6
Propeller Theory

6.1 Introduction

For certain missions and speed ranges the propeller-driven aircraft is the most efficient. Some examples are:

1) speed range of 0–450 kn
2) certain STOL missions
3) long endurance maritime reconnaissance missions

Recent NASA research has shown it possible to extend the cruise speed of propeller-driven aircraft to 0.8 Mach while maintaining a propeller efficiency in excess of 80%. Therefore for certain applications the propeller will be around indefinitely, and may be reintroduced in such areas as air transport.

6.2 Propeller Theory

There are three theories now used in the design of propellers. They are:

1) momentum theory
2) blade element theory
3) vortex theory

Until recently only the first two theories were in use. This is primarily due to the mathematical complexity of the vortex theory.

Most of today's propellers were designed using blade element theory. The problem for the designer is to find an existing propeller design that can be adapted to their requirements.

Since few propellers have been designed using vortex theory we will limit our discussion to momentum theory and blade element theory. Those interested in vortex theory should read Barnes W. McCormick's *Aerodynamics for V/STOL Flight*, Chapter 4 (Ref. 1).

6.2.1 Momentum Theory

Derived from Newton's second law:

$$F = m(\mathrm{d}V/\mathrm{d}t) \tag{6.1}$$

Assumptions:

1) Propeller does not add rotation to the air.
2) No profile losses.
3) Air is inviscid and incompressible.
4) Propeller has an infinite number of blades.

Since most of these assumptions are unrealistic, this theory is only useful in predicting ideal or maximum propeller efficiencies.

The mass of air passing through propeller (per unit time) is:

$$M = \rho A_1(V + v) \tag{6.2}$$

where
M = air mass
ρ = air density
A_1 = propeller disc area
V = true airspeed
v = velocity change through propeller

The thrust generated by the propeller is the mass per unit time multiplied times the total change in velocity per unit time through the control volume:

$$F = \rho A_1(V + v)2v \tag{6.3}$$

The power input to the propeller is:

$$P_i = F(V + v) \tag{6.4}$$

or the thrust times the velocity change the propeller. The power output by the propeller is:

$$P_o = FV \tag{6.5}$$

The ideal propeller efficiency is defined by:

$$\eta_{Pi} = P_o/P_i = FV/F(V + v) = V/(V + v) \tag{6.6}$$

6.2.2 Blade Element Theory

This theory considers the propeller blade to be a highly twisted wing. Most propellers in use today were designed using this theory. Since this theory considers the propeller to be a highly twisted wing, we will relate, where possible, the propeller parameters to more familiar wing parameters.

r = radius from hub to blade element
V = airplane velocity
β = geometric angle of attack of blade element
γ = reduction in blade angle of attack due to forward velocity

α = actual angle of attack
ω = rotational speed of propeller $\omega = 2\pi n$ where n = rps
u = propeller velocity due to rotation $u = r\omega = 2\pi rn$

$$\tan\gamma = V/u = V/r\omega = V/2\pi rn \tag{6.7}$$

Since β is fixed for a given blade element,

$$\alpha = \beta - \gamma \tag{6.8}$$

For a propeller α is different for each blade element and is not very useful. A better method is to evaluate the propeller at the tip since the twist is fixed. At the tip:

$$\tan\gamma_{tip} = 2V/\omega d \doteq V/\pi nd \tag{6.9}$$

where
d = propeller diameter

The dimension less quantity $V/nd = \pi \tan\gamma_{tip}$ is called the advance ratio J.

$$J = V/nd \tag{6.10}$$

This quantity in propeller theory is used to replace angle of attack in airfoil theory. Using this concept we know that the resultant velocity at a given blade section is proportional to nd, and that the aircraft velocity V is equal to the product of J and nd. We also know that the rotational velocity $u = (2\pi r/d)nd$. We can now arrive at the equation for the thrust generated by the propeller by using the laws of similarity: The propeller diameter as the reference length in determining Reynolds number; The thrust is then the product of the reference area d^2, the dynamic pressure, which corresponds to nd, and a thrust coefficient C_T, which is only a function of J, and the Reynolds number.

$$F = \rho(nd)^2 d^2 C_T = \rho n^2 d^4 C_T \tag{6.11}$$

This equation can be compared to the lift equation of airfoil theory where the constant 1/2 is absorbed into C_T.

In propeller theory we plot C_T vs J like we plot C_L versus α in airfoil theory. In momentum theory we did not consider the drag of the propeller. In blade element theory we do.

The drag goes to make up a part of the moment required to drive the propeller. This moment is the propeller torque Q:

$$Q = \rho n^2 d^5 C_Q \tag{6.12}$$

Where the torque coefficient C_Q is dependent upon J, R_e, and propeller shape.

The power into the propeller is a function of the torque Q and the rate of rotation ω:

$$P_i = \omega Q = 2\pi n Q = \rho n^3 d^5 2\pi C_Q \tag{6.13}$$

or

$$P_i = \rho n^3 d^5 C_p \tag{6.14}$$

where

$$C_P = 2\pi C_Q$$

Since C_Q was dependent upon J, R_e, and propeller shape, so is C_P. The power coefficient is similar to the drag coefficient of airfoil theory and is usually also plotted vs J. The two curves then resemble lift and drag coefficient plots.

To determine efficiency we consider the power out FV divided by the power in P_i:

$$\begin{aligned} \eta_P &= (TV)/P = (\rho n^2 d^4 C_T V)/(\rho n^3 d^5 C_P) \\ &= (C_T/C_P)(V/nd) = (C_T/C_F)(J) \end{aligned} \tag{6.15}$$

Figure 6.1 would be typical of a chart for fixed-pitch propellers. For this kind of propeller there is only one value of J where the efficiency is at a maximum, or since $J = V/nd$ only one flight condition where η_P is at a maximum. So the propeller for the fixed-pitch, propeller-driven airplane is optimized for

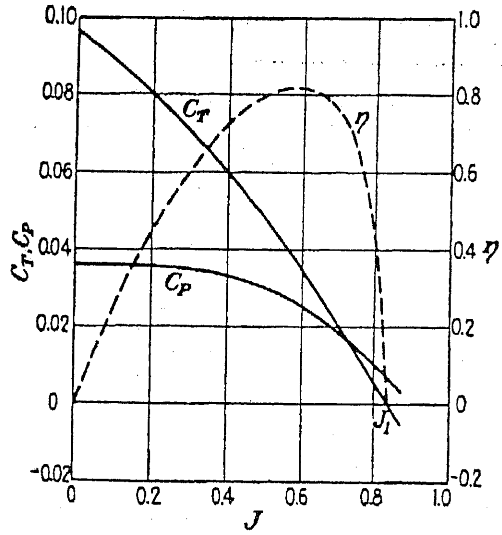

Fig. 6.1 Characteristic curves of a propeller at 15.5 deg blade pitch.[2]

max efficiency in either the climb or cruise condition depending upon its mission.

6.3 Propeller Polar Diagram

For certain flight tests of airplanes it is useful to define the propeller forces to airspeed squared rather than to $(nd)^2$. To do this C_P/J^2 is plotted versus C_T/J^2 such as is shown in Fig. 6.2 for the propeller of Fig. 6.1. From this figure we can see that in this form the propeller data is a straight line which is much easier to use. This curve can be obtained experimentally from flight test data.

6.4 Constant Speed or Controllable Propellers

The efficiency problem with fixed-pitch propellers led to the development of controllable and constant speed propellers. For these propellers there is a family of curves, one set of curves for each blade angle β. This allows us to maintain optimum efficiency over a wide range of J and corresponding flight conditions.

Although a propeller polar such as the one shown in Fig. 6.2 cannot be obtained for a constant speed propeller, Lowry[4] provides a generalized chart for constant speed propellers for light aircraft and Perkins and Hage[5] provide a normalized chart for larger aircraft that was developed by the Boeing Company.

6.5 Activity Factor

The ability of a propeller to absorb power is a function of the blade area as a ratio of the area of the propeller disk. This is expressed as a term called the "activity factor," which represents the integrated power absorption capability

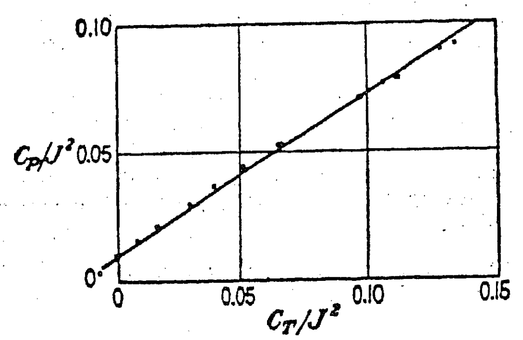

Fig. 6.2 Propeller polar for the propeller of Fig. 6.1 (Ref. 3).

of all the blade elements. The activity factor for a single blade can be found from:

$$A.F. = (100{,}000/16)\int_{.15}^{1.0}(C/D)x^3\,dx \qquad (6.16)$$

where $x = (r/R)$ or

$$A.F. = (100{,}000/16)[(C/D)(x^4/4)]_{.15}^{1.0} \qquad (6.17)$$

To determine the total activity factor for the propeller one should multiply the activity factor for a single blade by the total number of blades.

6.6 Propeller Noise

The efficiency of a propeller and the noise made by it are also functions of the tip Mach number. In order for a propeller to be efficient and quiet the tip Mach number should be well below 0.9 Mach. In addition, noise may be reduced by increasing the number of blades. However, as the number of blades increase the efficiency of the propeller decreases. A propeller with only one blade is the most efficient propeller.

References

[1]McCormick, Barnes W., *Aerodynamics of V/STOL Flight*, John Wiley and Sons, Inc., New York, 1967.

[2]Weick, Fred E., "Full-Scale Wind-Tunnel Tests with a Series of Propellers of Different Diameters on a Single Fuselage," NACA Technical Report 339, National Advisory Committee on Aeronautics, Langley Field, VA, Jan. 1931.

[3]Von Mises, Richard, *Theory of Flight*, Dover Publications, Inc., New York, 1959.

[4]Lowry, John T., *Performance of Light Aircraft*, AIAA Education Series, AIAA, Reston, VA, 1999.

[5]Perkins, Courtland D., and Hage, Robert E., *Airplane Performance Stability and Control*, John Wiley and Sons, Inc., New York, 1967.

Our list of possible corrections might well give the reader the impression that one can spend a lifetime studying one propeller. Especially if one bolts copies of that propeller onto different airplanes. Orville Wright was right: there is a lot going on up there. Propeller reality is certainly less sedate than the theory. We can leave the subject at this point, confident that this grounding in propeller culture is sufficient for our airplane performance purposes. We need to know something about propellers and their coefficients, but we will not be using complicated detailed calculational methods. We will do something a lot simpler involving the "propeller polar."

Example 6.3 This introduction to propeller theory may have given you more facility than you realize. As an example take the common experience of having revolutions per minute drop, as speed drops when one starts to climb, at constant throttle, out of level cruise. Here is one way that drop in engine speed with air speed can be explained. We only need three pieces of information:

1) The definition of C_P, Eq. (6.8);
2) The definition of advance ratio $J \equiv V/nd$; and
3) The fact that, in the "working range," graphs of $C_P(J)$ slope down to the right; i.e., $dC_P/dJ < 0$.

At constant throttle (constant manifold pressure), torque is constant. From assumption 1), plus the relation between power and torque, Eq. (6.5), that means there is a constant K such that $C_P = K/n^2$. Now when air speed V drops off, either engine speed n goes down or it does not. If we assume n decreases, we can not tell which way J might go; we are stymied. So let us (provisionally) assume the n increases or stays constant and see what we can figure out. Then, by assumption 2, because V decreases and n increases or stays constant, J decreases. But then, by assumption 3), C_P increases. But then, because $n^2 = K/C_P$, n decreases. So if n increases, then n decreases, which is absurd. So, when V decreases at constant torque, it must be true than n also decreases (just as we know it does).

A more constructive theory,[5] computing differentials with Jacobians, gives

$$\frac{dn}{dV} = \left[d\left(J - \frac{2C_P}{C_P'}\right)\right]^{-1} \tag{6.50}$$

where C_P' means dC_P/dJ. Again, the fact of negative C_P' ensures dn/dV is positive.

Propeller Polar Diagram

The propeller polar diagram is a graph of C_T/J^2 plotted as a function of C_P/J^2. Why would one care to make such a strange graph? Let us take a look at these new variables.

$$\frac{C_P}{J^2} = \frac{P}{n} \times \frac{1}{d^3 \rho V^2} \tag{6.51}$$

Equation (6.51) shows that propeller polar variable C_P/J^2 depends on engine torque and, at a given density altitude, on the true air speed. Engine torque is a quite linear function of manifold absolute pressure (MAP). And, for a given density altitude, MAP depends on throttle position. So for a fixed throttle and altitude, C_P/J^2 depends only on air speed. By selecting his air speed, the pilot thereby selects C_P/J^2. Let us take a similar look at the dependent variable:

$$\frac{C_T}{J^2} = \frac{T}{\rho d^2 V^2} \tag{6.52}$$

The important thing here is that this depends on thrust and air speed and *not* on engine revolutions per minute. That is a good thing because (without a lot of engine and propeller data) we do not have a good way of knowing revolutions per minute. (This is a defect to be remedied in the chapter on cruise and partial-throttle performance.) With an accurate propeller polar diagram, ignorance of revolutions per minute does not hurt much. The propeller polar lets us turn knowledge about throttle position (torque) and air speed into knowledge about thrust. We will use this line of reasoning when we discuss the *bootstrap approach* to computing fixed-pitch airplane performance.

The final interesting and useful fact about the propeller polar diagram or formula is that, over the main operating speeds of the airplane (excepting very low speeds early in the takeoff roll and very high speeds, say diving under power), *the propeller polar is linear*. Figure 6.18, for instance, is the polar for the McCauley 7557 propeller we investigated earlier. This example uses the correction-adjusted combined momentum theory and blade element calculation. Using McCauley's data, the polar is even more nearly linear. If we call the value of advance ratio where efficiency is maximum J_m then Fig. 6.18 uses only Js between 0.50 J_m and 1.17 J_m.

Finding the best straight line through the points of Fig. 6.18 gives us the "linearized" propeller polar (also in Fig. 6.18):

$$\frac{C_T}{J^2} = m\,\frac{C_P}{J^2} + b \tag{6.53}$$

with (in this particular case) slope $m = 2.086$ and intercept $b = -0.0670$.

There are theoretical reasons why propeller polar diagrams are nearly linear. A competing simplified theory, that of the "representative blade element," says there

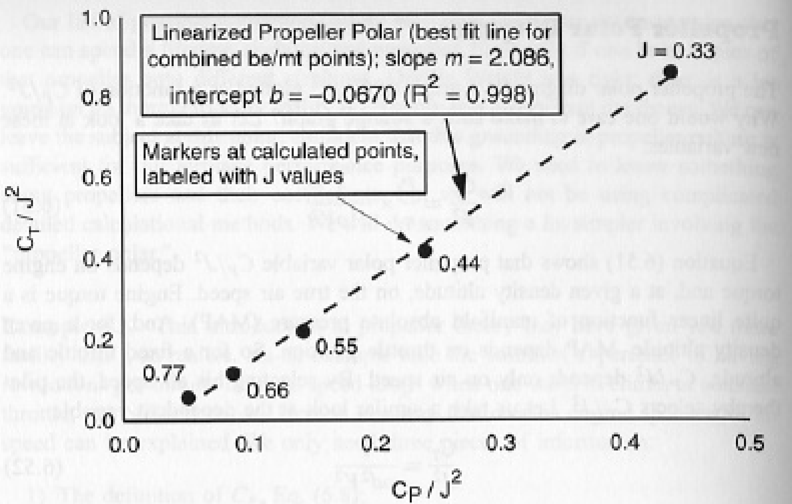

Figure 6.18 Propeller polar diagram from corrected combined momentum and blade element theories calculation.

is some station along the blade (not far from $x = 0.75$) that can "stand in" for the average propeller section. Von Mises[6] uses this representative blade element theory predominantly in his analytic propeller and aircraft performance sections. He includes a "proof" that the propeller polar is approximately linear. One can go farther and tie information about the propeller diameter and average width, together with the blade setting angle and drag coefficient of the representative blade element, to the two numbers (slope and intercept) characterizing the linear propeller polar. We will not do that here, however.

The best argument towards linearity of the propeller polar diagram, to our mind, is that whenever you construct such a diagram (restricted to the "working range") out of believable propeller characteristic charts $C_P(J)$ and $C_T(J)$, you do indeed get quite a straight line. So, for fixed-pitch propellers, we need propeller charts or, for this "reduced" description, propeller polar diagrams. It will turn out that bootstrap approach flight tests obtain the propeller polar experimentally. But what are we going to do about constant-speed propellers?

Constant-Speed Propeller

A proper pair of propeller characteristic graphs $C_P(J)$ and $C_T(J)$—or one of these plus a graph of $\eta(J)$—describes the action of any propeller with a fixed blade angle setting. We usually specify this setting by citing $\beta_{0.75}$, the blade angle at

$x = 0.75$. Often an entire family of related propellers differ only in that regard. But constant-speed propeller blades do not have fixed blade angle settings! Those blades twist in their automated hubs to meet (within limits) the torque and speed demands set by manifold pressure and the propeller speed control. For constant-speed propellers it looks like we will need a whole sheaf of characteristic graphs. If we had them, here is how we might proceed to find propeller efficiency η, the pivotal quantity we need:

- From V, n, and d, calculate J.
- Enter the POH cruise tables with pressure altitude h_p, OAT (giving density altitude h_ρ or its surrogates relative density σ or density ρ), manifold pressure and revolutions per minute, to find shaft power P (or P as a proportion of rated mean sea level power).
- From P, ρ, n, and d, calculate C_P.
- From graphs for C_P vs J for various $\beta_{0.75}$, determine $\beta_{0.75}$ by intersection (with perhaps a little interpolation).
- From the graph giving η vs J for the $\beta_{0.75}$ above (again, perhaps with interpolation), read off η.

The bad news is that you can seldom get your hands on that full set of charts for your particular propeller. The good news is that there is a much easier, though approximate, way that in a sense combines the last two steps above. This alternative has the additional advantage of being general, of not depending on all the details of your particular make and model constant-speed propeller.

General Propeller Charts

The momentum theory, recall, suggested that ideal propeller efficiency η_i is purely a function of $J/C_P^{1/3}$. As usual, reality (including blade drag, mutual interference among blades, and rotation of the slip-stream) is somewhat more complicated. Actual efficiency η is still pretty much a function of $J/C_P^{1/3}$, but the details of just *which* function depends on how big C_P is. And not only C_P but also a "power adjustment factor," X, which depends on the propeller's distribution of blade widths. The first use of this strategy was the general propeller chart pioneered by the Boeing Aircraft Company (BAC)[7] in its production of long-range bombers during World War II. A version of the BAC general propeller chart can be found in Perkins and Hage.[8]

But the difference in scale between say a Bonanza (285 hp) and a bomber or transport sporting four Pratt & Whitney Wasp Majors (2685 hp each) is just too great. The BAC general propeller chart does not give accurate results for general aviation airplanes with constant-speed propellers. What was needed was a new chart, similar to Boeing's but based on data for the smaller general-aviation-sized, constant-speed propeller: a *general aviation* general propeller chart. Data and measurements for a general aviation constant-speed propeller let us construct such a chart (see Fig. 6.19). This chart will play a pivotal role in the constant-speed version of the bootstrap approach. Let us see how it works.

Example 6.4 Our engine is turning over 2400 rpm ($n = 40$ rps) and is putting out 200 hp ($P = 110{,}000$ ft-lbf/s). We are cruising at density altitude of 5000 ft ($\sigma = 0.8617$, $\rho = 0.002048$ slug/ft^3), at 150 KTAS ($V = 253.2$ ft/s). Propeller diameter is $d = 7$ ft. Accordingly, advance ratio $J = V/nd = 0.9043$; $C_P = P/\rho n^3 d^5 = 0.04993$; $C_P^{1/3} = 0.36824$; $J/C_P^{1/3} = 2.456$. The eight curves of the general aviation general propeller chart are labeled by $C_{PX} \equiv C_P/X$, where X is the power adjustment factor (to be explained below), which depends on the total activity factor (TAF, also to be explained below) of our particular propeller. Assume our blade activity factor (BAF) = 100 and we have two of them; then TAF = 200. It turns out that $X = 0.246$. Hence, $C_{PX} = 0.2030$. We need the phantom graph about halfway between the graphs for $C_{PX} = 0.15$ and for $C_{PX} = 0.25$. Interpolating, we find $\eta = 0.764$ (the Boeing general propeller chart gives $\eta = 0.87$).

Now let us turn, as promised, to describing the propeller BAF and the power adjustment factor. There is actually one additional wrinkle to be ironed out, the SDEF, having to do with obstructed airflow through the propeller. Again, to be explained.

Activity Factor

The tangential air speed of the propeller section at relative station x is proportional to x. Aerodynamic forces vary as the area and as the square of the air speed and hence as $x^2 c(x)\,dx$. Absorbed power goes as force times speed, hence as $x^3 c(x)\,dx$. That is why BAF is defined as

$$\text{BAF} \equiv \frac{100{,}000}{32} \times \int_{0.2}^{1} x^3 \frac{c(x)}{R}\,dx \tag{6.54}$$

The denominator R (blade radius) is to make BAF dimensionless. The factor in front of the integral sign is to give BAFs reasonable numerical sizes. For commonly shaped blades, BAFs vary from about 70 to 140. Total activity factor, for the entire propeller, is defined as

$$\text{TAF} = \text{number of blades} \times \text{BAF} \tag{6.55}$$

Normally, you do not have a formula for blade width $c(x)$ as a function of relative station x and so evaluate the integral in Eq. (6.54) numerically. Table 6.6 and Eq. (6.55) are the ticket. First you must measure your propeller blade's widths at relative stations 0.20, 0.25, 0.30, ..., 0.90, 0.95, 1.00. If early stations (0.20,

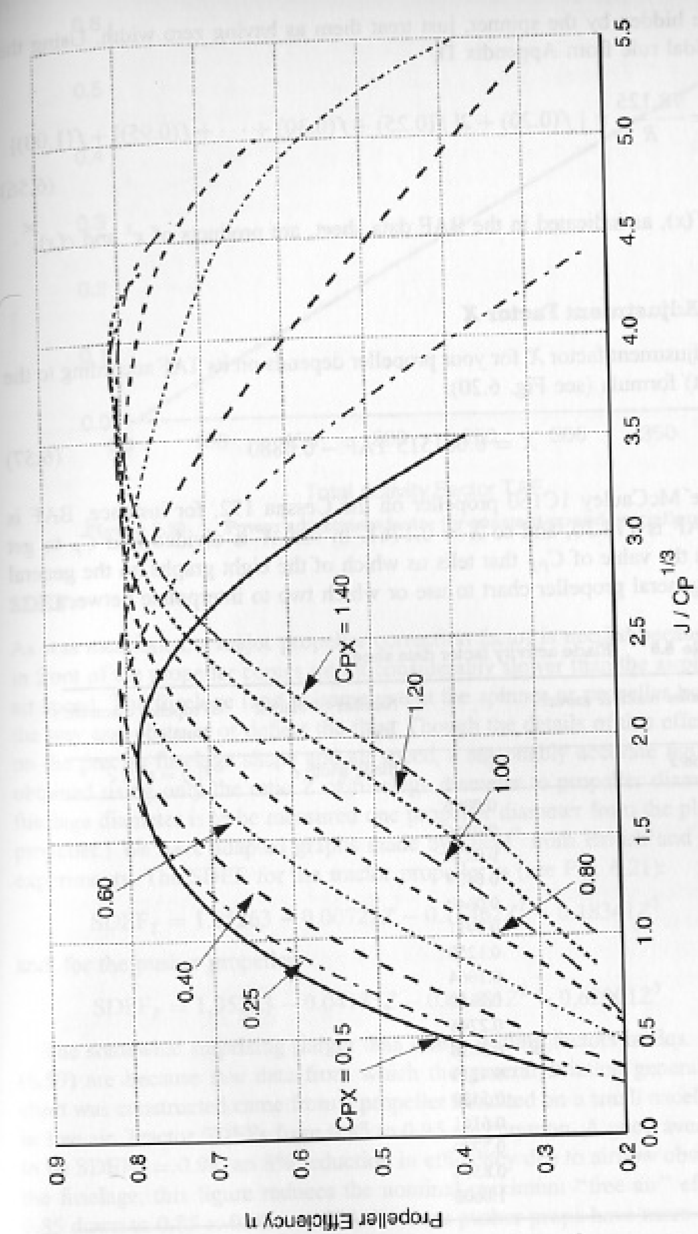

Figure 6.19 The general aviation general propeller chart.

etc.) are hidden by the spinner, just treat them as having zero width. Using the trapezoidal rule from Appendix D,

$$\text{BAF} = \frac{78.125}{R} \times \{ f(0.20) + 2[f(0.25) + f(0.30) + \cdots + f(0.95)] + f(1.00)\} \tag{6.56}$$

Values $f(x)$, as indicated in the BAF data sheet, are products of x^3 and $c(x)$.

Power Adjustment Factor *X*

Power adjustment factor X for your propeller depends on its TAF according to the (curve-fit) formula (see Fig. 6.20):

$$X = 0.001515\ \text{TAF} - 0.0880 \tag{6.57}$$

For the McCauley 1C160 propeller on the Cessna 172, for instance, BAF is 87.15, TAF is 174.30, and so $X = 0.1761$. In use, X is divided into C_P to get C_{PX}. It is the value of C_{PX} that tells us which of the eight graphs on the general aviation general propeller chart to use or which two to interpolate between.

Table 6.6 Blade activity factor data sheet

Propeller make & model:		Number of blades:	Propeller diameter:
BAF:		TAF:	Note:
Station x	x^3	Blade width c	$f(x) = x^3 \times c$
0.20	0.0080		
0.25	0.0156		
0.30	0.0270		
0.35	0.0429		
0.40	0.0640		
0.45	0.0911		
0.50	0.1250		
0.55	0.1664		
0.60	0.2160		
0.65	0.2746		
0.70	0.3430		
0.75	0.4219		
0.80	0.5120		
0.85	0.6141		
0.90	0.7290		
0.95	0.8574		
1.00	1.0000		

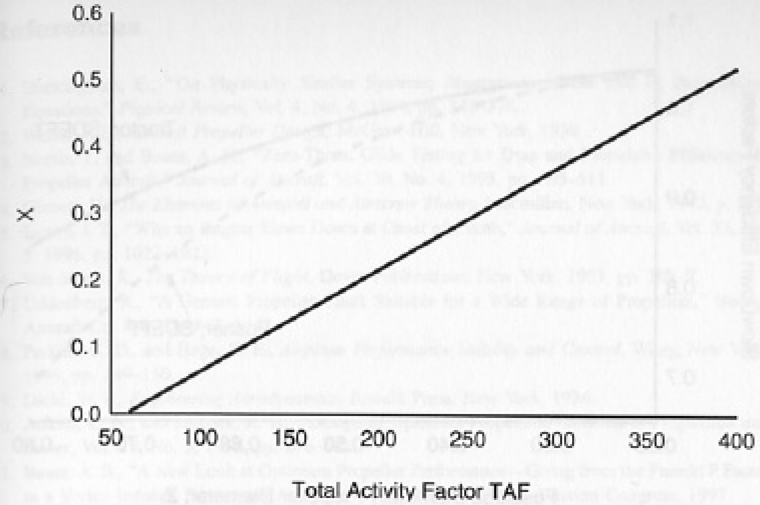

Figure 6.20 Power adjustment factor for constant-speed propellers.

SDEF

As was mentioned, a major propeller correction factor is needed because air just in front of the propeller comes into it considerably slower than the airplane's true air speed. The fuselage (and to some extent the spinner or propeller boss) get in the way and obstruct or deflect the flow. Though the details of this effect depend on the precise fuselage shape and air speed, a reasonably accurate figure can be obtained using only the ratio Z of fuselage diameter to propeller diameter. (The fuselage diameter is to be measured one propeller diameter from the plane of the propeller.) We have adapted graphs made by Diehl[9] from British and American experiments. The SDEF for the tractor propeller is (see Fig. 6.21):

$$\text{SDEF}_T = 1.05263 - 0.00722Z - 0.16462Z^2 - 0.18341Z^3 \tag{6.58}$$

and, for the pusher propeller,

$$\text{SDEF}_P = 1.05263 - 0.04185Z - 0.01481Z^2 - 0.62001Z^3 \tag{6.59}$$

The somewhat surprising (larger than unity) leading factors in Eqs. (6.58) and (6.59) are because raw data from which the general aviation general propeller chart was constructed came from a propeller mounted on a small nacelle, not one in free air. Tractor SDEFs from 0.85 to 0.95 are common. A good average seems to be $\text{SDEF}_T = 0.92$, an 8% reduction in efficiency due to airflow obstruction by the fuselage; this figure reduces the nominal maximum "free air" efficiency of 0.85 down to $0.85 \times 0.92 = 0.782$. Because pusher props have more fuselage in their way, their SDEFs are smaller than for tractors.

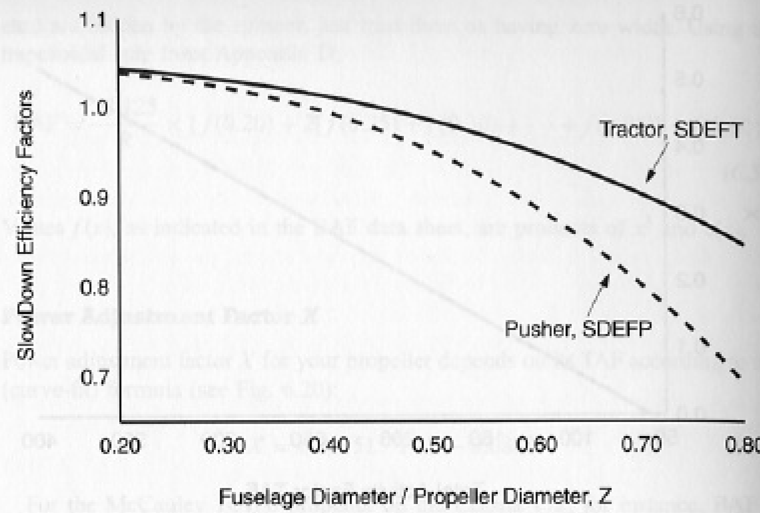

Figure 6.21 SDEF for constant-speed propellers.

The final constant-speed propulsive efficiency approximation is

$$\eta = \text{SDEF}(Z) \times \eta(J/C_P^{1/3}, C_{PX}) \tag{6.60}$$

Conclusion

As promised, a considerable struggle. We see why even Buckingham[1] wanted to avoid "venturing further into the chaos of screw-propeller theory" (along with Orville Wright). Retreating to the caves of theory to construct finely tuned and well-corrected propeller charts for ordinary performance prediction is usually impractical. For instance, we avoided the question how one might design an *optimum* propeller for a particular propulsion job. The reader interested in that study should consult the papers of Adkins and Liebeck,[10] of Bauer,[11] and of Ribner and Foster.[12] On the other hand, understanding at least a modicum of the theory behind propeller action is a major key to understanding performance of propeller-driven aircraft. Conveniently assuming some constant propeller efficiency—80 or 85% is often cited—can never get that job done. With the linearized propeller polar diagram and the general aviation general propeller chart we have realistic approximations, a middle road that will serve us well.

References

1. Buckingham, E., "On Physically Similar Systems; Illustrations of the Use of Dimensional Equations," *Physical Review*, Vol. 4, No. 4, 1914, pp. 345–376.
2. Weick, F. E., *Aircraft Propeller Design*, McGraw-Hill, New York, 1930.
3. Norris, J., and Bauer, A. B., "Zero-Thrust Glide Testing for Drag and Propulsive Efficiency of Propeller Aircraft," *Journal of Aircraft*, Vol. 30, No. 4, 1993, pp. 505–511.
4. Glauert, H., *The Elements of Aerofoil and Airscrew Theory*, Macmillan, New York, 1943, p. 215.
5. Lowry, J. T., "Why an Engine Slows Down at Onset of Climb," *Journal of Aircraft*, Vol. 33, No. 5, 1996, pp. 1022–1023.
6. Von Mises, R., *The Theory of Flight*, Dover Publications, New York, 1959, pp. 308 ff.
7. Uddenberg, R., "A General Propeller Chart Suitable for a Wide Range of Propellers," Boeing Aircraft Co. Rep. D-4842, 1943.
8. Perkins, C. D., and Hage, R. E., *Airplane Performance Stability and Control*, Wiley, New York, 1949, pp. 149–150.
9. Diehl, W. S., *Engineering Aerodynamics*, Ronald Press, New York, 1936.
10. Adkins, C. N., and Liebeck, R. H., "Design of Optimum Propellers," *Journal of Propulsion and Power*, Vol. 10, No. 5, 1994, pp. 676–682.
11. Bauer, A. B., "A New Look at Optimum Propeller Performance—Going from the Prandtl F Factor to a Vortex-Induced Downwash Analysis," Paper 975560, World Aviation Congress, 1997.
12. Ribner, H. S., and Foster, S. P., "Ideal Efficiency of Propellers: Theodorsen Revisited," *Journal of Aircraft*, Vol. 27, No. 9, 1990, pp. 810–819.