Input-output finite-time stabilization for a class of hybrid systems

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Outline

Input–output finite–time stability vs classic IO stability

IO stability

A system is said to be IO $\mathcal{L}_p$-stable if for any input of class $\mathcal{L}_p$, the system exhibits a corresponding output which belongs to the same class

IO–FTS

A system is defined to be IO-FTS if, given a class of norm bounded input signals over a specified time interval T, the outputs of the system do not exceed an assigned threshold during T

Main features of IO–FTS

IO–FTS:

- involves signals defined over a finite time interval
- does not necessarily require the inputs and outputs to belong to the same class
- specifies a *quantitative* bounds on both inputs and outputs

IO stability and IO-FTS are independent concepts

IO–FTS and state FTS

The definition of IO–FTS is fully consistent with the definition of (state) FTS given in [1, 2, 3], where the state of a zero-input system, rather than the input and the output, are involved.

P. Dorato
Short time stability in linear time-varying systems
Proc. IRE Int. Convention Record Pt. 4, 1961

F. Amato, M. Ariola, P. Dorato
Finite-time control of linear systems subject to parametric uncertanties and disturbances
Automatica, 2001

Y. Shen
Finite-time control of linear parameter-varying systems with norm-bounded exogenous disturbance
J. Contr. Theory Appl., 2008

Contribution of the paper

In this work we extend the work done in [1] to a class of hybrid systems: Impulsive Dynamical Linear Systems (IDLS).

IDLS are LTV continuous-time systems whose state undergoes finite jump discontinuities at discrete instants of time.

State jumps can be:

- *time-dependent*, if the state jumps are time-driven
- *state-dependent*, if the state jumps occur when the trajectory reaches an assigned subset of the state space, the so-called *resetting set*

F. Amato, R. Ambrosino, C. Cosentino, G. De Tommasi
Input to Output Finite Time Stability of Linear systems
Automatica, 2010

Impulsive Dynamical Linear Systems

IDLS ([1])

$$\dot{x}(t) = A(t)x(t) + G(t)w(t)\,, \quad x(t_0) = 0\,, \quad \big(t, x(t)\big) \notin \mathcal{S} \tag{1a}$$

$$x(t^+) = J(t)x(t)\,, \quad \big(t, x(t)\big) \in \mathcal{S} \tag{1b}$$

$$y(t) = C(t)x(t) \tag{1c}$$

where $A(\cdot)\,, J(\cdot)\,:\,\mathbb{R}_0^+ \mapsto \mathbb{R}^{n\times n}$, $G(\cdot)\,:\,\mathbb{R}_0^+ \mapsto \mathbb{R}^{n\times r}$, and $C(\cdot)\,:\,\mathbb{R}_0^+ \mapsto \mathbb{R}^{m\times n}$ are piecewise continuous matrix-valued functions and $\mathcal{S} \subset \mathbb{R}_0^+ \times \mathbb{R}^n$ is called the *resetting set*.

W. M. Haddad, V. Chellaboina, S. G. Nersesov
Impulsive and Hybrid Dynamical Systems
Princeton Univ. Press, 2006

Time–dependent and state–dependent IDLS

TD–IDLS

Time-dependent IDLS (TD-IDLS): in this case, given a set $\mathcal{T} := \{t_1, t_2, \dots\}$, $\mathcal{S}$ is defined as $\mathcal{S} = \mathcal{T} \times \mathcal{X}\big(w(\cdot), \mathcal{T}\big)$, where

$$\mathcal{X}\big(w(\cdot), \mathcal{T}\big) = \{x(\bar{t}) \,:\, \bar{t} \in \mathcal{T}\} \subset \mathbb{R}^n .$$

In this case the resetting set is defined by a prescribed sequence of time instants, which are independent of the state $x(\cdot)$ and input $w(\cdot)$;

SD–IDLS

State-dependent IDLS (SD-IDLS): in this case, given a set $\mathcal{X} \subset \mathbb{R}^n$, $\mathcal{S}$ is defined as $\mathcal{S} = \mathcal{T}\big(w(\cdot), \mathcal{X}\big) \times \mathcal{X}$, where

$$\mathcal{T}\big(w(\cdot), \mathcal{X}\big) = \{\bar{t} : x(\bar{t}) \in \mathcal{X}\} \subset \mathbb{R}_0^+ .$$

In this case the resetting set is defined by a region in the state space, which does not depend on the time.

Basic assumptions

In order to assure well-posedness of the resetting times and to prevent Zeno behavior, the following assumption are made

Assumption 1

For all $t \in \left[0, +\infty\right[$ such that $\big(t, x(t)\big) \in \mathcal{S}$,

$$\exists\, \varepsilon > 0 : \big(t+\delta\,, x(t+\delta)\big) \notin \mathcal{S}, \quad \forall \delta \in \left]0, \varepsilon\right]$$

Assumption 2

Given a compact interval $[t_0\,, t_0 + T]$, it includes only a finite number of resetting times. It follows that the resetting set to be considered in the time interval $[t_0\,, t_0 + T]$ is given by

$$\mathcal{S} = \mathcal{T} \times \mathcal{X} \subset [t_0\,, t_0 + T] \times \mathbb{R}^n\,,$$

with $\mathcal{T} = \left\{t_1\,, t_2\,, \ldots\,, t_r\right\}$

Definition of IO–FTS

Given a positive scalar T, a class of input signals $\mathcal{W}$ defined over $[t_0\,, t_0 + T]$, a positive definite matrix-valued function $Q(\cdot)$ defined over $[0\,, T]$, system (1) is said to be IO-FTS with respect to $\big(\mathcal{W}\,, Q(\cdot)\,, t_0\,, T\big)$ if

$$w(\cdot) \in \mathcal{W} \;\Rightarrow\; y^T(t)Q(t - t_0)y(t) < 1, \quad t \;\in\,]t_0\,, t_0 + T]\,.$$

IO finite–time stabilization via state–feedback

Problem SF

Consider the IDLS

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + G(t)w(t)\,, \quad x(t_0) = 0\,, \quad \big(t, x(t)\big) \notin \mathcal{S} \tag{2a}$$

$$x(t^+) = J(t)x(t)\,, \quad \big(t, x(t)\big) \in \mathcal{S} \tag{2b}$$

$$y(t) = C(t)x(t) \tag{2c}$$

where $u(\cdot)$ is the control input and $w(\cdot)$ is the exogenous input. Given a positive scalar T, a class of disturbances $\mathcal{W}$ defined over $[t_0\,, t_0 + T]$ and a positive definite matrix-valued function $Q(\cdot)$ defined over $[0\,, T]$, find a state feedback control law $u(t) = K(t - t_0)x(t)$, where $K(\cdot)$ is a piecewise continuous matrix-valued function defined over $[0\,, T]$, such that the closed-loop system is IO-FTS with respect to $\big(\mathcal{W}, Q(\cdot), t_0, T\big)$, where $A_{\mathrm{cl}}(t) = \big(A(t) + B(t)K(t - t_0)\big)$.

Considered class of input signals

$\mathcal{W}_2$ signals

Norm bounded square integrable signals over $[t_0\,, t_0 + T]$, defined as follows

$$\mathcal{W}_2(t_0\,, T\,, R) := \left\{ w(\cdot) \in \mathcal{L}_{2,[t_0\,,t_0+T]} \;:\; \|w\|_{[t_0\,,t_0+T],R} \leq 1 \right\}\,.$$

$\mathcal{W}_\infty$ signals

Uniformly bounded signals over $[t_0\,, t_0 + T]$, defined as follows

$$\mathcal{W}_\infty(t_0\,, T\,, R) := \left\{ w(\cdot) \in \mathcal{L}_{\infty,[t_0\,,t_0+T]} \;:\; w^T(t) R w(t) \leq 1\,, t \in [t_0\,, t_0 + T] \right\}.$$

IO–FTS of IDLS for $\mathcal{W}_2$ input signals

Given system (1), a positive definite matrix-valued function $Q(\cdot)$ define over $[0, T]$, and $t \in]t_0, t_0 + T]$, the condition

$$w(\cdot) \in \mathcal{W}_2 \Rightarrow y^T(t)Q(t-t_0)y(t) < 1$$

is satisfied if there exists a piecewise continuously differentiable symmetric solution $P(\cdot)$ defined over the interval $]t_0\,, t]$ such that the following conditions are satisfied

$$\dot{P}(\tau) + A(\tau)^T P(\tau) + P(\tau)A(\tau) + P(\tau)G(\tau)R^{-1}G(\tau)^T P(\tau) < 0\,, \quad \tau \in]t_0, t]\,, \tau \notin \mathcal{T} \tag{3a}$$

$$x^T(t_k)\big(J^T(t_k)P(t_k^+)J(t_k) - P(t_k)\big)x(t_k) \leq 0\,, \quad \big(t_k\,, x(t_k)\big) \in \mathcal{S} \tag{3b}$$

$$P(t) \geq C^T(t)Q(t-t_0)C(t) \tag{3c}$$

Comments

- In principle, conditions (3) should be checked for any $t \in]t_0, t_0 + T]$ in order to establish IO–FTS of IDLS wrt $\big(\mathcal{W}_2, Q(\cdot), t_0, T\big)$.
- The feasibility of infinitely many optimization problems should be checked (which is obviously an impossible task) !
- By means of the previous result it is possible to prove a theorem which requires to check the feasibility of a single Difference-Differential Linear Matrix Inequality with terminal condition (D/DLMI, [1]).
- When the structure of the optimization matrix is fixed *a priori* the feasibility problem can be turned into a classical optimization problem involving LMIs.

U. Shaked and V. Suplin
A New Bounded Real Lemma Representation for the Continuous-Time Case
IEEE Trans. on Automatic Control, 2001

IO–FTS of TD–IDLS for $\mathcal{W}_2$ input signals

Assume that the following D/DLMI with terminal condition

$$\begin{pmatrix} \dot{P}(\tau) + A(\tau)^T P(\tau) + P(\tau)A(\tau) & P(\tau)G(\tau) \\ G(\tau)^T P(\tau) & -R \end{pmatrix} < 0\,, \qquad \forall\ \tau \in]t_0, t_0 + T]\,, \tau \notin \mathcal{T} \tag{4a}$$

$$J^T(t_k)P(t_k^+)J(t_k) - P(t_k) \leq 0\,, \quad \forall\ t_k \in \mathcal{T} \tag{4b}$$

$$P(t) \geq C(t)^T Q(t-t_0) C(t)\,, \quad \forall\ t \in]t_0, t_0 + T] \tag{4c}$$

admits a piecewise continuously differentiable positive definite solution $P(\cdot)$, then the time-driven IDLS (1) is IO-FTS with respect to $\big(\mathcal{W}_2, Q(\cdot), t_0, T\big)$.

IO–FTS of SD–IDLS for $\mathcal{W}_2$ input signals

Assume that the following D/DLMI with terminal condition

$$\begin{pmatrix} \dot{P}(t) + A(t)^T P(t) + P(t)A(t) & P(t)G(t) \\ G(t)^T P(t) & -R \end{pmatrix} < 0\,, \quad \forall\, t \in]t_0, t_0 + T] \tag{5a}$$

$$x^T(t)\big(J^T(t)P(t^+)J(t) - P(t)\big)x(t) \leq 0\,, \quad \forall\, t \in]t_0, t_0 + T]\,,\ \forall\, x \in \mathcal{X} \tag{5b}$$

$$P(t) \geq C(t)^T Q(t - t_0)C(t)\,, \quad \forall\, t \in]t_0, t_0 + T] \tag{5c}$$

admits a piecewise continuously differentiable positive definite solution $P(\cdot)$, then the SD-IDLS (1) is IO-FTS with respect to $\big(\mathcal{W}_2, Q(\cdot), t_0, T\big)$.

S–procedure

S–procedure [1] can be applied in order to turn conditions (5b) into LMIs.

Cases in which S–procedure does not introduce additional conservatism have been considered in [2].

V. A. Jakubovič
The S-procedure in linear control theory
Vestnik Leningrad Univ. Math., 1977

R. Ambrosino, F. Calabrese, C. Cosentino, G. De Tommasi
Sufficient Conditions for Finite-Time Stability of Impulsive Dynamical Systems
IEEE Trans. on Automatic Control, 2009

IO–FTS wrt $\mathcal{W}_\infty$ input signals

Sufficient conditions for IO–FTS of both TD and SD–IDLS are given by substituting $Q(t)$ with

$$\widetilde{Q}(t) = tQ(t)\,, \quad \forall\ t \in [t_0\,, t_0 + T]$$

in (4c) and (5c).

Comments

- For a sufficiently large value of T the condition $\widetilde{Q}(t) = tQ(t)$ may lead to ill-conditioned problems.
- However, using a finite-time stability approach makes sense especially when dealing with time horizons that are less then the settling time of the considered system. In practical application T does not assume large values.
- If it is needed to deal with time horizons much larger than the settling time of the system, then it is probably more opportune to rely on infinite time horizon approaches.

IO finite–time stabilization of TD–IDLS via SF

Given the class of disturbances $\mathcal{W}_2$, Problem SF is solvable if there exist a positive definite and piecewise continuously differentiable matrix-valued function $\Pi(\cdot)$, and a matrix-valued function $L(\cdot)$ such that the following D/DLMI with terminal condition

$$\begin{pmatrix} \Upsilon(\tau) & G(\tau) \\ G(\tau)^T & -R \end{pmatrix} < 0\,, \quad \forall\ \tau \in]t_0\,, t_0+T]\,, \tau \notin \mathcal{T} \tag{6a}$$

$$\begin{pmatrix} \Pi(t_k) & \Pi(t_k)J^T(t_k) \\ J(t_k)\Pi(t_k) & \Pi(t_k^+) \end{pmatrix} \geq 0\,, \quad \forall\ t_k \in \mathcal{T} \tag{6b}$$

$$\begin{pmatrix} \Pi(t) & \Pi(t)C(t)^T \\ C(t)\Pi(t) & \Xi(t) \end{pmatrix} \geq 0\,, \quad \forall\ t \in]t_0\,, t_0+T] \tag{6c}$$

is satisfied, where

$$\Upsilon(t) = -\dot{\Pi}(t) + \Pi(t)A(t)^T + A(t)\Pi(t) + B(t)L(t) + L(t)^T B(t)^T\,,$$
$$\Xi(t) = Q(t-t_0)^{-1}\,.$$

In this case the a controller gain which solves Problem SF for the input class $\mathcal{W}_2$ is $K(t-t_0) \;=\; L(t)\Pi(t)^{-1}$.

Example

Consider the second order TD-IDLS with the continuous-time dynamic defined by

$$A = \begin{pmatrix} -2.5 & -6.25 \\ 4 & 0 \end{pmatrix}, G = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 3.125 \end{pmatrix},$$

and with the resetting law defined by

$$J = \begin{pmatrix} -0.8 & 0 \\ 0 & -0.8 \end{pmatrix}. \tag{7}$$

Given the resetting times set

$$\mathcal{T} = \{0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75\}, \tag{8}$$

and letting

$$R = 1, Q = 0.1, t_0 = 0, \text{and } T = 2,$$

we check IO-FTS of the given TD-IDLS wrt $(\mathcal{W}_\infty, 0.1, 0, 2)$.

Choice of the $P(\cdot)$ matrix–valued function

In order to recast the D/DLMI conditions in terms of LMIs, the matrix-valued function $P(\cdot)$ has been assumed piecewise linear with jumps in correspondence of the resetting times.
In particular, let consider the i-th time interval between two resetting times t_k and t_{k+1}. In this time interval the $P(\cdot)$ function is assumed equal to

$$P(t) = \begin{cases} P_i + \Theta_{i\,,1}\,(t - t_k)\;, & t \in [t_k, t_k + T_s]\;, \\ P_i + \sum_{h=1}^{j} \Theta_{i\,,h} T_s + \Theta_{i\,,j+1}\,(t - jT_s - t_k)\;, & \\ & t \in \,]t_k + jT_s, t_k + (j+1)T_s] \\ & j = 1, \ldots, J_i \end{cases}$$

where $J_i = \max\{j \in \mathbb{N} : j < (t_{k+1} - t_k)/T_s\}$, $T_s \ll T$ and P_i, $\Theta_{i\,,j}$, are the optimization variables.
In correspondence of a resetting time t_k, the $P(\cdot)$ function jumps between

$$P_{i-1} + \sum_{h=1}^{J_{i-1}} \Theta_{i-1\,,h} T_s + \Theta_{i-1\,,J_{i-1}+1}(t_k - J_{i-1} T_s - t_{k-1})$$

and P_i.

Choice of the $P(\cdot)$ matrix–valued function

Such a piecewise function can approximate a generic continuous $P(\cdot)$ with adequate accuracy, provided that the length of T_s is sufficiently small.

Example

Exploiting standard optimization tools such as the Matlab LMI Toolbox or TOMLAB, it is possible to find a matrix function $P(\cdot)$ that verifies the sufficient conditions. Hence the considered TD-IDLS is IO-FTS wrt $(\mathcal{W}_\infty, 0.1, 0, 2)$.

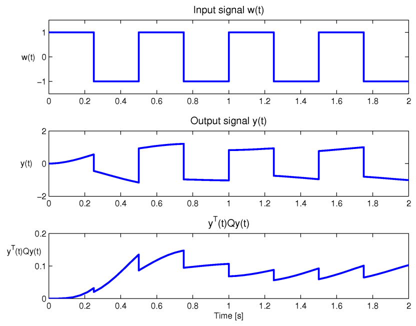

Figure: Time evolution of the exogenous input, of the output and of the weighted output

Example

Let consider the same IDLS with

$$R = 1\,,\, Q = 1\,,\, t_0 = 0\,, \text{and } \; T = 2\,.$$

In this case the the system is not IO-FTS for $w(\cdot) \;\in\; \mathcal{W}_\infty$.
We can add a control input $u(\cdot)$ with the corresponding matrix equal to

$$B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\,,$$

so as to exploit the D/DLMI condition to design a state–feedback control law $u(t) = K(t)x(t)$, such that the closed loop system is IO–FTS wrt $\big(\mathcal{W}_\infty\,, 1\,, 0\,, 2\big)$.

Example

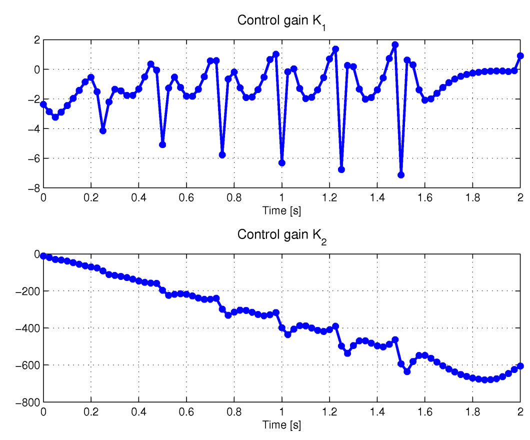

Figure: State–feedback controller gains.

Example

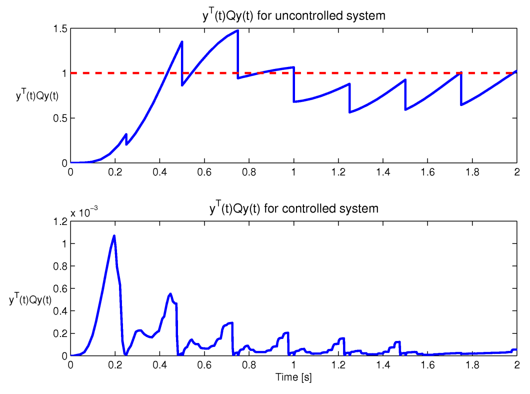

Figure: Weighted output without and with state–feedback control.

Conclusions

- Concept of IO–FTS has been extended to Impulsive Dynamical Linear Systems
- Sufficient conditions for IO–FTS of IDLS have been given, when the two classes of input signals $\mathcal{W}_2$ and $\mathcal{W}_\infty$ are considered
- The effectiveness of the approach has been illustrated by means of numerical examples
- Application to DC/DC converters are envisaged

Thank you!