Necessary and Sufficient Conditions for Input-Output Finite-Time Stability of Linear Time-Varying Systems

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Outline

Input-output finite-time stability vs classic IO stability

IO stability

A system is said to be IO $\mathcal{L}_p$-stable if for any input of class $\mathcal{L}_p$, the system exhibits a corresponding output which belongs to the same class

IO-FTS

A system is defined to be IO-FTS if, given a class of norm bounded input signals over a specified time interval T, the outputs of the system do not exceed an assigned threshold during T

Main features of IO-FTS

IO-FTS:

- involves signals defined over a finite time interval
- does not necessarily require the inputs and outputs to belong to the same class
- specifies a *quantitative* bounds on both inputs and outputs

IO stability and IO-FTS are independent concepts

Contribution of the paper

- In this paper we show that, in the case of $\mathcal{L}_2$ inputs, the sufficient condition given in

  F. Amato, R. Ambrosino, G. De Tommasi, C. Cosentino
  Input-output finite-time stabilization of linear systems
  Automatica, 2010

  is also **necessary**.
- To prove this result, a machinery involving the *teachability gramian* is used.

Notation

- $\mathcal{L}_p$ denotes the space of vector-valued signals whose p-th power is absolutely integrable over $[0, +\infty)$.
- The restriction of $\mathcal{L}_p$ to $\Omega := [t_0\,, t_0 + T]$ is denoted by $\mathcal{L}_p(\Omega)$.
- Given the time interval Ω, a symmetric positive definite matrix-valued function $R(\cdot)$, bounded on Ω, and a vector-valued signal $s(\cdot) \in \mathcal{L}_p(\Omega)$, the weighted signal norm

$$\left(\int_\Omega \left[s^T(\tau) R(\tau) s(\tau) \right]^{\frac{p}{2}} d\tau \right)^{\frac{1}{p}} ,$$

will be denoted by $\|s(\cdot)\|_{p\,,R}$. If $p = \infty$

$$\|s(\cdot)\|_{\infty\,,R} = \operatorname*{ess\,sup}_{t \in \Omega} \left[s^T(t) R(t) s(t) \right]^{\frac{1}{2}} .$$

LTV systems as Linear Operator

Let us consider a LTV system in the form

$$\Gamma : \begin{cases} \dot{x}(t) = A(t)x(t) + G(t)w(t)\,, \quad x(t_0) = 0 \\ y(t) = C(t)x(t) \end{cases} \tag{1}$$

Γ can be viewed as a linear operator mapping input signals ($w(\cdot)$'s) into output signals ($y(\cdot)$'s).

$\Phi(t\,,\tau)$ denotes the state transition matrix of system (1).

Reachability Gramian

The *reachability Gramian* of system (1) is defined as

$$W_r(t, t_0) \triangleq \int_{t_0}^{t} \Phi(t, \tau) G(\tau) G^T(\tau) \Phi^T(t, \tau) d\tau .$$

$W_r(t, t_0)$ is symmetric and positive semidefinite for all $t \geq t_0$.

Given system (1), $W_r(t, t_0)$ is the unique solution of the matrix differential equation

$$\dot{W}_r(t, t_0) = A(t) W_r(t, t_0) + W_r(t, t_0) A^T(t) + G(t) G^T(t) , \tag{2a}$$

$$W_r(t_0, t_0) = 0 \tag{2b}$$

IO-FTS of LTV systems

Given a positive scalar T, a class of input signals $\mathcal{W}$ defined over $\Omega = [t_0 , t_0 + T]$, a positive definite matrix-valued function $Q(\cdot)$ defined in Ω, system (1) is said to be IO-FTS with respect to $\big(\mathcal{W}, Q(\cdot), \Omega\big)$ if

$$w(\cdot) \in \mathcal{W} \Rightarrow y^T(t)Q(t)y(t) < 1 , \quad t \in \Omega .$$

In this work we consider the class of norm bounded square integrable signals over Ω

$$\mathcal{W}_2\big(\Omega , R(\cdot)\big) := \big\{w(\cdot) \in \mathcal{L}_2(\Omega) : \|w\|_{2,R} \leq 1\big\} ,$$

where $R(\cdot)$ denotes a continuous positive definite matrix-valued function.

Linear operator

The LTV system (1) is regarded as a linear operator that maps signals from the space $\mathcal{L}_2(\Omega)$ to the space $\mathcal{L}_\infty(\Omega)$

$$\Gamma : w(\cdot) \in \mathcal{L}_2(\Omega) \mapsto y(\cdot) \in \mathcal{L}_\infty(\Omega)\,. \tag{3}$$

If we equip the $\mathcal{L}_2(\Omega)$ and $\mathcal{L}_\infty(\Omega)$ spaces with the weighted norms $\|\cdot\|_{2,R}$ and $\|\cdot\|_{\infty,Q}$, respectively, the induced norm of the linear operator (3) is given by

$$\|\Gamma\| = \sup_{\|w(\cdot)\|_{2,R}=1} \left[\|y(\cdot)\|_{\infty,Q} \right],$$

Theorem 1

Given a time interval Ω, the class of input signals $\mathcal{W}_2$, and a continuous positive definite matrix-valued function $Q(\cdot)$, system (1) is IO-FTS with respect to $\left(\mathcal{W}_2\,, Q(\cdot)\,, \Omega\right)$ *if and only if* $\|\Gamma\| < 1$.

Dual operator

Given the linear operator (3), its dual operator is

$$\bar{\Gamma} : z(\cdot) \in \mathcal{L}_1(\Omega) \mapsto v(\cdot) \in \mathcal{L}_2(\Omega)\,,$$

with

$$\|\bar{\Gamma}\| = \sup_{\|z(\cdot)\|_{1,Q}=1} \left[\|v(\cdot)\|_{2,R} \right] .$$

By definition it holds

$$\|\Gamma\| = \|\bar{\Gamma}\|\,, \tag{4}$$

and

$$\langle z\,, \Gamma w \rangle = \langle \bar{\Gamma} z\,, w \rangle\,, \tag{5}$$

where $z(\cdot) \in \mathcal{L}_1(\Omega)$ and $w(\cdot) \in \mathcal{L}_2(\Omega)$.

Theorem 2

Given the LTV system (1), the norm of the corresponding linear operator (3) is given by

$$\|\Gamma\| = \operatorname*{ess\,sup}_{t\in\Omega} \lambda_{\max}^{\frac{1}{2}}\left(Q^{\frac{1}{2}}(t)C(t)W(t\,,t_0)C^T(t)Q^{\frac{1}{2}}(t)\right), \quad (6)$$

for all $t \in \Omega$, where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue, and $W(t\,,t_0)$ is the positive semidefinite matrix-valued solution of

$$\dot{W}(t\,,t_0) = A(t)W(t\,,t_0) + W(t\,,t_0)A^T(t) + G(t)R(t)^{-1}G^T(t) \quad (7a)$$

$$W(t_0\,,t_0) = 0 \quad (7b)$$

Sketch of proof - 1

- For the sake of simplicity, the weighting matrices $R(t)$ and $Q(t)$ are set equal to the identity; it follows that the solution of (7) is given by the reachability gramian $W_r(t\,,t_0)$;
- Considering the dual operator $\bar{\Gamma}$, proving (6) is equivalent to show

$$\|\bar{\Gamma}\| = \operatorname*{ess\,sup}_{t\in\Omega} \lambda_{\max}^{\frac{1}{2}}\Big(C(t)W_r(t\,,t_0)C^T(t)\Big)\,.$$

- We denote with

$$\bar{H}(t\,,\tau) = G^T(t)\Phi^T(\tau\,,t)C^T(\tau)\delta_{-1}(\tau-t)$$

the impulsive response of the dual system

$$\bar{\Gamma}:\begin{cases}\dot{\tilde{x}}(t) = -A^T(t)\tilde{x}(t) - C^T(t)z(t)\\ v(t) = G^T(t)\tilde{x}(t)\end{cases}\,.$$

Sketch of proof - 2

- Using $\bar{H}(t,\tau)$ it is possible to show that

$$\|v(\cdot)\|_2 = \left\| \int_\Omega \bar{H}(\cdot,\tau) z(\tau) d\tau \right\|_2 \leq \operatorname*{ess\,sup}_{t\in\Omega} \lambda_{\max}^{\frac{1}{2}} \left(C(t) W_r(t,t_0) C^T(t) \right) \cdot \|z(\cdot)\|_1$$

- Hence

$$\|\bar{\Gamma}\| \leq \operatorname*{ess\,sup}_{t\in\Omega} \lambda_{\max}^{\frac{1}{2}} \left(C(t) W_r(t,t_0) C^T(t) \right)$$

- Exploiting similar arguments as in

  D. A. Wilson
  Convolution and hankel operator norms for linear
  IEEE Trans. on Auto. Contr., 1989

  it is possible to show that

$$\|\bar{\Gamma}\| = \operatorname*{ess\,sup}_{t\in\Omega} \lambda_{\max}^{\frac{1}{2}} \left(C(t) W_r(t,t_0) C^T(t) \right),$$

which proofs the theorem.

Remark

If the system matrices in (1) and the weighting matrices $R(\cdot)$ and $Q(\cdot)$ are assumed to be continuous, in the closed time interval Ω the condition (6) is equivalent to

$$\|\Gamma\| = \max_{t\in\Omega} \lambda_{\max}^{\frac{1}{2}}\left(Q^{\frac{1}{2}}(t)C(t)W(t\,,t_0)C^T(t)Q^{\frac{1}{2}}(t)\right).$$

Theorem 3

The following statements are equivalent:

i) System (1) is IO-FTS with respect to $\left(\mathcal{W}_2\,,Q(\cdot)\,,\Omega\right)$.

ii) The inequality

$$\lambda_{\max}\left(Q^{\frac{1}{2}}(t)C(t)W(t\,,t_0)C^T(t)Q^{\frac{1}{2}}(t)\right) < 1 \tag{8}$$

holds for all $t \in \Omega$, where $W(\cdot,\cdot)$ is the positive semidefinite solution of the Differential Lyapunov Equality (DLE) (7).

iii) The coupled DLMI/LMI

$$\begin{pmatrix} \dot{P}(t) + A^T(t)P(t) + P(t)A(t) & P(t)G(t) \\ G^T(t)P(t) & -R(t) \end{pmatrix} < 0 \tag{9a}$$

$$P(t) > C^T(t)Q(t)C(t)\,, \tag{9b}$$

admits a positive definite solution $P(\cdot)$ over Ω.

Sketch of proof

- The equivalence of the three statements is proved by showing that **i)** $\Rightarrow$ **ii)**, **ii)** $\Rightarrow$ **iii)**, and **iii)** $\Rightarrow$ **i)**.
- A technical lemma is exploited to show that solving the DLE is equivalent to solve a matrix inequality.

Comparison

- The conditions stated in Theorem 3 are all necessary and sufficient.
- The numerical implementation of such conditions introduces some conservativeness.
- In order to compare each other, from the computational point of view the output weighting matrix is left as a free parameter.
- We define Q_{max} as the maximum value of the matrix Q such that a system is IO-FTS.
- To recast the DLMI condition (9) in terms of LMIs, the matrix-valued functions $P(\cdot)$ has been assumed piecewise linear. In particular, the time interval Ω has been divided in $n = T/T_s$ subintervals.

Results

In the paper we have considered the system

$$A(t) = \begin{pmatrix} 0.5 + t & 0.1 \\ 0.4 & -0.3 + t \end{pmatrix}, \; G = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 1 \end{pmatrix},$$

together with the following IO-FTS parameters:

$$R = 1 \, , \; \Omega = \left[0 \, , 0.5\right] .$$

Maximum values of Q satisfying Theorem 3. The results have been obtained by using a PC equipped with an Intel i7-720QM processor and 4 GB of RAM.

IO-FTS condition	Sample Time (T_s)	Estimate of Q_{max}	Computation time [s]
DLMI (9)	0.05	0.2	2.5
	0.025	0.25	12.7
	0.0125	0.29	257
	0.00833	0.3	1259
Solution of (7) and inequality (8)	0.003	0.345	6

Conclusions

- Necessary and sufficient conditions for IO-FTS have been presented in this paper for the class of $\mathcal{W}_2$ input signals.
- We are currently trying to find a necessary and sufficient condition for finite-time stability (FTS)

(Again) Thank you!