# Input-Output Finite-Time Stabilization with Constrained Control Inputs

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# Outline

# Input-output finite-time stability vs classic IO stability

## IO stability

A system is said to be IO $\mathcal{L}_p$-stable if for any input of class $\mathcal{L}_p$, the system exhibits a corresponding output which belongs to the same class

## IO-FTS

A system is defined to be IO-FTS if, given a class of norm bounded input signals over a specified time interval $T$, the outputs of the system do not exceed an assigned threshold during $T$

# Main features of IO-FTS

- IO-FTS:
  - involves signals defined over a finite time interval
  - does not necessarily require the inputs and outputs to belong to the same class
  - specifies a *quantitative* bounds on both inputs and outputs

IO stability and IO-FTS are independent concepts

# Contribution of the paper

- In this paper we provide extend the *classical* definition of IO-FTS to the one of structured IO-FTS
- Structured IO-FTS permits to incorporate amplitude constraints on the control input variables in the definition of the stabilization problem
- A necessary and sufficient condition is given for the solution of the IO finite-stabilization problem, when the input signals belong to $\mathcal{L}_2$
- A sufficient condition is given for the solution of the IO finite-stabilization problem, when the inputs belong to $\mathcal{L}_\infty$

# Notation

- $\mathcal{L}_p$ denotes the space of vector-valued signals whose $p$-th power is absolutely integrable over $[0, +\infty)$
- The restriction of $\mathcal{L}_p$ to $\Omega := [t_0\,, t_0 + T]$ is denoted by $\mathcal{L}_p(\Omega)$
- Given the time interval $\Omega$, a symmetric positive definite matrix-valued function $R(\cdot)$, bounded on $\Omega$, and a vector-valued signal $s(\cdot) \in \mathcal{L}_p(\Omega)$, the weighted signal norm

$$\left( \int_\Omega \left[ s^T(\tau) R(\tau) s(\tau) \right]^{\frac{p}{2}} d\tau \right)^{\frac{1}{p}} ,$$

will be denoted by $\|s(\cdot)\|_{p\,,R}$. If $p = \infty$

$$\|s(\cdot)\|_{\infty\,,R} = \operatorname*{ess\,sup}_{t \in \Omega} \left[ s^T(t) R(t) s(t) \right]^{\frac{1}{2}}$$

# Structured IO-FTS of LTV systems

- Let
  - $\mathcal{W}$ be a class of input signals defined over $\Omega = [t_0 \; , \; t_0 + T]$
  - $Q(t) := \text{diag}(Q_1(t), \ldots, Q_\alpha(t))$, with $Q_i(t) \in \mathbb{R}^{m_i \times m_i}$, $i = 1, \ldots, \alpha$, a positive definite matrix-valued function
- The system

$$\dot{x}(t) = A(t)x(t) + G(t)w(t)\,, \quad x(t_0) = 0 \tag{1a}$$

$$y(t) = C(t)x(t) + F(t)w(t) \tag{1b}$$

is said to be structured IO–FTS with respect to $(\mathcal{W}\,, Q(\cdot)\,, \Omega)$ if

$$w(\cdot) \in \mathcal{W} \;\Rightarrow\; y_i^T(t)Q_i(t)y_i(t) < 1\,, \qquad t \in \Omega\,, \qquad i = 1, \ldots, \alpha\,,$$

where the output vector $y(t)$ is partitioned as follows

$$y(t) = \left(y_1^T(t) \cdots y_\alpha^T(t)\right)^T\,, \quad t \in \Omega\,.$$

# The finite-time stabilization problem

- In the finite-time stabilization problem we consider the LTV system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + G(t)w(t)\,, \quad x(t_0) = 0 \tag{2a}$$
$$y(t) = C(t)x(t) + F(t)w(t) \tag{2b}$$

where $u(\cdot) : \ \Omega \mapsto \mathbb{R}^q$ is the control input and $w(\cdot)$ is the disturbance (exogenous) input

- Similarly to what has been done for the output, the control input vector $u(t)$ is partitioned as

$$u(t) = \left(u_1^T(t) \cdots u_\beta^T(t)\right)^T$$

# The IO finite-time stabilization problem via state feedback - 1

- Consider $\beta$ positive definite weighting matrix-valued functions $T_i(t) \in \mathbb{R}^{q_i \times q_i}$, $i = 1, \ldots, \beta$, and define

$$T(t) := \mathrm{diag}(T_1(t), \ldots, T_\beta(t))$$

# The IO finite-time stabilization problem via state feedback - 2

- Given a positive scalar $T$, the class of signals $\mathcal{W}$, and the weighting matrices $Q(\cdot)$, $T(\cdot)$, find a state feedback control law

$$u(t) = K(t)x(t)\,,$$

where $K(\cdot):\ \Omega \mapsto \mathbb{R}^{q\times n}$, such that the system

$$\dot{x}(t) = \left(A(t) + B(t)K(t)\right)x(t) + G(t)w(t) \tag{3a}$$

$$\begin{pmatrix} y(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} C_1(t) \\ \vdots \\ C_\alpha(t) \\ K_1(t) \\ \vdots \\ K_\beta(t) \end{pmatrix} x(t) + \begin{pmatrix} F_1(t) \\ \vdots \\ F_\alpha(t) \\ 0 \end{pmatrix} w(t) \tag{3b}$$

is structured IO–FTS with respect to $(\mathcal{W}, \operatorname{diag}\left(Q(\cdot), T(\cdot)\right), \Omega)$.

# The IO finite-time stabilization problem via state feedback - 3

Note that the partition

$$T(t) := \mathrm{diag}(T_1(t), \ldots, T_\beta(t))$$

induces the following structure for the controller gain

$$K(t) = \left(K_1^T(t) \cdots K_\beta^T(t)\right)^T , \quad t \in \Omega . \tag{4}$$

# Considered class of input signals

**$\mathcal{W}_2$ signals**

Norm bounded square integrable signals over $\Omega$, defined as follows

$$\mathcal{W}_2\,(\Omega\,, R(\cdot)) := \{w(\cdot) \in \mathcal{L}_2(\Omega)\ :\ \|w\|_{2,R} \leq 1\}\ .$$

**$\mathcal{W}_\infty$ signals**

Uniformly bounded signals over $\Omega$, defined as follows

$$\mathcal{W}_\infty\,(\Omega\,, R(\cdot)) := \{w(\cdot) \in \mathcal{L}_\infty(\Omega)\ :\ \|w\|_{\infty,R} \leq 1\}\,.$$

# Previous results

The analysis results presented in **Amato et al., Automatica 2010** and **Amato et al., TAC 2012** have been extended to the case of structured IO-FTS

F. Amato et al.
Input-output Finite-Time Stabilization of Linear Systems
*Automatica*, 2010

F. Amato et al.
Input-Output Finite-Time Stability of Linear Systems: Necessary and Sufficient Conditions
*IEEE Transactions on Automatic Control*, 2012

# Proper and strictly-proper linear systems

- For the class of $\mathcal{W}_2$ signals we consider a strictly proper system, i.e. $F(\cdot) = 0$, otherwise the concept of structured IO-FTS **would be ill-posed**.

  $\mathcal{W}_2$ includes signals that are unbounded on a zero measure interval included in $\Omega$. For those signals, if $F(\cdot) \neq 0$ then there exists at least one time instant where the output would be unbounded

- For the class of $\mathcal{W}_\infty$ signals we consider proper system, i.e. $F(\cdot) \neq 0$

# Structured IO-FTS for $\mathcal{W}_2$ signals

Given system (1) with $F(\cdot) = 0$, the class of inputs $\mathcal{W}_2$, a continuous positive definite matrix–valued function $Q(\cdot)$, and the time interval $\Omega$, the following statements are equivalent:

- **i)** System (1) is structured IO–FTS with respect to $\big(\mathcal{W}_2\,,\, Q(\cdot)\,,\,\Omega\big)$.
- **ii)** The inequality

$$\lambda_{\max}\Big(Q_i^{\frac{1}{2}}(t)C_i(t)W(t,t_0)C_i^T(t)Q_i^{\frac{1}{2}}(t)\Big) < 1 \tag{5}$$

holds for all $t \in \Omega$ and $i = 1,\dots,\alpha$, where $W(\cdot,\cdot)$ is the positive semidefinite solution of the DLE

$$\dot{W}(t\,,t_0) = A(t)W(t\,,t_0) + W(t\,,t_0)A^T(t) + G(t)R(t)^{-1}G^T(t) \tag{6a}$$

$$W(t_0\,,t_0) = 0 \tag{6b}$$

- **iii)** The coupled DLMI/LMI

$$\begin{pmatrix} \dot{P}(t) + A^T(t)P(t) + P(t)A(t) & P(t)G(t) \\ G^T(t)P(t) & -R(t) \end{pmatrix} < 0 \tag{7a}$$

$$P(t) \geq C_i^T(t)Q_i(t)C_i(t)\,, \quad i = 1\,,\dots\,,\alpha\,, \tag{7b}$$

admits a positive definite solution $P(\cdot)$ over $\Omega$.

# Structured IO-FTS for $\mathcal{W}_\infty$ signals

Let $\widetilde{Q}_i(t) = (t - t_0)Q_i(t)$; if there exist a positive definite and continuously differentiable matrix-valued function $P(\cdot)$ and $\alpha$ scalar functions $\theta_1(\cdot), \dots, \theta_\alpha(t) > 1$ such that the coupled DLMI/LMI

$$\begin{pmatrix} \dot{P}(t) + A^T(t)P(t) + P(t)A(t) & P(t)G(t) \\ G^T(t)P(t) & -R(t) \end{pmatrix} < 0\,, \tag{8a}$$

$$\theta_i(t)R(t) - R(t) \geq 2\,\theta_i(t)F_i^T(t)Q_i(t)F_i(t)\,, \quad i = 1, \dots, \alpha\,, \tag{8b}$$

$$P(t) \geq 2\,\theta_i(t)C_i(t)^T\widetilde{Q}_i(t)C_i(t)\,, \quad i = 1, \dots, \alpha\,, \tag{8c}$$

are fulfilled over $\Omega$, then system (1) is IO–FTS with respect to $(\mathcal{W}_\infty\,, Q(\cdot)\,, \Omega)$.

# IO finite-time stabilization for $\mathcal{W}_2$ signals

Given the class of disturbances $\mathcal{W}_2$ and $F(\cdot) = 0$, the IO finite-time stabilization problem via state feedback is solvable **if and only if** there exist a positive definite and continuously differentiable matrix–valued function $\Pi(\cdot)$, and $\beta$ continuously differentiable matrix–valued functions $L_1(\cdot), \dots, L_\beta(\cdot)$ such that,

$$\begin{pmatrix} \Theta(t) & G(t) \\ G^T(t) & -R(t) \end{pmatrix} < 0 \,, \tag{9a}$$

$$\begin{pmatrix} \Pi(t) & \Pi(t)C_i^T(t) \\ C_i(t)\Pi(t) & \Xi_i(t) \end{pmatrix} \geq 0 \,, \quad i = 1, \dots, \alpha \tag{9b}$$

$$\begin{pmatrix} \Pi(t) & L_j^T(t) \\ L_j(t) & \Upsilon_j(t) \end{pmatrix} \geq 0 \,, \quad j = 1, \dots, \beta \tag{9c}$$

for all $t \in \Omega$, with

$$\Theta(t) := -\dot{\Pi}(t) + \Pi(t)A^T(t) + A(t)\Pi(t) + B(t)\left(L_1^T(t)\cdots L_\beta^T(t)\right)^T + \left(L_1^T(t)\cdots L_\beta^T(t)\right)B^T(t) \,,$$

$\Xi_i(t) := Q_i^{-1}(t)$, and $\Upsilon_j(t) := T_j^{-1}(t)$.

The controller gain which solves the IO finite-time stabilization problem via state feedback is given by (4) with $K_j(t) = L_j(t)\Pi^{-1}(t)\,, j = 1, \dots, \beta$.

# Sketch of proof - 1

Conditions (7) for the augmented output closed–loop system (3) read

$$\begin{pmatrix} \dot{P}(t) + A_{cl}^T(t)P(t) + P(t)A_{cl}(t) & P(t)G(t) \\ G^T(t)P(t) & -R(t) \end{pmatrix} < 0\,, \tag{10a}$$

$$P(t) \geq C_i^T(t)Q_i(t)C_i(t)\,, \quad i = 1\,, \ldots\,, \alpha \tag{10b}$$

$$P(t) \geq K_j^T(t)T_j(t)K_j(t)\,, \quad j = 1\,, \ldots\,, \beta\,, \tag{10c}$$

where

$$A_{cl}(\cdot) = A(\cdot(+B(\cdot)K(\cdot)$$

# Sketch of proof - 2

Let $\Pi(t) = P^{-1}(t)$. By pre- and post-multiplying (10a) by $\begin{pmatrix} \Pi(t) & 0 \\ 0 & I \end{pmatrix} > 0$, and by pre- and post-multiplying (10b) and (10c) by $\Pi(t)$, we have

$$\begin{pmatrix} -\dot{\Pi}(t) + \Pi(t)A_{cl}^T(t) + A_{cl}(t)\Pi(t) & G(t) \\ G^T(t) & -R(t) \end{pmatrix} < 0\,, \tag{11a}$$

$$\begin{pmatrix} \Pi(t) & \Pi(t)C_i^T(t) \\ C_i(t)\Pi(t) & \Xi_i(t) \end{pmatrix} \geq 0\,, \quad i = 1\,, \ldots\,, \alpha \tag{11b}$$

$$\begin{pmatrix} \Pi(t) & \Pi(t)K_j^T(t) \\ K_j(t)\Pi(t) & \Upsilon_j(t) \end{pmatrix} \geq 0\,, \quad j = 1\,, \ldots\,, \beta \tag{11c}$$

where (11b) and (11c) are obtained by applying the Schur complements. The proof of the theorem then readily follows by letting $L_j(t) = K_j(t)\Pi(t)$ for $j = 1\,, \ldots\,, \beta$.

# IO finite-time stabilization for $\mathcal{W}_\infty$ signals

Given the class of disturbances $\mathcal{W}_\infty$, the IO finite-time stabilization problem via state feedback is solvable **if** there exist a positive definite and continuously differentiable matrix–valued function $\Pi(\cdot)$, $\beta$ continuously differentiable matrix–valued functions $L_1(\cdot), \dots, L_\beta(\cdot)$, and $\alpha$ strictly positive functions $\lambda_1(\cdot), \dots, \lambda_\alpha(\cdot) < 1$ such that (9a) and

$$R(t) - \lambda_i(t)R(t) \geq 2\, F_i^T(t)Q_i(t)F_i(t)\,, \quad i = 1, \dots, \alpha \tag{12a}$$

$$\begin{pmatrix} \Pi(t) & \Pi(t)C_i^T(t) \\ C_i(t)\Pi(t) & \frac{\lambda_i(t)}{2}\widetilde{\Xi}_i(t) \end{pmatrix} \geq 0\,, \quad i = 1, \dots, \alpha \tag{12b}$$

$$\begin{pmatrix} \Pi(t) & L_j^T(t) \\ L_j(t) & \widetilde{\Upsilon}_j(t) \end{pmatrix} \geq 0\,, \quad j = 1, \dots, \beta \tag{12c}$$

hold, when $t \in \Omega$, with $\widetilde{\Xi}_i(t) := \left((t - t_0)Q_i(t)\right)^{-1}$, and $\widetilde{\Upsilon}_j(t) := \left((t - t_0)T_j(t)\right)^{-1}$. The controller gain which solves the IO finite-time stabilization problem via state feedback is given by (4) with $K_j(t) = L_j(t)\Pi^{-1}(t)\,, j = 1, \dots, \beta$.

# Quarter car suspension model

- $M_s$ sprung mass
- $M_u$ unsprung mass
- $B_s$ suspension damping coefficient
- $K_s$ suspension spring elastic coefficient
- $K_u$ elastic coefficient that models tire deflection
- $u_f$ active force generated by the hydraulic actuator $S$

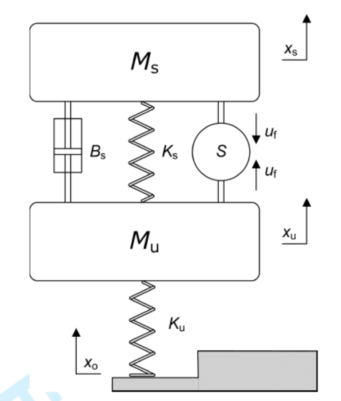


**Figure:** Schematic representation of the active suspension system.

# Model

Letting

- $x_s$ and $x_u$ the vertical displacement of the sprung and unsprung masses, respectively
- $x_o$ the vertical ground displacement caused by road unevenness

and choosing as state variables

- the suspension stroke $x_s - x_u$
- the tire deflection $x_u - x_o$

and their derivatives The resulting open-loop dynamical model reads

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -\frac{K_s}{M_s} & -\frac{B_s}{M_s} & 0 & \frac{B_s}{M_s} \\ 0 & 0 & 0 & 1 \\ \frac{K_s}{M_u} & \frac{B_s}{M_u} & -\frac{K_u}{M_u} & -\frac{B_s}{M_u} \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \frac{u_{\max}}{M_s} \\ 0 \\ -\frac{u_{\max}}{M_u} \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} w(t), \quad (13)$$

where the normalized active force $u(\cdot) = u_f(\cdot)/u_{\max}$ is the control input and the exogenous input $w(\cdot) = \dot{x}_o(\cdot)$ represents the disturbance caused by the road roughness.

# Design constraints

When designing a controller a number of constraints should be considered.

- To ensure a firm uninterrupted contact of wheels to road, the dynamic tire load should not exceed the static one

$$k_u|x_3(t)| < (m_s + m_u)\, g \quad \forall\ t \geq 0\,. \tag{14}$$

- The suspension stroke should fulfill the following constraint

$$|x_1(t)| \leq SS\,, \quad \forall\ t \geq 0\,. \tag{15}$$

In order to cast the control design problem in the IO-FTS framework, we consider the following system outputs

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = \begin{pmatrix} \dot{x}_2(t) \\ \frac{x_1(t)}{SS} \\ \frac{k_u x_3(t)}{g(m_s+m_u)} \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} x(t) + \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} u\,, \tag{16}$$

where

$$C_1 = \begin{pmatrix} -\frac{k_s}{m_s} & -\frac{c_s}{m_s} & 0 & \frac{c_s}{m_s} \end{pmatrix}, \qquad D_1 = \frac{u_{\max}}{m_s}\,,$$

$$C_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}, \qquad D_2 = 0\,,$$

$$C_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}, \qquad D_3 = 0\,.$$

# Model parameters

The following values for the model parameters have been taken from **Chen and Guo, TCST, 2005**

$$M_s = 320\ kg\,, \qquad K_s = 18\ \frac{kN}{m}\,,$$

$$B_s = 1\ \frac{kN \cdot s}{m}\,, \qquad K_u = 200\ \frac{kN}{m}\,,$$

$$M_u = 40\ kg\,, \qquad u_{\max} = 1.5\ kN\,,$$

$$SS = 0.08\ m\,,$$

# Actuator saturation

Due to actuator saturation, the active force is bounded by $u_{\max}$, i.e. the normalized force has to satisfy

$$|u(t)| \leq 1\,, \quad \forall\; t \geq 0\,. \tag{17}$$

In order to frame the problem of designing the active suspension control system in the context of structured IO finite-time stabilization let us rewrite the output equation as

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} \dot{x}_2(t) \\ \frac{x_1(t)}{SS} \\ \frac{k_u x_3(t)}{g(m_s+m_u)} \\ K(t)x(t) \end{pmatrix} = \begin{pmatrix} C_1 + D_1 K(t) \\ C_2 + D_2 K(t) \\ C_3 + D_3 K(t) \\ K(t) \end{pmatrix} x(t)\,. \tag{18}$$

# Reference disturbance

We will design the time–varying state feedback $K(t)$ that *optimize* the response to an isolated bump modeled as the $\mathcal{W}_2$ disturbance

$$w(t) = \begin{cases} \frac{M}{2}\left(1 - \cos\left(\frac{2\pi V}{L} t\right)\right), & 0 \le t \le \frac{L}{V} \\ 0, & t > \frac{L}{V} \end{cases} \quad (19)$$

where $M = 0.1\ m$, $L = 5\ m$ are the bump height and width, respectively, while $V = 45\ km/h$ is the vehicle forward velocity.

In particular, given the bump (19) we want to minimize the body acceleration $y_1(t) = \dot{x}_2(t)$ fulfilling the constraints (14)–(17).

# IO-FTS parameters

We consider the following IO-FTS parameters

$$T = 2\ s\,, \quad R = 8\,.$$

Furthermore, given the selected outputs (18), the two outputs weighting matrices

$$Q_2 = Q_3 = 1\,,$$

allows to take into account the constraints (14) and (15), while the input weighting matrix is

$$T_1 = 0.15\,,$$

which allows to exploit the full scale of the control input when (19) is considered.

In order to minimize the body acceleration it is possible to exploit Theorem 1 and solve the following optimization problem

$$\begin{array}{l} \text{minimize } \Xi_1 \\ \text{subject to } (9) \end{array} \qquad (20)$$

where $\Xi_1 = Q_1^{-1}$.

# Solving the problem

- Assuming the two matrix–valued functions $\Pi(\cdot)$ and $L(\cdot)$ to be piecewise linear, it is possible to recast problem (20) in the LMIs framework.
- By solving (20), we get $\Xi_{1_{\min}} = 7.22$ and the two feasible matrix–valued functions $\Pi(\cdot)$ and $L(\cdot)$; the time–varying controller $K(t)$ is then given by $K(t) = L(t)\Pi(t)^{-1}$.

# Results - 1

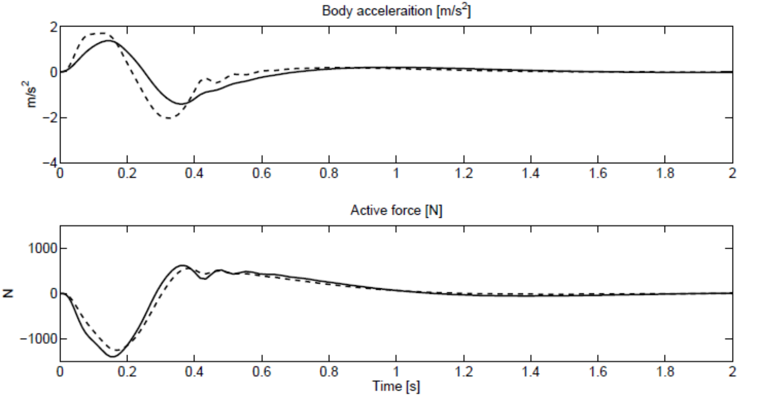


**Figure:** Bump response: IO-FTS time-varying controller (–), constrained $\mathcal{H}_\infty$ controller (- -, Chen and Guo, TCST, 2005).

# Results - 2

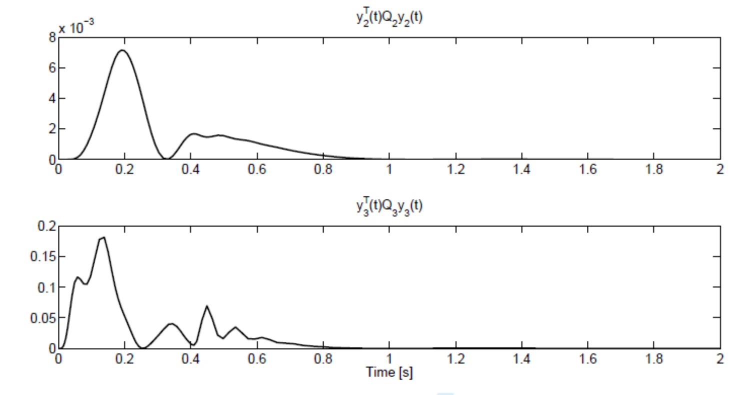


**Figure:** Bump response: time behavior of the weighted outputs $y_2(t)^T Q_2 y_2(t)$ and $y_3(t)^T Q_3 y_3(t)$ when the IO-FTS time–varying controller is considered.

# Conclusions

- The notion of structured IO-FTS has been introduced
- Structured IO-FTS allows to take into account design constraint on the control input
- Conditions for IO finite-time stabilization (in the *structured context*) of LTV systems via state feedback have been given
- The effectiveness of the approach has been illustrated by means of an engineering case-study

**Thank you!**