Expressiveness, decidability, and undecidability of Interval Temporal Logic

ITL - Beyond the end of the light

Ph.D. Defence

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Udine - April 1, 2011
At the beginning...

At the beginning, it was the darkness...

Then, logicians made the light,
they became curious,
and moved toward the darkness...

... as close as they could
Outline

Introduction
  The Halpern and Shoham’s logic HS

Expressiveness of HS

The satisfiability problem for HS
  Undecidability

Classical extensions
  Metric extensions
  Hybrid extensions
  First-order extensions

Summary and perspectives
Outline

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Summary and perspectives
Interval-based temporal reasoning: reasoning about time, where the primary concept is ‘time interval’, rather than ‘time instant’.

Origins:

- **Philosophy**, in particular philosophy and ontology of time.
- **Linguistics**: analysis of progressive tenses, semantics of natural languages.
- **Artificial intelligence**: temporal knowledge representation, temporal planning, theory of events, etc.
- **Computer science**: specification and design of hardware components, concurrent real-time processes, temporal databases, etc.
Motivations

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- it is true over a precise interval of time
- it is not true over any other interval (starting/ending interval, inner interval, ecc.)
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- Zeno’s flying arrow paradox (“if at each instant the flying arrow stands still, how is movement possible?”)
- The dividing instant dilemma (“if the light is on and it is turned off, what is its state at the instant between the two events?”)
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The truth of a formula over an interval does not necessarily depend on its truth over subintervals.
Interval temporal reasoning and temporal ontologies

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- Can intervals be unbounded?
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- discrete or dense?
- with or without beginning/end?, etc.

New issues arise regarding the nature of the intervals:

- Can intervals be unbounded?
- Are intervals with coinciding endpoints admissible or not?
Intervals and interval structures

\[ \mathbb{D} = \langle D, < \rangle : \text{partially ordered set.} \]

An interval in \( \mathbb{D} \): ordered pair \([a, b]\), where \(a, b \in D\) and \(a \leq b\).
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In this talk I will restrict attention to linear interval structures, i.e. interval structures over linear orders.
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In this talk I will restrict attention to linear interval structures, i.e. interval structures over linear orders.

In particular, standard interval structures on \( \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \) and \( \mathbb{R} \) with their usual orders.
Binary interval relations on linear orders

J. F. Allen
Maintaining knowledge about temporal intervals.
Binary interval relations on linear orders

Later

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After (right neighbour)

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6 relations + their inverses + equality = 13 Allen’s relations.

J. F. Allen
Maintaining knowledge about temporal intervals.
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Every interval relation gives rise to a modal operator over relational interval structures.
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J. Halpern and Y. Shoham

A propositional modal logic of time intervals.

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**non-strict semantics:**

All modalities are definable in terms of $\langle B \rangle, \langle E \rangle, \langle \neg B \rangle, \langle \neg E \rangle$

$HS \equiv BE \neg BE \neg BE$
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$$HS \equiv B \overline{E} B \overline{E}$$

**strict semantics:**
Also needed additional modalities $\langle A \rangle, \langle \overline{A} \rangle$

$$HS \equiv B \overline{E} B \overline{E} + A \overline{A}$$
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All modalities are definable in terms of \(\langle B \rangle \), \(\langle E \rangle \), \(\langle \overline{B} \rangle \), \(\langle \overline{E} \rangle\)

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**strict semantics:**
Also needed additional modalities \(\langle A \rangle \), \(\langle \overline{A} \rangle\)

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**Syntax** of Halpern-Shoham’s logic, hereafter called \(HS\):

\[ \phi ::= p \mid \neg \phi \mid \phi \land \psi \mid \langle B \rangle \phi \mid \langle E \rangle \phi \mid \langle \overline{B} \rangle \phi \mid \langle \overline{E} \rangle \phi \mid (\mid \langle A \rangle \phi \mid \langle \overline{A} \rangle \phi) . \]
Models for propositional interval logics

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Non-strict interval model:

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M^+ = \langle \mathbb{I}(\mathbb{D})^+, V \rangle,
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where \( V : \mathcal{AP} \rightarrow 2^{\mathbb{I}(\mathbb{D})^+} \).
Models for propositional interval logics

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where \( V : \mathcal{AP} \rightarrow 2^{\mathbb{I}(\mathcal{D})^+} \).

**Strict interval model:**

\[ M^- = \langle \mathbb{I}(\mathcal{D})^-, V \rangle, \]

where \( V : \mathcal{AP} \rightarrow 2^{\mathbb{I}(\mathcal{D})^-} \).
**Formal semantics of HS**

\( \langle B \rangle : \ M, [d_0, d_1] \models \langle B \rangle \phi \) iff there exists \( d_2 \) such that \( d_0 \leq d_2 < d_1 \) and \( M, [d_0, d_2] \models \phi \).

\( \langle \overline{B} \rangle : \ M, [d_0, d_1] \models \langle \overline{B} \rangle \phi \) iff there exists \( d_2 \) such that \( d_1 < d_2 \) and \( M, [d_0, d_2] \models \phi \).
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**current interval:**

\( \langle E \rangle \phi \):

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\( \langle A \rangle : \) \( M, [d_0, d_1] \models \langle A \rangle \phi \) iff there exists \( d_2 \) such that \( d_1 < d_2 \) and \( M, [d_1, d_2] \models \phi \).

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\( \phi \)
Formal semantics of HS - contd’

\[ \langle L \rangle : \quad M, [d_0, d_1] \vDash \langle L \rangle \phi \text{ iff there exists } d_2, d_3 \text{ such that } d_1 < d_2 < d_3 \text{ and } M, [d_2, d_3] \vDash \phi. \]

\[ \langle \bar{L} \rangle : \quad M, [d_0, d_1] \vDash \langle \bar{L} \rangle \phi \text{ iff there exists } d_2, d_3 \text{ such that } d_2 < d_3 < d_0 \text{ and } M, [d_2, d_3] \vDash \phi. \]
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\( \langle D \rangle : \) \( M, [d_0, d_1] \models \langle D \rangle \phi \) iff there exists \( d_2, d_3 \) such that \( d_0 < d_2 < d_3 < d_1 \) and \( M, [d_2, d_3] \models \phi \).

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\[ \langle O \rangle: M, [d_0, d_1] \models \langle O \rangle \phi \text{ iff there exists } d_2, d_3 \text{ such that } d_0 < d_2 < d_1 < d_3 \text{ and } M, [d_2, d_3] \models \phi. \]

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**current interval:**

- \( \langle O \rangle \phi: \) 
  
- \( \langle O \rangle \phi: \) 

\[ \langle \overline{O} \rangle: M, [d_0, d_1] \models \langle \overline{O} \rangle \phi \text{ iff there exists } d_2, d_3 \text{ such that } d_2 < d_0 < d_3 < d_1 \text{ and } M, [d_2, d_3] \models \phi. \]
Defining the other interval modalities in HS

A useful new symbol is the modal constant $\pi$ for point-intervals:

$$M, [d_0, d_1] \models \pi \text{ iff } d_0 = d_1.$$
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It is definable as either $[B] \perp$ or $[E] \perp$, so it is only needed in weaker fragments of HS.
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A useful new symbol is the modal constant $\pi$ for point-intervals:

$$M, [d_0, d_1] \models \pi \text{ iff } d_0 = d_1.$$ 

It is definable as either $[B] \bot$ or $[E] \bot$, so it is only needed in weaker fragments of HS.

In general, it is possible defining HS modalities in terms of others
The zoo of fragments of HS

Technically, there are $2^{12} = 4096$ fragments of HS
Of them, several hundreds are of essentially different expressiveness
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Of them, several hundreds are of essentially different expressiveness
Each of these, considered with respect to some parameters:

1. over special classes of interval structures (all, dense, discrete, finite, etc.)
2. with strict or non-strict semantics
3. including or excluding $\pi$ operator (whenever it cannot be defined)
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Comparing the expressiveness of fragments of HS

Expressiveness classification problem: classify the fragments of HS with respect to their expressiveness, relative to important classes of interval models.
The problem of comparing expressive power of HS fragments

$L_1, L_2$ HS-fragments

$L_1$  

$L_2$
The problem of comparing expressive power of HS fragments

$L_1, L_2$ HS-fragments

$L_1 \{\prec, \equiv, \succ, \not\approx\} L_2$
The problem of comparing expressive power of HS fragments

$L_1, L_2$ HS-fragments

$L_1 \{\prec, \equiv, \succ, \not\approx\} L_2$

does $L_1$ translate into $L_2$?

- yes
- no

- yes
- no

- yes
- no

- yes
- no
Truth-preserving translation

There exists a truth-preserving translation of $L_1$ into $L_2$ iff $L_2$ is at least as expressive as $L_1$ ($L_1 \preceq L_2$)
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$2^{12}$ fragments... \( \frac{2^{12} \cdot (2^{12} - 1)}{2} \) comparisons
Inter-definability equations

Notation: $X_1 X_2 \ldots X_n$ will denote the fragment of HS containing the modalities $\langle X_1 \rangle, \langle X_2 \rangle, \ldots, \langle X_n \rangle$.

\[
\langle L \rangle p \equiv \langle A \rangle \langle A \rangle p \\
\langle O \rangle p \equiv \langle E \rangle \langle \overline{B} \rangle p \\
\langle D \rangle p \equiv \langle E \rangle \langle B \rangle p
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\langle L \rangle p \equiv \langle A \rangle \langle A \rangle p \quad \langle L \rangle \sqsubseteq A
\]
\[
\langle O \rangle p \equiv \langle E \rangle \langle B \rangle p \quad \langle O \rangle \sqsubseteq EB
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\langle D \rangle p \equiv \langle E \rangle \langle B \rangle p \quad \langle D \rangle \sqsubseteq EB
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\langle L \rangle p \equiv \langle A \rangle \langle A \rangle p \quad \quad \quad \quad \quad \langle L \rangle \sqsubseteq A \quad \quad \quad \langle \overline{L} \rangle \sqsubseteq \overline{A}
\]
\[
\langle O \rangle p \equiv \langle E \rangle \langle \overline{B} \rangle p \quad \quad \quad \quad \quad \langle O \rangle \sqsubseteq \overline{EB} \quad \quad \quad \langle \overline{O} \rangle \sqsubseteq \overline{EB}
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\]
\[
\langle L \rangle p \equiv \langle \overline{B} \rangle [E] \langle \overline{B} \rangle \langle E \rangle p \quad \quad \quad \quad \quad \langle L \rangle \sqsubseteq \overline{BE} \quad \quad \quad \langle \overline{L} \rangle \sqsubseteq \overline{BE}
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\[ \langle D \rangle p \equiv \langle E \rangle \langle B \rangle p \]

\[ \langle L \rangle p \equiv \langle \overline{B} \rangle [E] \langle \overline{B} \rangle \langle E \rangle p \]

\[ \langle L \rangle \sqsubseteq A \quad \langle L \rangle \sqsubseteq \overline{A} \]
\[ \langle O \rangle \sqsubseteq E B \quad \langle O \rangle \sqsubseteq \overline{E B} \]
\[ \langle D \rangle \sqsubseteq E B \quad \langle D \rangle \sqsubseteq \overline{E B} \]
\[ \langle L \rangle \sqsubseteq \overline{B} E \quad \langle L \rangle \sqsubseteq \overline{B} \overline{E} \]

Soundness and completeness???
Inter-definability equations

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\langle L \rangle p \equiv \langle A \rangle \langle A \rangle p \\
\langle O \rangle p \equiv \langle E \rangle \langle \overline{B} \rangle p \\
\langle D \rangle p \equiv \langle E \rangle \langle B \rangle p
\]

\[
\langle L \rangle \subseteq A \\
\langle O \rangle \subseteq \overline{EB} \\
\langle D \rangle \subseteq \overline{EB}
\]

\[
\langle L \rangle p \equiv \langle \overline{B} \rangle [E] \langle \overline{B} \rangle \langle E \rangle p \\
\langle L \rangle \subseteq \overline{BE} \\
\langle L \rangle \subseteq B \overline{E}
\]

Soundness and completeness???

Soundness: all equations are valid
Inter-definability equations

Notation: $X_1 X_2 \ldots X_n$ will denote the fragment of HS containing the modalities $\langle X_1 \rangle, \langle X_2 \rangle, \ldots, \langle X_n \rangle$

$$\langle L \rangle p \equiv \langle A \rangle \langle A \rangle p$$
$$\langle O \rangle p \equiv \langle E \rangle \langle B \rangle p$$
$$\langle D \rangle p \equiv \langle E \rangle \langle B \rangle p$$

$$\langle L \rangle p \equiv \langle B \rangle [E] \langle B \rangle \langle E \rangle p$$

Soundness and completeness???

Soundness: all equations are valid

SIMPLE
Inter-definability equations

Notation: $X_1 X_2 \ldots X_n$ will denote the fragment of HS containing the modalities $\langle X_1 \rangle, \langle X_2 \rangle, \ldots, \langle X_n \rangle$

\[
\langle L \rangle p \equiv \langle A \rangle \langle A \rangle p \quad \langle L \rangle \sqsubseteq A \quad \langle L \rangle \sqsubseteq \overline{A}
\]
\[
\langle O \rangle p \equiv \langle E \rangle \langle \overline{B} \rangle p \quad \langle O \rangle \sqsubseteq \overline{EB} \quad \langle O \rangle \sqsubseteq \overline{EB}
\]
\[
\langle D \rangle p \equiv \langle E \rangle \langle B \rangle p \quad \langle D \rangle \sqsubseteq EB \quad \langle D \rangle \sqsubseteq EB
\]
\[
\langle L \rangle p \equiv \langle \overline{B} \rangle [E] \langle \overline{B} \rangle \langle E \rangle p \quad \langle L \rangle \sqsubseteq \overline{BE} \quad \langle L \rangle \sqsubseteq \overline{BE}
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Soundness and completeness???

Soundness: all equations are valid
Completeness: there are no more inter-definability equations

SIMPLE
Inter-definability equations

Notation: \( X_1X_2\ldots X_n \) will denote the fragment of HS containing the modalities \( \langle X_1 \rangle, \langle X_2 \rangle, \ldots, \langle X_n \rangle \).

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\langle L \rangle p \equiv \langle A \rangle \langle A \rangle p \\
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\langle L \rangle p \equiv \langle B \rangle [E] \langle B \rangle \langle E \rangle p
\]

Soundness and completeness???

Soundness: all equations are valid

Completeness: there are no more inter-definability equations
Bisimulation between interval structures

\[ Z \subseteq M_1 \times M_2 \text{ is a bisimulations wrt the fragment } X_1X_2\ldots X_n \text{ iff } \]
Bisimulation between interval structures

$Z \subseteq M_1 \times M_2$ is a bisimulations wrt the fragment $X_1X_2\ldots X_n$ iff

1. $Z$-related intervals satisfy the same propositional letters, i.e.:

$$(i_1, i_2) \in Z \Rightarrow (p \text{ is true over } i_1 \iff p \text{ is true over } i_2)$$
Bisimulation between interval structures

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   \]

2. the bisimulation relation is “preserved” by modal operators, i.e., for every modal operator \( \langle X \rangle \):

\[
\begin{align*}
M_1 & \quad i_1 & \quad i_1' \\
M_2 & \quad i_2
\end{align*}
\]

\[
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2. the bisimulation relation is “preserved” by modal operators, i.e., for every modal operator $\langle X \rangle$:

   $(i_1, i_2) \in Z 
   (i_1, i'_1) \in X

\[\begin{array}{c}
M_1 \\
\hdashline
Z \\
\hdashline
M_2 \\
\end{array}\]

\[\begin{array}{c}
i_1 \\
\hdashline
X \\
\hdashline
i'_1 \\
\end{array}\]

\[\begin{array}{c}
i_2 \\
\hdashline
\end{array}\]
Bisimulation between interval structures

$Z \subseteq M_1 \times M_2$ is a bisimulations wrt the fragment $X_1X_2 \ldots X_n$ iff

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   $(i_1, i_2) \in Z \Rightarrow (p \text{ is true over } i_1 \iff p \text{ is true over } i_2)$

2. the bisimulation relation is “preserved” by modal operators, i.e., for every modal operator $\langle X \rangle$:
   
   $\begin{cases} (i_1, i_2) \in Z \\ (i_1, i'_1) \in X \end{cases} \Rightarrow \exists i'_2 \text{ s.t.}$
Bisimulation between interval structures

$Z \subseteq M_1 \times M_2$ is a bisimulations wrt the fragment $X_1X_2\ldots X_n$ iff

1. $Z$-related intervals satisfy the same propositional letters, i.e.:
   
   $$(i_1, i_2) \in Z \Rightarrow (p \text{ is true over } i_1 \iff p \text{ is true over } i_2)$$

2. the bisimulation relation is “preserved” by modal operators, i.e., for every modal operator $\langle X \rangle$:

   $$\left\{ (i_1, i_2) \in Z \right\} \cap \left\{ (i_1', i_2') \in X \right\} \Rightarrow \exists i_2' \text{ s.t. } \left\{ (i_1', i_2') \in Z \right\} \cap \left\{ (i_2, i_2') \in X \right\}$$

\[\text{Diagram:}\]

\[\begin{array}{c}
\text{M}_1
\end{array}\]

\[\begin{array}{c}
\text{Z}
\end{array}\]

\[\begin{array}{c}
\text{M}_2
\end{array}\]
Theorem Let \( Z \) be a bisimulation between \( M_1 \) and \( M_2 \) for the language \( \mathcal{L} \) and let \( i_1 \) and \( i_2 \) be intervals in \( M_1 \) and \( M_2 \), respectively. Then, truth of \( \mathcal{L} \)-formulae is preserved by \( Z \), i.e.,

If \( (i_1, i_2) \in Z \), then for every formula \( \varphi \) of \( \mathcal{L} \):

\[
M_1, i_1 \models \varphi \ \text{iff} \ M_2, i_2 \models \varphi
\]
How to use bisimulations to disprove definability

Suppose that we want to prove:

\(<X>\) is not definable in terms of \(\mathcal{L}\)
Suppose that we want to prove:

\[
\langle X \rangle \text{ is not definable in terms of } \mathcal{L}
\]

We must provide:

1. two models \( M_1 \) and \( M_2 \)
How to use bisimulations to disprove definability

Suppose that we want to prove:

\[ \langle X \rangle \text{ is not definable in terms of } \mathcal{L} \]

We must provide:

1. two models \( M_1 \) and \( M_2 \)
2. a bisimulation \( Z \subseteq M_1 \times M_2 \) wrt fragment \( \mathcal{L} \)
How to use bisimulations to disprove definability

Suppose that we want to prove:

\[ \langle X \rangle \text{ is not definable in terms of } \mathcal{L} \]

We must provide:

1. two models \( M_1 \) and \( M_2 \)
2. a bisimulation \( Z \subseteq M_1 \times M_2 \) wrt fragment \( \mathcal{L} \)
3. two interval \( i_1 \in M_1 \) and \( i_2 \in M_2 \) such that
   a. \( i_1 \) and \( i_2 \) are \( Z \)-related
   b. \( M_1, i_1 \models \langle X \rangle p \) and \( M_2, i_2 \models \neg \langle X \rangle p \)
How to use bisimulations to disprove definability

Suppose that we want to prove:

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   b. \( M_1, i_1 \models \langle X \rangle p \) and \( M_2, i_2 \models \neg \langle X \rangle p \)

By contradiction
If \( \langle X \rangle \) is definable in terms of \( \mathcal{L} \), then \( \langle X \rangle p \) is.
How to use bisimulations to disprove definability

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   b. \( M_1, i_1 \models \langle X \rangle p \) and \( M_2, i_2 \models \neg \langle X \rangle p \)

By contradiction
If \( \langle X \rangle \) is definable in terms of \( \mathcal{L} \), then \( \langle X \rangle p \) is.
Truth of \( \langle X \rangle p \) should have been preserved by \( Z \), but \( \langle X \rangle p \) is true in \( i_1 \) (in \( M_1 \)) and false in \( i_2 \) (in \( M_2 \))
How to use bisimulations to disprove definability

Suppose that we want to prove:

$\langle X \rangle$ is not definable in terms of $\mathcal{L}$

We must provide:

1. two models $M_1$ and $M_2$
2. a bisimulation $Z \subseteq M_1 \times M_2$ wrt fragment $\mathcal{L}$
3. two interval $i_1 \in M_1$ and $i_2 \in M_2$ such that
   a. $i_1$ and $i_2$ are $Z$-related
   b. $M_1, i_1 \models \langle X \rangle p$ and $M_2, i_2 \models \neg \langle X \rangle p$

**By contradiction**

If $\langle X \rangle$ is definable in terms of $\mathcal{L}$, then $\langle X \rangle p$ is.

Truth of $\langle X \rangle p$ should have been preserved by $Z$, but $\langle X \rangle p$ is true in $i_1$ (in $M_1$) and false in $i_2$ (in $M_2$) $\Rightarrow$ contradiction
An example: the operator \( \langle D \rangle \)

Semantics:

\[
M, [a, b] \models \langle D \rangle \varphi \iff \exists c, d \text{ such that } a < c < d < b \text{ and } M, [c, d] \models \varphi
\]
An example: the operator $\langle D \rangle$

Semantics:

$M, [a, b] \models \langle D \rangle \varphi \overset{\text{def}}{\iff} \exists c, d \text{ such that } a < c < d < b \text{ and } M, [c, d] \models \varphi$

Operator $\langle D \rangle$ is definable in terms of BE

$\langle D \rangle \varphi \equiv \langle B \rangle \langle E \rangle \varphi$
An example: the operator $\langle D \rangle$

Semantics:

$M, [a, b] \vdash \langle D \rangle \varphi \overset{\text{def}}{\iff} \exists c, d \text{ such that } a < c < d < b \text{ and } M, [c, d] \vdash \varphi$

\[\frac{\langle D \rangle \varphi}{\varphi}\]

Operator $\langle D \rangle$ is definable in terms of BE

$\langle D \rangle \varphi \equiv \langle B \rangle \langle E \rangle \varphi$

To prove that $\langle D \rangle$ is not definable in terms of any other fragment, we must prove that:

1) $\langle D \rangle$ is not definable in terms of ALBO

2) $\langle D \rangle$ is not definable in terms of ALEO
\[ \langle D \rangle \] is not definable in terms of A
A bisimulation wrt fragment A but not D

Bisimulation wrt A (\( \mathcal{AP} = \{p\} \)):

- models: \( M_1 = \langle \mathbb{I}(\mathbb{N}), V_1 \rangle, M_2 = \langle \mathbb{I}(\mathbb{N}), V_2 \rangle \)
\[ \langle D \rangle \text{ is not definable in terms of A} \]

\[ \text{A bisimulation wrt fragment A but not D} \]

Bisimulation wrt A (\( \mathcal{AP} = \{ p \} \)):

- models: \( M_1 = \langle \mathbb{I}(\mathbb{N}), V_1 \rangle, M_2 = \langle \mathbb{I}(\mathbb{N}), V_2 \rangle \)
  - \( V_1(p) = \{[1, 2]\} \)
\[\langle D \rangle\] is not definable in terms of \(A\)

A bisimulation \(\text{wrt fragment } A\) but not \(D\)

Bisimulation \(\text{wrt } A\) \((\mathcal{AP} = \{p\})\):

- models: \(M_1 = \langle \mathbb{I}(\mathbb{N}), V_1 \rangle, M_2 = \langle \mathbb{I}(\mathbb{N}), V_2 \rangle\)
  - \(V_1(p) = \{[1, 2]\}\)
  - \(V_2(p) = \emptyset\)
⟨D⟩ is not definable in terms of A
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  - \(V_2(p) = \emptyset\)
- bisimulation relation \(Z\): \(([x, y], [w, z]) \in Z\) iff

\[
\begin{array}{ccccccccccc}
0 & 1 & p & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{array}
\]
$\langle D \rangle$ is not definable in terms of $A$

A bisimulation wrt fragment $A$ but not $D$

Bisimulation wrt $A$ ($AP = \{p\}$):
- models: $M_1 = \langle \mathbb{I}(\mathbb{N}), V_1 \rangle, M_2 = \langle \mathbb{I}(\mathbb{N}), V_2 \rangle$
  - $V_1(p) = \{[1, 2]\}$
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- bisimulation relation $Z$: $([x, y], [w, z]) \in Z$ iff
  1. $[x, y] = [w, z] = [0, 3]$
\langle D \rangle \text{ is not definable in terms of A}

A bisimulation wrt fragment A but not D

Bisimulation wrt A ($A\mathcal{P} = \{p\}$):

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- bisimulation relation $Z$: $([x, y], [w, z]) \in Z$ iff
  1. $[x, y] = [w, z] = [0, 3]$
  2. $[x, y] = [w, z]$ and $x \geq 3$

\[ \begin{array}{ccccccccc}
0 & 1 & p & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
0 & 1 & \neg p & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{array} \]
\( \langle D \rangle \) is not definable in terms of A

A bisimulation wrt fragment A but not D

Bisimulation wrt A (\( \mathcal{AP} = \{p\} \)):
  - models: \( M_1 = \langle \mathbb{I}(\mathbb{N}), V_1 \rangle, M_2 = \langle \mathbb{I}(\mathbb{N}), V_2 \rangle \)
    - \( V_1(p) = \{[1, 2]\} \)
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    1. \( [x, y] = [w, z] = [0, 3] \)
    2. \( [x, y] = [w, z] \text{ and } x \geq 3 \)

\[ M_1, [0, 3] \models \langle D \rangle p \text{ and } M_2, [0, 3] \models \neg\langle D \rangle p \]
\( \langle D \rangle \) is not definable in terms of A

A bisimulation wrt fragment A but not D

Bisimulation wrt A \((\mathcal{AP} = \{p\})\):

- models: \( M_1 = \langle \mathbb{I}(\mathbb{N}), V_1 \rangle, M_2 = \langle \mathbb{I}(\mathbb{N}), V_2 \rangle \)
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\end{array} \]

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\end{array} \]

\( M_1, [0, 3] \models \langle D \rangle p \) and \( M_2, [0, 3] \models \neg \langle D \rangle p \)

\( \Rightarrow \) the thesis
Outline

Introduction
  The Halpern and Shoham’s logic HS

Expressiveness of HS

The satisfiability problem for HS
  Undecidability

Classical extensions
  Metric extensions
  Hybrid extensions
  First-order extensions

Summary and perspectives
The satisfiability problem for HS

**Satisfiability problem for a logic** $\mathcal{L}$  Given an $\mathcal{L}$-formula $\varphi$, is $\varphi$ satisfiable, i.e., there exists a model and an interval in which $\varphi$ is true?
The satisfiability problem for HS

Satisfiability problem for a logic $L$ Given an $L$-formula $\varphi$, is $\varphi$ satisfiable, i.e., there exists a model and an interval in which $\varphi$ is true?

$L$ is decidable (wrt the satisfiability problem).

iff

for each formula it is possible to answer the question
The satisfiability problem for HS

**Satisfiability problem for a logic** \( \mathcal{L} \) Given an \( \mathcal{L} \)-formula \( \varphi \), is \( \varphi \) satisfiable, i.e., there exists a model and an interval in which \( \varphi \) is true?

\( \mathcal{L} \) is decidable (wrt the satisfiability problem) iff

for each formula it is possible to answer the question there exists a terminating algorithm that answer yes / not for any \( \varphi \)
The satisfiability problem for HS

**Satisfiability problem for a logic** $\mathcal{L}$ Given an $\mathcal{L}$-formula $\varphi$, is $\varphi$ satisfiable, i.e., there exists a model and an interval in which $\varphi$ is true?

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for each formula it is possible to answer the question there exists a terminating algorithm that answer yes / not for any $\varphi$

Expressive enough, yet decidable, HS fragments
The satisfiability problem for HS

Satisfiability problem for a logic $\mathcal{L}$ Given an $\mathcal{L}$-formula $\varphi$, is $\varphi$ satisfiable, i.e., there exists a model and an interval in which $\varphi$ is true?

$\mathcal{L}$ is decidable (wrt the satisfiability problem) iff for each formula it is possible to answer the question there exists a terminating algorithm that answer yes / not for any $\varphi$

Expressive enough, yet decidable, HS fragments

Classification of all HS fragments wrt (un)decidability
Maximal decidable HS fragments

- PNL ($\equiv A\bar{A}$) in general case

D. Bresolin, V. Goranko, A. Montanari, G. Sciavicco

*Propositional Interval Neighborhood Logic: Decidability, Expressiveness, and Undecidable Extensions.*

Maximal decidable HS fragments

- PNL ($\equiv A\overline{A}$) in general case
- $AB\overline{BL}$ (and $\overline{AE}\overline{EL}$) in general case

D. Bresolin, A. Montanari, P. Sala, G. Sciavicco

*What’s decidable about Halpern and Shoham’s interval logic?*
*The maximal fragment ABBL.*

Maximal decidable HS fragments

- PNL (≡ AÄ) in general case
- ABBL (and ÄEEL) in general case
- ABBA (and ÄEEA) over finite structures

A. Montanari, G. Puppis, P. Sala

*Maximal Decidable Fragments of Halpern and Shoham’s Modal Logic of Intervals.*

Maximal decidable HS fragments

- PNL ($\equiv A\bar{A}$) in general case
- $AB\bar{BL}$ (and $\bar{A}E\bar{E}\bar{L}$) in general case
- $AB\bar{BA}$ (and $\bar{A}E\bar{E}\bar{E}A$) over finite structures
- $D\bar{DB}\bar{B}\bar{L}\bar{L}$ over $\mathbb{Q}$

P. Sala

PhD thesis

2010
Weakest undecidable HS fragments

$AD, A\overline{D}, \overline{A}D, \overline{AD}$
Weakest undecidable HS fragments

$\text{AD, AD, } \overline{\text{AD, } \overline{\text{AD}}}$

$\text{BE, } \overline{\text{BE, } \overline{\text{BE, } \overline{\text{BE}}}}$
Weakest undecidable HS fragments

AD, A\overline{D}, \overline{A}D, \overline{AD}

BE, B\overline{E}, \overline{B}E, \overline{BE}

O (and \overline{O})
Weakest undecidable HS fragments

$\text{AD, AD, } \overline{\text{AD}}, \overline{\text{AD}}$

$\text{BE, BE, } \overline{\text{BE}}, \overline{\text{BE}}$

$\text{O (and } \overline{\text{O}})$

$\text{D (and } \overline{\text{D}} \text{) over discrete}$
Weakest undecidable HS fragments

$\text{AD}, \overline{\text{AD}}, \overline{\text{AD}}, \overline{\text{AD}}$ [in this thesis]

$\text{BE}, \overline{\text{BE}}, \overline{\text{BE}}, \overline{\text{BE}}$ [in this thesis]

$\text{O} \text{ (and } \overline{\text{O}}) \text{)}$ [in this thesis]

$\text{D} \text{ (and } \overline{\text{D}} \text{) over discrete}$ [Michaliszyn, Marcinkowski]

*The Ultimate Undecidability Result*

*for the Halpern-Shoham Logic*

*LICS 2011*
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Summary and perspectives
The Octant Tiling Problem

This is the problem of establishing whether a given finite set of tile types \( \mathcal{T} = \{t_1, \ldots, t_k\} \) can tile the 2nd octant of the integer plane:

\[
\mathcal{O} = \{(i, j) : i, j \in \mathbb{N} \land 0 \leq i \leq j\},
\]

while respecting the color constraints.
The Octant Tiling Problem

This is the problem of establishing whether a given finite set of tile types $\mathcal{T} = \{t_1, \ldots, t_k\}$ can tile the 2nd octant of the integer plane:

$$\mathcal{O} = \{(i,j) : i, j \in \mathbb{N} \land 0 \leq i \leq j\},$$

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The Octant Tiling Problem

This is the problem of establishing whether a given finite set of tile types $\mathcal{T} = \{t_1, \ldots, t_k\}$ can tile the 2nd octant of the integer plane:

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while respecting the color constraints.

Proposition The Octant Tiling Problem is undecidable.
The Octant Tiling Problem

This is the problem of establishing whether a given finite set of tile types $\mathcal{T} = \{t_1, \ldots, t_k\}$ can tile the 2nd octant of the integer plane:

$$\mathcal{O} = \{(i, j) : i, j \in \mathbb{N} \land 0 \leq i \leq j\},$$

while respecting the color constraints.

Proposition  The Octant Tiling Problem is undecidable.

Proof: by reduction from the tiling problem for $\mathbb{N} \times \mathbb{N}$, using König’s Lemma.
Undecidability of the interval logics via tiling: generic construction

1. Encoding of the octant
2. Encoding of the neighborhood relations
   - Right-neighborhood relation SIMPLE
   - Above-neighborhood relation HARD
Undecidability of the interval logics via tiling: generic construction

1. Encoding of the octant
2. Encoding of the neighborhood relations
   ▶ Right-neighborhood relation SIMPLE
   ▶ Above-neighborhood relation HARD

Encoding of the octant

▶ Force the existence of a unique infinite chain of unit-intervals on the linear order, which covers an initial segment of the interval model. (propositional letter u)

Unit intervals are used to place tiles and delimiting symbols.
Undecidability of the interval logics via tiling: generic construction

1. Encoding of the octant
2. Encoding of the neighborhood relations
   - Right-neighborhood relation SIMPLE
   - Above-neighborhood relation HARD

Encoding of the octant

- Force the existence of a unique infinite chain of unit-intervals on the linear order, which covers an initial segment of the interval model. (propositional letter u)

Unit intervals are used to place tiles and delimiting symbols.

- ID-intervals are then introduced to represent the layers of tiles. (propositional letter Id)
Undecidability of the interval logics via tiling: generic construction cont’d
Undecidability of the interval logics via tiling: generic construction cont’d

Each ID-interval must have the right number of tiles
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The most challenging part usually is to ensure that the consecutive ID-intervals match vertically: the Above-Neighbour relation.
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For that, auxiliary propositional letter up_rel can be used to connecting (endpoints of) two intervals representing tiles that are above connected in the octant.
Undecidability of the interval logics via tiling: generic construction completed

Eventually, we encode the given Octant tiling problem by specifying the matching conditions between intervals that are right-connected or above-connected.
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The specific part of the construction is to use the given fragment of HS to set the chain of unit intervals and to express all necessary properties of IDs, the propositional letters for correspondence intervals, and the tile matching conditions.
In summary: interval logics are generally undecidable, even under very weak assumptions.
Summary of (un)decidability results and outlook

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Summary of (un)decidability results and outlook

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- More statistics are available on the web page: https://itl.dimi.uniud.it/content/logic-hs
Outline

Introduction
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Expressiveness of HS

The satisfiability problem for HS
   Undecidability

Classical extensions
   Metric extensions
   Hybrid extensions
   First-order extensions

Summary and perspectives
Outline

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Summary and perspectives
PNL: syntax and semantics

Syntax

- PNL: \( \phi ::= p \mid \neg \phi \mid \phi \lor \phi \mid \langle A \rangle \phi \mid \langle \overline{A} \rangle \phi \)
Syntax

- PNL: $\varphi ::= p \mid \neg \varphi \mid \varphi \lor \varphi \mid \langle A \rangle \varphi \mid \langle \overline{A} \rangle \varphi$

Semantics

- Operators *meets* ($\langle A \rangle$) and *met-by* ($\langle \overline{A} \rangle$):

  ```plaintext
  meets:
  \[ \langle A \rangle \varphi \]
  ```
PNL: syntax and semantics

Syntax

PNL: $\varphi ::= p \mid \neg \varphi \mid \varphi \lor \varphi \mid \langle A\rangle \varphi \mid \langle \overline{A} \rangle \varphi$

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\[
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Semantics

- Operators *meets* \((\langle A \rangle)\) and *met-by* \((\langle \overline{A} \rangle)\):
  
  ```
  meets: \[
  \begin{array}{c}
  \langle A \rangle \varphi \\
  \varphi
  \end{array}
  \]
  
  met-by: \[
  \begin{array}{c}
  \varphi \\
  \langle \overline{A} \rangle \varphi
  \end{array}
  \]
  ```
Two types of metric extensions of interval logics over the integers:
Metric interval logics

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1. Extensions of the modal operators: $\langle A \rangle =^k, \langle A \rangle >^k, \langle A \rangle [^k, k']$, \ldots
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The former are definable in terms of the latter in PNL, e.g.:

$$\langle A \rangle^>k p := \langle A \rangle(p \land \text{len}^>k).$$
Two types of metric extensions of interval logics over the integers:

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**MPNL**: PNL extended with integer constraints for interval lengths.
Decidability of metric interval logic

**Theorem** Satisfiability in MPNL on $\mathbb{N}$ is decidable. It is \textit{NEXPTIME-complete} if the metric constraints are represented in unary, and \textit{in between EXPSPACE and 2NEXPTIME} if they are represented in binary.

D. Bresolin, D. Della Monica, V. Goranko, A. Montanari, G. Sciavicco

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D. Bresolin, D. Della Monica, V. Goranko, A. Montanari, G. Sciavicco

Exact complexity is an open problem
Relative expressive power of logics in MPNL

\[
\begin{align*}
\text{MPNL}^\geq & \quad \text{MPNL}^\triangleright \\
\text{MPNL}^\triangleright & \quad \text{MPNL}^0 \\
\text{MPNL}^0 & \quad \text{MPNL}^\leq \\
\text{MPNL}^\leq & \quad \text{MPNL}^\triangleleft
\end{align*}
\]
Relative expressive power of logics in MPNL
Relative expressive power of logics in MPNL

\[ \text{MPNL} \equiv \text{PNL} \]

In 2\text{NEXPTIME} \quad \text{EXPSPACE-hard}

\[ \text{MPNL} <,\varepsilon \quad \text{MPNL} -,\varepsilon \quad \text{MPNL} >,\varepsilon \]

\[ \text{MPNL} \equiv \text{PNL} \quad \text{MPNL} <,> \quad \text{MPNL} (^{()},\varepsilon) \quad \text{MPNL} ^{()},\varepsilon \]

\[ \text{MPNL} _{I} \quad \text{MPNL} _{I} ^{(())} \quad \text{MPNL} ^{()-},\varepsilon \]

\[ \text{MPNL} _{I} ^{(),\varepsilon} \quad \text{MPNL} _{I} ^{(),\varepsilon} \quad \text{MPNL} _{I} ^{(),-,\varepsilon} \quad \text{MPNL} _{I} ^{(),-,\varepsilon} \]

\[ \text{NEXPTIME}-complete \]
Relative expressive power of logics in MPNL

Decidability of $MPNL$: by small model property
Comparing expressiveness of metric fragments: by bisimulations
Outline

Introduction
The Halpern and Shoham’s logic HS

Expressiveness of HS

The satisfiability problem for HS
Undecidability

Classical extensions
Metric extensions
Hybrid extensions
First-order extensions

Summary and perspectives
Extending PNL

PNL
Extending PNL

NEXPTIME-co

PNL
Extending PNL

PNL + any HS operator

NEXPTIME-co → PNL
Extending PNL

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Undecidable

NEXPTIME-co

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NEXPTIME-co

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- NEXPTIME-co
- MPNL
  - between EXPSPACE and 2NEXPTIME
  - Undecidable
Extending PNL

- \(\text{PNL} + \) any HS operator
- NEXPTIME-co
- MPNL \(\rightarrow\) MPNL
- Undecidable
- between EXPSPACE and 2NEXPTIME
Extending PNL

- PNL + any HS operator
- NEXPTIME-co
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- MPNL
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- Undecidable
Extending PNL

- PNL + any HS operator
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- Hybrid Extensions
- MPNL → MPNL⁺
- Undecidable
Extending PNL

- PNL + any HS operator
- NEXPTIME-co
- MPNL
- Hybrid Extensions
- MPNL+ 
- Undecidable
- between EXPSPACE and 2NEXPTIME
- Undecidable
- ???
Possible hybrid extension of PNL and MPNL

Nominals are definable in PNL

(*Basic Hybrid PNL*)
Possible hybrid extension of PNL and MPNL

Binders over state variables (intervals) (Strongly Hybrid MPNL) lead to undecidability

Nominals are definable in PNL (Basic Hybrid PNL)
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Nominals are definable in PNL (Basic Hybrid PNL)
Possible hybrid extension of PNL and MPNL

Binders over state variables (intervals) (Strongly Hybrid MPNL) lead to undecidability

Binders over length of intervals (Weakly Hybrid MPNL)

Nominals are definable in PNL (Basic Hybrid PNL)
Weakly Hybrid MPNL (WHMPNL)

Metric constraints of MPNL use constants

\[ \text{len}_{\geq 5}, \text{len}>2, \ldots \]

WHMPNL allows one to store the length of the current interval and to refer to it in sub-formulae

\[ \downarrow_x (\ldots | = |x), \downarrow_x (\ldots | \leq |x), \ldots \]
Remark

- Constant metric constraints are inter-definable
- Hybrid metric constraints ARE NOT!!!
  (e.g.: you cannot define $\text{len} \leq x$ in terms of $\text{len} \geq x$)
Remark

- Constant metric constraints are inter-definable
- Hybrid metric constraints ARE NOT!!!
  (e.g.: you cannot define $\text{len}_{\leq x}$ in terms of $\text{len}_{=x}$)

Possible choices:

1. which subset of hybrid constraints among $\{<, \leq, =, \geq, >\}$
WHMPNL fragments

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1. which subset of \textit{hybrid} constraints among \{<, \leq, =, \geq, >\}
2. constant metric constraints are allowed or not (WHPNL or WHMPNL)
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  (e.g.: you cannot define \(\text{len}_{\leq x}\) in terms of \(\text{len}_{= x}\))

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3. how many length variables
WHMPNL fragments

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<table>
<thead>
<tr>
<th></th>
<th>set of hybrid constraints</th>
<th>constant constraints</th>
<th># of length variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>WHMPNL(&lt;,\leq,=,\geq,&gt;)</td>
<td>{&lt;, \leq, =, \geq, &gt;}</td>
<td>YES</td>
<td>unbounded</td>
</tr>
<tr>
<td>WHPNL(&lt;,=)</td>
<td>{&lt;, =}</td>
<td>NO</td>
<td>unbounded</td>
</tr>
<tr>
<td>WHPNL(&lt;)1</td>
<td>{&lt;}</td>
<td>NO</td>
<td>1</td>
</tr>
</tbody>
</table>
### WHMPNL fragments

**Remark**

- **Constant** metric constraints are inter-definable
- **Hybrid** metric constraints **ARE NOT!!!**
  (e.g.: you cannot define \( \text{len} \leq x \) in terms of \( \text{len} = x \))

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</tr>
<tr>
<td>WHPNL((&lt;))_1</td>
<td>( {&lt;} )</td>
<td>NO</td>
<td>1</td>
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</table>
The fragment $WHPNL(=)_1$

Reduction from the Finite Tiling Problem

This is the problem of establishing whether, for a given finite set of tile types $\mathcal{T} = \{t_1, \ldots, t_k\}$, there exists a finite rectangle $\mathcal{R}$ having the border colored with a fixed color such that $\mathcal{T}$ can tile $\mathcal{R}$ respecting the color constraints.
The fragment $WHPNL(\equiv)_1$

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Introduction
The Halpern and Shoham’s logic HS

Expressiveness of HS

The satisfiability problem for HS
Undecidability

Classical extensions
Metric extensions
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Summary and perspectives
Extending PNL

PNL +
any HS operator

Undecidable

NEXPTIME-co

between EXPSPACE and 2NEXPTIME

Hybrid Extensions

Metric PNL

MPNL$^+$

Undecidable
Extending PNL

- PNL + any HS operator
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Undecidable
First-Order together with Propositional

F O R P N L

First-Order Right Propositional Neighborhood Logic
First-Order together with Propositional Neighborhood Logic
First-Order together with Propositional Neighborhood Logic
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First-Order together with Propositional

1. Propositional (modal) setting
First-Order together with Propositional

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   - predicates over elements
   - existential and universal quantifications
First-Order together with Propositional

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F O R P N L

First-Order Right Propositional Neighborhood Logic

1. Propositional (modal) setting
2. First-Order setting
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3. Propositional (modal) + First-Order setting
Parameters of the logic

- Temporal domain: discrete, dense, finite, bounded, unbounded, ...
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- First-order constructs:
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  - individual variables $x, y, \ldots$
  - individual constants $a, b, \ldots$
  - function $f(\ldots), g(\ldots), \ldots$
  - quantifiers
  - terms $t_1, t_2, \ldots$ (variables, constants, and functions)
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    \[ \text{terms} = \text{variables} \]
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Terms = variables

For tight undecidability only 1 variable (no free variables)
Undecidability of FORPNL

Reduction from the Finite Tiling Problem
This is the problem of establishing whether, for a given finite set of tile types $\mathcal{T} = \{t_1, \ldots, t_k\}$, there exists a finite rectangle $R$ having the border colored with a fixed color ■ such that $\mathcal{T}$ can tile $R$ respecting the color constraints.

\[
\begin{array}{|c|c|c|}
\hline
\multicolumn{3}{|c|}{\text{Finite Rectangle}} \\
\hline
\end{array}
\]
Reduction from the Finite Tiling Problem

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It is possible to simulate HS operators \( \langle B \rangle \langle E \rangle \langle D \rangle \).
Extending PNL: the final picture

PNL + any HS operator

First-Order Extensions

PNL

Hybrid Extensions

PNL

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Introduction
   The Halpern and Shoham’s logic HS

Expressiveness of HS

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Classical extensions
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- Expressiveness of HS fragments
Summary and perspectives

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Not discussed, and not yet explored, but important:

- Model checking of Interval logics
- Automata-based techniques for interval logics
Exams and attended courses

- **Exams**
  - “Constraint Programming and NMR Constraints for Determining Protein Structure”, A. Dovier
  - GAMES Spring School 2009
  - “Systems Biology”, A. Policriti/M. Miculan
  - “Computational Complexity (Complessità computazionale)”, R. Rizzi
  - “Introduction to Software Configuration Management”, L. Bendix

- **Other courses**
  - “(Meta-)Modeling with UML and OCL”, M. Gogolla
  - “Data Mining and Mathematical Programming”, P. Serafini
  - “Sistemi Reattivi: automi, logica, algoritmi” (Master Course), A. Montanari
  - English course for academic purposes (CLAV)
Other activities

- **Summer school**
  - International Lipari Summer School 2008 on “Algorithms: Science and Engineering”
  - GAMES Spring School 2009 (Bertinoro)

- **Visiting**
  - Oct - Dec 2009: University of Murcia - Murcia, Spain (G. Sciavicco)
  - Sept - Nov 2010: Technical University of Denmark (DTU) - Lyngby, Copenhagen, Denmark (V. Goranko)

- **Events organization**
  - Annual Workshop of the ESF Networking Programme on Games for Design and Verification (GAMES 2009)
  - First International Symposium on Games, Automata, Logics and Formal Verification (GandALF 2010)
  - Second International Symposium on Games, Automata, Logics and Formal Verification (GandALF 2011)
Publications


**The end.**