Input-Output Finite-Time Stabilization with Constrained Control Inputs

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Outline

1 Motivations

2 Structured Input-Output Finite-Time Stability
   - Notation
   - Problem Statement

3 Analysis Results

4 Synthesis Results
   - Theorem 1
   - Theorem 2

5 Example
Input-output finite-time stability vs classic IO stability

**IO stability**

A system is said to be IO $\mathcal{L}_p$-stable if for any input of class $\mathcal{L}_p$, the system exhibits a corresponding output which belongs to the same class.

**IO-FTS**

A system is defined to be IO-FTS if, given a class of norm bounded input signals over a specified time interval $T$, the outputs of the system do not exceed an assigned threshold during $T$. 
Main features of IO-FTS

- IO-FTS:
  - involves signals defined over a finite time interval
  - does not necessarily require the inputs and outputs to belong to the same class
  - specifies a *quantitative* bounds on both inputs and outputs

IO stability and IO-FTS are independent concepts
Motivations

Contribution of the paper

- In this paper we provide extend the *classical* definition of IO-FTS to the one of *structured* IO-FTS.
- Structured IO-FTS permits to incorporate *amplitude constraints on the control input variables* in the definition of the stabilization problem.
- A *necessary and sufficient condition* is given for the solution of the IO finite-stabilization problem, when the input signals belong to $\mathcal{L}_2$.
- A *sufficient condition* is given for the solution of the IO finite-stabilization problem, when the inputs belong to $\mathcal{L}_\infty$. 
Structured IO-FTS

Notation

- $\mathcal{L}_p$ denotes the space of vector-valued signals whose $p$-th power is absolutely integrable over $[0, +\infty)$.
- The restriction of $\mathcal{L}_p$ to $\Omega := [t_0, t_0 + T]$ is denoted by $\mathcal{L}_p(\Omega)$.
- Given the time interval $\Omega$, a symmetric positive definite matrix-valued function $R(\cdot)$, bounded on $\Omega$, and a vector-valued signal $s(\cdot) \in \mathcal{L}_p(\Omega)$, the weighted signal norm

$$\left( \int_{\Omega} \left[ s^T(\tau) R(\tau) s(\tau) \right]^{\frac{p}{2}} d\tau \right)^{\frac{1}{p}},$$

will be denoted by $\|s(\cdot)\|_{p,R}$. If $p = \infty$

$$\|s(\cdot)\|_{\infty,R} = \text{ess sup}_{t \in \Omega} \left[ s^T(t) R(t) s(t) \right]^{\frac{1}{2}}.$$
Structured IO-FTS of LTV systems

Let

- $\mathcal{W}$ be a class of input signals defined over $\Omega = [t_0, t_0 + T]$.
- $Q(t) := \text{diag}(Q_1(t), \ldots, Q_\alpha(t))$, with $Q_i(t) \in \mathbb{R}^{m_i \times m_i}$, $i = 1, \ldots, \alpha$, a positive definite matrix-valued function.

The system

$$
\dot{x}(t) = A(t)x(t) + G(t)w(t), \quad x(t_0) = 0 \quad (1a)
$$
$$
y(t) = C(t)x(t) + F(t)w(t) \quad (1b)
$$

is said to be structured IO–FTS with respect to $(\mathcal{W}, Q(\cdot), \Omega)$ if

$$
w(\cdot) \in \mathcal{W} \Rightarrow y_i^T(t)Q_i(t)y_i(t) < 1, \quad t \in \Omega, \quad i = 1, \ldots, \alpha,
$$

where the output vector $y(t)$ is partitioned as follows

$$
y(t) = (y_1^T(t) \cdots y_\alpha^T(t))^T, \quad t \in \Omega.
$$
The finite-time stabilization problem

- In the finite-time stabilization problem we consider the LTV system

\[
\dot{x}(t) = A(t)x(t) + B(t)u(t) + G(t)w(t), \quad x(t_0) = 0 \tag{2a}
\]

\[
y(t) = C(t)x(t) + F(t)w(t) \tag{2b}
\]

where \( u(\cdot) : \Omega \mapsto \mathbb{R}^q \) is the control input and \( w(\cdot) \) is the disturbance (exogenous) input.

- Similarly to what has been done for the output, the control input vector \( u(t) \) is partitioned as

\[
u(t) = (u_1^T(t) \cdots u_\beta^T(t))^T
\]
The IO finite-time stabilization problem via state feedback - 1

Consider $\beta$ positive definite weighting matrix-valued functions $T_i(t) \in \mathbb{R}^{q_i \times q_i}, i = 1, \ldots, \beta$, and define

$$T(t) := \text{diag}(T_1(t), \ldots, T_\beta(t))$$
The IO finite-time stabilization problem via state feedback - 2

Given a positive scalar $T$, the class of signals $\mathcal{W}$, and the weighting matrices $Q(\cdot), T(\cdot)$, find a state feedback control law

$$u(t) = K(t)x(t),$$

where $K(\cdot) : \Omega \mapsto \mathbb{R}^{q \times n}$, such that the system

$$\dot{x}(t) = (A(t) + B(t)K(t))x(t) + G(t)w(t)$$

is structured IO–FTS with respect to $(\mathcal{W}, \text{diag}(Q(\cdot), T(\cdot)), \Omega)$.
The IO finite-time stabilization problem via state feedback - 3

Note that the partition

\[ T(t) := \text{diag}(T_1(t), \ldots, T_\beta(t)) \]

induces the following structure for the controller gain

\[ K(t) = (K_1^T(t) \cdots K_\beta^T(t))^T, \quad t \in \Omega. \] (4)
Considered class of input signals

\( \mathcal{W}_2 \) signals

Norm bounded square integrable signals over \( \Omega \), defined as follows

\[ \mathcal{W}_2 (\Omega, R(\cdot)) := \{ w(\cdot) \in L_2(\Omega) : \| w \|_{2,R} \leq 1 \} \cdot \]

\( \mathcal{W}_\infty \) signals

Uniformly bounded signals over \( \Omega \), defined as follows

\[ \mathcal{W}_\infty (\Omega, R(\cdot)) := \{ w(\cdot) \in L_\infty(\Omega) : \| w \|_{\infty,R} \leq 1 \} \cdot \]
The analysis results presented in Amato et al., Automatica 2010 and Amato et al., TAC 2012 have been extended to the case of structured IO-FTS.

- F. Amato et al.
  Input-output Finite-Time Stabilization of Linear Systems
  *Automatica, 2010*

- F. Amato et al.
  Input-Output Finite-Time Stability of Linear Systems: Necessary and Sufficient Conditions
  *IEEE Transactions on Automatic Control, 2012*
Proper and strictly-proper linear systems

- For the class of $\mathcal{W}_2$ signals we consider a strictly proper system, i.e. $F(\cdot) = 0$, otherwise the concept of structured IO-FTS would be ill-posed.

$\mathcal{W}_2$ includes signals that are unbounded on a zero measure interval included in $\Omega$. For those signals, if $F(\cdot) \neq 0$ then there exists at least one time instant where the output would be unbounded.

- For the class of $\mathcal{W}_\infty$ signals we consider proper system, i.e. $F(\cdot) \neq 0$
Structured IO-FTS for $\mathcal{W}_2$ signals

Given system (1) with $F(\cdot) = 0$, the class of inputs $\mathcal{W}_2$, a continuous positive definite matrix–valued function $Q(\cdot)$, and the time interval $\Omega$, the following statements are equivalent:

i) System (1) is structured IO–FTS with respect to $(\mathcal{W}_2, Q(\cdot), \Omega)$.

ii) The inequality

$$
\lambda_{\max}\left(Q_i^{\frac{1}{2}}(t)C_i(t)W(t, t_0)C_i^T(t)Q_i^{\frac{1}{2}}(t)\right) < 1
$$

holds for all $t \in \Omega$ and $i = 1, \ldots, \alpha$, where $W(\cdot, \cdot)$ is the positive semidefinite solution of the DLE

$$
\dot{W}(t, t_0) = A(t)W(t, t_0) + W(t, t_0)A^T(t) + G(t)R(t)^{-1}G^T(t)
$$

$W(t_0, t_0) = 0$ (6a) (6b)

iii) The coupled DLMI/LMI

$$
\begin{pmatrix}
P(t) + A^T(t)P(t) + P(t)A(t) & P(t)G(t) \\
G^T(t)P(t) & -R(t)
\end{pmatrix} < 0
$$

$$
P(t) \geq C_i^T(t)Q_i(t)C_i(t), \quad i = 1, \ldots, \alpha
$$

admits a positive definite solution $P(\cdot)$ over $\Omega$. 
Structured IO-FTS for $\mathcal{W}_\infty$ signals

Let $\tilde{Q}_i(t) = (t - t_0)Q_i(t)$; if there exist a positive definite and continuously differentiable matrix-valued function $P(\cdot)$ and $\alpha$ scalar functions $\theta_1(\cdot), \ldots, \theta_{\alpha}(t) > 1$ such that the coupled DLMI/LMI

$$
\begin{pmatrix}
\dot{P}(t) + A^T(t)P(t) + P(t)A(t) & P(t)G(t) \\
G^T(t)P(t) & -R(t)
\end{pmatrix} < 0,
$$

$$
\theta_i(t)R(t) - R(t) \geq 2 \theta_i(t)F_i^T(t)Q_i(t)F_i(t),
$$

$$
P(t) \geq 2 \theta_i(t)C_i(t)^T\tilde{Q}_i(t)C_i(t), \quad i = 1, \ldots, \alpha,
$$

are fulfilled over $\Omega$, then system (1) is IO–FTS with respect to $(\mathcal{W}_\infty, Q(\cdot), \Omega)$. 
Theorem 1

IO finite-time stabilization for $\mathcal{W}_2$ signals

Given the class of disturbances $\mathcal{W}_2$ and $F(\cdot) = 0$, the IO finite-time stabilization problem via state feedback is solvable if and only if there exist a positive definite and continuously differentiable matrix–valued function $\Pi(\cdot)$, and $\beta$ continuously differentiable matrix–valued functions $L_1(\cdot), \ldots, L_\beta(\cdot)$ such that,

\[
\begin{pmatrix}
\Theta(t) & G(t) \\
G^T(t) & -R(t)
\end{pmatrix} < 0, \tag{9a}
\]

\[
\begin{pmatrix}
\Pi(t) & \Pi(t)C_i^T(t) \\
C_i(t)\Pi(t) & \Xi_i(t)
\end{pmatrix} \geq 0, \quad i = 1, \ldots, \alpha \tag{9b}
\]

\[
\begin{pmatrix}
\Pi(t) & L_j^T(t) \\
L_j(t) & \Upsilon_j(t)
\end{pmatrix} \geq 0, \quad j = 1, \ldots, \beta \tag{9c}
\]

for all $t \in \Omega$, with

$\Theta(t) := -\dot{\Pi}(t) + \Pi(t)A^T(t) + A(t)\Pi(t) + B(t)\left(L_1^T(t) \cdots L_\beta^T(t)\right)^T + \left(L_1^T(t) \cdots L_\beta^T(t)\right)B^T(t),$ \\

$\Xi_i(t) := Q_i^{-1}(t),$ and $\Upsilon_j(t) := T_j^{-1}(t).$

The controller gain which solves the IO finite-time stabilization problem via state feedback is given by (4) with $K_j(t) = L_j(t)\Pi^{-1}(t), j = 1, \ldots, \beta.$
Sketch of proof - 1

Conditions (7) for the augmented output closed-loop system (3) read

\[
\begin{pmatrix}
\dot{P}(t) + A_{cl}^T(t)P(t) + P(t)A_{cl}(t) & P(t)G(t) \\
G^T(t)P(t) & -R(t)
\end{pmatrix} < 0, \\
P(t) \geq C_i^T(t)Q_i(t)C_i(t), \quad i = 1, \ldots, \alpha \\
P(t) \geq K_j^T(t)T_j(t)K_j(t), \quad j = 1, \ldots, \beta,
\]

where

\[A_{cl}(\cdot) = A(\cdot + B(\cdot)K(\cdot))\]
Sketch of proof - 2

Let $\Pi(t) = P^{-1}(t)$. By pre- and post-multiplying (10a) by 
$\begin{pmatrix} \Pi(t) & 0 \\ 0 & I \end{pmatrix} > 0$, and by pre- and post-multiplying (10b) and (10c) 
by $\Pi(t)$, we have

\begin{align*}
\begin{pmatrix} -\dot{\Pi}(t) + \Pi(t)A_{cl}^T(t) + A_{cl}(t)\Pi(t) & G(t) \\ G^T(t) & -R(t) \end{pmatrix} & < 0, \\
\begin{pmatrix} \Pi(t) & \Pi(t)C_i^T(t) \\ C_i(t)\Pi(t) & \Xi_i(t) \end{pmatrix} & \geq 0, \quad i = 1, \ldots, \alpha \\
\begin{pmatrix} \Pi(t) & \Pi(t)K_j^T(t) \\ K_j(t)\Pi(t) & \Upsilon_j(t) \end{pmatrix} & \geq 0, \quad j = 1, \ldots, \beta \end{align*}

(11a) (11b) (11c)

where (11b) and (11c) are obtained by applying the Schur complements. The proof of the theorem then readily follows by letting 
$L_j(t) = K_j(t)\Pi(t)$ for $j = 1, \ldots, \beta$. 
IO finite-time stabilization for $\mathcal{W}_\infty$ signals

Given the class of disturbances $\mathcal{W}_\infty$, the IO finite-time stabilization problem via state feedback is solvable if there exist a positive definite and continuously differentiable matrix–valued function $\Pi(\cdot)$, $\beta$ continuously differentiable matrix–valued functions $L_1(\cdot), \ldots, L_\beta(\cdot)$, and $\alpha$ strictly positive functions $\lambda_1(\cdot), \ldots, \lambda_\alpha(\cdot) < 1$ such that (9a) and

$$R(t) - \lambda_i(t)R(t) \geq 2 F_i^T(t) Q_i(t) F_i(t), \quad i = 1, \ldots, \alpha$$

$$(12a)$$

$$\begin{pmatrix} \Pi(t) & \Pi(t) C_i^T(t) \\ C_i(t) \Pi(t) & \frac{\lambda_i(t)}{2} \tilde{\Xi}_i(t) \end{pmatrix} \geq 0, \quad i = 1, \ldots, \alpha$$

$$(12b)$$

$$\begin{pmatrix} \Pi(t) & L_j^T(t) \\ L_j(t) & \tilde{\Upsilon}_j(t) \end{pmatrix} \geq 0, \quad j = 1, \ldots, \beta$$

$$(12c)$$

hold, when $t \in \Omega$, with $\tilde{\Xi}_i(t) := (\{(t - t_0)Q_i(t)\})^{-1}$, and $\tilde{\Upsilon}_j(t) := (\{(t - t_0)T_j(t)\})^{-1}$.

The controller gain which solves the IO finite-time stabilization problem via state feedback is given by (4) with $K_j(t) = L_j(t) \Pi^{-1}(t), j = 1, \ldots, \beta$. 


Quarter car suspension model

- $M_s$ sprung mass
- $M_u$ unsprung mass
- $B_s$ suspension damping coefficient
- $K_s$ suspension spring elastic coefficient
- $K_u$ elastic coefficient that models tire deflection
- $u_f$ active force generated by the hydraulic actuator $S$

**Figure:** Schematic representation of the active suspension system.
Letting $x_s$ and $x_u$ the vertical displacement of the sprung and unsprung masses, respectively

- $x_o$ the vertical ground displacement caused by road unevenness

and choosing as state variables

- the suspension stroke $x_s - x_u$
- the tire deflection $x_u - x_o$

and their derivatives The resulting open-loop dynamical model reads

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -\frac{K_s}{M_s} & -\frac{B_s}{M_s} & 0 & \frac{B_s}{M_s} \\ 0 & 0 & \frac{B_s}{M_u} & -\frac{B_s}{M_u} \\ \frac{K_s}{M_u} & \frac{B_s}{M_u} & -\frac{K_u}{M_u} & -\frac{B_s}{M_u} \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \frac{u_{\text{max}}}{M_s} \\ 0 \\ -\frac{u_{\text{max}}}{M_u} \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} w(t), \quad (13)$$

where the normalized active force $u(\cdot) = u_f(\cdot)/u_{\text{max}}$ is the control input and the exogenous input $w(\cdot) = \dot{x}_o(\cdot)$ represents the disturbance caused by the road roughness.
Design constraints

When designing a controller a number of constraints should be considered.

- To ensure a firm uninterrupted contact of wheels to road, the dynamic tire load should not exceed the static one

\[ k_u |x_3(t)| < (m_s + m_u) g \quad \forall \ t \geq 0. \]  
\[ (14) \]

- The suspension stroke should fulfill the following constraint

\[ |x_1(t)| \leq SS, \quad \forall \ t \geq 0. \]  
\[ (15) \]

In order to cast the control design problem in the IO-FTS framework, we consider the following system outputs

\[
\begin{pmatrix}
  y_1(t) \\
  y_2(t) \\
  y_3(t)
\end{pmatrix} = \begin{pmatrix}
  \dot{x}_2(t) \\
  x_1(t) \\
  SS \\
  k_u x_3(t) \\
  g(m_s + m_u)
\end{pmatrix} = \begin{pmatrix}
  C_1 & C_2 & C_3
\end{pmatrix} x(t) + \begin{pmatrix}
  D_1 \\
  D_2 \\
  D_3
\end{pmatrix} u,
\]

\[ (16) \]

where

\[ C_1 = \begin{pmatrix}
  -k_s/m_s & -c_s/m_s & 0 & c_s/m_s
\end{pmatrix}, \quad D_1 = \frac{u_{\text{max}}}{m_s}, \]
\[ C_2 = \begin{pmatrix}
  1 & 0 & 0 & 0
\end{pmatrix}, \quad D_2 = 0, \]
\[ C_3 = \begin{pmatrix}
  0 & 0 & 1 & 0
\end{pmatrix}, \quad D_3 = 0. \]
Model parameters

The following values for the model parameters have been taken from Chen and Guo, TCST, 2005

\[ M_s = 320 \text{ kg}, \quad K_s = 18 \frac{kN}{m}, \]
\[ B_s = 1 \frac{kN \cdot s}{m}, \quad K_u = 200 \frac{kN}{m}, \]
\[ M_u = 40 \text{ kg}, \quad u_{max} = 1.5 \text{ kN}, \]
\[ SS = 0.08 \text{ m}. \]
Actuator saturation

Due to actuator saturation, the active force is bounded by $u_{\text{max}}$, i.e. the normalized force has to satisfy

$$|u(t)| \leq 1, \quad \forall \ t \geq 0.$$  \hfill (17)

In order to frame the problem of designing the active suspension control system in the context of structured IO finite-time stabilization let us rewrite the output equation as

\[
\begin{pmatrix}
y_1(t) \\
y_2(t) \\
y_3(t) \\
u(t)
\end{pmatrix}
= \begin{pmatrix}
\dot{x}_2(t) \\
\frac{x_1(t)}{SS} \\
\frac{k_u x_3(t)}{g(m_s+m_u)} \\
K(t) x(t)
\end{pmatrix}
= \begin{pmatrix}
C_1 + D_1 K(t) \\
C_2 + D_2 K(t) \\
C_3 + D_3 K(t) \\
K(t)
\end{pmatrix} x(t).
\]  \hfill (18)
Reference disturbance

We will design the time–varying state feedback $K(t)$ that optimize the response to an isolated bump modeled as the $\mathcal{W}_2$ disturbance

$$w(t) = \begin{cases} \frac{M}{2} (1 - \cos \left( \frac{2\pi V}{L} t \right)), & 0 \leq t \leq \frac{L}{V} \\ 0, & t > \frac{L}{V} \end{cases}$$

(19)

where $M = 0.1 \ m$, $L = 5 \ m$ are the bump height and width, respectively, while $V = 45 \ km/h$ is the vehicle forward velocity.

In particular, given the bump (19) we want to minimize the body acceleration $y_1(t) = \dot{x}_2(t)$ fulfilling the constraints (14)–(17).
We consider the following IO-FTS parameters

\[ T = 2 \, \text{s}, \quad R = 8. \]

Furthermore, given the selected outputs (18), the two outputs weighting matrices

\[ Q_2 = Q_3 = 1, \]

allows to take into account the constraints (14) and (15), while the input weighting matrix is

\[ T_1 = 0.15, \]

which allows to exploit the full scale of the control input when (19) is considered.

In order to minimize the body acceleration it is possible to exploit Theorem 1 and solve the following optimization problem

\[
\begin{align*}
\text{minimize} \quad & \Xi_1 \\
\text{subject to} \quad & (9)
\end{align*}
\]

(20)

where \( \Xi_1 = Q_1^{-1} \).
Solving the problem

- Assuming the two matrix–valued functions $\Pi(\cdot)$ and $L(\cdot)$ to be piecewise linear, it is possible to recast problem (20) in the LMIs framework.

- By solving (20), we get $\Xi_{1_{\text{min}}} = 7.22$ and the two feasible matrix–valued functions $\Pi(\cdot)$ and $L(\cdot)$; the time–varying controller $K(t)$ is then given by $K(t) = L(t)\Pi(t)^{-1}$. 
Results - 1

Figure: Bump response: IO-FTS time-varying controller (−), constrained $\mathcal{H}_\infty$ controller (− −, Chen and Guo, TCST, 2005).
**Figure:** Bump response: time behavior of the weighted outputs $y_2(t)^TQ_2y_2(t)$ and $y_3(t)^TQ_3y_3(t)$ when the IO-FTS time–varying controller is considered.
Conclusions

- The notion of structured IO-FTS has been introduced.
- Structured IO-FTS allows to take into account design constraint on the control input.
- Conditions for IO finite-time stabilization (in the structured context) of LTV systems via state feedback have been given.
- The effectiveness of the approach has been illustrated by means of an engineering case-study.

Thank you!