economic Statistics*. In my view, homothetic separability is likely to be the *least* attractive of the alternatives given here; it is rarely sensible to *maintain* without testing that subgroup demands have unit group expenditure elasticities. In many cases, prices will be sufficiently collinear for the problem (122) to given an acceptably accurate representation. And if not, additivity between broad groups together with the very flexible Gorman generalized polar form should provide an excellent alternative. Even failing these possibilities, there are other types of separability with useful empirical properties, see Blackorby, Primont and Russell (1978) and Deaton and Muellbauer (1980, Chapter 5).

One final issue related to separability is worth noting. As pointed out by Blackorby, Primont and Russell (1977), flexible functional forms do not in general remain flexible under the global imposition of separability restrictions. Hence, a specific functional form which offers a local second-order approximation to an arbitrary utility function may not be able to similarly approximate, say, an arbitrary additive utility function once its parameters are restricted to render it globally additive. For example, Blackorby et al. show that weak separability of the translog implies either strong separability or homothetic separability so that the translog cannot model non-homothetic weak separability. The possibility of imposing and testing restrictions locally (say, at the sample mean) remains, but this is less attractive since it is difficult to discriminate between properties of the data generation process and the approximating properties of the functional form.

4.2. *Strong separability and additivity*

Strong separability restricts (114) to the case where the overall function is additive, i.e. for some monotone increasing f

$$u = f\left(\sum_R v_R(q^R)\right) \tag{124}$$

If each of the groups q^R contains a single good, preferences are said to be *additive*, or that wants are independent. I deal with this case for simplicity since all the additional features over weak separability occur between groups rather than within them. The central feature of additivity is that *any* combination of goods forms a separable set from any other, so that (116) must hold without the G, H labels on μ_{GH}, i.e. for some μ and for all i, j in different groups ($i \neq j$ under additivity)

$$s_{ij} = \mu \frac{\partial q_i}{\partial x} \frac{\partial q_j}{\partial x}. \tag{125}$$

The budget constraint (or homogeneity) can be used to complete this for all i and j; in elasticity terms, the relationship is, Frisch (1959), Houthakker (1960)

$$e_{ij} = \phi \delta_{ij} e_i - e_i w_j (1 + \phi e_j), \tag{126}$$

for some scalar ϕ, (uncompensated) cross-price elasticity e_{ij}, and total expenditure elasticity e_i. This formula shows immediately the strengths and weaknesses of additivity. Apart from the data w_i, knowledge of the $(n-1)$ independent e_i's together with the quantity ϕ (obtainable from knowledge of one single price elasticity) is sufficient to determine the whole $(n \times n)$ array of price elasticities. Additivity can therefore be used to estimate price elasticities on data with little or no relative price variation, e.g. on cross-sections, on short-time series, or in centrally planned economies where relative prices are only infrequently altered. This was first realised by Pigou (1910) and the idea has a distinguished history in the subject, see Frisch (1932), (1959) and the enormous literature on the (additive) linear expenditure system [for Eastern European experience, see Szakolczai (1980) and Fedorenko and Rimashevskaya (1981)]. Conversely, however, there is very little reason to suppose that (126) is empirically valid. Note, in particular, that for w_i small relative to e_i (as is usually the case), $e_{ii} \simeq \phi e_i$ (as Pigou pointed out) and there seems no grounds for such a proportionality relationship to be generally valid. Indeed such tests as have been carried out, Barten (1969), Deaton (1974b) (1975a) (1975b), Theil (1975b), suggest that additivity is generally not true, even for broad categories of goods. Nevertheless, the assumption continues to be widely used, for example in the interesting cross-country work of Theil and Suhm (1982), no doubt because of its economy of parametrization (= high level of restrictiveness). There is also a substantial industry in collecting estimates of the parameter ϕ under the (entirely baseless) supposition that it measures the inverse of the elasticity of the marginal utility of money.

Few of the practical objections to additivity apply to its use in an intertemporal context and it is standard practice to specify life-time preferences by (124) where the R's refer to time periods, an example being Lluch's (1973) intertemporal linear expenditure system (ELES), although this is also additive within periods. On elegant way of exploiting additivity is again due to Gorman (1976) and utilizes the concept of a "consumer profit function". Define $\pi(p,r)$ by

$$\pi(p,r) = \max_q \{ -p\cdot q + r\cdot u;\ u = v(q) \}, \tag{127}$$

for concave $v(q)$, so that the consumer sells utility (to him or herself) at a price r (= the reciprocal of the marginal utility of money) using inputs q at prices p. Now if $v(q)$ has the explicitly additive form $\Sigma v_R(q^R)$, so will $\pi(p,r)$, i.e.

$$\pi(p,r) = \sum_R \pi_R(r, p_R). \tag{128}$$

Now $\pi(p,r)$ also has the derivative property $q = -\nabla_p \pi(p,r)$ so that for i belonging to group R,

$$q_i = -\frac{\partial \pi_R(r, p_R)}{\partial p_{Ri}}, \tag{129}$$

which depends only on within group prices and the single price of utility r which is common to all groups and provides the link between them. In the intertemporal context, r is the price of lifetime utility, which is constant under certainty or follows (approximately) a random walk under uncertainty, while p_R is within period prices. Hence, as realized by MaCurdy and utilized in Heckman (1978), Heckman and MaCurdy (1980), and MaCurdy (1981), eq. (129) can be implemented on panel data by treating r as a fixed effect so that only data on current magnitudes are required. Since these are typically the only data available, the technique is of considerable importance. See Browning, Deaton and Irish (1984) for further discussion of profit functions and additivity and for an application to British data (in which the simple life-cycle model of the simultaneous determination of consumption and labor supply has some difficulty in dealing with the evidence.)

Another important use of separability in general and of additivity in particular is as a vehicle for the structuring and interpretation of preference patterns. For example, in the "characteristics" model of consumer behaviour pioneered by Gorman (1956, 1980), Stone (1956) and Lancaster (1966), and recently estimated by Pudney (1981a), it is a transformation of the goods which generates utility, and it may be quite plausible to assume that preferences are separable or even additive in the transformed characteristics (food, shelter, mate, etc.) rather than in the market goods which have no direct role in satisfying wants. One possibility, extensively explored by Theil and his co-workers, e.g. Theil (1976) and Theil and Laitinen (1981) for a review, is that preferences are additive over characteristics given by a linear transform of the market goods. Theil and Laitinen use the Rotterdam model and, by a technique closely related to factor analysis, rotate the axes in goods space to obtain the "preference independence transform". Applied to the demand for beef, pork and chicken in the U.S., the model yields the transformed goods "inexpensive meat", "beef/pork contrast" and "antichicken", Theil (1976, p. 287). These characteristics may indeed reflect real aspects of preference structures in the U.S., but as is often the case with factor analytical techniques (see e.g. Armstrong (1967) for an amusing cautionary tale) there is room for some (largely unresolvable) scepticism about the validity and value of any specific interpretations.

5. Aggregation over consumers

Clearly, on micro or panel data, aggregation is not an issue, and as the use of such data increases, the aggregation problem will recede in importance. However, much demand analysis is carried out on macroeconomic aggregate or per capita data, and it is an open question as to whether this makes sense or not. The topic is a large one and I present only the briefest discussion here, see Deaton and Muellbauer (1980a, Chapter 6) for further discussion and references. At the most general level, average aggregate demand $\bar{q}_i$ is given by

$$\bar{q}_i = G_i(x^1, x^2, \dots, x^h, \dots, x^H, p), \tag{130}$$

for the H outlays x^h of household h. The function G_i can be given virtually any properties whatever depending on the configuration of individual preferences. If, however, the outlay distribution were fixed in money terms, $x^h = k^h \bar{x}$ for constants k^h, (130) obviously gives

$$\bar{q}_i = G_i^*(\bar{x}, p), \tag{131}$$

although without restrictions on preferences, see e.g. Eisenberg (1961), Pearce (1964), Chipman (1974), and Jerison (1984), there is no reason to suppose that the G_i^* functions possess any of the usual properties of Marshallian demands. Of course, if the *utility* (real outlay) distribution is fixed, Hicksian demands aggregate in the same way as (130) and (131) and there exist macro demand functions with all the usual properties. There is very little relevant empirical evidence on the movement over time of either the outlay or the utility distribution, but see Simmons (1980) for some conjectures for the U.K.

If the distribution of outlay is not to be restricted in any way, formulae such as (131) can only arise if mean preserving changes in the x-distribution have no effect on aggregate demand, i.e. if all individuals have identical marginal propensities to spend on each of the goods. This condition, of parallel linear Engel curves, dates back to Antonelli (1886), but is usually (justly) credited to Gorman (1953) (1961). As he showed, utility maximizing consumers have parallel linear Engel curves if and only if the individual cost functions have the form

$$c^h(u^h, p) = a^h(p) + b(p)u^h, \tag{132}$$

a specification known as the "Gorman polar form". Suitable choice of the $a^h(p)$ and $b(p)$ functions permits (132) to be a flexible functional form, Diewert (1980a), but the uniformity across households implied by the need for all Engel curves to be parallel seems implausible. However, it should be noted that a single cross-section is insufficient to disprove the condition since, in principle, and without the use of panel data, variation in the $a^h(p)$ functions due to non-outlay factors cannot be distinguished from the direct effects of variations in x^h. A somewhat weaker form of the aggregation condition, emphasized by Theil (1954) (1975 Chapter 4) is that the marginal propensities to consume be distributed independently of the x^h, see also Shapiro (1976) and Shapiro and Braithwait (1979). Note finally that if aggregation is to be possible for all possible income distributions, including those for which some people have zero income, then the parallel linear Engel curves must pass through the origin so that $a^h(p)$ in (132) is zero and preferences are identical and homothetic.

If, however, the casual evidence against any form of linear Engel curves is taken seriously exact aggregation requires the abandonment of (131), at least in principle. One set of possibilities has been pursued by Muellbauer (1975b) (1976a) (1976b) who examines conditions under which the aggregate budget share of each good can be expressed as a function of prices and a single indicator of x, not necessarily the mean. If, in addition, this indicator is made independent of prices, the cost functions must take the form

$$c^h(u^h, p) = k^h\{a(p)^\alpha(1-u^h)+b(p)^\alpha u^h\}^{1/\alpha}, \tag{133}$$

called by Muellbauer, "price-independent generalised linearity" (PIGL). With $\alpha=1$, PIGL is essentially the Gorman polar form and the Engel curves are linear; otherwise, α controls the curvature of the Engel curves with, for example, the AIDS and Working-Leser forms as special cases when $\alpha=0$. The macro relationships corresponding to (133) render $\bar{q}_i$ a function of both x and of the mean of order $(1-\alpha)$ of the outlay distribution. Hence, if $\alpha=-1$, the Engel curves are quadratic and the average aggregate demands depend upon the mean and variance of x. This opens up two new possibilities. On the one hand, the presumed (or estimated) curvature of the Engel curves can be used to formulate the appropriate index of dispersion for inclusion in the aggregate demands, see e.g. the papers by Berndt, Darrough and Diewert (1977) and by Simmons (1980) both of which use forms of (133). On the other hand, the income and hence outlay distribution changes very little over time, such models allow the dispersion terms to be absorbed into the function and justify the use of (131) interpreted as a conventional Marshallian demand function, see e.g. Deaton and Muellbauer (1980b). This position seems defensible in the light of the many studies which, using one technique or another, have failed to find any strong influence of the income distribution on consumer behaviour.

Recent theoretical work on aggregation has suggested that the generalized linearity and price independent generalised linearity forms of preference have a more fundamental role to play in aggregation than solving the problem posed by Muellbauer. Jerison (1984) has shown that the generalized linearity conditions are important for aggregation with fixed income distribution, while Freixas and Mas-Colell (1983) have proved the necessity of PIGL for the weak axiom of revealed preference to hold in aggregate if the income distribution is unrestricted. (Note that Hildenbrand's (1983) proof that WARP holds on aggregate data requires that the density of the income distribution be monotone declining and have support $(0,\infty)$, so that modal income is zero!).

In a more empirical vein, Lau (1982) has considered a more general form of aggregation than that required by (131). Lau considers individual demand functions of the form $g^h(x^h, p, a^h)$ for budget x^h, prices p and attributes (e.g. demographics) a^h. His first requirement is that $\Sigma g^h(x^h, p, a^h)$ be symmetric in the H x^h's and a^h's, i.e. be invariant to who has what x and what a. This alone is sufficient to restrict demands to the form

$$g^h(x^h, p, a^h) = g(x^h, p, a^h) + k^h(p), \tag{134}$$

i.e. to be identical up to the addition of a function of prices alone. Lau then derives the conditions under which aggregate demands are a function of not the H x's and a's, but of a smaller set of m indices, $m < H$. Lau shows that

$$\Sigma g^h(x^h, p, a^h) = G\{p, f_1(x, a), f_2(x, a), \dots, f_m(x, a)\}, \tag{135}$$

with $f_i(x, a)$ non-constant symmetric functions of the H-vectors x and a, implies that

$$g^h(x^h, p, a^h) = \sum_{k=1}^{m} h_k(p)\phi_k(x^h, a^h) + k^h(p), \tag{136}$$

Gorman's (1981) theorem, see 3(a) above, tells us what form the ϕ_k functions can take, while Lau's theorem makes Gorman's results the more useful and important. Lau's theorem provides a useful compromise between conventional aggregation as represented by (131) on the one hand and complete agnosticism on the other. Distributional effects on demand are permitted, but in a limited way. Gorman's results tell us that to get these benefits, polynomial specifications are necessary which either link quantities to outlays or shares to the logarithms of outlays. The latter seem to work better in practice and are therefore recommended for use.

Finally, mention must be made of the important recent work of Stoker who, in a series of papers, particularly (1982) (1984), has forged new links between the statistical and economic theories of aggregation. This work goes well beyond demand analysis per se but has implications for the subject. Stoker (1982) shows that the estimated parameters from cross-section regressions will estimate the corresponding macro-effects not only under the Gorman perfect aggregation conditions, but also if the independent variables are jointly distributed within the exponential family of distributions. In the context of demand analysis, the marginal propensity to consume from a cross-section regression would consistently estimate the impact of a change in mean income on mean consumption either with linear Engel curves or with non-linear Engel curves and income distributed according to some exponential family distribution. Since one of the reasons we are interested in aggregation is to be able to move from micro to macro in this way, these results open up new possibilities. Stoker (1984) also carries out the process in reverse and derives completeness (or identification) conditions on the distribution of exogenous variables that allow recovery of micro behavior from macro relationships.

Much of the work reported in this section, by Muellbauer, Lau and Stoker, can be regarded as developing the appropriate techniques of allowing for the impacts of distribution on aggregate demand functions. That such effects could be potentially important has been known for a long time, see de Wolff (1941) for an early contribution. What still seems to be lacking so far is empirical evidence that such effects are actually important.

6. Behavior under quantity constraints

The existence and consequences of quantity constraints on purchases has recently been given much attention in the literature and the question of whether (or how) the labor market clears remains of central importance for much of economic analysis, see Ashenfelter (1980) for a good discussion in which rationing is taken seriously. If empirical studies of consumer behavior are to contribute to this discussion, they must be able to model the effects of quantity rationing on purchases in other markets and be able to test whether or not quantity constraints exist. Perhaps the most famous work on the theory of quantity constraints traces back to Samuelson's (1947) *Foundations* and the enunciation of the Le Chatelier principle by which substitution possibilities in all markets are reduced by the imposition of quantity restrictions in any. These effects were further studied in the later papers of Tobin and Houthakker (1951) and surveyed in Tobin (1952). All the results obtained are essentially local, given the effects on deviations or elasticities of imposition or changes in quantity restrictions. Applied work, however, requires theory which generates functional forms and, for this, global relationships between rationed and unrationed demands are required. In the presentation here, I follow the work of Neary and Roberts (1980) and Deaton (1981b).

The commodity vector q is partitioned into (q^0, q^1) where q^0 may or may not be constrained to take on values z. These may be outside impositions or they may essentially be "chosen" by the consumer. An example of the latter is when a consumer decides not to participate in the labor force; since hours cannot be negative, the commodity demand functions conditional on non-participation are those which arise from a quantity restriction of zero hours worked. The simplest case arises if q^1 forms a separable group, so that without quantity restrictions on q^0, it is possible to write

$$q_i^1 = g_i^1(x - p^0 \cdot q^0, p^1) \tag{137}$$

see eq. (115) above. Clearly, rationing makes no difference to (137) except that z replaces q^0, so that testing for the existence of the quantity restrictions can be carried out by testing for the endogeneity of q^0 using a Wu (1973) or Hausman (1978) test with p^0 as the necessary vector of exogenous instruments not appearing in (137). Without separability matters are more complicated and, in addition to the variables in (137), the demand for q^1 depends on z so that without quantity restrictions

$$q_i^1 = g_i^F(x, p^0, p^1), \tag{138}$$

while, under rationing,

$$q_i^1 = g_i^R(x - p^0 \cdot z, p^1, z). \tag{139}$$

Efficient estimation and testing requires that the relationship between g^F and g^R be fully understood. Once again, the cost function provides the answer. If $c(u, p^0, p^1)$ is the unrestricted cost function, i.e. that which generates (138), the *restricted* cost function $c^*(u, p^0, p^1, z)$ is defined by

$$\begin{aligned} c^*(u, p^0, p^1, z) &= \min\{ p^0\cdot q^0 + p^1\cdot q^1;\ v(q^0, q^1) = u, q^0 = z \} \\ &= p^0\cdot z + \gamma(u, p^1, z), \end{aligned} \tag{140}$$

where γ does not depend upon p^0. Define the "virtual prices", $\tilde{p}^0$, Rothbarth (1941), as a function $\zeta^0(u, p^1, z)$ by the relation

$$\frac{\partial c\{ u, \zeta^0(u, p^1, z), p^1 \}}{\partial p_i^0} = z_i, \tag{141}$$

so that $\tilde{p}^0$ is the vector of prices which at u and p^1 would cause z to be freely chosen. At these prices, restricted and unrestricted costs must be identical, i.e.

$$c(u, \tilde{p}^0, p) = \tilde{p}^0\cdot z + \gamma(u, p^1, z), \tag{142}$$

is an identity in u, p^1 and z with $\tilde{p}^0 = \zeta^0(u, p^1, z)$. Hence, combining (140) and (142)

$$c^*(u, p^0, p^1, z) = (p^0 - \tilde{p}^0)\cdot z + c(u, \tilde{p}^0, p). \tag{143}$$

With $\tilde{p}^0$ determined by (141), this equation is the bridge between restricted and unrestricted cost functions and, since (138) derives from differentiating $c(u, p^0, p)$ and (139) from differentiating $c^*(u, p^0, p^1, z)$, it also gives full knowledge of the relationship between g^F and g^R. This can be put to good theoretical use, to prove all the standard rationing results and a good deal more besides.

For empirical purposes, the ability to derive g^R from g^F allows the construction of a "matched pair" of demand functions, matched in the sense of deriving from the same preferences, and representing both free and constrained behavior. A first attempt, applied to housing expenditure in the U.K., and using the Muellbauer cost function (117) is given in Deaton (1981b). In that study I also found that allowing for quantity restrictions using a restricted cost function related to that for the AIDS, removed much of the conflict with homogeneity on post-war British data. Deaton and Muellbauer (1981) have also derived the matched functional form g^F and g^R for commodity demands for the case where there is quantity rationing in the labor market and where unrestricted labor supply equations take the linear functional forms frequently assumed in the labor supply literature.

7. Other topics

In a review of even this length, only a minute fraction of demand analysis can be covered. However, rather than omit them altogether, I devote this last section to an acknowledgement of the existence of three areas closely linked to the preceeding analysis (and which many would argue are central), intertemporal demand analysis, the analysis of quality, and the use of demand analysis in welfare economics.

7.1. *Intertemporal demand analysis*

Commodity choices over a lifetime can perhaps be modelled using the utility function

$$u = V\{q^1, q^2, \dots, q^\tau, \dots q^L, B/\pi^L\}, \tag{144}$$

where the q^τ represent vectors of commodity demands for period τ, B is bequests at death which occurs with certainty at the end of period L, and π^L is some appropriate price index to be applied to B. Utility is maximized subject to the appropriate constraint, i.e.

$$\sum_1^L \hat{p}^\tau \cdot q^\tau + \hat{\pi}^L(B/\pi^L) = W, \tag{145}$$

where a denotes discounting and W is the discounted present value at 0 of present and future financial assets and either full income, if labor supply is included, or labor income, if labor supply is taken as fixed.

Clearly (144) (145) are together formally identical to the usual model so that the whole apparatus of cost functions, duality, functional forms and so on can be brought into play. However, the problem is nearly always given more structure by assuming (144) to be additively separable between periods so that demand analysis proper applies to the more disaggregated stage of two stage budgeting, while the allocation to broad groups (i.e. of expenditure between the periods) becomes the province of the consumption function, or more strictly, the life-cycle model. The apparatus of Section 4.2 can be brought into play to yield the new standard life-cycle results, see Browning, Deaton and Irish (1985), Hall (1981), Bewley (1977). Even a very short review of this consumption function literature would double the length of this chapter.

The presence of durable goods can also be allowed for by entering stocks at various dates into the intertemporal model (144). Under the assumption of perfect capital markets, constant proportional physical depreciation, and no divergence between buying and selling prices, these stocks can be priced at "user cost" defined by

$$p_t^* = \left[p_t - p_{t+1}(1-\delta)/(1+r_{t+1}) \right], \tag{146}$$

when p_t is the price of the good at time t, δ is the rate of physical depreciation and r_t is the interest rate, see Diewert (1974b) or Deaton and Muellbauer (1980a Chapter 13) for full discussions of this model. If user cost pricing is followed, (although note the expectational element in p_{t+1}), durable goods can be treated like any other good with $p_t^*S_t$ (for stock S_t) as a dependent variable in a demand system, and x_t (including $p_t^*S_t$ *not* the purchase of durables) and all prices and user costs as independent variables. The model is a very useful benchmark, but its assumptions are more than usually unrealistic and it is not surprising that it appears to be rejected in favour of alternative specifications, see Muellbauer (1981a). However, no fully satisfactory alternative formulation exists, and the literature contains a large number of quite distinct approaches. In many of these, commodity demands are modelled conditional on the stocks which, in turn, evolve with purchases, so that dynamic formulations are created in which long-run and short-run responses are distinct. The stock-adjustment models of Stone and Rowe (1957) (1958) and Chow (1957) (1960) are of this form, as is the very similar "state" adjustment model of Houthakker and Taylor (1966) who extend the formulation to all goods while extending the concept of stocks to include "stocks" of habits (since in these models, stocks are substituted out, it makes little difference what name is attached to them). There are also more sophisticated models in which utility functions are defined over instantaneous purchases and stocks, e.g. Phlips' (1972) "dynamic" linear expenditure system, and further refinements in which intertemporal functions are used to model the effects of current purchases on future welfare via their effects on future stocks, Phlips (1974, 1983 Part II). These models are extremely complicated to estimate and it is not clear that they capture any essential features not contained in the stock-adjustment model, on the one hand, and the user cost model on the other, see in particular the results of Spinnewyn (1979a) (1979b). It remains for future work to tackle the very considerable task of constructing models which can deal, in manageable form, with the problems posed by the existence of informational asymmetries [lemons, Akerlof (1970)], borrowing constraints, indivisibilities, technological diffusion, and so on.

7.2. *Choice of qualities*

The characteristics model of consumer behavior is a natural way of analysing choice of qualities and, indeed, Gorman's (1956, 1980) classic paper is concerned with quality differentials in the Iowa egg market. By specifying a technology linking quality with market goods, the model naturally leads to the characterization of shadow prices for qualities and these have played a central role in the "new household economics", see in particular, Becker (1976). A related but more direct method of dealing with quality was pioneered in the work of Fisher and Shell (1971), see also Muellbauer (1975a) and Gorman (1976) for reformulations and extensions. The model is formally identical to the Barten model of household composition discussed in Section 3 above with the m's now interpreted as quality parameters "augmenting" the quantities in consumption. Under either formulation, competition between goods manufacturers will, under appropriate assumptions, induce a direct relationship between the price of each good (or variety) and an index of its quality attributes. These relationships are estimated by means of "hedonic" regressions in which (usually the logarithm of) price is regressed on physical attributes across different market goods, see e.g. Burstein (1961) and Dhrymes (1971) for studies of refrigerator prices, and Ohta and Griliches (1976), Cowling and Cubbin (1971) (1972), Cubbin (1975) and Deaton and Muellbauer (1980a p. 263–5) for results on car prices. These techniques date back to Griliches (1961) and ultimately to Court (1939). Choice among discrete varieties involves many closely related techniques, see Chapter 24 of this handbook.

Empirical studies of consumer demand for housing are a major area where quality differences are of great importance. However, until recently, much of the housing literature has consisted of two types of study, one regressing quantities of housing services against income and some index of housing prices, either individual or by locality, while the other follows the hedonic approach, regressing prices on the quantities of various attributes, e.g. number of rooms, size, presence of and type of heating, distance from transport, shops and so on. Serious attempts are currently being made to integrate these two approaches and this is a lively field with excellent data, immediate policy implications, and some first-rate work being done. Lack of space prevents my discussing it in detail; for a survey and further references see Mayo (1978).

7.3. Demand analysis and welfare economics

A large proportion of the results and formulae of welfare economics, from cost benefit analysis to optimal tax theory, depend for their implementation on the results of empirical demand analysis, particularly on estimates of substitution responses. Since the coherence of welfare theory depends on the validity of the standard model of behavior, the usefulness of applied demand work in this context depends crucially on the eventual solution of the problems with homogeneity (possible symmetry) and global regularity discussed in Section 2 above. But even without such difficulties, the relationship between the econometric estimates and their welfare application is not always clearly appreciated. In consequence, I review briefly here the estimation of three welfare measures, namely consumer surplus, cost-of-living indices, and equivalence scales.

I argued in Section 1 that it was convenient to regard the cost function as the centrepiece of applied demand analysis. It is even more convenient to do so in welfare analysis. Taking *consumer surplus* first, the compensating variation (CV) and equivalent variation (EV) are defined by, respectively,

$$\mathrm{CV} = c(u^0, p^1) - c(u^0, p^0), \tag{147}$$

$$\mathrm{EV} = c(u^1, p^1) - c(u^1, p^0), \tag{148}$$

so that both measure the money costs of a welfare affecting price change from p^0 to p^1, CV using u^0 as reference (compensation returns the consumer to the original welfare level) and EV using u^1 (it is equivalent to the change to u^1). Base and current reference true *cost-of-living index numbers* are defined analogously using ratios instead of differences, hence

$$P(p^1, p^0; u^0) = c(u^0, p^1)/c(u^0, p^0), \tag{149}$$

$$P(p^1, p^0; u^1) = c(u^1, p^1)/c(u^1, p^0), \tag{150}$$

are the base and current true indices. Note the CV, EV and the two price indices depend in no way on how utility is measured; they depend only on the indifference curve indexed by u, which could equally well be replaced by $\phi(u)$ for any monotone increasing ϕ. Even so, the cost function is not observed directly and a procedure must be prescribed for constructing it from the (in principle) observable Marshallian demand functions. If the functional forms for these are known, and if homogeneity, symmetry and negativity are satisfied, the cost function can be obtained by solving the partial differential equations (12), often analytically, see e.g. Hausman (1981). Unobserved constants of integration affect only the measurability of u so that complete knowledge of the Marshallian demands is equivalent to complete knowledge of consumer surplus and the index numbers. If analytical integration is impossible or difficult, numerical integration is straightforward (provided homogeneity and symmetry hold) and algorithms exist in the literature, see e.g. Samuelson (1948) and in much more detail, Vartia (1983). If the integrability conditions fail, consumer behavior is not according to the theory and it is not sensible to try to calculate the welfare indices in the first place, nor is it possible to do so. Geometrically, calculating CV or EV is simply a matter of integrating the area under a *Hicksian* demand curve; there is no valid theoretical or practical reason for ever integrating under a *Marshallian* demand curve. The very considerable literature discussing the practical difficulties of doing so (the path-dependence of the integral, for example) provides a remarkable example of the elaboration of secondary nonsense which can occur once a large primary category error has been accepted; the emperor with no clothes, although quite unaware of his total nakedness, is continuously distressed by his inability to tie his shoelaces. A much more real problem is the assumption that the functional forms of the Marshallian demands are known, so that working with a specific model inevitably understates the margin of ignorance about consumer surplus or index numbers. The tools of non-parametric demand analysis, as discussed in Section 2.7, can, however, be brought to bear to give bounding relationships on the cost function and hence on the welfare measures themselves, see Varian (1982b).

The construction of empirical scales is similar to the construction of price indices although there are a few special difficulties. For household characteristics a^h, the equivalence scale $M(a^h, a^0;\ u, p)$ is defined by

$$M(a^h, a^0, u, p) = c(u, p, a^h)/c(u, p, a^0), \tag{151}$$

for reference household characteristics a^0 and suitably chosen reference welfare level u and price vector p. Models such as those discussed in Section 3.2 yield estimates of the parameters of $c(u, p, a)$ so that scales can be evaluated. However, the situation is not quite the same as for the price indices (149) and (150). For these, $c(u, p)$ only is required and this is identified by the functional forms for its tangents $h_i(u, p) = g_i\{c(u, p), p\}$. But for $c(u, p, a)$, we observe only the p-tangents together with their derivatives with respect to a, i.e. $\partial q_i/\partial a_j$, the demographic effects on demand, and this information is insufficient to identify the function. In particular, as emphasized by Pollak and Wales (1979), the cost functions $c(\phi(u, a), p, a)$ and $c(u, p, a)$ have identical behavioral consequences if $\partial\phi/\partial u > 0$ while giving quite different equivalence scales. Since $c(u, p, a)$ is formally identical to the restricted cost function discussed in Section 6 above, its derivatives with respect to a can be interpreted as shadow prices [differentiate eq. (143)]. These could conceivably be measured from "economic" studies of fertility, in which case the equivalence scale would be fully identified just as are the price indices from $c(u, p)$. Failing such evidence, it is necessary to be very explicit about exactly what prior information is being used to identify the scales. In Deaton and Muellbauer (1981), the identification issue is discussed in detail and it is shown that the same empirical evidence yields systematically different scales for different models, e.g. those of Engel, Barten and Rothbarth discussed in 3.2. It is also argued that plausible identification assumptions can be made, so that demand analysis may, after all, have something to say about the economic costs of children.

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