GEOMETRIC ASPECTS OF PDE'S AND FUNCTIONAL INEQUALITIES

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Virginia Agostiniani

The electrostatic potential: monotonicity formulas and geometric inequalities

The aim of this talk is to report on some recent progress - obtained in collaboration with L. Mazzieri - in the study of some geometric aspects of potential theory. In particular, we show how the technique introduced in some previous work, based on a conformal reformulation of the electrostatic problem jointly with some splitting principles, allows us to single out certain relevant quantities associated with the level set flow of the electrostatic potential, which are proved to be monotone along the flow. As a consequence of this monotonicity, we derive some old and new geometric inequalities, whose equality cases are characterized in terms of the rotational symmetry of the potential function and sphere-type theorems. If the time permits, we will also show that similar results hold for the problem of the static metrics in general relativity.

Gisella Croce

Symmetry and asymmetry of minimizers of a class of noncoercive functionals

In this talk we will present a symmetry result for minimizers of a non coercive functional defined on the class of Sobolev functions with zero mean value. We will prove that the minimizers are foliated Schwarz symmetric, i.e., they are axially symmetric with respect to an axis passing through the origin and nonincreasing in the polar angle from this axis. In the two dimensional case we will show a symmetry breaking. This work has been obtained in collaboration with F. Brock, O. Guibé and A. Mercaldo.

Giuseppina di Blasio

Anisotropic Hardy inequalities

We discuss about some Hardy-type inequalities involving a general norm in $\mathbb{R}^n$. More precisely, let F be a smooth norm of $\mathbb{R}^n$, we investigate the validity of the following Hardy-type inequalities

$$\int_\Omega F(\nabla u)^2\,dx \geq C_F(\Omega)\int_\Omega \frac{u^2}{d_F^2}\,dx, \qquad \forall u \in H_0^1(\Omega), \tag{1}$$

where Ω is a domain of $\mathbb{R}^n$, and d_F is the anisotropic distance to the boundary with respect to the dual norm. The best possible constant for which (1) holds, in the sense that

$$C_F(\Omega) = \inf_{u\in H_0^1(\Omega)} \frac{\displaystyle\int_\Omega F(\nabla u)^2\,dx}{\displaystyle\int_\Omega \frac{u^2}{d_F^2}\,dx},$$

will also presented. This is a joint work with F. Della Pietra and N. Gavitone.

Norisuke Ioku

Existence and nonexistence of solutions for a heat equation with a superlinear source term

We consider a heat equation with a superlinear source term:

$$\text{(P)} \qquad \begin{cases} \partial_t u = \Delta u + f(u) & \text{in} \quad \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) \geq 0 & \text{in} \quad \mathbb{R}^N, \end{cases}$$

where $\partial_t = \partial/\partial t$, $N \geq 1$, $T > 0$, u_0 is a nonnegative initial function and f is a positive monotonically increasing function in $(0,\infty)$ with superlinear growth. We consider the case $u_0 \notin L^\infty(\mathbb{R}^N)$ and investigate local in time existence and nonexistence of solutions for problem (P) without any concrete assumption on the growth rate of f. In particular, we reveal the threshold integrability of u_0 to classify existence and nonexistence of solutions for problem (P). This is a joint work with Professor Yohei Fujishima(Shizuoka University).

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James Kennedy

Eigenvalue estimates for quantum graphs

A quantum graph is a metric graph – a collection of intervals of varying lengths, connected at a set of vertices – on which a differential operator such as the Laplacian acts. As simple objects which can display surprisingly complicated behaviour often reminiscent of higher dimensional PDEs, they are becoming increasingly popular as ‘toy’ problems within the PDE community. However, despite this seemingly simple nature, there seems to be very little known linking the ‘geometry’ of a graph to the spectrum of the operator.

We will give an overview of a prototypical case: estimates on the first non-trivial eigenvalue, i.e. the spectral gap, of the Laplacian with so-called Kirchhoff, or natural, conditions at the vertices, a natural generalisation or analogue of the familiar Neumann condition, in terms of basic properties of the graph such as its total length, diameter, number of edges and so on. Although we will (largely) only use elementary variational and comparison arguments, which however tend to become more powerful in one dimension, we hope to demonstrate that such problems can display surprisingly complex, variegated and non-obvious behaviour.

This is largely based on joint work with Pavel Kurasov (Stockholm), Gabriela Malenova (KTH Stockholm) and Delio Mugnolo (Hagen).

Hynek Kovařík

Optimizing the first eigenvalue of some quasilinear operators with respect to boundary conditions

We consider a class of quasilinear operators on a bounded domains and address the question of optimizing the first eigenvalue with respect to the boundary conditions, which are of the Robin-type. We describe the optimizing boundary conditions and establish upper and lower bounds on the respective maximal and minimal eigenvalue. This is a joint work with F. Della Pietra and N.Gavitone.

Jimmy Lamboley

New results about shape derivatives and convexity constraints

The results we are focusing on in this talk are motivated by the analysis of optimal shapes in the framework of convexity constraints, namely, analyzing shapes, solutions to the following type of problem:

$$\min\left\{J(\Omega),\ \Omega \text{ convex in } \mathbb{R}^N\right\}.$$

The convexity constraint makes it difficult to write optimality conditions for this problem. However, in many examples, some very useful information on optimal shapes can be obtained from the use of second order shape derivatives.
With this motivation in mind, we obtained new estimates for (first and second) order shape derivatives for classical energy functionals (based on elliptic PDE), valid under very mild assumption on the regularity of the shapes, and which are sharp when the set is convex. This rely on deep but well-known results about elliptic regularity.
This is a common work with A. Novruzi (Ottawa) and M. Pierre (Rennes)

Michele Marini

Sharp estimates for the anisotropic torsional rigidity and the anisotropic principal frequency of a convex domain

Abstract: As it is well known, for a convex domain, there are sharp upper and lower bounds for the torsional rigidity and the first eigenvalue of the Dirichlet Laplacian in terms of the volume and the inradius. In the talk we consider the h-anisotropic torsional rigidity and principal frequency and we prove sharp estimates, generalizing the known ones in the linear case, involving the volume and a suitable concept of anisotropic inradius.

Dario Mazzoleni

Geometric properties of optimal sets for some spectral optimization problems

We consider the problem of minimizing convex combinations of the first two eigenvalues of the Dirichlet-Laplacian among open set of $\mathbb{R}^N$ of fixed measure. We show that, by purely elementary arguments, based on the minimality condition, it is possible to obtain informations on the *geometry* of the minimizers of convex combinations: we study, in particular, when these minimizers are no longer convex and the minimality of balls. Moreover we will discuss some symmetry problems for optimal sets.

Our techniques involve explicit constants in quantitative inequalities as well as a purely geometrical problem: the minimization of the Fraenkel 2-asymmetry among *convex* sets of fixed measure.

The talk is based on joint works with Davide Zucco.

Enea Parini

On a shape functional related to Cheeger's inequality

We prove the sharp inequality

$$J(\Omega) := \frac{\lambda_1(\Omega)}{h_1(\Omega)^2} < \frac{\pi^2}{4},$$

where Ω is any planar, convex set, $\lambda_1(\Omega)$ is the first eigenvalue of the Laplacian under Dirichlet boundary conditions, and $h_1(\Omega)$ is the Cheeger constant of Ω. The value on the right-hand side is optimal, and any sequence of convex sets with fixed volume and diameter tending to infinity is a maximizing sequence. Morever, we discuss the minimization of J in the same class of subsets: we provide a lower bound which improves the generic bound given by Cheeger's inequality, we show the existence of a minimizer, and we give some optimality conditions.

Stefania Patrizi

On a long range segregation problem

Segregation phenomena occurs in many areas of mathematics and science: from equipartition problems in geometry, to social and biological process (cells, bacteria, ants, mammals) to finance (sellers and buyers). There is a large body of literature studying segregation models where the interaction between species is punctual. There are many processes though, where the growth of a population at a point is inhibited by the populations in a full area surrounding that point. The work we present is a first attempt to study the properties of such a segregation process. This is a joint paper with Luis Caffarelli and Véronica Quitalo.

Nicolas Popoff

Ground state energy of the Robin Laplacian

I will consider the problem of the asymptotics of the first eigenvalue for the Laplacian with Robin boundary condition, when the Dirichlet parameter gets large. This problem can be linked to the question of the best constant in some Ehrling?s type lemma. I will focus on the case where the domain belongs to a general class of corner domains. I will introduce singular chains associate with tangent geometries of a corner domain, and show that the asymptotics is given at first order by the minimization of a function, called "local energy??, defined on singular chains. Using a multiscale analysis, we give an estimate of the remainder. I will also provide a more precise asymptotics when the domain is regular, and I will derive Faber-Krahn type inequalities for the first eigenvalue.

Yannick Privat

Optimal shape and location of actuators or sensors in PDE models

We investigate the problem of optimizing the shape and location of actuators or sensors for evolution systems driven by a partial differential equation, like for instance a wave equation, a Schrdinger equation, or a parabolic system, on an arbitrary domain Omega, in arbitrary dimension, with boundary conditions if there is a boundary, which can be of Dirichlet, Neumann, mixed or Robin. This kind of problem is frequently encountered in applications where one aims, for instance, at maximizing the quality of reconstruction of the solution, using only a partial observation. From the mathematical point of view, using probabilistic considerations we model this problem as the problem of maximizing what we call a randomized observability constant, over all possible subdomains of Omega having a prescribed measure. The spectral analysis of this problem reveals intimate connections with the theory of quantum chaos. More precisely, if the domain Omega satisfies some quantum ergodic assumptions then we provide a solution to this problem. These works are in collaboration with Emmanuel Trlat (Univ. Paris 6) and Enrique Zuazua (BCAM Bilbao, Spain).

Berardo Ruffini

Compact Sobolev embedding and torsion function

Given a regular open set of finite measure, it is well known that the Sobolev space $W_0^{1,p}$ of such a set is compactly embedded inside some integrability space L^q. In general this does not hold true if the set has not finite measure. In this seminar we show a way to weaken this hypothesis. Namely we see how integrability properties of the torsion function of a set are equivalent to the compact immersion of $W_0^{1,p}$ into suitable integrability spaces. The talk is based on a joint work with Lorenzo Brasco.

Lenka Slavíková

Higher-order Sobolev embeddings and isoperimetric inequalities

We present a recent result, obtained jointly with Andrea Cianchi and Luboš Pick, showing that optimal higher-order Sobolev type embeddings follow from isoperimetric inequalities. This establishes a higher-order analogue of a well-known link between first-order Sobolev embeddings and isoperimetric inequalities. We reduce Sobolev type embeddings of any order, involving arbitrary rearrangement-invariant norms, on open sets in $\mathbb{R}^n$, possibly endowed with a measure density, to much simpler one-dimensional inequalities for suitable integral operators depending on the isoperimetric function of the relevant sets. We also discuss the related question of compactness of these embeddings. Our results can be applied, in particular, to any-order Sobolev embeddings in regular (John) domains of the Euclidean space, in Maz'ya classes of (possibly irregular) Euclidean domains described in terms of their isoperimetric function, and in families of product probability spaces, of which the Gauss space is a classical instance.

Chao Xia

Inverse anisotropic curvature flow and Minkowski's inequality

In this talk, I will first discuss the inverse anisotropic mean curvature flow from a star-shaped hypersurface and show such flow exists for long time and converges to a rescaled Wulff shape. Next I will discuss general inverse anisotropic curvature flows from convex hypersurfaces. As an application, we prove Minkowski's inequality for mixed volumes.