Necessary and Sufficient Conditions for Input-Output Finite-Time Stability of Impulsive Dynamical Systems

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Outline

Input-output finite-time stability vs classic IO stability

IO stability

A dynamical system is said to be IO $\mathcal{L}_p$-stable if for any input of class $\mathcal{L}_p$, the system exhibits a corresponding output which belongs to the same class

IO-FTS

A dynamical system is defined to be IO-FTS if, given a class of norm bounded input signals over a specified time interval T, the outputs of the system do not exceed an assigned threshold during T

Main features of IO-FTS

IO-FTS:

- involves signals defined over a finite time interval
- does not necessarily require the inputs and outputs to belong to the same class
- specifies a *quantitative* bounds on both inputs and outputs

IO stability and IO-FTS are independent concepts

Examples of application

- G. Ambrosino, M. Ariola, G. De Tommasi, A. Pironti
  Plasma Vertical Stabilization in the ITER Tokamak via Constrained Static Output Feedback
  IEEE Trans. Contr. Tech., 2011
- F. Amato, G. Carannante, G. De Tommasi, A. Pironti
  Input-Output Finite-Time Stabilization of Linear Systems with Input Constraints
  IET Contr. Theory Appl., 2014

Contribution of the paper

- In this paper we show that the sufficient condition to check IO-FTS of time-dependent Impulsive Dynamical Linear Systems (IDLS), which is expressed in terms of a coupled difference/differential LMI (D/DLMI) feasibility problem, and which was originally given in

  F. Amato, G. Carannante, G. De Tommasi
  Input-output Finite-Time Stabilisation of a class of Hybrid Systems via Static Output Feedback
  Int. J. Contr., 2011

  is also **necessary**.
- An alternative, **and numerically more efficient**, necessary and sufficient condition for IO-FTS is proved, which requires the solution of a coupled difference/differential Lyapunov equation (D/DLE).

Notation

- $\mathcal{L}_p$ denotes the space of vector-valued signals whose p-th power is absolutely integrable over $[0, +\infty)$.
- The restriction of $\mathcal{L}_p$ to the time interval $\Omega := [t_0\, , t_0 + T]$ is denoted by $\mathcal{L}_p(\Omega)$.
- Given the time interval Ω, a symmetric positive definite matrix-valued function $R(\cdot)$, bounded on Ω, and a vector-valued signal $s(\cdot) \in \mathcal{L}_p(\Omega)$, the weighted signal norm

$$\left(\int_\Omega \left[s^T(\tau) R(\tau) s(\tau) \right]^{\frac{p}{2}} d\tau \right)^{\frac{1}{p}} ,$$

  will be denoted by $\|s(\cdot)\|_{p\,,R}$. If $p = \infty$

$$\|s(\cdot)\|_{\infty\,,R} = \operatorname*{ess\,sup}_{t \in \Omega} \left[s^T(t) R(t) s(t) \right]^{\frac{1}{2}} .$$

Impulsive Dynamical Linear Systems

The class of time dependent Impulsive Dynamical Linear Systems is described by

$$\Gamma : \begin{cases} \dot{x}(t) = A(t)x(t) + G(t)w(t)\,, \quad x(t_0) = 0\,, \quad t \notin \mathcal{T} & \text{(1a)} \\ x^+(t_i) = J(t_i)x(t_i)\,, \quad t_i \in \mathcal{T} & \text{(1b)} \\ y(t) = C(t)x(t)\,, \quad \forall\ t\,, & \text{(1c)} \end{cases}$$

- $J(\cdot)$ is the matrix-valued function that describes the *resetting law* of the system.
- The elements of the set $\mathcal{T} = \{t_1\,, t_2\,, \ldots\}$ are called *resetting times*.
- According to the continuous-time dynamics (1a) and the resetting law (1b), an IDLS presents a left-continuous trajectory with a finite jump from $x(t_i)$ to $x^+(t_i)$ at each resetting time $t_i \in \mathcal{T}$.
- Being interested in the dynamic behaviour of the IDLS in the time interval Ω, the number of resetting times in Ω is assumed equal to N.
- It is also assumed that the first resetting time $t_1 \in \mathcal{T}$ is such that $t_1 > t_0$.

The state transition matrix $\Phi(t\,,\tau)$ of an IDLS

The following properties for the transition matrix $\Phi(t\,,\tau)$ of (1) hold

$$\Phi(t_0\,,t_0) = I\,, \tag{2a}$$

$$\frac{\partial}{\partial t}\Phi(t\,,t_0) = A(t)\Phi(t\,,t_0)\,, \quad t \notin \mathcal{T} \tag{2b}$$

$$\Phi^{+}(t_i\,,t_0) = J(t_i)\Phi(t_i\,,t_0)\,, \quad t_i \in \mathcal{T}\,. \tag{2c}$$

Given $t_k < t < t_{k+1}$, with $t_k\,,t_{k+1} \in \mathcal{T}$, it is (see also Medina 2007)

$$\Phi(t\,,t_0) = \Phi_{k+1}(t\,,t_k)J(t_k)\Phi_k(t_k\,,t_{k-1})J(t_{k-1})\cdots J(t_2)\Phi_2(t_2\,,t_1)J(t_1)\Phi_1(t_1\,,t_0)\,, \tag{3}$$

where for $j = 1\,,\ldots\,,N$, $\Phi_j(\cdot,\cdot)$ satisfies

$$\frac{\partial}{\partial t}\Phi_j(t\,,t_{j-1}) = A(t)\Phi_j(t,t_{j-1})\,, \quad t \in [t_{j-1}\,,t_j[\,, \quad \Phi_j(t_{j-1}\,,t_{j-1}) = I\,,$$

and $\Phi_{N+1}(\cdot,\cdot)$,

$$\frac{\partial}{\partial t}\Phi_{N+1}(t\,,t_N) = A(t)\Phi_{N+1}(t,t_N)\,, \quad t \in [t_N\,,t_0+T]\,, \quad \Phi_{N+1}(t_N\,,t_N) = I\,.$$

Given (3), it is straightforward to verify that the impulsive response of (1), is given by

$$H(t\,,\tau) = C(t)\Phi(t\,,\tau)G(\tau)\delta_{-1}(t-\tau)\,.$$

Reachability Gramian of IDLSs

- Also the reachability Gramian $W_r(\cdot\,,\cdot)$ of an IDLS can be recursively defined
- In Medina and Lawrence 2009 it has been shown that $W_r(\cdot\,,\cdot)$ is the unique symmetric and positive semidefinite solution of the following D/DLE

$$\dot{W}_r(t\,,t_0) = A(t)W_r(t\,,t_0) + W_r(t\,,t_0)A^T(t) + G(t)G^T(t)\,, \quad t \notin \mathcal{T} \tag{4a}$$

$$W_r^+(t_i\,,t_0) = J(t_i)W_r(t_i\,,t_0)J^T(t_i)\,, \quad t_i \in \mathcal{T} \tag{4b}$$

$$W_r(t_0\,,t_0) = 0 \tag{4c}$$

Formal definition of IO-FTS for IDLSs

IO-FTS of IDLSs

Given a positive scalar T, a class of input signals $\mathcal{W}$ defined over $\Omega = [t_0 \; , \; t_0 + T]$, a continuous, positive definite matrix-valued function $Q(\cdot)$ defined in Ω, system (1) is said to be IO-FTS with respect to $(\mathcal{W}\,, Q(\cdot)\,, \Omega)$ if

$$w(\cdot) \in \mathcal{W} \;\Rightarrow\; y^T(t)Q(t)y(t) < 1\,, \quad \forall t \in \Omega\,.$$

Class square integrable disturbances - $\mathcal{W}_2$

$$\mathcal{W}_2\left(\Omega, R(\cdot)\right) := \left\{w(\cdot) \in \mathcal{L}_2\left(\Omega\right) \,:\, \|w\|_{2,R} \leq 1\right\}\,.$$

IDLSs as linear operators

- The IDLS (1) can be regarded as a linear operator

$$\Gamma : w(\cdot) \in \mathcal{L}_2(\Omega) \mapsto y(\cdot) \in \mathcal{L}_\infty(\Omega)\,, \tag{5}$$

- Equipping the $\mathcal{L}_2(\Omega)$ and $\mathcal{L}_\infty(\Omega)$ spaces with the weighted norms $\|\cdot\|_{2,R}$ and $\|\cdot\|_{\infty,Q}$, respectively, the induced norm of (5) is equal to

$$\|\Gamma\| = \sup_{\|w(\cdot)\|_{2,R}=1} \left[\|y(\cdot)\|_{\infty,Q} \right] .$$

Theorem 1

Given a time interval Ω, the class of input signals $\mathcal{W}_2$, and a continuous positive definite matrix-valued function $Q(\cdot)$, system (1) is IO-FTS with respect to $\left(\mathcal{W}_2\,, Q(\cdot)\,, \Omega\right)$ *if and only if* $\|\Gamma\| < 1$.

Norm of the linear operator Γ

Theorem 2

Given the IDLS (1), the norm of the corresponding linear operator (5) is given by

$$\|\Gamma\| = \operatorname*{ess\,sup}_{t\in\Omega} \lambda_{\max}^{\frac{1}{2}}\left(Q^{\frac{1}{2}}(t)C(t)W(t)C^T(t)Q^{\frac{1}{2}}(t)\right), \qquad (6)$$

for all $t \in \Omega$; $W(\cdot)$ is the piecewise continuously differentiable positive semidefinite matrix-valued solution of

$$\dot{W}(t) = A(t)W(t) + W(t)A^T(t) + G(t)R(t)^{-1}G^T(t), \quad t \notin \mathcal{T} \qquad (7a)$$

$$W^+(t_i) = J(t_i)W(t_i)J^T(t_i), \quad t_i \in \mathcal{T} \qquad (7b)$$

$$W(t_0) = 0. \qquad (7c)$$

Main result

Theorem 3

The following statements are equivalent:

i) The IDLS (1) is IO finite-time stable with respect to $(\mathcal{W}_2\,, Q(\cdot)\,, \Omega)$.

ii) The inequality $\operatorname{ess\,sup}_{t\in\Omega} \lambda_{\max}\left(Q^{\frac{1}{2}}(t)C(t)W(t)C^T(t)Q^{\frac{1}{2}}(t)\right) < 1$ holds, where $W(\cdot)$ is the solution of (7).

iii) The coupled D/DLMI

$$\begin{pmatrix} \dot{P}(t) + A^T(t)P(t) + P(t)A(t) & P(t)G(t) \\ G^T(t)P(t) & -R(t) \end{pmatrix} < 0$$

$$t \notin \mathcal{T}\,, \tag{8a}$$

$$J^T(t_i)P^+(t_i)J(t_i) - P(t_i) < 0\,, \quad t \in \mathcal{T}\,, \tag{8b}$$

$$P(t) > C^T(t)Q(t)C(t)\,, \quad t \in \Omega \tag{8c}$$

admits a piecewise continuously differentiable positive definite solution $P(\cdot)$ over Ω.

Comparison of the computational burden 1/3

Let us consider the time-varying IDLS

$$A = \begin{pmatrix} -2.5 + 0.2 \cdot t & -6.3 \\ 4 & 0.2 \cdot t \end{pmatrix}, \quad G = \begin{pmatrix} 2 \\ 0 \end{pmatrix},$$
$$C = \begin{pmatrix} 1.2 & 3.2 \end{pmatrix}, \quad J = \begin{pmatrix} 1.1 & 0 \\ 0 & -0.8 \end{pmatrix}. \tag{9}$$

The time interval we consider in this example is $\Omega = [0\,, 2]$, while the resetting times are

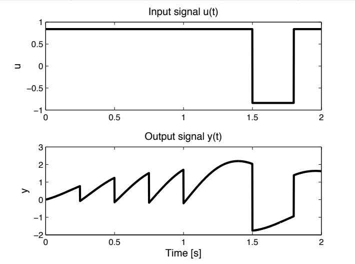
$$\mathcal{T} = \{0.25\,, 0.5\,, 0.75\,, 1\,, 1.5\,, 1.8\}\ .$$

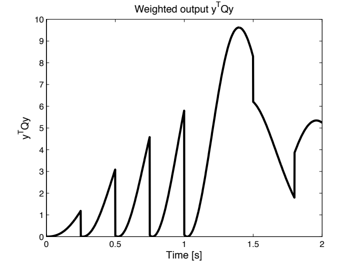
The input weighting matrix is taken constant and equal to

$$R = 0.7\,.$$

Comparison of the computational burden 2/3

Time response of the IDLS (9) in the interval Ω when an input in $\mathcal{W}_2\left([0\,,2]\,,0.7\right)$ is considered, and when **Q is taken equal to 2**.





Comparison of the computational burden 3/3

Table: Values of $Q_{\max}$ obtained exploiting condition **ii)** in Theorem 3 for the IDLS system (9)

Sample Time (T_s) [ms]	$Q_{\max}$	Computation time for the solution of the D/DLE (7) [s]
10	0.0900	0.19
1	0.0910	0.22
0.1	0.0918	0.7

Table: Values of $Q_{\max}$ obtained exploiting condition **iii)** in Theorem 3 for the IDLS system (9)

Sample Time (T_s) [ms]	$Q_{\max}$	Average computation time for a single iteration [s]
50	0.0740	2.6
25	0.0796	14.2
10	0.0837	298.4

Conclusions

- Necessary and sufficient conditions for IO-FTS of IDLSs have been presented for the class of $\mathcal{W}_2$ disturbances
- The D/DLMI formulation can be extended to solve the IO finite-time stabilization problem either via state-feedback, or via output-feedback

Thank you!