GEOMETRIC DESCRIPTION OF LIFTING MONOMIAL IDEALS#

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By focusing our attention on the set of monomials outside a given monomial ideal, we tackle the study of the geometric configurations of (reduced) unions of projective linear varieties arising from lifting monomial ideals via a classic lifting procedure, called t-lifting, and a more general lifting procedure, called pseudo-t-lifting. We observe that, in contrast to the Artinian case, in the positive dimensional case we may not obtain generalized stick figures also via a generic pseudo-t-lifting. In particular, in dimension 1 a generic pseudo-1-lifting produces a seminormal union of lines. Then we give conditions to obtain generalized stick figures by means of pseudo-t-liftings of non-Artinian monomial ideals.

INTRODUCTION

In this paper we study the geometric configurations of (reduced) unions of projective linear varieties that are obtained by lifting monomial ideals. In general, one is interested in finding “nice” projective schemes with given Hilbert functions and minimal free resolutions. Since monomial ideals, suitable and easy to handle, do not define significant schemes, the lifting procedure was introduced in order to obtain configurations of projective linear varieties, reflecting algebraic and geometric properties of the given monomial ideals.

To place the question, we first recall that Macaulay (1927) lifted an Artinian lex-segment ideal in order to exhibit a set of points with a given Hilbert function. We also recall that Hartshorne (1966) proved the connectedness of the Hilbert scheme by using monomial ideals, in particular Borel ideals (balanced ideals),

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1483
liftings (distractions) of monomial ideals, and unions of linear varieties (fans). Hartshorne’s results are exposed in modern and clear language in the more recent article of Reeves (1995).

Migliore and Nagel (2000, 2002) studied further properties concerning the process of lifting monomial ideals. They considered a classic lifting procedure, called $t$-lifting, where $t$ is the number of the added variables, and a more general lifting procedure called pseudo-lifting. Among many other results they proved in particular that a monomial ideal $J$ and a pseudo-lifting $I$ of $J$ have the same graded Betti numbers and the same Hilbert functions. Moreover, $I$ is saturated and $I$ defines an arithmetically Cohen–Macaulay scheme if and only if $J$ does so. More recently, Bigatti et al. (preprint), considering a similar procedure (called distraction), gave different proofs of these facts.

In this paper we describe some geometric properties of the configurations of linear varieties arising from $t$-liftings and pseudo-liftings of monomial ideals, and we correct some assertions of Theorem 3.4 of Migliore and Nagel (2000). We obtain our results by focusing our attention on the set of monomials outside the given monomial ideal, following in this Macaulay (1927) and pursuing the results obtained by Marinari and Ramella (2002).

In Section 1 we recall some definitions and known results about a pseudo-$t$-lifting $I$ of a monomial ideal $J \subseteq \mathbb{K}[x_1, \ldots, x_n]$ induced by a matrix $A$ of linear forms. Furthermore, we illustrate conditions on the matrix $A$ so that the pseudo-$t$-lifting $I$ is a reduced ideal with the same height as $J$. In this case we can describe the minimal primes of $I$ by means of the minimal generators of $J$ (Proposition 1.14).

In Section 2 we give a geometric description of a monomial ideal $J$, pointing out a relation between the irreducible components of the corresponding order ideal $\mathcal{N}(J) \subseteq \mathbb{N}^n \subseteq \mathbb{A}^n_\mathbb{K}$ and the irreducible components of the projective scheme $V$ defined by a pseudo-$t$-lifting $I$ of $J$ (Proposition 2.5). So we can study in depth the above relation when $J$ is Artinian and in particular when $J$ is a lex-segment or strongly stable (Theorem 2.7), by generalizing a result of Migliore and Nagel (2000).

In Sections 3 and 4 we study under which conditions $V$ satisfies the geometric properties of some good configurations of linear varieties, considering separately the case in which $J$ is Artinian and the case in which $J$ is non-Artinian.

Notice that the sets of points obtained by pseudo-$t$-liftings of Artinian lex-segment ideals adding one single variable are a special type of the so-called $k$-configurations (see Geramita et al., 2000, 2001).

In Section 3 we observe that if $J$ is an Artinian monomial ideal and $V$ is the projective scheme defined by either a $t$-lifting of $J$ satisfying certain conditions of genericity or by a generic pseudo-$t$-lifting of $J$ with $\dim(V) \geq 1$, then $V$ is a generalized stick figure, i.e., $V$ is reduced and equidimensional, and the intersection of any three components of $V$ has dimension at most $\dim(V) - 2$ (Theorems 3.3 and 3.4). Moreover, we exhibit reduced $t$-liftings of Artinian ideals that are not generalized stick figures (Remark 3.2), so we furnish counterexamples to Theorem 3.4 (c) of Migliore and Nagel (2000).

For the non-Artinian case, in Section 4 we give monomial ideals $J$ such that every pseudo-$t$-lifting of $J$ is not a generalized stick figure (Examples 4.1 and 4.2, Proposition 4.10); thus we point out that the assertions (b) and (d) of Theorem 3.4 of Migliore and Nagel (2000) do not hold.
Moreover, we give conditions for obtaining generalized stick figures by means of pseudo-\(t\)-liftings of non-Artinian monomial ideals, based on the shape of the set of monomials outside the given monomial ideals (Proposition 4.10(b)).

Furthermore, given an unmixed monomial ideal \(J\) with \(\dim K[x_1, \ldots, x_n]/J = d \geq 1(n \leq 2)\) and a generic pseudo-\(t\)-lifting \(V\) of \(J\), the intersection of any \(n-d+2\) components of \(V\) has dimension at most \(\dim(V) - 2\) (Proposition 4.3). We give a large class of examples attaining this estimate (Example 4.4). In particular, if \(d = 1\) and \(V\) is a generic pseudo-\(t\)-lifting of \(J\) obtained by adding a single variable, we show that \(V\) is a seminormal union of lines, i.e., at every intersection point the lines through that point have linearly independent directions (Corollary 4.7).

1. NOTATION AND PRELIMINARIES

In this section we recall and illustrate same definitions and results given in Migliore and Nagel (2000).

**Definition 1.1.** Let \(R\) be any (commutative) ring in which \(u_1, \ldots, u_t \in R\) form an \(R\)-regular sequence, and let \(S := R/(u_1, \ldots, u_n)\). Let \(B\) be an \(S\)-module and \(C\) an \(R\)-module. If \(u_1, \ldots, u_t\) is a \(C\)-regular sequence and \(C/(u_1, \ldots, u_t)C \cong B\), then \(C\) is said to be a \(t\)-lifting of \(B\) to \(R\). If \(t = 1\), then \(C\) is just called a lifting of \(B\).

From now on \(K\) will be an infinite field and \(x_1, \ldots, x_n, u_1, \ldots, u_t\) will be indeterminates, \(n \in \mathbb{N}^*\), \(t \in \mathbb{N}\). In particular, we let \(R := K[x_1, \ldots, x_n, u_1, \ldots, u_t]\) and \(S := K[x_1, \ldots, x_n]\). Moreover, unless contrarily specified, \(A = (L_{j,i})_{1 \leq j \leq n, i \in \mathbb{N}}\) will denote a matrix whose entries \(L_{j,i}\) are linear forms in \(R\). Then, via \(A\), we can associate

To every monomial \(m = x_1^{z_1} \cdots x_n^{z_n} \in S\) the homogeneous polynomial

\[
\tilde{m} = \prod_{j=1}^{n} \left( \prod_{i=1}^{z_j} L_{j,i} \right) \in R;
\]

To each monomial ideal \(J = (m_1, \ldots, m_s) \subset S\) (note that from now on \(\{m_1, \ldots, m_s\}\) is the minimal system of monic monomial-generators) the homogeneous ideal

\[
I = (\tilde{m}_1, \ldots, \tilde{m}_s) \subset R.
\]

In Migliore and Nagel (2000), under the assumption

\[
(*) \quad \forall j \in \{1, \ldots, n\}, \forall i \in \mathbb{N}^*, \text{ the linear forms } L_{j,i}, \text{ actually belong to } K[x_j, u_1, \ldots, u_t] \subset R, \text{ and } x_j \text{ has a nonzero coefficient.}
\]

Note the above ideal \(I\) is a \(t\)-lifting of \(J\). So a matrix \(A\) fulfilling the assumption \((*)\) is called a \(t\)-lifting matrix, the homogeneous polynomial \(\tilde{m} \in R\) is called a \(t\)-lifting of \(m \in S\) by \(A\), and the homogeneous ideal \(I \subset R\) is called a \(t\)-lifting of \(J\) induced by \(A\).
Example 1.2. In $S = K[x_1, x_2, x_3]$ consider the monomial ideal $J = (x_1 x_2^2, x_2^3 x_3, x_1 x_2 x_3, x_1 x_2 x_3, x_2^3 x_2)$. Let $A = (L_{ji})$ be a $t$-lifting matrix. Then the $t$-lifting of $J$ induced by $A$ is the homogeneous ideal $I = (L_{1,1} L_{3,1} L_{3,2} L_{2,1} L_{2,2} L_{3,1}, L_{1,1} L_{2,1} L_{3,1}, L_{1,1} L_{1,2} L_{2,1})$.

As noted by Migliore and Nagel (2000), p. 5682 and p. 5691, only finitely many $L_{ji}$ need be chosen as entries of a $t$-lifting matrix $A$. In fact, for any given monomial ideal $J \subset S$, let $N_j$, $1 \leq j \leq n$, be the largest power of $x_j$ occurring as a factor of any of the minimal generators of $J$, and let $N := \max\{N_1, \ldots, N_n\}$. Then the given $J$ can be $t$-lifted using a matrix $A$ (consisting of linear forms $L_{ji} \in K[x_j, u_1, \ldots, u_t]$ verifying $(\star)$) of size $n \times r$, with $r \geq N$.

In order to generalize $t$-liftings of monomial ideals induced by matrices with entries $L_{ji}$ in $K[x_j, u_1, \ldots, u_t] \subset R$, in Migliore and Nagel (2000) matrices with entries $L_{ji}$ in the whole ring $R$ and satisfying the following conditions are considered.

Condition (z). The polynomials $F_j = \prod_{i=1}^m L_{ji}, 1 \leq j \leq n$, define $F = (F_1, \ldots, F_n) \subset R$ a complete intersection of height $n$ (Migliore and Nagel, 2000, Remark 2.21).

Condition (z'). The polynomials $F_j = \prod_{i=1}^m L_{ji}, 1 \leq j \leq n$, define $F = (F_1, \ldots, F_n) \subset R$ a reduced complete intersection of height $n$ (Migliore and Nagel, 2000, introduction of Section 4).

Condition (β). The entries of the matrix $A$ are generic linear forms in $R$, i.e., for any choice of $k$ entries of $A$, say $L_{1,1}, \ldots, L_{k,1} \in R$ from $A$, we have $\dim_K \langle L_{1,1}, \ldots, L_{k,1} \rangle = \min\{n + t, k\}$ (Migliore and Nagel, 2000, Remark 3.2).

Definition 1.3. Let $J = (m_1, \ldots, m_r) \subset S$ be a monomial ideal. Let $A$ be an $n \times r$ matrix of linear forms $L_{ji} \in R$ satisfying Condition (z). Let $I = (\tilde{m}_1, \ldots, \tilde{m}_s) \subset R$ be the homogeneous ideal obtained via $A$ from the given $J$ as in the case of an $t$-lifting.

Then $I$ is called a pseudo-$t$-lifting of $J$, and $A$ is called a pseudo-$t$-lifting matrix (Migliore and Nagel, 2000, Definition 2.22).

Remark 1.4.

(a) A $t$-lifting matrix is also a pseudo-$t$-lifting matrix.
(b) The complete intersection $F$ involved in Condition (z) is the pseudo-$t$-lifting of the Artinian ideal $J = (x_1', \ldots, x_r')$.
(c) A pseudo-$t$-lifting with Condition (β) is called “generic” (Migliore and Nagel, 2000, Remark 3.2).

Example 1.5. Let $J$ be the ideal $(x_1^2, x_2^3)$ in $S = K[x_1, x_2]$ and $A$ the pseudo-$2$-lifting matrix

$$A = \begin{pmatrix} x_1 + u_1 & x_2 + 2u_1 \\ x_1 - x_2 + 3u_1 & x_1 + x_2 + 4u_2 \end{pmatrix}.$$ 

Then the pseudo-$2$-lifting of $J$ is the ideal $I = ((x_1 + u_1)(x_2 + 2u_1), (x_1 - x_2 + 3u_1) (x_1 + x_2 + 4u_2)) \subset K[x_1, x_2, u_1, u_2]$. Note that $I$ is not a $2$-lifting of $J$, namely $(I, u_1, u_2)/(u_1, u_2) = (x_1 x_2, x_1^2 - x_2^2) \not\subset J$. 

Equation 4.1 of Migliore and Nagel (2000) obtained for Artinian monomial ideals $J$ and reduced pseudo-$t$-lifting $I$ of $J$ can be generalized to any monomial ideal $J$ and pseudo-$t$-lifting $I$ of $J$ in the following way (see also Proposition 1.14).

**Lemma 1.6.** Let $J = (m_1, \ldots, m_s) \subset S$ be a monomial ideal, $A = (L_{i,j})$ an $n \times r$ pseudo-$t$-lifting matrix of linear forms of $R$, and $I = (\tilde{m}_1, \ldots, \tilde{m}_s)$ the homogeneous ideal obtained via $A$ from $J$. Then $\sqrt{I}$ is the intersection of the ideals $(\ell_{i,j,1}, \ldots, \ell_{i,j,r})$, where $1 \leq k \leq n$, and for every generator $m_\ell$ of $J$ there exists $h \in \{1, \ldots, k\}$ such that $x_{j_h}^{m_\ell}$ divides $m_\ell$ and for every $h \in \{1, \ldots, k\}$ there exists a generator $m_\ell$ of $J$ such that $x_{j_h}^{m_\ell}$ divides $m_\ell$.

**Proof.** Every $(\ell_{j_1,i_1}, \ldots, \ell_{j_k,i_k})$ is a prime ideal, since each $\ell_{j,i} \in R$ is a linear form, and contains $I$ by the definition of $\tilde{m}_\ell$, $\ell \in \{1, \ldots, s\}$. Moreover, $R$ being a unique factorization domain, every prime ideal containing $I$ must contain an ideal of the considered type. Finally, as $\sqrt{I}$ is the intersection of all the prime ideals containing $I$, our contention is proved. \qed

The following proposition points out the meaning of Conditions $(x)$ and $(x')$.

**Proposition 1.7.** Let $A = (L_{i,j})$ be an $n \times r$ matrix of linear forms of $R$ and $F = (F_1, \ldots, F_n)$, where $F_j = \prod_{i=1}^{r} L_{j,i}$, for every $1 \leq j \leq n$. Then

(a) $\sqrt{F} = \bigcap_{1 \leq i_1 \leq r} (L_{1,i_1}, \ldots, L_{n,i_n})$.

(b) $F$ is a complete intersection of height $n$ if and only if for every $1 \leq i_1, \ldots, i_n \leq r$ the $n$ linear forms $L_{1,i_1}, \ldots, L_{n,i_n}$ are linearly independent.

(c) $F$ is a reduced complete intersection of height $n$ if and only if for every $1 \leq i_1, \ldots, i_n \leq r$ the vector spaces $\langle L_{1,i_1}, \ldots, L_{n,i_n} \rangle$ are $n$-dimensional and pairwise distinct.

**Proof.** (a) The first assertion follows by Lemma 1.6.

(b) If $F$ defines a complete intersection of height $n$, then in particular $R/F$ is a Cohen–Macaulay ring and so it is height unmixed. Thus the ideals $(L_{1,i_1}, \ldots, L_{n,i_n})$ are of height $n$, i.e., the linear forms $L_{1,i_1}, \ldots, L_{n,i_n}$ are linearly independent. Vice versa, it is enough to prove that $F_1, \ldots, F_n$ is a regular sequence. For $n = 1$, $F_1$ is a non-zero-divisor of $R$. Suppose the assertion true for $n - 1$. If $F_n$ were a zero-divisor on $R/(F_1, \ldots, F_{n-1})$ then $F_n$ would belong to a prime $P = (L_{1,i_1}, \ldots, L_{n-1,i_{n-1}})$ and so far some index $i_n$ the linear form $L_{n,i_n}$ would belong to $P$, which is impossible by assumption.

(c) Assume that the vector spaces $\langle L_{1,i_1}, \ldots, L_{n,i_n} \rangle$ are $n$-dimensional and pairwise distinct. By (b) we must show that $F = \sqrt{F}$.

Let $\mathfrak{a}$ be a primary ideal associated with $F$. As $\sqrt{\mathfrak{a}} = (L_{1,i_1}, \ldots, L_{n,i_n})$ for some $n$-tuple $(\tilde{i}_1, \ldots, \tilde{i}_n) \in \{1, \ldots, r\}^n$, it is enough to show that actually $\mathfrak{a} = (L_{1,i_1}, \ldots, L_{n,i_n})$. Assume that $L_{1,i_1} \notin \mathfrak{a}$. Since $F_1 = \prod_{i=1}^{r} L_{1,i} \in \mathfrak{a}$, then there exists some $s \in \mathbb{N}$ such that $(L_{1,i_1} \cdot \ldots \cdot L_{1,i_s})^s \in \mathfrak{a} \subset (L_{1,i_1}, \ldots, L_{n,i_n})$. So there exists $i \neq \tilde{i}_1$ such that $L_{1,i} \in (L_{1,i_1}, \ldots, L_{n,i_n})$. This contradicts the assumption that
the vector spaces \( \langle L_{1,1}, \ldots, L_{n,n} \rangle \) are \( n \)-dimensional and pairwise distinct. Thus
\[ L_{1,1} \in \mathfrak{a}. \]
By repeating the same reasoning for \( L_{2,2}, \ldots, L_{n,1} \), we have that \( a = (L_{1,1}, \ldots, L_{n,n}) \) as claimed. Hence, \( F = \sqrt{F} \).

Now suppose that \( F \) is a complete intersection of height \( n \), but two vector spaces, for example \( \langle L_{1,1}, \ldots, L_{n,n} \rangle \) and \( \langle L_{1,2}, \ldots, L_{n,2} \rangle \), are equal. Put \( L_{n,2} = \sum_{j=1}^{n} a_j L_{j,j} \). Note that \( a_n \neq 0 \), since \( F \) is a complete intersection. Thus \( F = (L_{1,1} L_{1,2} G_1, \ldots, L_{n-1,1} L_{n-1,2} G_{n-1}) \), where \( G_j = L_{j,1} \cdots L_{j,r} \). So \( (L_{1,1}, \ldots, L_{n-1,1}, L_{n,1}) \), with \( s \geq 2 \), is a primary ideal associated with \( F \), and \( F \) is not radical. \( \square \)

**Remark 1.8.** Over an infinite field every pseudo-\( t \)-lifting matrix (and, by Remark 1.5, also every \( t \)-lifting matrix) can be completed to a distraction matrix, in accordance with the definition of distraction matrix given in Bigatti et al. (preprint), and the ideals obtained via both constructions coincide.

**Remark 1.9.** (a) While a \( t \)-lifting matrix satisfies Condition \((\alpha)\), it may not verify Condition \((\alpha')\). For instance, if \( J = (x_1^2, x_2^2) \) in \( S = K[x_1, x_2] \) and \( A = (L_{j,i}) \) is a \( 2 \times 2 \) matrix of linear forms of \( R \) such that \( L_{1,2} \) is a multiple of \( L_{1,1} \), then \( I = (F_1, F_2) = (L_{1,1}^2, L_{2,1}, L_{2,2}) \) is not reduced. So we can obtain a nonreduced \( t \)-lifting, and we point out that Theorem 3.4 (a) of Migliore and Nagel (2000) does not hold without Condition \((\alpha')\).

(b) By Proposition 1.7, we have that a \( t \)-lifting matrix \( A = (L_{j,i}) \) satisfies Condition \((\alpha')\) if and only if for every \( 1 \leq j \leq n \) no \( L_{j,i} \) is a scalar multiple of an \( L_{j,k}, i \neq k \) (see Migliore and Nagel, 2000, end of p. 5682).

(c) Note that if \( r > t + 1 \) then Condition \((\beta)\) does not make sense for a \( t \)-lifting matrix. Consider the following example: let \( J = (x_1^4, x_2^4) \subset K[x_1, x_2] \) and let \( A \) be a \( 2 \)-lifting matrix; we have that \( \dim_K \langle L_{1,1}, L_{1,2}, L_{1,3}, L_{1,4} \rangle \leq 3 \), which is different from \( \min\{n + t, k\} = 4 \). However, Condition \((\beta)\) makes sense if the linear forms \( L_{j,i} \) are taken in the whole ring \( R \).

**Proposition 1.10.** If \( A = (L_{j,i}) \) is an \( n \times r \) matrix of linear forms of \( R \) such that Condition \((\beta)\) holds, then Condition \((\alpha)\) is satisfied, and moreover if \( t \geq 1 \), also Condition \((\alpha')\) is satisfied.

**Proof.** For any \( n \) entries \( L_1, \ldots, L_n \) of the matrix \( A \), we have \( \dim_K \langle L_1, \ldots, L_n \rangle = n \), i.e., the linear forms \( L_1, \ldots, L_n \) are linearly independent; so by Proposition 1.7 (b), \( F \) is a complete intersection.

If \( t \geq 1 \), any \( n + 1 \) entries of the matrix \( A \) are linearly independent, so two different \( n \)-tuples generate different vector spaces. So by Proposition 1.7(c), \( F \) is a reduced complete intersection. \( \square \)

**Remark 1.11.** The pseudo-\( t \)-lifting of a monomial ideal \( j = Q_j \cap \cdots \cap Q_s \) is \( I = \overline{Q_j} \cap \cdots \cap \overline{Q_s} \), where \( \overline{Q_j} \) is the pseudo-\( t \)-lifting of \( Q_j \) (see Migliore and Nagel, 2000; Bigatti et al., preprint). So, since a monomial ideal \( J \) is an intersection of ideals of type \( (x_{j_1}, \ldots, x_{j_h}) \), a pseudo-\( t \)-lifting \( I \) is the intersection of pseudo-\( t \)-liftings of complete intersections \( (x_{j_1}', \ldots, x_{j_h}') \).
We want to describe in an explicit way the minimal primes associated with a (reduced) pseudo-$t$-lifting $I$ of any monomial ideal $J$, generalizing again Equation 4.1 of Migliore and Nagel (2000) obtained for Artinian monomial ideals. The family of prime ideals described in Lemma 1.6 is too wide, as the following example shows.

**Example 1.12.** Let $J = (x_1, x_2^3) \subseteq K[x_1, x_2]$, $R = K[x_1, x_2, u_1, \ldots, u_t]$, $t > 0$, and let $A = (L_{j,i})$ be a $2 \times 2$ matrix of linear forms of $R$. Then $I = (L_{1,1}, L_{2,1}, L_{1,1}L_{1,2})$. The prime ideals arising from Lemma 1.6 are $(L_{1,1}), (L_{1,1}, L_{1,2}), (L_{2,1}, L_{1,1})$, and $(L_{2,1}, L_{1,2})$, where $(L_{1,1}, L_{1,2})$ and $(L_{2,1}, L_{1,1})$ are superfluous.

Inspired by the way of determining a minimal primary decomposition of a monomial ideal, we introduce the family $\mathcal{P}(I)$ of prime ideals containing $I$, defined as

**Definition 1.13.** Let $J = (m_1, \ldots, m_s) \subseteq S$ be a monomial ideal, $A = (L_{j,i})$ an $n \times r$ matrix of linear forms of $R$, and $I = (\tilde{m}_1, \ldots, \tilde{m}_t) \subseteq R$ the pseudo-$t$-lifting of $J$ induced by $A$. A prime ideal $(L_{j_1,i_1}, \ldots, L_{j_k,i_k}) \subseteq R$ belongs to $\mathcal{P}(I)$ if

(i) $j_1, \ldots, j_k, 1 \leq k \leq n$, are distinct.

(ii) For every generator $m_\ell$ of $J$ there exists $h \in \{1, \ldots, k\}$ such that $x_{j_h}^x$ divides $m_\ell$, and for every $h \in \{1, \ldots, k\}$ there exists a generator $m_\ell$ of $J$ such that $x_{j_h}^x$ divides $m_\ell$.

(iii) There exists $k$ (distinct) generators of $J$ of type $m_\ell_h = x_{j_h}^x \mu_h$, $h \in \{1, \ldots, k\}$, such that $\mu_h$ is a monomial nondivisible by $x_{j_p}^x$ for every $p \in \{1, \ldots, k\}, p \neq h$.

**Proposition 1.14.** Let $J = (m_1, \ldots, m_s) \subseteq S$ be a monomial ideal, $A = (L_{j,i})$ an $n \times r$ matrix of linear forms of $R$, and $I = (\tilde{m}_1, \ldots, \tilde{m}_t) \subseteq R$ the ideal obtained via $A$.

Then Condition (z) implies $ht(I) = ht(J)$ (Migliore and Nagel, 2000, Lemma 2.9). Furthermore Condition (z') implies $\sqrt{I} = I$ and in this case the family $\mathcal{P}(I)$ of Definition 1.13 corresponds to the family of the minimal primes of $I$.

**Proof.** The family $\mathcal{P}(I)$ is contained in the family of primes given by the Lemma 1.6. Conditions (i) and (iii) of Definition 1.13 are introduced in order to get rid of the redundant prime ideals furnished by Lemma 1.6. Thus $\sqrt{I}$ is the intersection of the prime ideals of $\mathcal{P}(I)$.

The remaining part of the proposition can be proved by using Proposition 1.7, since $I$ is an intersection of pseudo-$t$-liftings of complete intersection monomial ideals of type $(x_{j_1}^x, \ldots, x_{j_k}^x)$, as pointed out in Remark 1.11. $\square$

From now on let $J \subseteq S$ be a monomial ideal and $I \subseteq R$ be the ideal obtained via a pseudo-$t$-lifting matrix. Let $W = V(J) \subseteq \mathbb{P}^{n-1}$ and $W_a = V_a(J) \subseteq \mathbb{A}^n$ be the schemes defined by $J$; and let $V = V(I) \subseteq \mathbb{P}^{n+t-1}$ be the scheme defined by $I$. We note explicitly that the irreducible components of $V$ are linear varieties. Let $\text{codim}(W)$, $\text{codim}(W_a)$, and $\text{codim}(V)$ be, respectively, the codimension of $W$ in $\mathbb{P}^{n-1}$, the codimension of $W_a$ in $\mathbb{A}^n$, and the codimension of $V$ in $\mathbb{P}^{n+t-1}$. If $S/J$ is an Artinian ring, then $W$ is empty and we set $\text{codim}(W) := \text{codim}(W_a) = n$. By Proposition 1.14 we have $\text{codim}(V) = \text{codim}(W)$. Moreover, if Condition (z') holds, then $V$ is reduced.
Since a monomial ideal \( J \) is intersection of ideals of type \( (x_1^i, \ldots, x_n^i) \), the irreducible components of \( V \) are always components of a pseudo-\( t \)-lifting of an ideal of type \( (x_1^r, \ldots, x_n^r) \). Note that if \( J \) is Artinian, then \( V \) is contained in the scheme defined by the pseudo-\( t \)-lifting of \( (x_1^1, \ldots, x_n^1) \).

### 2. ORDER IDEALS AND PSEUDO-\( t \)-LIFTINGS

We begin this section by recalling some definitions concerning monomial ideals.

**Definition 2.1.** Let \( > \) denote lex term ordering on the set of terms (= monic monomials) of \( S \), for which \( x_1^{a_1} \cdots x_n^{a_n} > x_1^{b_1} \cdots x_n^{b_n} \) if the first nonzero coordinate of \((a_1 - b_1, \ldots, a_n - b_n)\) is positive.

(a) A monomial ideal \( J \subset S \) is strongly stable (or Borel in case \( \text{char} \ K = 0 \)) if \( m = x_1^{a_1} \cdots x_i^{a_i} \cdots x_n^{a_n} \in J \) implies \( m' = x_1^{a_1} \cdots x_i^{a_i+1} \cdots x_n^{a_n} \in J \) for all \( i > j \) (notice that, with respect to the given lex term ordering, we have \( m' > m \)).

(b) A monomial ideal \( J \subset S \) is a lex-segment ideal if for all \( m, m' \in S \), with \( \deg(m) = \deg(m') \), \( m' > m \), and \( m \in J \), \( m' \in J \) holds.

**Definition 2.2.** (1) A set \( N \subseteq \mathbb{N}^n \) is an order ideal if and only if \((a_1, \ldots, a_n) \in N \) and \((b_1, \ldots, b_n) \in \mathbb{N}^n \) with \( b_j \leq a_j \) for every \( 1 \leq j \leq n \) implies \((b_1, \ldots, b_n) \in N \).

(2) A set \( J \subset \mathbb{N}^n \) is a semigroup ideal if and only if for every \((a_1, \ldots, a_n) \in J \) then \((b_1, \ldots, b_n) \in J \) whenever \( b_i \geq a_i \) for every \( 1 \leq i \leq n \).

Substituting to the multiplicative semigroup of terms of \( S \) the additive semigroup \( \mathbb{N}^n \), via the identification of the term \( x_1^{a_1} \cdots x_n^{a_n} \in S \) with the point \((a_1, \ldots, a_n) \in \mathbb{N}^n \), to a monomial ideal \( J \subset S \) corresponds the semigroup ideal \( J(J) \subset \mathbb{N}^n \) generated by the set consisting of the \( n \)-tuples of the exponents of the minimal monomial system of generators of \( J \). To the set of all the terms outside \( J \) corresponds the order ideal \( N(J) \subset \mathbb{N}^n \) given by \( \mathbb{N}^n \setminus J(J) \). In this context, we introduce the following.

**Definition 2.3.** For every monomial ideal \( j \subset S \), the order ideal \( N(j) := \{(a_1, \ldots, a_n) \in \mathbb{N} : x_1^{a_1} \cdots x_n^{a_n} \not\in J \} \) is called sous-écalier of \( J \) (see Marinari and Ramella, 2002).

**Remark 2.4.** As far as we know, the first appearance of the set consisting of all terms outside a monomial ideal \( J \subset S \) is in Macaulay (1927), in relation to the bound of the growth of the Hilbert function, and the term sous-écalier for it goes back to Galligo (1974), which exploited important features in the case of Borel ideals.

We will show now that the irreducible components of the scheme \( V(I) \) defined by a pseudo-\( t \)-lifting \( I \subset R \) of a monomial ideal \( J \subset S \) can be read off from \( J \) looking at \( N(J) \subset \mathbb{N}^n \).

The next proposition generalizes Lemma 4.2 of Migliore and Nagel (2000) to any monomial ideal \( J \), by considering also non-Artinian ideals.
Proposition 2.5. Let $J \subset S$ be a monomial ideal, $I$ a pseudo-t-lifting induced by an $n \times r$ matrix $A = (L_{j,i})$, and $\mathcal{V}(I) \subseteq \mathbb{P}^{n+r-1}$.

1) $(L_{j_1,i_1}, \ldots, L_{j_k,i_k})$ (with $j_1, \ldots, j_k$ distinct) gives a linear space contained in $V$ if and only if $(a_1, \ldots, a_n) \in \mathbb{N}^n : a_j = i_j - 1, \ldots, a_k = i_k - 1 \subset \mathcal{N}(J)$.

2) $(L_{j_1,i_1}, \ldots, L_{j_k,i_k})$ (with $j_1, \ldots, j_k$ distinct) gives an irreducible component of $V$ (i.e., is a prime ideal of the family $\mathcal{P}(I)$ described in Definition 1.13) iff $(a_1, \ldots, a_n) \in \mathbb{N}^n : a_j = i_j - 1, \ldots, a_k = i_k - 1 \subset \mathcal{N}(J)$ (here $\mathbb{N}^n$ is considered a subspace of $\mathbb{A}_0^n$ with the Zariski topology).

3) If Condition (x') holds, then $V$ is reduced and there is a bijection between the irreducible components of $V$ and the ones of $\mathcal{N}(J)$.

Proof. Let $J = (m_1, \ldots, m_s) \text{ and } I = (\bar{m}_1, \ldots, \bar{m}_s)$. The prime ideal $(L_{j_1,i_1}, \ldots, L_{j_k,i_k})$ defines a linear space contained in $V$ if and only if $\bar{m}_i, 1 \leq \ell \leq s$, is divisible by at least one of $L_{j_1,i_1}, \ldots, L_{j_k,i_k}$, i.e., any $m_i, 1 \leq \ell \leq k$, is divisible by at least one of $x_{j_1}^{i_1}, \ldots, x_{j_k}^{i_k}$ (Lemma 1.6). That means $x_{j_1}^{i_1-1} \cdots x_{j_k}^{i_k-1} \cdot m \notin J$ for every monomial $m$ noninvolving the variables $x_{j_1}, \ldots, x_{j_k}$ if and only if there exists a generator of $J, m_\ell = x_{j_1}^{i_1} \cdots x_{j_k}^{i_k}$, such that $i_1 > \lambda_{j_1}, \ldots, i_k > \lambda_{j_k}$; thus $m_\ell$ is not divisible by any $x_{j_1}^{i_1}, \ldots, x_{j_k}^{i_k}$. So the first assertion is proved.

The ideal $(L_{j_1,i_1}, \ldots, L_{j_k,i_k})$ defines an irreducible component of $V$ if and only if it defines a linear space contained in $V$ and by removing a generator we do not obtain a linear space with this property (that characterizes the prime ideals of the family $\mathcal{P}(I)$). So the subset of $\mathbb{N}^n$ determined in (1) by the above ideal is an irreducible component of $\mathcal{N}(J)$. Part (3) follows by Proposition 1.14. □

Remark 2.6. Suppose $\text{char}(K) = 0$; let $j \subset S$ be a monomial ideal and consider the scheme $V \subset \mathbb{P}^n$ defined by the 1-lifting induced by the matrix $A = (L_{j,i})$, where $L_{j,i} = x_i - (i - 1)u$. By identifying $\mathbb{A}^n$ with the affine open set of $\mathbb{P}^n$ defined by $u \neq 0$, we obtain $V \cap \mathbb{A}^n \cap \mathbb{N}^n = \mathcal{N}(J)$, the order ideal of $J$.

Theorem 4.7 of Migliore and Nagel (2000) characterized the configurations of linear varieties that arise as pseudo-t-liftings (or t-liftings) of Artinian monomial ideals and in particular those that arise as pseudo-t-liftings of Artinian lex-segment monomials ideals. Similar characterizations can be given also by considering strongly stable Artinian monomial ideals, by using the techniques developed by Marinari and Ramella (2002).

Theorem 2.7. Let $V$ be a reduced union of linear varieties of codimension $n$ in $\mathbb{P}^{n+r-1}$. Let $A = (L_{j,i})$ be a matrix of linear forms in $R$ satisfying Condition (x'), and suppose that $V$ is contained in the reduced complete intersection defined by $(F) = (\ldots, \prod_{j=1} L_{j,i}, \ldots)$. Then

1) $V$ is the pseudo-t-lifting via $A$ of an Artinian monomial ideal in $S$ if and only if, for $1 \leq j \leq n$ and $i_j \geq 2$, $\mathcal{V}(L_{1,i_1}, \ldots, L_{j,i_j}, \ldots, L_{n,i_n}) \subseteq V$ implies $\mathcal{V}(L_{1,i_1}, \ldots, L_{j,i_{j-1}}, \ldots, L_{n,i_n}) \subseteq V$ (see Migliore and Nagel, 2000, Lemma 4.3).

2) $V$ is the pseudo-t-lifting via $A$ of an Artinian lex-segment monomial ideal in $S$ if and only if, for $1 \leq j \leq n$ and $i_j \geq 2$, $\mathcal{V}(L_{1,i_1}, \ldots, L_{j,i_j}, \ldots, L_{n,i_n}) \subseteq V$ implies
Corollary 2.8. As noted in Migliore and Nagel (2000), the configurations of points arising from reduced pseudo-1-lifting of Artinian lex-segment ideals are special kinds of $k$-configurations as defined, for example, in Geramita et al. (2000, 2001). Moreover, the so-called standard $k$-configurations are the $l$-lifting of Artinian lex-segment ideals induced by the matrices described in Remark 2.6.

The geometry of a $k$-configuration of points in $\mathbb{P}^n$ is described in Geramita et al. (2000, 2001) by means of so-called $n$-type vectors (a sequence of positive integers and parentheses), which give immediately the coordinates of the points of the associated standard $k$-configuration (see Geramita et al., 2000) and the monomials of the order ideal $N(J)$ of the associated Artinian lex-segment ideal $J$; see Marinari and Ramella (2002).

In Marinari and Ramella (2002), monomial ideals $J$ are described by means of numerical strings, denoted $\epsilon(J)$, consisting of sequences of symbols $\infty$, positive integers and $/$’s. The string $\epsilon(J)$, called the $\epsilon$-vector of $J$, describes the geometric configuration of the order ideal $N(J)$ of $J$. Thus, by Proposition 2.5, $\epsilon(J)$ can describe the geometric configuration of a reduced pseudo-$t$-lifting of $J$. In Marinari and Ramella (2002), features of the $\epsilon$-vector $\epsilon(J)$ in the stable, strongly stable, and lex-segment cases are fully determined, and they can be translated to describe pseudo-$t$-liftings of stable, strongly stable, and lex-segment ideals.

Note that the $\epsilon$-vectors of Artinian lex-segment ideals correspond exactly to $n$-type vectors quoted above.

3. PSEUDO-$t$-LIFTING OF AN ARTINIAN MONOMIAL IDEAL

In the last two sections we described the geometry of the pseudo-$t$-lifting of monomial ideals in both the Artinian and the non-Artinian case. Before doing this, we recalled the following definition, which has two parts: the first is classical and the second has been introduced in Migliore and Nagel (2000) to generalize the first one.
**Definition 3.1** (Migliore and Nagel, 2000). A (reduced) union $V$ of lines in $\mathbb{P}^m$ with at most double points is a *stick figure*. More in general, a (reduced) union $V$ of linear subvarieties of $\mathbb{P}^m$ of the same dimension $d$ is a *generalized stick figure* if the intersection of any three components of $V$ has dimension at most $d - 2$ (the empty set is taken to have dimension $-1$).

We use the following notation. If $A$ is any $n \times r$ matrix of linear forms $L_{j,i}$ of $R$, then for every $j \in \{1, \ldots, n\}$ let $M_j \in M_{r,n+t}(K)$ be the $r \times (n+t)$ matrix whose $i$th row consists of the coefficients of $L_{j,i}$ and let $M$ be the $nr \times (n+t)$ matrix

$$M = \begin{pmatrix} M_1 \\ \vdots \\ M_n \end{pmatrix}.$$ 

Such a matrix $M$ with entries in $K$ can be associated with any matrix $A$ of linear forms. Condition $(\beta)$ means that $k$ rows of $M$ form a maximal rank matrix.

**Remark 3.2.** The following two examples furnish counterexamples to Theorem 3.4 (c) of Migliore and Nagel (2000): we see that the hypothesis of reduced $t$-lifting is not sufficient to obtain a generalized stick figure from an Artinian monomial ideal.

Let $J = (x_1^3) \subset K[x_1]$, and let $I$ be the 2-lifting of $J$ with

$$M = \begin{pmatrix} 1 & a & b \\ 1 & c & d \\ 1 & \frac{a+c}{2} & \frac{b+d}{2} \end{pmatrix}$$

as matrix of the coefficients of the linear forms of the 2-lifting. Hence the ideal $I = (L_{1,1}, L_{1,2}, L_{1,3})$ defines three lines in $\mathbb{P}^2$ which intersect in the point $\mathcal{V}(L_{1,1}, L_{1,2})$, because $L_{1,3} = \frac{1}{2}(L_{1,1} + L_{1,2})$.

Another example with $n = 2$ and $t = 2$ is given by $j = (x_1^2, x_2^2) \subset K[x_1, x_2]$ with

$$M = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 1 & 0 & 3 & 4 \\ 0 & 1 & 4 & 5 \\ 0 & 1 & 3 & 4 \end{pmatrix}$$

as matrix of coefficients of the linear forms of the 2-lifting. The ideal $I = (L_{1,1}, L_{1,2}, L_{2,1}, L_{2,2})$ defines four lines in $\mathbb{P}^3$ through the point $[1, 1, 1, -1]$.

Now we introduce another condition on matrices $A$ of linear forms (Condition $(\gamma)$) to obtain generalized stick figures from $t$-liftings of Artinian monomial ideals. We use the matrix $M$ defined above.

**Condition $(\gamma)$.** (a) For each $j \in \{1, \ldots, n\}$, every three rows of $M_j$ are linearly independent. (b) For every $j, k \in \{1, \ldots, n\}$, such that $j \neq k$, two rows of $M_j$ and two rows of $M_k$ form a system of four independent vectors.

Note that, unlike Condition $(\beta)$, this condition can be satisfied also by a $t$-lifting for every $t \geq 2$. 
Condition (γ), weaker than Condition (β), is the “minimal” genericity condition for a $t$-lifting to be added to the hypothesis of Theorem 3.4 (c) of Migliore and Nagel (2000) in order to have the assertion, as we prove in the following theorem.

**Theorem 3.3.** Let $J \subset S$ be an Artinian monomial ideal, $t \geq 2$, and $I \subset R$ a $t$-lifting of $J$ such that Condition (γ) holds. Then $V = \mathcal{V}(I)$ is a generalized stick figure.

**Proof.** A $t$-lifting matrix with Condition (γ) verifies Condition ($\gamma'$) (see Remark 1.9), so $V$ is reduced and the family $\mathcal{P}(I)$ of Definition 1.13 gives the irreducible components of $V$ (Proposition 1.14). If $J$ is Artinian, the order ideal $\mathcal{N}(J)$ consists of a finite set of points; then by Proposition 2.5 the scheme $V$ is equidimensional of dimension $t - 1$ in $\mathbb{P}^{n+t-1}$.

Let $(\mathbb{P}^{n+t-1})^\vee$ be the dual space of $\mathbb{P}^{n+t-1}$. Recall that the irreducible components of $V$ are linear spaces, and let $\bar{V} \subset (\mathbb{P}^{n+t-1})^\vee$ be the union of the duals of these components. Then $\dim(\bar{V}) = n - 1$. We have to prove that any three $(n - 1)$-planes of $\bar{V}$ generate a linear space of dimension $\geq n + 1$.

Let $M$ be the matrix with entries in $K$ associated to the given $t$-lifting matrix and note that the shape of $M$ is such that every submatrix $M_j$ has the entries equal to 1 in the column corresponding to $x_j$ and equal to 0 in the columns corresponding to $x_i$ with $\ell \neq j$. If an irreducible component of $V$ is given by $(L_{1,i_1}, L_{2,i_2}, \ldots, L_{n,i_n})$, then the corresponding irreducible component of $\bar{V}$ is generated by the points corresponding to the $i_1$th row of $M_1$, the $i_2$th row of $M_2$, $\ldots$, the $i_n$th row of $M_n$.

Take the union of three different components of $\bar{V}$ and note that there are two possibilities. The first is that this union contains at least the points corresponding to three different rows of the same matrix $M_j$ and to one row of every other matrix $M_s$, with $s \in \{1, \ldots, n\} \setminus \{j\}$. These $n + 2$ points must be independent by part (a) of Condition (γ) and by the shape of the matrix $M$. The second possibility is that the union of three different components of $V$ contains at least the points corresponding to two rows of a matrix $M_j$, two rows of a matrix $M_s$, with $k \neq j$, and one row of every other matrix $M_t$, with $s \in \{1, \ldots, n\} \setminus \{j, k\}$. These $n + 2$ points are linearly independent by part (b) of Condition (γ) and by the shape of the matrix $M$. \quad \Box

With regard to pseudo-$t$-liftings, we obtain the following result.

**Theorem 3.4.** Let $J$ be an Artinian monomial ideal, $t \geq 2$, and $I$ a pseudo-$t$-lifting of $J$ such that Condition (β) holds. Then $V = \mathcal{V}(I)$ is a generalized stick figure.

**Proof.** We proceed as in the proof of Theorem 3.3. $V \subset \mathbb{P}^{n+t-1}$ is reduced and equidimensional of dimension $t - 1$. We consider the union $\bar{V} \subset (\mathbb{P}^{n+t-1})^\vee$ of the duals of the components of $V$. As in Theorem 3.3 we must to prove that any three $(n - 1)$-planes of $\bar{V}$ generate a linear space of dimension $\geq n + 1$.

Take the union of three different components of $\bar{V}$ and note that there are two possibilities: either this union contains at least the points corresponding to three rows of a matrix $M_j$ and one row of every other matrix $M_s$, with $s \in \{1, \ldots, n\} \setminus \{j\}$; or this union contains at least the points corresponding to two rows of a matrix $M_j$, two rows of a matrix $M_k$, $k \neq j$, and one row of every other matrix $M_t$, with $s$ in
Remark 3.5. Let $A = (L_{j,i})$ be a $2 \times 2$ matrix of linear forms of $R = K[x_1, x_2, u_1, \ldots, u_t]$, $t \geq 3$, such that Condition (β) holds. Then the pseudo-$t$-lifting of $J = (x_1^2, x_2^2)$ is the ideal $I = (L_{1,1}, L_{2,1}) \cap (L_{1,2}, L_{2,2}) \cap (L_{1,1}, L_{2,1}) \cap (L_{1,2}, L_{2,2})$, and the intersection of the four components of $V$ contains the $(t - 3)$-plane $\mathcal{V}(L_{1,1}, L_{1,2}, L_{2,1}, L_{2,2})$. So, for a pseudo-$t$-lifting with Condition (β) it does not make sense to ask that four components of $V$ intersect in dimension $\leq \dim(V) - 3$, five components intersect in dimension $\leq \dim(V) - 4$, etc., as has been already observed in Remark 3.6 of Migliore and Nagel (2000).

4. PSEUDO-$t$-LIFTING OF A NON-ARTINIAN MONOMIAL IDEAL

In this section, $J \subseteq S$ is a monomial unmixed ideal such that $\dim(S/J) = d \geq 1$ and $n \geq 2$; $A = (L_{j,i})$ is a matrix of linear forms belonging to $R$, with $t \geq 1$ and $V = \mathcal{V}(I)$, where $I$ is the pseudo-$t$-lifting via $A$ of $J$. Hence the order ideal $\mathcal{N}(J) \subseteq \mathbb{N}^n$ is equidimensional of dimension $d$, and $V \subseteq \mathbb{P}^{n+t-1}$ is equidimensional of dimension $d + t - 1$ (Proposition 2.5).

The following examples give pseudo-$t$-liftings that are not generalized stick figures and furnish counterexamples to Theorem 3.4 (b) and (d) of Migliore and Nagel (2000).

Example 4.1. The monomial ideal $J = (x_1, x_2) \cap (x_1, x_3) \cap (x_2, x_3) = (x_1x_2, x_1x_3, x_2x_3) \subseteq K[x_1, x_2, x_3]$ gives a simple example. Note that $\dim(S/J) = 1$, $J$ is unmixed and radical with components of degree 1, and $W = W_{\text{red}} = \{[1, 0, 0], [0, 1, 0], [0, 0, 1]\} \subseteq \mathbb{P}^2$. The order ideal $\mathcal{N}(J) \subseteq \mathbb{N}^3$ consists of three lines meeting in a point; every reduced pseudo-$t$-lifting ($t \geq 1$) $\mathcal{V}(I)$ consists of three $t$-planes in $\mathbb{P}^{2+t}$ meeting in $\mathcal{V}(L_{1,1}, L_{2,1}, L_{3,1})$, which is of dimension $t - 1 > t - 2$. Then $\mathcal{V}(I)$ is not a generalized stick figure, and moreover the intersection $\mathcal{V}(L_{1,1}, L_{2,1}, L_{3,1})$ is away from $W_{\text{red}}$ for every $t$-lifting and for generic pseudo-$t$-lifting.

From this last example we can get a large class of unmixed monomial ideals $J$ for which conditions (b) and (d) of Theorem 3.4 of Migliore and Nagel (2000) do not hold. For example, if $J \subseteq (x_1x_2, x_1x_3, x_2x_3, x_4, \ldots, x_k) \subseteq S$, with $\dim(S/J) = d = n - k + 1 \geq 1$, every reduced pseudo-$t$-lifting $V = \mathcal{V}(I)$ of $J$ is not a generalized stick figure; in fact, $V$ has three irreducible components $\mathcal{V}(L_{1,1}, L_{2,1}, L_{4,1}, \ldots, L_{k,1})$, $\mathcal{V}(L_{1,1}, L_{3,1}, L_{4,1}, \ldots, L_{k,1})$, $\mathcal{V}(L_{2,1}, L_{3,1}, L_{4,1}, \ldots, L_{k,1})$, of dimension $n + t - k$ ($= \dim(V)$) meeting in the linear space $\mathcal{V}(L_{1,1}, L_{2,1}, L_{3,1}, L_{4,1}, \ldots, L_{k,1})$ whose dimension is $n + t - k - 1 = \dim(V) - 1$. This fact arises from the behavior of the irreducible components of $\mathcal{N}(J)$: the intersection of the three components $\{(a_1, a_2, \ldots, a_k, \ldots, a_n) \in \mathbb{N}^n : a_2 = \cdots = a_k = 0, \quad \{(a_1, a_2, \ldots, a_k) \in \mathbb{N}^n : a_1 = a_3 = \cdots = a_k = 0, \quad \{(a_1, a_2, \ldots, a_n) \in \mathbb{N}^n : a_1 = a_2 = a_4 = \cdots = a_k = 0\}$ is $\{(a_1, a_2, \ldots, a_{n-k}) \in \mathbb{N}^n : a_1 = \cdots = a_k = 0\}$ whose dimension is $n - k = d - 1$.

Example 4.2. Let $J = (x_1^2, x_2^2) \cap (x_2^2, x_3^2) \cap (x_2^2, x_3^2) = (x_1^2x_2^2, x_1^2x_3^2, x_1x_2x_3, x_2^2x_3)$ \subseteq $K[x_1, x_2, x_3]$. Note that $\dim(S/J) = 1$, $J$ is unmixed with components of degree 2, and $W_{\text{red}} = \{[1, 0, 0], [0, 1, 0], [0, 0, 1]\} \subseteq \mathbb{P}^2$. Also in this case assertions (b) and (d) of Theorem 3.4 of Migliore and Nagel (2000) do not hold. Consider a reduced
pseudo-$t$-lifting $I$ of $J$:

$$I = (L_{1,1}L_{3,2}, L_{2,1}L_{3,1}, L_{1,1}L_{2,1}L_{3,1}, L_{1,1}L_{1,2}L_{2,1})$$

$$= (L_{1,1}, L_{2,1}) \cap (L_{1,1}, L_{2,2}) \cap (L_{1,1}L_{3,1}) \cap (L_{1,2}, L_{3,1}) \cap (L_{2,1}, L_{3,1}) \cap (L_{2,1}, L_{1,2}).$$

Hence $I$ is the ideal of the following pairwise distinct $t$-planes of $\mathbb{P}^{2+t}$:

$$r_1 = \mathcal{V}(L_{1,1}, L_{2,1}), r_2 = \mathcal{V}(L_{1,1}, L_{2,2}), r_3 = \mathcal{V}(L_{1,1}, L_{3,1}),$$

$$r_4 = \mathcal{V}(L_{1,2}, L_{3,1}), r_5 = \mathcal{V}(L_{2,1}, L_{3,1}), r_6 = \mathcal{V}(L_{2,1}, L_{3,2}).$$

Note that $r_1 \cap r_2 \cap r_3 \neq \emptyset$ is of dimension $t - 1 > t - 2$; moreover, for every $t$-lifting and for generic pseudo-$t$-lifting this intersection is away from $W_{\text{red}}$.

For generic pseudo-$t$-liftings of unmixed (non-Artinian) monomial ideals we obtain

**Proposition 4.3.** Let $J \subset S$ be an unmixed monomial ideal with $\dim(S/J) = d \geq 1$, and let $V \subset \mathbb{P}^{n+t-1}$ be the scheme defined by a pseudo-$t$-lifting satisfying Condition $(\beta)(t \geq 1)$. If $s$ irreducible components of $V$ intersect in dimension $\dim(V) - 1 = d + t - 2$, then $2 \leq s \leq n - d + 1$. So, in particular, any $n - d + 2$ components of $V$ intersect in dimension less than or equal to $\dim(V) - 2 = d + t - 3$.

**Proof.** $V$ is reduced by Condition $(\beta)$ and equidimensional of dimension $d + t - 1$ in $\mathbb{P}^{n+t-1}$: it is obvious that, for any $s \geq 2$, $s$ distinct irreducible components of $V$ intersect in dimension $\leq \dim(V) - 1$. We have to prove that $n - d + 2$ (distinct) irreducible components of $V$ intersect in dimension $\leq \dim(V) - 2$.

As usual, $(\mathbb{P}^{n+t-1})^\vee$ denotes the dual space of $\mathbb{P}^{n+t-1}$ (so that a linear from $L_{j,i} \subset R$ is identified with a point of $(\mathbb{P}^{n+t-1})^\vee$) and $\bar{V}$ denotes the union of the duals of the linear spaces given by the irreducible components of $V$, so $\dim(\bar{V}) = n - d - 1$.

Note that two distinct irreducible components of $\bar{V}$, which are $(n - d - 1)$-planes, generate at least an $(n - d)$-plane, since they involve (at least) $n - d + 1$ points $L_1, \ldots, L_{n-d+1}$ corresponding to $n - d + 1$ entries of the matrix $A$, which are independent by Condition $(\beta)$. These points may involve at most $n - d + 1$ irreducible components of $\bar{V}$, $\bar{V}_k = \langle L_1, \ldots, \tilde{L}_k, \ldots, L_{n-d+1} \rangle$ for $1 \leq k \leq n - d + 1$. Thus $n - d + 2$ distinct components of $\bar{V}$ involve at least $n - d + 2$ entries of the matrix $A$, and by Condition $(\beta)$ the corresponding points in $(\mathbb{P}^{n+t-1})^\vee$ generate a $(n - d + 1)$-plane. \hfill $\square$

**Example 4.4.** Take an unmixed monomial ideal

$$J \subseteq (x_1x_2, x_1x_3, \ldots, x_1x_k, x_2x_3, \ldots, x_2x_k, \ldots, x_{k-1}x_k) \subset S$$

with $k = n - d + 1$ and $\dim(S/J) = d \geq 1$. Any reduced pseudo-$t$-lifting $V = \mathcal{V}(J)$ of $J$ has $n - d + 1$ distinct irreducible components $V_i := \mathcal{V}(L_{1,1}, \ldots, \tilde{L}_{i,t}, \ldots, L_{1,n-d+1}), 1 \leq i \leq n - d + 1$, meeting at $\mathcal{V}(L_{1,1}, \ldots, L_{1,n-d+1}),$ which is of dimension
dim(V) − 1. This example shows that Proposition 4.3 cannot be improved. Of course V is not a generalized stick figure for n − d ≥ 2.

Note that the ideal J = (x₁x₂, x₁x₃, x₂x₃) of Example 4.1 is of the above type, for d = 1 and n = 3.

**Definition 4.5.** Let X be a (reduced) union of lines in \( \mathbb{P}^n \). We will say that X is a seminormal configuration of lines if at every intersection point the lines through that point have linearly independent directions.

**Remark 4.6.** Let A be a reduced noetherian ring and \( Q(A) \) be its total quotient ring; the ring A is called seminormal if \( t \in Q(A), t^2, t^3 \in A \) imply \( t \in A \) (see, e.g., Swan, 1980). For a union X of lines we have that X is a seminormal configuration if and only if for every \( x \in X \) the local ring \( O_{X,x} \) is seminormal (see Davis, 1978, Corollary 4).

For an Artinian monomial ideal J, a 2-lifting with Condition (γ) and a pseudo-2-lifting with Condition (β) define a (reduced) union V of lines in \( \mathbb{P}^{n+1} \) that is a stick figure (Theorems 3.3 and 3.4); then V is a seminormal configuration of lines. For a non-Artinian monomial ideal J we have the following result.

**Corollary 4.7.** Let \( J \subset S \) be an unmixed monomial ideal with \( \dim(S/J) = 1 \) and \( V = V(J) \) a pseudo-1-lifting of J with Condition (β). Then V is a seminormal configuration of lines.

**Proof.** By Proposition 4.3, for every integer s such that \( 2 \leq s \leq n \), \( s \) lines of V intersect in at most a point. Moreover the intersection of \( n+1 \) lines of V is empty. We must only prove that if \( s \) lines of V, with \( 2 \leq s \leq n \), intersect in a point, then they generate an \( s \)-plane. As usual we consider the union \( \tilde{V} \subset \mathbb{P}^{n'} \) of the duals of the lines of V. \( \tilde{V} \) is equidimensional of dimension \( n − 2 \). Note that s lines of V intersect in a point if and only if the corresponding \( (n−2) \)-planes of \( \tilde{V} \) generate an \((n−1)\)-plane, determinated by \( n \) points \( L_1, \ldots, L_n \in \mathbb{P}^{n'} \) given by \( n \) entries of the pseudo-1-lifting matrix A (which are independent by Condition (β)). The above \((n−2)\)-planes are generated by \( n−1 \) points among \( L_1, \ldots, L_n \). Then they intersect in an \((n−s−1)\)-plane, and the corresponding \( s \) lines of V \( \subset \mathbb{P}^n \) generate an \( s \)-plane. □

**Remark 4.8.** If a monomial ideal J ⊂ S is either Artinian or unmixed with \( \dim(S/J) = 1 \), then \( S/J \) is Cohen–Macaulay, and for any pseudo-t-lifting \( I \subset R \) of J also \( R/I \) is Cohen–Macaulay (Migliore and Nagel, 2000). Thus the seminormal configurations of lines obtained by means of 2-liftings of Theorem 3.3, pseudo-2-liftings of Theorem 3.4, and pseudo-1-liftings of the above Corollary are arithmetically Cohen–Macaulay, and then, by Corollary 5.9 of Geramita and Weibel (1985), their homogeneous coordinate rings are seminormal.

Any pseudo-t-lifting satisfying Condition (β) of the monomial ideal \( J = (x_1^2, x_2x_3) \subset K[x_1, x_2, x_3] \) is a generalized stick figure, although in this case we have \( n − d + 2 > 3 \).

Now we give conditions for obtaining generalized stick figures by means of pseudo-t-liftings of non-Artinian monomial ideals.
Definition 4.9. Let $J \subset S$ be an unmixed monomial ideal with $\dim(S/J) = d \geq 1$. The order ideal $\mathcal{N}(J) \subset \mathbb{N}^n \subset \mathbb{A}_K^n$, which is equidimensional of dimension $d$, is called a generalized stick figure if the intersection of any three components of $\mathcal{N}(J)$ has dimension at most $d - 2$.

Proposition 4.10. Let $J \subset S = K[x_1, \ldots, x_n]$ be an unmixed monomial ideal with $d = \dim(S/J) \geq 1$, and let $V \subset \mathbb{P}^{n+t-1}$ be the scheme defined by a pseudo-$t$-lifting with $t \geq 1$.

(a) If the order ideal $\mathcal{N}(J)$ is not a generalized stick figure, then the pseudo-$t$-lifting $V$ is not a generalized stick figure.

(b) If the order ideal $\mathcal{N}(J)$ is a generalized stick figure, and the pseudo-$t$-lifting satisfies Condition ($\beta$), then $V$ is a generalized stick figure.

Proof. Let $A = (L_{j_t})$ be the pseudo-$t$-lifting matrix.

(a) If $\mathcal{N}(J)$ is not a generalized stick figure, then there exist three components $N, N', N''$ of $\mathcal{N}(J)$ such that $\dim(N \cap N' \cap N'') = d - 1$. These components are of type

$$N = \{(a_1, \ldots, a_n) \in \mathbb{N}^n : a_{j_1} = i_1 - 1, \ldots, a_{j_k} = i_k - 1\},$$

$$N' = \{(a_1, \ldots, a_n) \in \mathbb{N}^n : a_{j_1} = i_1 - 1, \ldots, a_{j_{k-1}} = i_{k-1} - 1, a_{j_k} = i_{k+1} - 1\},$$

$$N'' = \{(a_1, \ldots, a_n) \in \mathbb{N}^n : a_{j_1} = i_1 - 1, \ldots, a_{j_{k-2}} = i_{k-2} - 1, a_{j_k} = i_k - 1, \quad \times a_{j_{k+1}} = i_{k+1} - 1\},$$

with $k = n - d$ and $j_1, \ldots, j_{k+1}$ pairwise distinct.

By Proposition 2.5, this equivalently means that the corresponding linear varieties $W = \langle L_{j_1}, \ldots, L_{j_k} \rangle$, $W' = \langle L_{j_1}, \ldots, L_{j_{k-1}}, L_{j_k+i_k-1} \rangle$, $W'' = \langle L_{j_1}, \ldots, L_{j_{k-2}}, i_{k-2}, L_{j_k+i_k}, L_{j_{k+1}} \rangle$ define irreducible (and distinct, otherwise $V$ would not be reduced) components of $V$, and they intersect in dimension $n + t - 2 = \dim(V) - 1$. So $V$ is not generalized stick figure.

(b) As usual we consider the linear forms $L_{j_t}$ as points of $(\mathbb{P}^{n+t-1})^\vee$. Note that $V$ is reduced and equidimensional. If $V$ is not a generalized stick figure, then $V \subset \mathbb{P}^{n+t-1}$ has three irreducible components, say $W, W', W''$, intersecting in dimension $\dim(V) - 1 = d + t - 2$. Then the duals $\tilde{W}, \tilde{W}', \tilde{W}''$ of $W, W', W''$ are $(n - d - 1)$-dimensional and generate an $(n - d)$-plane $\Sigma$.

Put $k = n - d$ and let $L_{j_1, i_1}, \ldots, L_{j_k, i_k}$ be the point of $(\mathbb{P}^{n+t-1})^\vee$ corresponding to entries of the matrix $A$ generating $\tilde{W}$.

Observe that, by Condition ($\beta$), among $k + 2$ entries $L_1, \ldots, L_k, L_{k+1}, L_{k+2}$ from the matrix $A$, such that $\dim(L_1, \ldots, L_k, L_{k+1}, L_{k+2}) = k + 1$, there are only $k + 1$ different linear forms. Since $\Sigma = \langle W, W' \rangle$ and $\dim(\Sigma) = n - d$, by applying repeatedly the above observation on the points corresponding to the entries from $A$ generating $\tilde{W}$ we obtain that there is only one of these points, say for example $L_{j_k+i_k, i_k}$, that is different from the point generating $\tilde{W}$ and such that $\Sigma = \langle L_{j_1, i_1}, \ldots, L_{j_k, i_k}, L_{j_{k+1}, i_{k+1}} \rangle$. So, for example, it is $\tilde{W}' = \langle L_{j_1, i_1}, \ldots, L_{j_{k-1}, i_{k-1}}, L_{j_{k+1}, i_{k+1}} \rangle$. Since moreover $\Sigma = \langle \tilde{W}, \tilde{W}', \tilde{W}'' \rangle$, by the same
observation, we can conclude that the points corresponding to the entries of $A$ generating $\tilde{W}''$ must belong to $\{L_{j_1,i_1}, \ldots, L_{j_k,i_k}, L_{j_{k+1},i_{k+1}}\}$ and so, for example, $\tilde{W}'' = \langle L_{j_1,i_1}, \ldots, L_{j_{k-2},i_{k-2}}, L_{j_{k-1},i_{k-1}}, L_{j_{k+1},i_{k+1}} \rangle$.

By Proposition 2.5, $W, W', W''$ give three components of $N(J)$ intersecting in dimension $n - k - 1$, which is absurd because $N(J)$ is a generalized stick figure. □

Remark 4.11. Let $j \subseteq S$ be an unmixed monomial ideal with $\dim(S/J) = d \geq 1$. Note that by the definition of order ideal it follows that $N(J) \subseteq \mathbb{N}^n \subseteq \mathbb{A}^n$ has three components intersecting in dimension $d - 1$ if and only if there exist $k + 1$ distinct indices $j_1, \ldots, j_{k-1}, j_k, j_{k+1}$, with $k = n - d$ such that

$$N = \{(a_1, \ldots, a_n) \in \mathbb{N}^n : a_{j_1} = \cdots = a_{j_k} = 0\},$$

$$N' = \{(a_1, \ldots, a_n) \in \mathbb{N}^n : a_{j_1} = \cdots = a_{j_{k-1}} = a_{j_{k+1}} = 0\},$$

$$N'' = \{(a_1, \ldots, a_n) \in \mathbb{N}^n : a_{j_1} = \cdots = a_{j_{k-2}} = a_{j_k} = a_{j_{k+1}} = 0\}$$

are components of $N(J)$.

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REFERENCES


