CHAPTER 2

A Technical Framework for Probability-based Demand and Capacity Factor Design (DCFD) Seismic Formats

F. Jalayer and C. A. Cornell (2003). RMS Report No. 43- Part I Foundation Development

2.1 Abstract

This chapter presents an analytical foundation for probability-based formats for seismic design and assessment of structures. These formats are designed to be suitable for code and guideline implementation. The framework rests on non-linear, dynamic seismic analysis. The formats can be used to ensure that the structural seismic design can be expected to satisfy specified probabilistic performance objectives, and perhaps (more novel) that it does so with a desired, guaranteed degree of confidence. Performance objectives are presumed to be expressed as the annual probability of exceeding a structural performance level. Structural performance levels are in turn defined as specified structural parameters (e.g., ductility, strength, maximum drift ratio, etc.) reaching a structural limit state (e.g. onset of yield, collapse, etc.). The degree of confidence in meeting the specified performance objective may be quantified through the upper confidence bound on the (uncertain) probability. In order to make such statements, aleatory (random) uncertainty and epistemic (knowledge limited) uncertainty must be distinguished. The single seismic design foundation can be formatted into the alternative conventional design methods such as LRFD design and fragility-hazard design. Versions of the new developments reported here are already in place in recently completed seismic guidelines (see FEMA 350-352, and Banon et al., 2001).

2.2 Introduction

This chapter is the first part to a report (Jalayer and Cornell 2003) that develops a framework for probability-based seismic demand and capacity design and assessment procedures. In this framework, the natural randomness inherent in seismic phenomena and the uncertainty involved in the evaluation of parameters related to such phenomena are both considered.

This chapter (part 1 of the report) develops the necessary formal foundation for this probability-based design framework; producing the annual probability of exceeding a prescribed performance level as the main output. The foundation derivation is performed in two sections. The first considers only the variability due to randomness. In the second section the uncertainty due to imperfect models and model parameters are introduced into the foundation derivation.

The next chapter (part 2 of the report) derives various alternative design formats that stem from the probability-based foundation developed in the former part of the report. Most of these formats are analogous to Load and Resistance Factor Design (LRFD) procedures associated with static, force-based structural engineering, e.g., the AISC LRFD Code, but due to the generalizations here to a non-linear, dynamic displacement basis we refer to these new formats as DCFD (Demand and Capacity Factor Design). The choice among these alternative formats must be made on grounds such as familiarity, practicality, etc., because in many cases they are technically equivalent.

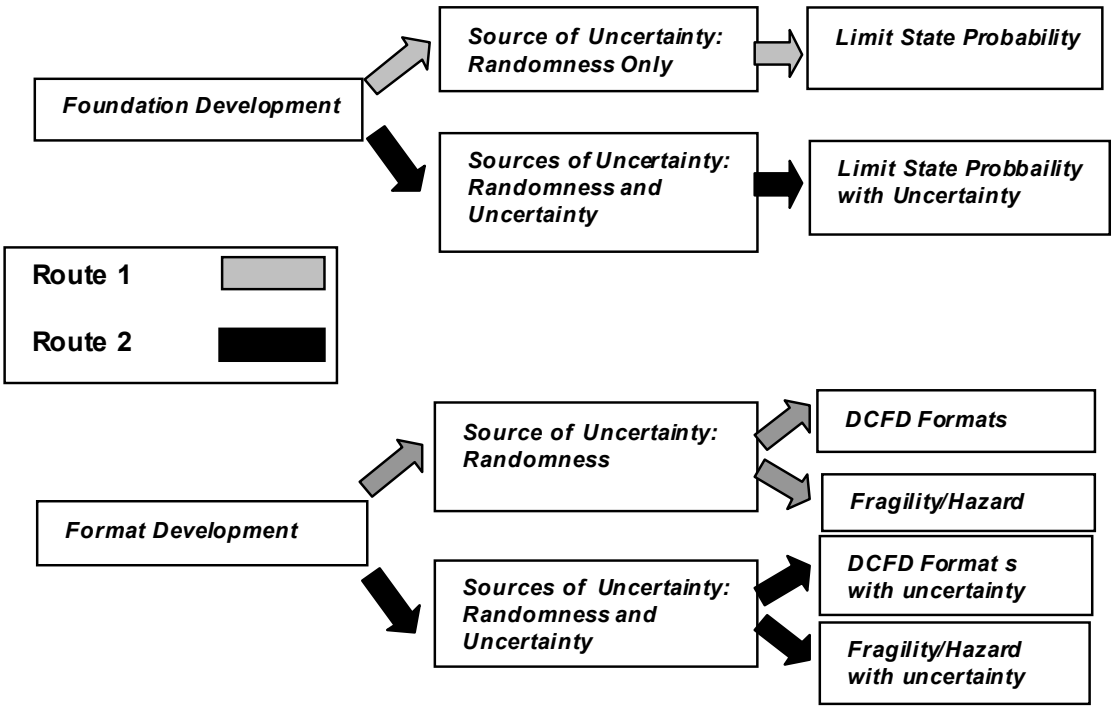
2.3 Sources of Uncertainty in Engineering Problems

Sources of uncertainty in engineering safety problems are classified into two major groups known, confusingly and unfortunately, by various pairs of words in the broader reliability community, for example, *randomness* and *statistical uncertainty*, *aleatory uncertainty* and *epistemic uncertainty*, *frequency* and *probability*, and simply *Type I* and *Type II*. Moreover, there is still some inconsistency between the researchers regarding the nature of uncertainty each group identifies. However, in the present work, the first term identifies the more familiar "natural variability" such as the times and magnitudes of future earthquakes in a region, record-to -record variability in acceleration time-history amplitudes and phases, etc. The second term of each pair signifies the limited knowledge and data that the profession currently has about, for example the modeling of structural systems in the highly non-linear range and exact numerical values of parameters of physical and random (stochastic) models, e.g., the *median* value of the maximum inter-story drift of a particular model frame under a population of future ground motions of specified intensity. This second kind of uncertainty can be reduced by more data (larger sample sizes) and/or by more research. In the following text we shall typically use the simple pair of words "randomness" and "uncertainty". Therefore we shall be using the second word in the more restrictive sense of epistemic uncertainty and not in the broad, common sense, as we used it in the title of this section. Occasionally, for example when precision is imperative, we shall use the

longer unambiguous terms "aleatory uncertainty" and "epistemic uncertainty", which are now quite common in seismic hazard analysis.

2.4 Document Map

This report contains a complete analytical background for the probability-based seismic design procedure. For pedagogical reasons the development of the text follows a detailed stepwise manner that makes it somewhat long. However, it is possible to bypass some sections without losing the general picture. The document map below illustrates two possible routes the patient reader can follow.



Route 1 goes through the entire development of the framework taking in to account the randomness source of variability only. Route 2 goes through a more generalized derivation that considers uncertainty also.

2.5 Foundation Development

This part of the report is dedicated to developing an analytical foundation for the probability-based framework. This foundation involves the entire endeavor that leads to the estimation of probability of exceeding a specified limit state for a given structural system. In other words, the final product of this section is the annual limit state probability that is calculated taking into

account the uncertainty in the various elements involved in the seismic design of the structural system.

Estimation of the limit state probability (a short way of saying "the probability of exceeding a specified limit state") will be presented in two parts. In the first part, limit state probability is derived considering only the variability due to randomness. In the second part, the more generalized form of limit state probability is introduced which accounts for both randomness and uncertainty.

2.5.1 Structural Limit States

Structural limit states are thresholds for structural behavior defined in different ways in various codes. One of the most commonly used structural limit states is the global collapse limit state. The foundation derivation represented in this text applies to virtually any limit state. However for simplicity and clarity, this report focuses on the global collapse limit state C.

2.5.2 Structural Demand Variable (State Variable)

Demand or state variable is normally chosen as a displacement-based structural response representative of structural dynamic and nonlinear behavior. The most common examples for buildings include: roof displacement or inter-story drift.

For demonstration here we have chosen the maximum inter-story drift ratio (MIDR) as the demand variable. (The maximum is the peak in the response time histories over all stories in the building.) MIDR is particularly relevant to global collapse prediction (FEMA 350). Maximum inter-story drift values may be obtained from the results of structural analyses for various ground motion intensities.

We have chosen to refer to the maximum inter-story drift variable as D. This will keep the future derivations general with respect to a generic demand variable D. It is also suggestive of the displacement-based nature of the demand variable.

2.5.3 Structural Capacity Variable (Limit State Variable)

The capacity or limit state variable as the name suggests, is a limit (threshold) for acceptable structural behavior. We have already introduced the demand (state) variable for describing the

structural behavior. The capacity (limit state) variable describes the limiting value for the demand (state) variable. Obviously, it will be represented on the same basis as the structural demand value; maximum inter-story drift ratio in this case. The capacity might be a pre-specified inter-story drift ratio, e.g., 2% (which FEMA 350 uses for an "onset of damage limit state"), or capacity with respect to connection failure that might be modeled as a random variable based on test data. In this chapter, we shall focus on global (dynamic) collapse capacities determined for IDA (Incremental Dynamic Analysis) curves corresponding to a set of ground motion records applied to the structure (see Vamvatsikos and Cornell, 2001).

In order to keep the derivations general, we have used the generic notation C for the random inter-story drift capacity. This will also be consistent with the demand variable denoted as D.

2.5.4 Limit State Probability P_{LS}

The final product of the proposed probabilistic procedure is called the probability of exceeding a structural limit state, where the limit state is the condition that, $D > C$. In order to be brief, we will refer to it as the limit state probability P_{LS}. For the case that we are mainly interested in, i.e. the collapse limit state, it is also reasonable to call this quantity the failure probability. Therefore, we seek:

$$P_{LS} = P[D > C]$$

2.5.5 General Solution Strategy

In order to determine $P_{LS} = P[D > C]$, we are going to decompose the problem into more tractable pieces and then reassemble it. First, we introduce a ground motion intensity measure IM (such as the spectral acceleration , S_a , at say 1 second period), because the level of ground motion is the major determinant of the demand D and because this permits us to separate the problem into a seismological part and a structural engineering part. To do this, we make use of a standard tool in applied probability, The Total Probability Theorem (see Appendix B), TPT which permits the following decomposition of P_{LS} with respect to an interface variable (here, the spectral acceleration):

$$P_{LS} = P[D > C] = \sum_{\text{all } x} P[D > C \mid S_a = x] \cdot P[S_a = x] \qquad (2\text{-}1)$$

In Equation 2-1 we have introduced S_a as the intensity measure. In simple words, the problem of calculating the limit state probability has been decomposed into two problems that we already know how to solve. The first problem is to calculate the term $P[S_a = x]$ or the likelihood that the spectral acceleration will equal a specified level, *x*. This likelihood is a number we can get from a Probabilistic Seismic Hazard Analysis (PSHA) of the site. The second problem is to estimate the term $P[D > C \mid S_a = x]$ or the conditional limit state probability for a given level of ground motion intensity, here represented by, $S_a = x$. Estimating the conditional limit state probability, for a given ground motion intensity, requires an understanding of, for example, response/demand variability from record-to-record of the same intensity, which is an easier and "purely" structural problem to resolve. The TPT simply tells us how to re-combine these two pieces of the problem back into P_{LS}. The solution strategy outlined above, calculating the limit state probability by decomposing it with respect to spectral acceleration, shall be referred to as the "IM-based approach" hereafter.

An alternative solution strategy (which is the main strategy employed in this chapter) consists of decomposing the expression for the limit state probability in two steps and therefore employs two interface variables. The first step is to decompose the limit state probability with respect to the displacement-based demand (the first interface variable) using TPT:

$$P_{LS} = P[D > C] = \sum_{\text{all d}} P[D > C \mid D = d] \cdot P[D = d]$$

The second step is to decompose the term, $P[D = d]$, or the likelihood that the displacement-based demand is equal to a value *d*, with respect to the spectral acceleration (the second interface variable):

$$P_{LS} = P[D > C] = \sum_{\text{all d}} \sum_{\text{all x}} P[D > C \mid D = d] \cdot P[D = d \mid S_a = x] \cdot P[S_a = x] \qquad (2\text{-}2)$$

This two-step solution strategy, which employs the displacement-based demand as one of the interface variables, shall be referred to as the "displacement-based approach". Equation 2-2 is a special case of the framework equation used by Pacific Earthquake Engineering Research (PEER) as a basis for probabilistic design and assessments.

It should be noted that the equations introduced in this section are valid for discrete interface variables. However, here they solely serve as a schematic outline of the solution strategy. Later in this chapter, we are going to present parallel equations for continuous interface variables.

2.5.6 Ground Motion Intensity Measure

The ground motion intensity measure, IM, implemented in the solution strategies outlined in the previous section, serves as an interface between the seismicity characterization and structural behavior assessment. Ideally, such a variable should contain sufficient information about the ground motion to serve as an accurate and efficient predictor of structural response, and it should preferably be a variable for which the PSHA results are available (or readily obtainable). This problem has been studied by Shome et al. (1998) and by Luco and Cornell, (2003). It has been demonstrated by Shome and Cornell (1999) that, for short and moderate-period structures, the spectral acceleration at a period approximately equal to that of the fundamental mode of the structure satisfies the criteria mentioned above. In fact, the study of such "intensity measure" is the subject of significant current research by a variety of investigators within PEER. We shall use this variable here for specificity, but the resulting derivations will not change if spectral acceleration is replaced by any other scalar intensity measure, such as, for example, the inelastic spectral acceleration (see Luco and Cornell, 2003).

2.5.7 Randomness: The Only Source of Variability

The probability-based seismic assessment and design procedure presented here aims to evaluate the probability P_{LS} that the limit state variable exceeds a limit state threshold LS. Our first objective here is to derive the limit state probability assuming that randomness is the only source of uncertainty in the design variables.

We will follow the displacement-based solution strategy discussed in Section 2.5.5. in order to derive the limit state probability. The derivation are laid out in a step-by-step manner in order to make them easier to follow. At the end of this section we are going to briefly present the IM-based approach for deriving the limit state probability. We start by deriving the hazard values for our adopted seismic intensity measure, which is the spectral acceleration of the "first" structural mode. Then we use common probabilistic tools (e.g., TPT as explained previously) in order to first derive the hazard values for the displacement-based response, (here, maximum inter-story drift angle) and then to derive the limit state probability P_{LS}.

2.5.7.1 Spectral Acceleration Hazard:

The hazard corresponding to a specific value of the ground motion intensity measure is defined as the probability that the intensity measure for future ground motion events be greater than or equal to this specific value. Spectral acceleration hazard curves are normally provided by seismologists for a given site (e.g. The USGS website). Each curve provides the mean annual frequency of exceeding a particular spectral acceleration for a given period and damping ratio. It is advantageous to approximate such a curve in the region of interest by a power-law relationship (see DOE, 1994, and, Luco and Cornell 1998):

$$H_{S_a}(s_a) = P[S_a \geq x] = k_0 \cdot x^{-k} \qquad (2 - 3)$$

k_0 and k are parameters defining the shape of the hazard curve.

Figure 2-1 shows a typical hazard curve for a Southern California site that corresponds to a period of 1.8 seconds and damping ratio of 5%. As it can be seen from the figure, a line with slope k and intercept k_0 is fit to the hazard curve in the logarithmic scales around the region of interest (e.g., mean annual frequencies between 1/475 or 10% frequency of exceedance in 50 years and 1/2475 or 2% frequency of exceedance). Here, k =2.73 and k_0 =0.00012.

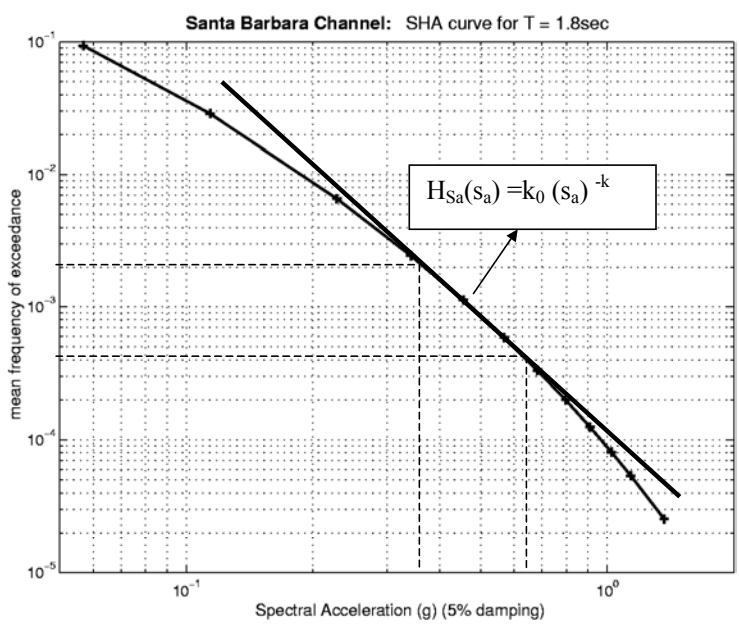


Figure 2-1 - A typical hazard curve for spectral acceleration. It corresponds to a damping ratio equal to 5% and a structural fundamental period of 1.8 seconds

It is important to note that the hazard values are usually provided in terms of the "mean rates" of exceedance over a certain time interval (usually a year) rather than the "probabilities" of exceedance. Therefore, it is more appropriate to refer to the hazard function as, for example, the "mean annual frequency" rather than the "annual probability" of exceeding a certain value. Nonetheless, for very small probability values, that are for example derived from a Poisson model, the average rate and the resulting probability value are almost the same. For simplicity, we are going to drop the "mean" term before the frequency. However, in the part where epistemic uncertainty is introduced into the problem, we will need to be more precise in how we refer to the hazard function.

2.5.7.2 Median Relationship between Spectral Acceleration and Inter-story Drift Demand

Observations of demand values are normally obtained from the result of structural time history analyses performed for various ground motion intensity levels. Figure 2-2 shows such results, e.g. maximum inter-story drift, D, versus S_a. For a given level of ground motion intensity, there will be variability in the displacement-based demand results over any suite of ground motion records applied to the structure. It is assumed here that this variability is a result of randomness in the seismic phenomena as the discussed before (later in Section 2.5.7 we are also going to take into account the epistemic uncertainties, such as modeling uncertainty, in addition to record-to-record variability). It is convenient to introduce a functional relationship between the load intensity measure and a central value, specifically the median η_D of the demand parameter based on the data available from such time history analyses.

In general, for a spectral acceleration equal to x, the functional relationship will be:

$$\eta_D(x) = g(x)$$

This is called the conditional median of D given S_a (more formally denoted as $\eta_{D|S_a}(x)$, but we shall keep the simpler notation). We can construct a full conditional probabilistic model of the variability displayed in Figure 2-2 by writing:

$$D = \eta_D(x) \cdot \varepsilon = g(x) \cdot \varepsilon$$

in which ε is a random variable with a median equal to unity and a probability distribution to be discussed below. At this point we introduce a particular functional relationship that both conforms to our perceptions of a structural performance curve and also helps simplifying the

future analytical efforts. We have used the linear regression in logarithmic space (i.e. $\ln \eta_D(x_a) = \ln a + b \ln x$) to fit a power-law curve, $a \cdot S_a{}^b$, to our collection of maximum inter story drift ratio and "first" mode spectral acceleration data pairs.

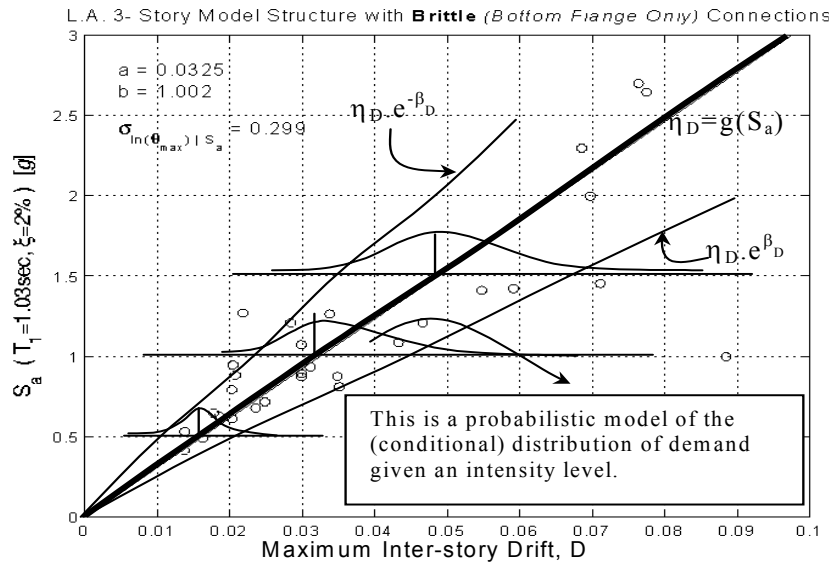


Figure 2-2 - A set of spectral acceleration and demand data pairs, showing the model fit to these data points.

It is not an objective here to describe the various ways $\eta_D(x)$ may be estimated. In design practice it is likely to come from one or more structural analysis procedures, perhaps previously calibrated to nonlinear dynamic results for similar structures (see FEMA 350, and Yun and Foutch, 2002). In assessment practice or research it can be obtained through one or more schemes of selecting and processing records and results (see Bazzurro et al, 1998, Luco and Cornell, 1998, Vamvatsikos and Cornell, 2001, Chapters 3 and 4 of this thesis). We shall see below that the number of required time history analyses may be quite small (e.g., of the order of 5 to 10 time history analyses). For a set of drift demand and spectral acceleration data points, such a regression in the logarithmic scale will result in the following relationship between spectral acceleration and (median) inter-story drift response:

$$\eta_D(x) = a \cdot x_a{}^b \qquad (2\text{-}4)$$

Figure 2-2 illustrates a typical power law relationship between the median maximum inter story drift demand and the spectral acceleration for a three-story steel frame building located in Los

Angeles. In this case, $b \cong 1$ which is consistent with the so-called "equal displacement rule" (Veletsos and Newmark, 1960).

2.5.7.3 Annual frequency of exceeding demand – Drift Hazard

We are going to break the displacement-based approach for deriving the limit state probability in Equation 2-2 into two parts. The first part is to derive the probability that the displacement-based demand exceeds a given value *d,* and the second part is to derive the probability that the displacement-based demand exceeds the capacity or the limit state probability. In simple words, the randomness in demand and capacity is modeled in two stages. This section describes the derivation of a closed-form expression for the annual probability of exceeding a certain demand value *d*, also known as the *Drift Hazard*, by modeling the randomness in the displacement-based demand.

Recalling from last section, the median demand versus spectral acceleration relationship was introduced as:

$$\eta_D(x) = a \cdot x^{\mathrm{b}} \qquad (2\text{-}5)$$

As shown above, the demand can be written in terms of the product of its median value and a lognormal random variable ε with the following characteristics:

$$D = \eta_D(x) \cdot \varepsilon \qquad (2\text{-}6)$$

We assume (based on observation of data) that ε can be represented by a lognormal distribution, in which case we define its parameters, the median and standard deviation of $\ln \varepsilon$, to be:

$$\begin{aligned} \eta_\varepsilon &= e^{mean(\ln(\varepsilon))} = 1 \\ \sigma_{\ln(\varepsilon)} &= \beta_{D|S_a} \end{aligned} \qquad (2\text{-}7)$$

where η_ε denotes the median value for ε. Note that what we call the "dispersion", i.e., $\beta_{D|S_a}$, will in general depend to some degree on the level of S_a. Here for analytical tractability, we assume it is constant; the value should be chosen for S_a values in the range of primary interest. If we replace $\eta_D(x)$ with its corresponding value from Equation 2-4, the following expression for drift demand as a function of spectral acceleration and lognormal random variable ε is obtained:

$$D = a \cdot x_a{}^b \cdot \varepsilon \qquad (2\text{-}8)$$

Since we have assumed that ε is a lognormal variable, we can also conclude that the displacement-based demand D is also a random variable with the following statistical properties:

$$\eta_{D|S_a}(x) = a \cdot x^b$$
$$\sigma_{\ln D|S_a}(x) = \beta_{D|S_a} \qquad (2\text{-}9)$$

where $\eta_{D|S_a}(x)$ and $\sigma_{\ln D|S_a}(x)$ are the conditional median and standard deviation of the natural logarithm for the displacement-based demand given spectral acceleration. As mentioned above, the conditional standard deviation of the natural logarithm $\sigma_{\ln D|S_a}(x)$ or the conditional "fractional" standard deviation $\beta_{D|S_a}(x)$ of demand given spectral acceleration is assumed to be constant. The conditional median demand for a given spectral acceleration $\eta_{D|S_a}(x)$ (or more briefly $\eta_D(x)$) is approximated as a power-law function of the spectral acceleration level, x. Figure 2-3 illustrates a graphical presentation of basic components for drift hazard evaluation in which the median drift curve, the variability of the displacement-based response around it, and the conditional lognormal distribution fit to the data (at any given S_a) are all plotted together with the spectral acceleration hazard. The median drift times $\exp(\beta_{D|S_a})$ is referred to as the "mean plus one sigma" curve as it corresponds to the 84th percentile of the data for a lognormal variable; this is illustrated in the figure as $\eta_D(x) \cdot \exp(\beta_{D|S_a})$. In a similar manner, the median drift times $\exp(-\beta_{D|S_a})$ is referred to as the "mean minus one sigma" curve as it corresponds to the 16th percentile of the data (for a lognormal variable) that is illustrated in the figure as $\eta_D(x) \cdot \exp(-\beta_{D|S_a})$. Our objective is to find the drift hazard curve, $H_D(d)$, the annual probability of exceeding a certain drift angle d (e.g., $P[D \geq d]$). The strategy outlined below for finding the P_{LS} involves decomposition and re-composition via the TPT.

We have two random variables of interest D and S_a, analogous to Equations 2-1 and 2-2 presented in Section 2.5.5. Under the general solution strategy, we can decompose the drift hazard into conditional probability of exceeding drift value d for a given spectral acceleration value x and the likelihood that the spectral acceleration is equal to the value x:

$$H_D(d) = \nu \cdot P[D > d] = \nu \cdot \sum_{\text{all x}} P[D > d \mid S_a = x] \cdot P[S_a = x] \qquad (2\text{-}10)$$

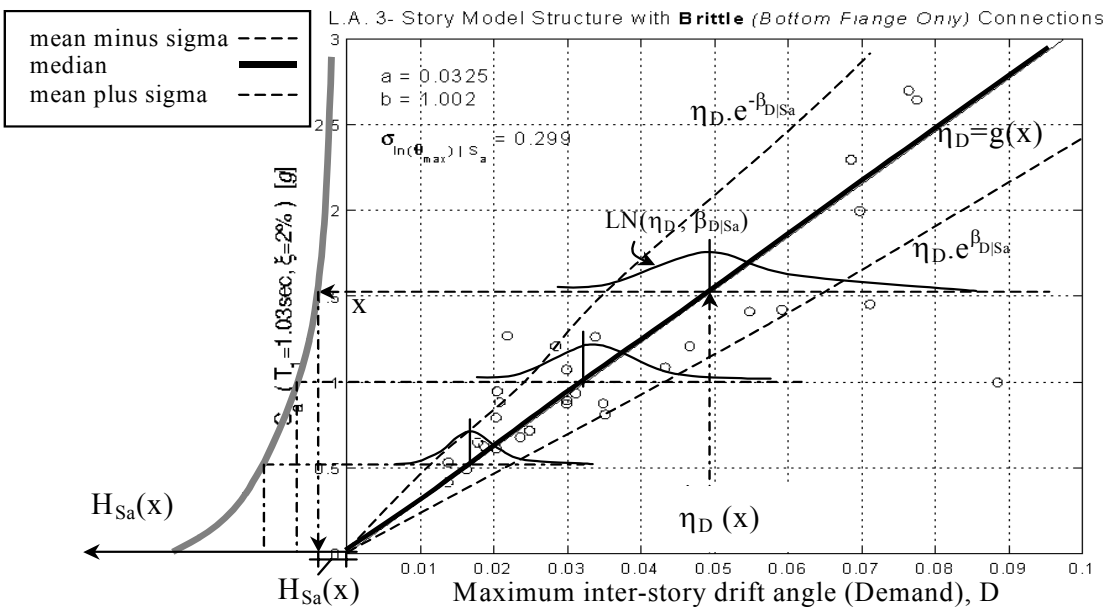


Figure 2-3 - Basic elements for drift hazard evaluation, $H_{S_s}(x)$, and the distribution of D given S_a characterized by $\eta_D(x)$ and $\beta_{D|S_a}$.

Where ν represents the (annual) rate of the occurrence of the events "of interest" (or rate of seismicity for brevity), e.g., events with magnitude greater than 5 in surrounding 200km. Thus, the drift hazard in Equation 2-10 is equal to $P[D > d]$ times the rate of occurrence of the earthquake events that interest us. Therefore, the drift hazard itself is expressed in terms of the "rate of exceedance" or the "mean annual frequency of exceedance" (however, in this chapter we may use the terms probability and frequency interchangeably).

We should note that the above expression involves discrete variables. However, since we are using analytic parameter estimations, we are going to base our derivations on an equivalent expression for the drift hazard derived for continuous variables:

$$H_D(d) = \nu \cdot P[D > d] = \int_0^\infty P[D > d \mid S_a = x] \cdot \nu \cdot f_{S_a}(x) \cdot dx = \int_0^\infty P[D > d \mid S_a = x] \cdot \left| \nu \cdot dG_{S_a}(x) \right|$$

(2-11)

where $f_{S_a}(x)$ is probability density function (PDF) at spectral acceleration value x, and, $G_{S_a}(x)$ is the complementary cumulative distribution function (CCDF) at $S_a = x$. It should be noted that the $\left| \nu \cdot dG_{S_a}(x) \right|$ term in Equation 2-11 is resulting from the following relationship between $F(\cdot)$

or the cumulative distribution function (CDF) and $f(\cdot)$ or the PDF for a random variable (e.g., spectral acceleration S_a):

$$f_{S_a}(x) = \lim_{\Delta x \to 0} \frac{P[x \le S_a \le x + \Delta x]}{\Delta x} = \lim_{\Delta x \to 0} \frac{P[S_a \le x + \Delta x] - P[S_a \le x]}{\Delta x} = \frac{dF_{S_a}(x)}{dx} = \left| \frac{dG_{S_a}(x)}{dx} \right| \quad (2\text{-}12)$$

The last equality is based on the fact that the CCDF is expressed in terms of the probability of exceedance whereas the CDF is expressed in terms of the probability of being less than or equal to. Therefore, their corresponding derivatives are equal in absolute values but will have opposite signs.

It should be noted that the spectral acceleration hazard $H_{S_a}(x)$ is equal to the spectral acceleration CCDF, $G_{S_a}(x)$ times the rate of seismicity ν:

$$H_{S_a}(x) = \nu \cdot G_{S_a}(x) \quad (2\text{-}13)$$

Therefore, we can re-write Equation 2-11 as a function of the spectral acceleration hazard:

$$H_D(d) = \int_0^\infty P[D > d \mid S_a = x] \cdot \left| \nu \cdot dG_{S_a}(x) \right| = \int_0^\infty P[D > d \mid S_a = x] \cdot \left| dH_{S_a}(x) \right| \quad (2\text{-}14)$$

Since we have assumed that the displacement-based demand is a lognormal variable, we can derive the term $P[D > d \mid S_a = x]$ using the tables that provide the CDF of a standardized normal variable (Rice 1995). In order to use the normal tables, we first need to transform the random variable D into a standardized normal variable:

$$P[D > d \mid S_a = x] = 1 - P[D \le d \mid S_a = x] = 1 - P[\frac{\ln D - mean \ln D}{\beta_{D|S_a}} \le \frac{\ln d - mean \ln D}{\beta_{D|S_a}} \mid S_a = x]$$
$$= 1 - P[\frac{\ln D - \ln \eta_{D|Sa}(x)}{\beta_{D|S_a}} \le \frac{\ln d - \ln \eta_{D|Sa}(x)}{\beta_{D|S_a}}] = 1 - \Phi\left(\frac{\ln(\frac{d}{a \cdot x^b})}{\beta_{D|S_a}} \right) \quad (2\text{-}15)$$

where $\Phi(\cdot)$ is the standardized Gaussian CDF. The above equation is derived based on the following property of a lognormal variable in which the mean of the logarithm is equal to the logarithm of the median (Benjamin and Cornell 1970)

$$mean \ln D = \ln \eta_D$$

If we substitute the standardized Gaussian representation of $P[D > d \mid S_a = x]$ in Equation 2-15 into Equation 2-14, the drift hazard will be expressed as:

$$H_D(d) = \int_0^\infty P[D > d \mid S_a = x] \cdot \left| dH_{S_a}(x) \right| = \int_0^\infty \{1 - \Phi\left(\frac{\ln(\frac{d}{a \cdot x^b})}{\beta_{D|S_a}} \right)\} \cdot \left| dH_{S_a}(x) \right| \qquad (2\text{-}16)$$

We are going to use integration by parts in order to re-arrange the above equation so that we can integrate it analytically. We first need to calculate the derivative of the first term in the integrand:

$$\frac{d}{dx}\{1 - \Phi\left(\frac{\ln(\frac{d}{a \cdot x^b})}{\beta_{D|S_a}} \right)\} = -\frac{d}{dx}\Phi\left(\frac{\ln d - \ln a \cdot x^b}{\beta_{D|S_a}} \right) = \frac{b}{x \cdot \beta_{D|S_a}} \cdot \phi\left(\frac{\ln d - \ln a \cdot x^b}{\beta_{D|S_a}} \right) \qquad (2\text{-}17)$$

where $\phi(\cdot)$ is the standardized Gaussian PDF which is equal to:

$$\phi(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} \qquad (2\text{-}18)$$

for any standardized normal variable u. The drift hazard in Equation 2-14 is re-arranged into the following form after applying the integration by parts assuming that the term $P[D > d \mid S_a = x] \cdot H_{S_a}(x)$ is close to zero for the integration limits, i.e., very small and very large S_a values. It should be noted that for a lognormal variable, the range of possible values vary from 0 to ∞.

$$H_D(d) = \int_0^\infty \frac{dP[D > d \mid S_a = x]}{dx} \cdot H_{S_a}(x) \cdot dx = \int_0^\infty \frac{b}{x \cdot \beta_{D|S_a}} \cdot \phi\left(\frac{\ln d - \ln a \cdot x^b}{\beta_{D|S_a}} \right) \cdot H_{S_a}(x) \cdot dx$$

(2-19)

Now we are going to replace the hazard term $H_{S_a}(x)$ by its power-law approximation from Equation 2-3 and also replace the Gaussian PDF by its analytical form in Equation 2-17:

$$H_D(d) = \int_0^\infty \frac{b}{x \cdot \beta_{D|S_a}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp(-\frac{1}{2}\left(\frac{\ln d - \ln a - b \ln x}{\beta_{D|S_a}}\right)^2) \cdot k_0 \cdot x^{-k} \cdot dx$$

(2 - 20)

In order to calculate the above integral analytically, we are going to form a square term (so that we can form a Gaussian PDF) inside the integral. This way we can calculate the integral by using the fact that the integral of a PDF function (over all possible values of the variable) is equal to unity. We begin by some simple algebraic manipulations in order to make the equation is a bit simplified:

$$H_D(d) = k_0 \int_0^\infty \frac{b}{x \cdot \beta_{D|S_a}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp(-\frac{1}{2}\left(\frac{\ln(\frac{d}{a})^{\frac{1}{b}} - \ln x}{\frac{\beta_{D|S_a}}{b}}\right)^2) \cdot \exp(-\frac{1}{2} \cdot 2k \ln x) \cdot dx$$

(2 - 21)

The next step is to form a full squared term inside the integral and also take all the constant terms out of the integral:

$$H_D(d) =$$

$$k_0 \exp(\frac{1}{2} k^2 (\frac{\beta_{D|S_a}}{b})^2) \cdot \exp(-k \ln(\frac{d}{a})^{\frac{1}{b}}) \int_0^\infty \frac{b}{x \cdot \beta_{D|S_a}} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}(\frac{\ln x - \{k(\frac{\beta_{D|S_a}}{b})^2 - \ln(\frac{d}{a})^{\frac{1}{b}}\}}{\frac{\beta_{D|S_a}}{b}})^2) dx$$

(2 - 22)

We can notice that the term inside the integral is indeed the PDF for the standardized Gaussian variable u with the derivative $\frac{du}{dx}$:

$$u = \frac{\ln x - \{k(\frac{\beta_{D|S_a}}{b})^2 - \ln(\frac{d}{a})^{\frac{1}{b}}\}}{\frac{\beta_{D|S_a}}{b}}$$

$$\frac{du}{dx} = \frac{b}{x \cdot \beta_{D|S_a}}$$

Therefore the expression for drift hazard can be also written as:

$$H_D(d) = k_0 \exp(\frac{1}{2}k^2(\frac{\beta_{D|S_a}}{b})^2) \cdot \exp(-k\ln(\frac{d}{a})^{\frac{1}{b}}) \int_0^\infty \frac{d}{dx}\{\Phi\left(\frac{\ln x - \{k(\frac{\beta_{D|S_a}}{b})^2 - \ln(\frac{d}{a})^{\frac{1}{b}}\}}{\frac{\beta_{D|S_a}}{b}}\right)\} \cdot dx$$

(2 - 23)

Noting that the integral of a normal PDF over all the possible values is equal to unity, the drift hazard can be written in the following simplified form:

$$H_D(d) = k_0 \exp(\frac{1}{2}k^2(\frac{\beta_{D|S_a}}{b})^2) \cdot \exp(-k\ln(\frac{d}{a})^{\frac{1}{b}}) = k_0(\frac{d}{a})^{\frac{-k}{b}} \cdot e^{\frac{1}{2}\cdot\frac{k^2}{b^2}\cdot\beta^2_{D|S_a}}$$

(2 - 24)

In order to have a more condensed formulation of the drift hazard, we introduce the notation S_a^d or *spectral acceleration "corresponding" to drift angle* d defined as:

$$S_a^d = \left(\frac{d}{a}\right)^{\frac{1}{b}}$$

This is also the solution of Equation 2-4 for a given value of d, i.e., if we read the corresponding S_a value[1] from $d = a \cdot S_a{}^b$ curve. The graphic interpretation of S_a^d can be seen from Figure 2-4. In simple words, this means that for a given drift demand value *d*, we find the corresponding Sa value from the median curve $\eta_D(S_a) = a \cdot S_a{}^b$.

The derivation can be further simplified by making use of the hazard curve definition in Equation 2-3:

$$H_D(d) = k_0 \cdot \left(\frac{d}{a}\right)^{\frac{-k}{b}} \cdot e^{\frac{1}{2}\cdot\frac{k^2}{b^2}\cdot\beta^2_{D|S_a}} = k_0 \cdot \left(S_a^d\right)^{-k} \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2_{D|S_a}}$$

$$\Rightarrow H_D(d) = H_{S_a}(S_a^d) \cdot e^{\frac{1}{2}\cdot\frac{k^2}{b^2}\cdot\beta^2_{D|S_a}} \qquad (2 - 25)$$

1 Note that S_a^d is not necessarily the median S_a for a given value of drift angle d. It is just the corresponding S_a value found from the curve. In other words, the fact that the D - S_a curve gives the median drift d for a given value of S_a does not mean that it will also provide the median S_a for a given value of drift d.

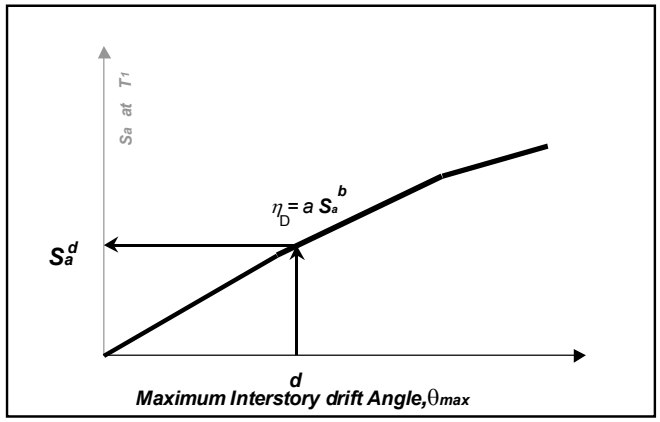


Figure 2-4 - Spectral acceleration corresponding to the demand value, *d*.

It can be seen by inspection of Equation 2-25 that the hazard curve for the drift demand $H_D(d)$ is equal to the hazard function $H_{S_a}(.)$ evaluated at the spectral acceleration corresponding to this drift demand times a factor related to dispersion in the drift demand for a given spectral acceleration. The first factor can be interpreted as a "first order" estimate; it is also the drift hazard if the dispersion $\beta_{D|S_a}$ is zero. Experience suggests that the second factor may typically have values in the order of 1.5 to 3. Note that in this form one can read the first factor directly from a given hazard curve without actually making the approximating fit $k_0 \cdot x^{-k}$. The log-log slope *k* of the approximation is needed for the second factor however.

2.5.7.3.1 Numerical Example

We would like to derive the drift hazard curve for a 3-story (model) structure with brittle connections located in Los Angeles. This structure is a typical 3-story steel moment resisting frame building used in the SAC project (Luco and Cornell 1998). A set of nonlinear dynamic analyses were conducted, and the resulting maximum inter-story drift ratios have been plotted versus the first mode spectral acceleration as it is illustrated in Figure 2-5. The hazard curve represented in Figure 2-5 corresponds to oscillators with a fundamental period around 1.0 sec and located in Los Angeles, thus we have used it as the spectral acceleration hazard curve for our model structure. In approximate analytical form it is:

$$H_{S_a}(s_a) = P[S_a \geq s_a] = 0.00124 \cdot s_a^{-3.03}$$

Note that the k value is nearly equal to 3.0. Our next step is to determine the median relationship between spectral acceleration and drift. This is done in Figure 2-5, by fitting a line to the data points in a log-log scale; which gives the following information:

$$\eta_D(S_a) = 0.0325 \cdot S_a^{1.002}$$
$$\beta_{D|S_a} = 0.299 \approx 0.3$$

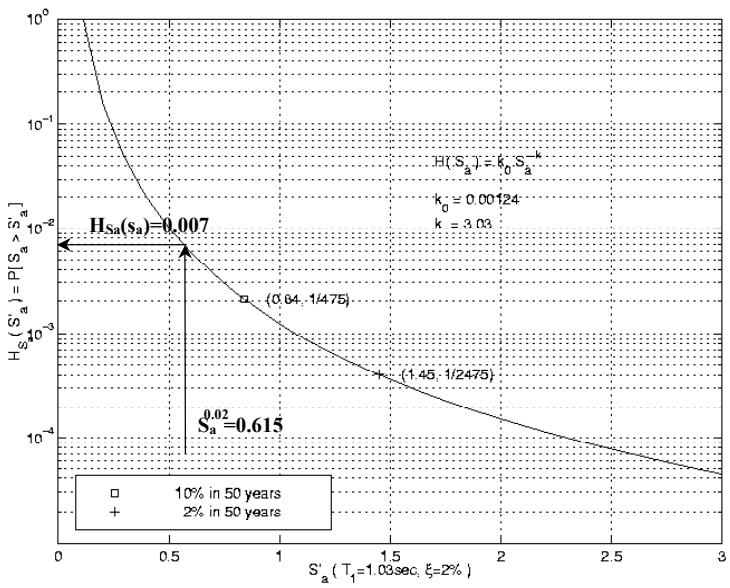


Figure 2-5 - Hazard curve for spectral acceleration values corresponding to an oscillator with a period equal 1.0 and damping ratio of 2%.

Note that $b \approx 1$ for this range of data, i.e., the median drift is approximately proportional to the S_a. It should be mentioned however that there may be a certain level of non-linearity (material or geometric) in which b is not close to 1.0 anymore. Linear behavior is limited in this structure to inter-story drifts less than about 0.01. We would like to evaluate the probability that the maximum inter-story drift angle exceeds a specific value, say 2%, $H_D(0.02)$. If we substitute 0.02 for *d* in Equation 2-25:

$$H_D(0.02) = P[D > 0.02] = H_{S_a}(S_a^{0.02}) \cdot e^{\frac{1}{2} \cdot \frac{k^2}{b^2} \cdot \beta^2_{D|S_a}}$$

Recall that S_a^d is equal to $S_a^d = \left(\frac{d}{a}\right)^{\frac{1}{b}}$ per definition:

$$S_a^{0.02} = \left(\frac{0.02}{0.0325}\right)^1 = 0.615 \quad [\text{g}]$$

Equivalently we could simply have read this value from median line in Figure 2-3 by entering at a drift value equal to 0.02. Now we look up for the value of $H_{S_a}(0.615)$ for the spectral acceleration hazard curve. As illustrated in Figure 2-5 it is equal to *0.007*. Hence, $H_D(0.02)$ can be derived as:

$$H_D(0.02) = P[D > 0.02] = 0.007 \cdot e^{(\frac{1}{2})(\frac{3^2}{1^2})(0.3^2)} = 0.007 \times 1.5 = 0.0105$$

Note that the factor $\exp(\frac{1}{2} \cdot \frac{3^2}{1^2} \cdot 0.3^2)$ is equal to 1.50.

We can repeat the above calculations for multiple drift values in order to obtain the drift hazard curve or we can find an analytical expression for the drift hazard. In general we can compute the drift hazard for a specified drift value, say *d* as follows:

$$H_D(d) = P[D > d] = H_{S_a}(S_a^d) \cdot e^{\frac{1}{2} \cdot \frac{k^2}{b^2} \cdot \beta^2_{D|S_a}}$$

Recalling that $\exp(\frac{1}{2} \cdot \frac{3^2}{1^2} \cdot 0.3^2)$ is equal to 1.5 and $S_a^d = \left(\frac{d}{a}\right)^{\frac{1}{b}} \cong \frac{d}{0.0325}$ the above equation becomes:

$$H_D(d) = H_{S_a}(S_a^d) \cdot e^{\frac{1}{2} \cdot \frac{k^2}{b^2} \cdot \beta^2_{D|S_a}} = 1.5 \cdot H_{S_a}(\frac{d}{0.0325})$$

Next we need to find the expression for the spectral acceleration hazard curve evaluated at $\frac{d}{0.0325}$. This is:

$$H_{S_a}(\frac{d}{0.0325}) = 0.00124 \cdot \left(\frac{d}{0.0325}\right)^{-3.03} \approx 4.25 \cdot 10^{-8} \cdot d^{-3}$$

Finally the drift hazard for a specified value of drift, *d* is derived as:

$$H_D(d) \approx 6.375 \cdot 10^{-8} \cdot d^{-3}$$

The above relationship is plotted in Figure 2-6.

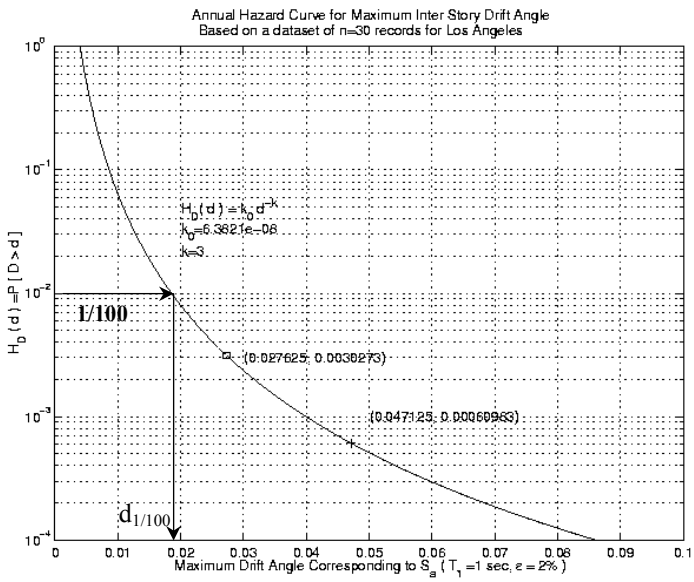


Figure 2-6 - Hazard curve derived for maximum inter-story drift values

One can use the above curve, to determine for example the 100 year return period drift, by setting $H_D(d_{1/100})$ to 1/100 and solving for $d_{1/100}$:

$$1/100 = 6.375 \cdot 10^{-8} \cdot (d_{1/100})^{-3}$$

solving for $d_{1/100}$: $d_{1/100} = 0.0185$. The same value can be also found simply from Figure 2-6.

2.5.7.4 Annual Frequency of Exceeding a Limit State

We have already derived the probability that the limit state variable exceeds a certain value. The next step is to find the probability that this limit state variable exceeds a specified *limit state threshold* or capacity, *C*. The difference in this case is due to the fact that the *limit state threshold* can be a random variable itself. For example in the SAC project (FEMA 350) modern "reduced beam section" (RBS) connections were concluded to have a median capacity of $\eta_C = 0.07$ (inter-story drift ratio) with a dispersion of $\beta_C = 0.2$ reflecting specimen-to-specimen variability in (hypothetical[2]) test results and even possible record-to-record variations in the drift failure due to sequence effects in the low-cycle fatigue suffered by the connection. Beyond the drift capacity the connection lost virtually all vertical load carrying (shear) capacity implying the potential collapse of floor above. In this section we will derive the expression for the probability of limit

state, $P_{LS} = P[D \le C]$, by introducing the variability in the limit state threshold. The basic elements involved in the derivation are illustrated in Figure 2-7.

Once again we use the total probability theorem to sum the joint probabilities that limit state variable exceeds the capacity variable for a given value of capacity, over the entire range of possible values for the capacity variable:

$$P_{LS} = P[D \ge C] = \sum_{all\ c} P[D \ge c \mid C = c] P[C = c] \qquad (2\text{-}26)$$

We next assume that demand and capacity are (statistically) independent, i.e. that:

$$P[D \ge C \mid C = c] = P[D \ge c] \qquad (2\text{-}27)$$

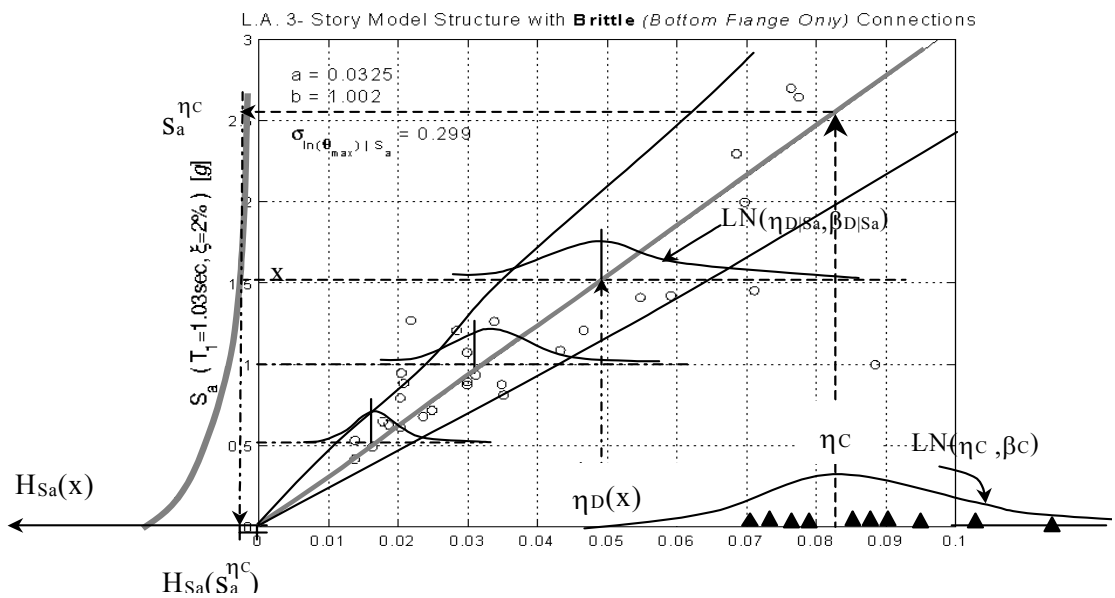


Figure 2-7 - Basic elements for limit state probability evaluation, $H_{S_s}(x)$, distribution of drift variable D given S_a characterized by $\eta_D(x)$ and $\beta_{D|S_a}$, distribution of capacity variable C characterized by η_C and β_C

The annual frequency of exceeding the limit state, H_{LS}, can be expressed as the limit state probability P_{LS} times the seismicity rate ν (as mentioned in the Section 2.5.7.4.):

[2] In fact no connections were able to be tested by the SAC project to such large drift ratios. The parameters

$$H_{LS} = \nu \cdot P_{LS} = \nu \cdot P[D \le C] \qquad (2\text{-}28)$$

Since we are going to base our derivation on the expression for drift hazard, our calculations are going to yield the mean annual frequency of exceedance (or limit state frequency in short), H_{LS}[3], as the end result. Therefore, the limit state frequency can be calculated by substituting Equations 2-26 and 2-27 into Equation 2-28:

$$H_{LS} = \nu \cdot P_{LS} = \nu \cdot P[D \ge C] = \sum_{all\ c} \nu \cdot P[D \ge c] P[C = c] \qquad (2\text{-}29)$$

The probability that drift demand exceeds drift capacity for a given value of drift capacity can be readily determined from the drift hazard curve:

$$H_D(c) = \nu \cdot P[D \ge c] \qquad (2\text{-}30)$$

Substituting the term $\nu \cdot P[D \ge c]$ in Equations 2-29 by $H_D(c)$ from the above equation,

$$H_{LS} = \nu \cdot P_{LS} = \sum_{all\ c} H_D(c) \cdot P[C = c] \qquad (2\text{-}31)$$

However, the above equation is valid for discrete variables; in the continuous form, the summation is replaced by an integral and the probability term, $P[C = c]$, is replaced by a probability density function term, $f_C(c) \cdot dc$:

$$H_{LS} = \nu \cdot P_{LS} = \int H_D(c) \cdot f_C(c) \cdot dc \qquad (2\text{-}32)$$

Substituting the drift hazard value for $H_D(c)$ from Equation 2-25 into Equation 2-32 results in:

$$H_{LS} = \int H_D(c) f_C(c) dc = \int H_{S_a}(S_a^c) \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2_{D|S_a}} f_C(c) dc \qquad (2\text{-}33)$$

From Equations 2-3 and 2-4, $H_{S_a}(S_a^c)$ is equal to $k_0 \cdot \left(\frac{c}{a}\right)^{\frac{-k}{b}}$. Thus, the limit state frequency is obtained by performing the following integration:

---

were estimated indirectly and are based on some level of expert opinion.

[3] In this chapter, we have used the notation $H(.)$ in order to refer to the mean annual rate of exceedance. However, in the next chapters we may use the notation $\lambda(.)$ instead.

$$H_{LS} = \int H_{S_a}(S_a^d) \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2_{D|S_a}} \cdot f_C(c) \cdot dc = \int k_0 \cdot \left(\frac{c}{a}\right)^{\frac{-k}{b}} \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2_{D|S_a}} \cdot f_C(c) \cdot dc$$

(2-34)

For the above integral to be evaluated, the probability density function of the random variable C, $f_C(c)$, has to be known. Here for tractability, it is assumed that C is a lognormal random variable with following characteristics:

$$median(C) = \eta_C$$
$$\sigma_{\ln(C)} = \beta_C$$

After some simple re-arrangements:

$$H_{LS} = \int k_0 \left(\frac{c}{a}\right)^{\frac{-k}{b}} e^{\frac{1}{2}\frac{k^2}{b^2}\cdot\beta^2_{D|S_a}} f_C(c)dc = k_0 \left(\frac{1}{a}\right)^{\frac{-k}{b}} \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2_{D|S_a}} \cdot \int c^{\frac{-k}{b}} f_C(c)dc \qquad (2-35)$$

It can be seen that the term inside the last integral equals expectation of $c^{-\frac{k}{b}}$. It has been shown in Appendix A that the expected value of log normal random variable Y (with median η_Y and dispersion $\sigma_{\ln Y}$), to the power of α equals to:

$$E(Y^a) = E(e^{a \ln Y}) = (\eta_Y)^{\mathrm{a}}.\mathrm{e}^{\frac{1}{2}a^2\sigma^2_{\ln Y}}$$

Since C is assumed to be a lognormal variable, the above property can be used to solve the integral in Equation 2-35 as follows (For further details regarding the integration scheme refer to Appendix A):

$$H_{LS} = k_0 \left(\frac{1}{a}\right)^{\frac{-k}{b}} \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2_{D|S_a}} \cdot \mathrm{E}(c^{\frac{-k}{b}}) = k_0 \left(\frac{1}{a}\right)^{\frac{-k}{b}} \cdot \eta_C^{\frac{-k}{b}} \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2_{D|S_a}} \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2_C}$$

We conclude that:

$$H_{LS} = \nu \cdot P[D > C] = k_0 \left(\frac{\eta_C}{a}\right)^{\frac{-k}{b}} \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2_{D|S_a}} \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2_C}$$

We can recognize in the above expression the spectral acceleration hazard from Equation 2-3 combined with the spectral acceleration-median drift relationship in Equation 2-4, $k_0(\eta_C / a)^{-k/b}$, which equals the hazard value for the spectral acceleration corresponding to median capacity, $S_a^{\eta_C}$:

$$H_{S_a}(S_a^{\eta_C}) = k_0 \left(\frac{\eta_C}{a} \right)^{\frac{-k}{b}} \qquad (2-36)$$

Thus:

$$k_0 \cdot \left(\frac{\eta_C}{a} \right)^{\frac{-k}{b}} \cdot e^{\frac{1}{2} \cdot \frac{k^2}{b^2} \cdot \beta^2_{D|S_a}} = H_{S_a}(S_a^{\eta_C}) \cdot e^{\frac{1}{2} \frac{k^2}{b^2} \beta^2_C} = H_D(\eta_C) \qquad (2-37)$$

where the last equality is based on the expression for drift hazard $H_D(.)$ at median capacity, η_C, from Equation 2-25, and $S_a^{\eta_C}$, as mentioned before, is the spectral acceleration "corresponding" to a drift value equal to η_C, i.e., $S_a^{\eta_C} = (\eta_C / a)^{b^{-1}}$. Finally, the probability of limit state is derived as:

$$H_{LS} = \nu \cdot P[D > C] = H_D(\eta_C) \cdot e^{\frac{1}{2} \cdot \frac{k^2}{b^2} \cdot \beta_C^2} = H_{S_a}(S_a^{\eta_C}) \cdot e^{\frac{1}{2} \cdot \frac{k^2}{b^2} \cdot \beta^2_{D|S_a}} \cdot e^{\frac{1}{2} \frac{k^2}{b^2} \cdot \beta_C^2} \qquad (2-38)$$

It can be observed that the probability of limit state (or the probability of drift demand exceeding the drift capacity) is equal to the hazard curve for the spectral acceleration corresponding to the median drift capacity times two coefficients accounting for the randomness in drift demand for a given spectral acceleration and the randomness in drift capacity itself. Again the first factor can be seen as a first-order approximation to the limit state frequency, H_{LS}.

2.5.7.4.1 Numerical Example:

Returning to our 3-story frame numerical example of the last section, now we assume that the median drift capacity and its dispersion parameter are given as:

$$median(C) = \eta_C \cong 0.07$$
$$\sigma_{\ln(C)} = \beta_C \cong 0.20$$

We first need to find $H_{S_a}(S_a^{\eta_C})$. We can do this graphically. $S_a^{\eta_C}$ can be calculated as the spectral acceleration corresponding to $\eta_C = 0.07$ from median-spectral acceleration curve in Figure 2-7 resulting in $S_a^{0.07} \cong 2.15g$ (note that the capacity points in the figure are only for schematic representation). We enter the hazard curve (Figure 2-8 below) with this value and find $H_{S_a}(2.15) \cong 0.00012$.

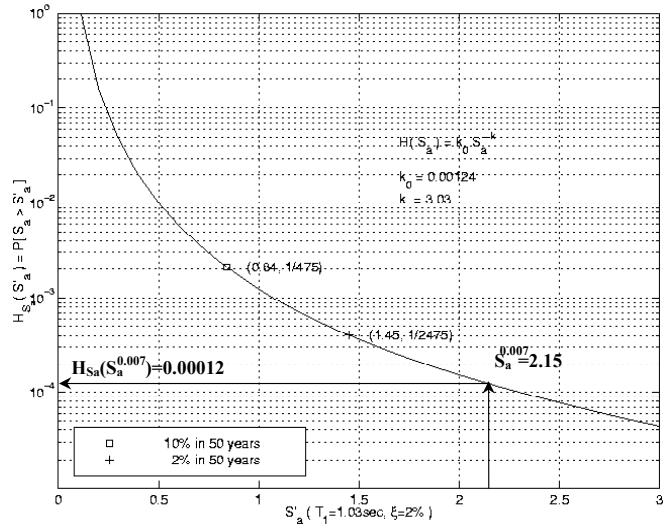


Figure 2-8 - The spectral acceleration hazard curve. The hazard value for a spectral acceleration equal to, 2.15, is shown on the figure.

Alternately we can use the analytical approximations. Using Equation 2-36 we can calculate the hazard for the spectral acceleration corresponding to the median drift capacity as:

$$H_{S_a}(S_a^{\eta_C}) = H_{S_a}(S_a^{0.07}) = 0.00124 \cdot \left(\frac{0.07}{0.0325}\right)^{-3} = 1.2\times10^{-4}$$

Also: $\exp(\frac{1}{2}\frac{k^2}{b^2}\beta_C{}^2) = \exp[\frac{1}{2}\cdot\frac{3^2}{1^2}\cdot 0.2^2] = \exp[0.18] = 1.19$

This value together with the 1.50 value for the coefficient $\exp(\frac{1}{2}\frac{k^2}{b^2}\beta^2{}_{D|S_a})$ calculated in Section 2.5.7.3.1 are used to calculate the annual frequency of exceeding the limit state from Equation 2-38 as follows:

$$H_{LS} = H_{S_a}(S_a^{\eta_C}) \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta_C^2} \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta_{D|S_a}^2} = 1.2\times10^{-4} \times 1.50 \times 1.19 = 2.2\times10^{-4}$$

It can be seen that in this example the randomness in the drift capacity and drift demand for a given spectral acceleration causes the limit state frequency to increase about a factor of 2 over the first order estimate of $H_{S_a}(S_a^{\eta_C}) = 1.2\times10^{-4}$.

2.5.7.5 Annual Frequency of Exceeding a Limit State, the IM-based approach

In this section we are going to derive the annual frequency of exceeding a limit state, H_{LS}, by following an IM-based approach. The total probability theorem (TPT) is used to decompose the expression for the limit state frequency into (conditional) frequencies of exceeding the limit state for a given spectral acceleration (the adopted IM), and composing the results by integration over all spectral acceleration values:

$$H_{LS} = \nu \cdot P[S_a \geq S_{a,C}] = \int P[S_a \geq S_{a,C} \mid S_a = x] \cdot \nu \cdot f_{S_a}(x) \cdot dx = \int P[x \geq S_{a,C}] \cdot \left| dH_{S_a}(x) \right|$$
$$(2-39)$$

where S_a represents the IM-based demand, $S_{a,C}$ represent the limit state capacity also expressed in spectral acceleration terms, and ν represents the seismicity rate (the reason for including it in the derivations is explained before for the displacement-based derivation). We have used Equation 2-12 in order to express the PDF of spectral acceleration in terms of the increment in the spectral acceleration hazard. We assume that the spectral acceleration capacity is a lognormal variable with the following statistical parameters:

$$median(S_{a,C}) = \eta_{S_{a,C}}$$
$$\sigma_{\ln(S_{a,C})} = \beta_{S_{a,C}}$$

We can observe that the first term in the integral $P[s_a \geq S_{a,C}]$ can be also interpreted as the CDF of the spectral acceleration capacity at $S_a = x$:

$$F_{S_{a,C}}(x) = P[x \geq S_{a,C}]$$

Since $S_{a,C}$ is assumed to be a lognormal variable, the corresponding CDF can be expressed in terms of the standardized normal CDF:

$$F_{S_{a,C}}(x) = P[x \geq S_{a,C}] = \Phi\left(\frac{\ln \frac{x}{\eta_{S_{a,C}}}}{\beta_{S_{a,C}}} \right) \qquad (2\text{-}40)$$

In order to be able to integrate Equation 2-39, we use integration by parts and transform the equation into the following form:

$$H_{LS} = \nu \cdot P[S_a \geq S_{a,C}] = -\int \Phi\left(\frac{\ln \frac{x}{\eta_{S_{a,C}}}}{\beta_{S_{a,C}}}\right) \cdot dH_{S_a}(x) = \int d\{\Phi\left(\frac{\ln \frac{x}{\eta_{S_{a,C}}}}{\beta_{S_{a,C}}}\right)\} \cdot H_{S_a}(x) \qquad (2\text{-}41)$$

Just as in Section 2.5.7.3., the derivative of the standard Normal CDF can be calculated as:

$$\frac{d}{dx}\{\Phi\left(\frac{\ln \frac{x}{\eta_{S_{a,C}}}}{\beta_{S_{a,C}}}\right)\} = \frac{1}{x \cdot \beta_{S_{a,C}}} \cdot \phi\left(\frac{\ln x - \ln \eta_{S_{a,C}}}{\beta_{S_{a,C}}}\right) \qquad (2\text{-}42)$$

After the derivative of the Normal CDF in Equation 2-42 is substituted in Equation 2-41, and the hazard term is replaced by the power-law approximation from Equation 2-3:

$$H_{LS} = \nu \cdot P[S_a \geq S_{a,C}] = \int \frac{k_0}{x \cdot \beta_{S_{a,C}}} \cdot \phi\left(\frac{\ln x - \ln \eta_{S_{a,C}}}{\beta_{S_{a,C}}}\right) \cdot x^{-k} \cdot dx \qquad (2\text{-}43)$$

If we substitute the expression for the Normal PDF in Equation 2-18 into the above equation:

$$H_{LS} = \nu \cdot P[S_a \geq S_{a,C}] = \int \frac{k_0}{\sqrt{2\pi} \cdot x \cdot \beta_{S_{a,C}}} \cdot \exp\frac{-1}{2}\left(\frac{\ln x - \ln \eta_{S_{a,C}}}{\beta_{S_{a,C}}}\right)^2 \cdot x^{-k} \cdot dx \qquad (2\text{-}44)$$

Similar to the derivation in Section 2.5.7.3, we transform the integrand into a complete square term and take all the constant terms outside of the integrand:

$$H_{LS} = \nu \cdot P[S_a \geq S_{a,C}] =$$

$$k_0 \exp(\frac{1}{2}k^2 \cdot \beta_{S_{a,C}}{}^2) \cdot \exp(-k \cdot \eta_{S_{a,C}}) \int_0^\infty \frac{1}{x \cdot \beta_{S_{a,C}}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp(-\frac{1}{2}\left(\frac{\ln x - \{k\beta_{S_{a,C}}{}^2 - \ln \eta_{S_{a,C}}\}}{\beta_{S_{a,C}}}\right)^2) \cdot dx$$

(2 - 45)

The term inside the integral is itself the derivative for a standard Normal CDF:

$$H_{LS} = \nu \cdot P[S_a \geq S_{a,C}] = k_0 \exp(\frac{1}{2}k^2 \cdot \beta_{S_{a,C}}{}^2) \cdot \exp(-k \cdot \eta_{S_{a,C}}) \int_0^\infty \frac{d}{dx}\{\Phi(\frac{\ln x - \ln \eta_{S_{a,C}} \cdot e^{k \cdot \beta_{S_{a,C}}{}^2}}{\beta_{S_{a,C}}})\} \cdot dx$$

(2 - 46)

Noting that the integral is equal to unity, the limit state probability can be derived as:

$$H_{LS} = P[S_a \geq S_{a,C}] = k_0 \cdot \exp(\frac{1}{2} k^2 \cdot \beta_{S_{a,C}}{}^2) \cdot \exp(-k \cdot \eta_{S_{a,C}}) = k_0 \cdot \eta_{S_{a,C}}{}^{-k} \cdot \exp(\frac{1}{2} k^2 \cdot \beta_{S_{a,C}}{}^2)$$
(2-47)

We can observe the power-law term outside the exponential is equal to the frequency of exceeding (i.e., hazard) a spectral acceleration equal to the median spectral acceleration capacity:

$$H_{LS} = P[S_a \geq S_{a,C}] = H_{S_a}(\eta_{S_{a,C}}) \cdot \exp(\frac{1}{2} k^2 \cdot \beta_{S_{a,C}}{}^2)$$
(2-48)

We can argue that $H_{S_a}(\eta_{S_{a,C}})$ is a first-order approximation to the limit state probability and the exponential term $\exp(k^2 \cdot \beta_{S_{a,C}}{}^2 / 2)$ is a magnifying factor that accounts for the sensitivity of the limit state probability to the randomness in the spectral acceleration capacity. If we compare the IM-based expression for the limit state frequency in Equation 2-48 to the displacement-based one in Equation 2-38, we can observe that the exponential term accounting for the dispersion in displacement-demand is missing. Also the slope parameter *b* that measures the gradient of the displacement-based demand with regard to spectral acceleration is absent. This is because the IM-based approach, when applying TPT to derive the limit state probability, does not employ the displacement-based demand as (one of the) an intermediate variable(s).

2.5.8 Randomness and uncertainty as the sources of variability

In previous sections, a closed-form expression for annual probability of exceeding a limit state (here, the collapse limit state) was derived. We saw that the hazard value for the load intensity measure corresponding to median drift capacity (i.e., the annual probability of exceeding the load intensity measure corresponding to median drift capacity) is a first-order approximation to the limit state frequency. This first-order approximation is multiplied by two second-order estimate coefficients accounting for the randomness in drift demand for a given spectral acceleration and the randomness in drift capacity itself.

Our objective here is to derive the limit state frequency when there is both randomness and uncertainty in the design variables such as spectral acceleration hazard, drift demand given spectral acceleration and drift capacity. Our derivations are going to be based on the assumption

that to a first approximation we can represent all the epistemic uncertainty in variable X by the uncertainty in its median. The model becomes:

$$X = \hat{\eta}_X \cdot \varepsilon_\eta \cdot \varepsilon_X \qquad (2\text{-}49)$$

where $\hat{\eta}_X$ is the current point estimate of the median of X, the unit-median random variable ε_η represents the epistemic uncertainty as to the true value the median of X, and the unit-median random variable ε_X represents the aleatory randomness of X. We are also going to assume that the deviation from median, ε_η, can be properly modeled by a lognormal distribution. In general, of course, the epistemic uncertainty in β_X should also be taken into account. Also, the shape of the distribution of X may not be lognormal.

As in the previous section, we start by deriving the hazard values for the load intensity variable, spectral acceleration of the "first" structural mode. We use some probabilistic tools (e.g., TPT as explained previously) to derive the hazard values for the limit state variable, maximum inter-story drift, and then complete the derivation by obtaining the limit state probability P_{LS}. Whenever possible we will make use of the results obtained in the previous part and generalize them to the case where there is both randomness and uncertainty in the design variables.

2.5.8.1 Spectral Acceleration Hazard:

The concept of hazard curves for the load intensity measure was introduced in the previous section. Our focus was on the spectral acceleration hazard curves which are normally provided by seismologists for a given site condition and its location with respect to a fault. The hazard curve estimation involves many scientific assumptions (see Kramer 1996). In other words there is uncertainty in the evaluation of a hazard curve. That's why spectral acceleration hazard curves are normally provided as mean and 84th percentile hazard curves (As shown in Figure 2-9). Here we are going to take into account the uncertainty in the evaluation of the spectral acceleration hazard.

In the previous sections, we found it advantageous to approximate the hazard curve by a power-law relationship as proposed by Kennedy and Short (1994) and Luco and Cornell (1998):

$$H_{S_a}(s_a) = k_0 \cdot x^{-k}$$

where k_0 and k are parameters defining the shape of the hazard curve. We are going to let an equation of the same form as the one above represent the median estimate of the uncertain hazard curve: $\hat{H}_{S_a}(x) = k_0 \cdot x^{-k}$ (2-50)

Further we introduce the random variable ε_{UH} that represents the uncertainty in the spectral acceleration hazard, so that we have: $H_{S_a}(x) = \hat{H}_{S_a}(x) \cdot \varepsilon_{UH}$ (2-51).

Here we have assumed that ε_{UH} is a lognormal random variable whose statistical parameters have the following characteristics:

$$median(\varepsilon_{UH}) = \eta_{\varepsilon_{UH}} = e^{mean(\ln(\varepsilon))} = 1$$
$$\sigma_{\ln(\varepsilon_{UH})} = \beta_{UH}$$
$$(2-52)$$

where β_{UH} reflects the degree of uncertainty in the PSHA estimation. We recognize the spectral acceleration hazard itself as an uncertain (random) variable, $\widetilde{H}_{S_a}(x)$, which can be represented as the median ("best") estimate times this uncertain deviation, $\widetilde{\varepsilon}_{UH}$:

$$\widetilde{H}_{S_a}(x) = \widehat{H}_{S_a}(x) \cdot \widetilde{\varepsilon}_{UH} \qquad (2\text{-}53)$$

Note the use of a "tilda" to denote a random variable, when clarity is needed. Considering our assumption about $\widetilde{\varepsilon}_{UH}$ being log-normal, we can observe from the above equation that the hazard for any value of S_a itself is a lognormal random variable (i.e., instead of having a single deterministic value assigned to it, it has a probability distribution). We can write the spectral acceleration hazard as:

$$\widetilde{H}_{S_a}(x) = \widehat{H}_{S_a}(x) \cdot \varepsilon_{UH} = k0 \cdot x^{-k} \cdot \varepsilon_{UH} \qquad (2\text{-}54)$$

Where $\widetilde{H}_{S_a}(x)$ is a lognormal random variable with its median equal to $\widehat{H}_{S_a}(x)$ from Equation 2-50 and its dispersion measure (i.e., the standard deviation of the natural logarithm or fractional standard deviation) equal to β_{UH}. The mean hazard curve can be written as:

$$\overline{H}_{S_a}(x) = \widehat{H}_{S_a}(x) \cdot mean(\varepsilon_{UH}) = \widehat{H}_{S_a}(x) \cdot e^{\frac{1}{2}\beta_{UH}^2} \qquad (2\text{-}55)$$

This equation is based on a property of the lognormal variables; their expected value is equal to the median times the exponential of half of the squared standard deviation (see Appendix A). Figure 2-9 shows the 16th percentile, median, mean and 84th percentile hazard curves for a California site that corresponds to a period of 1.8 seconds and damping ratio of 5%. Note that the median curve in the figure is the same hazard curve we used in the previous section. The 84th percentile is given by $\hat{H}_{S_a}(x) \cdot e^{\beta_{UH}}$.

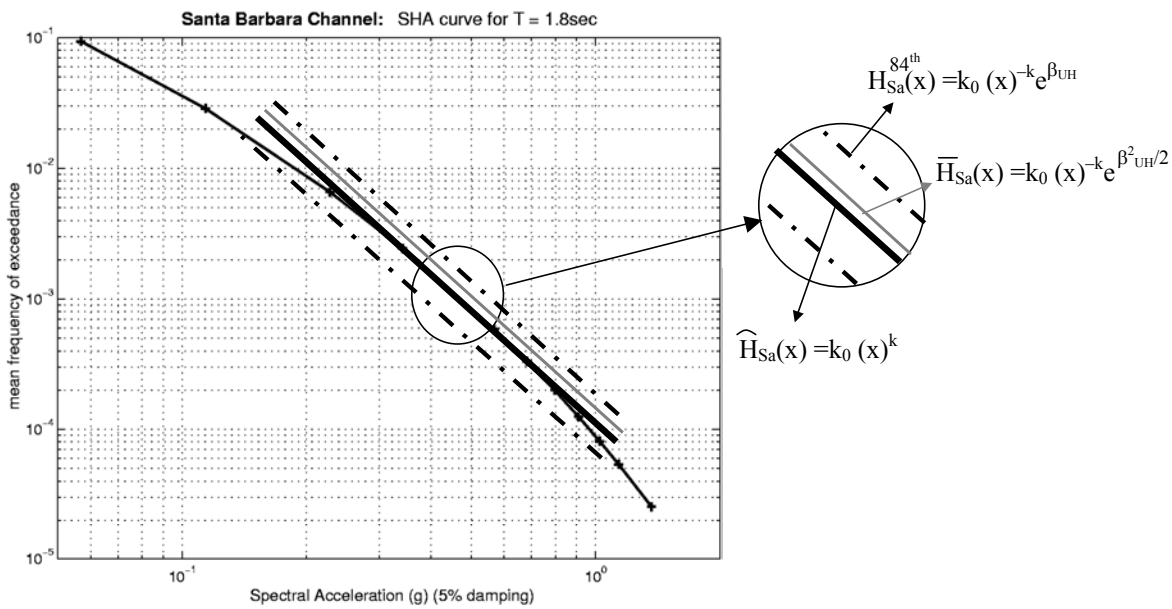


Figure 2-9 - 16th, median and 84th percentile spectral acceleration hazard corresponding to a damping ratio equal to 5% and a structural fundamental period of 1.8 seconds.

Figure 2-10 shows the basic components of drift hazard evaluation when there is uncertainty (due to limited knowledge and data) in the estimation of the spectral acceleration hazard $\tilde{H}_{S_a}(s_a)$. The probability density for uncertainty in hazard is plotted with solid black lines. The two hazard curves on the graph correspond to the median estimate of hazard, $\hat{H}_{S_a}(s_a)$, and hazard curve for a given value of deviation, ε_{UH}, in the estimation of hazard curve, $\hat{H}_{S_a}(s_a) \cdot \varepsilon_{UH}$, respectively.

2.5.8.2 Probability of Exceeding a Drift Demand value – Drift Hazard

Recalling from the previous section, the drift demand variable (given a specified S_a level) was introduced as the median demand value times a random variable ε representing the random variation (e.g., record-to-record) around the median value. We assumed that ε has a lognormal distribution:

$$D = \eta_D(x) \cdot \varepsilon \qquad (2-56)$$

Randomness is assumed to be the only source of variability in the above expression. In general, the median drift demand is also an uncertain quantity. The uncertainty in the median drift demand is caused by the limited knowledge and data about modeling and analysis of the structural system especially in the highly non-linear range and/or exact numerical values of the parameters of structural model.

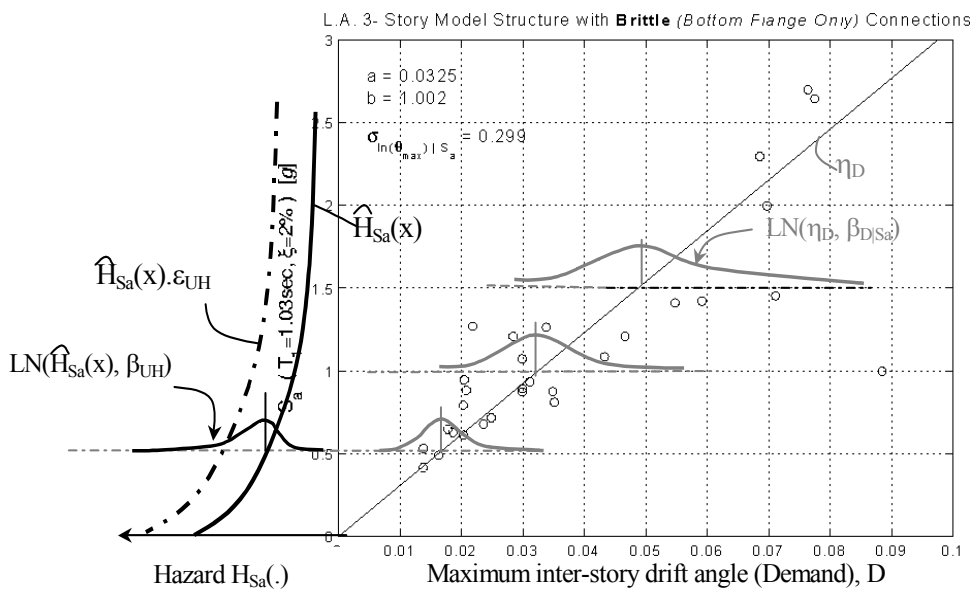


Figure 2-10 - Basic components for the evaluation of drift hazard with uncertainty in the estimation of spectral acceleration hazard, $\tilde{H}_{S_a}(x)$

The uncertainty is also caused by using a finite number of non-linear analyses to estimate the median value. The scatter of the displacement-based response in Figure 2-2 indicating record-to-record variability implies that the estimate of the median, $\hat{\eta}_D(x)$, can depend on the particular sample of records used and its size. In order to distinguish this type of uncertainty from the one that we considered in the previous section, we refer to it as *epistemic* uncertainty. The median

inter-story drift can be expressed as the product of its median estimate, $\hat{\eta}_D(x)$ and a random variable ε_{UD} (*UD* stands for the uncertainty in evaluation of *D*) representing the uncertainty involved in the evaluation of $\eta_D(x)$:

$$\eta_D(x) = \hat{\eta}_D(x) \cdot \varepsilon_{UD} \qquad (2\text{-}57)$$

Replacing $\eta_D(x)$ in Equation 2-56 with its representation in Equation 2-57, the drift demand can be written as:

$$D = \hat{\eta}_D(x) \cdot \varepsilon_{UD} \cdot \varepsilon \qquad (2-58)$$

In order to be consistent with ε_{UD}, we now subscript ε with *RD*, standing for the randomness (aleatory uncertainty) in drift demand evaluation. Finally the drift demand is represented as:

$$D = \eta_D(x) \cdot \varepsilon_{UD} = \hat{\eta}_D(x) \cdot \varepsilon_{UD} \cdot \varepsilon_{RD} \qquad (2\text{-}59)$$

where ε_{RD} and ε_{UD} are assumed to be independent and to have log-normal distributions with following characteristics:

$$\begin{aligned} &\eta_{\varepsilon_{RD}} = \eta_{\varepsilon_{UD}} = e^{mean(\ln(\varepsilon))} = 1 \\ &\sigma_{\ln(\varepsilon_{RD})} = \beta_{RD} \\ &\sigma_{\ln(\varepsilon_{UD})} = \beta_{UD} \\ &(2-60) \end{aligned}$$

Our objective in this section is to derive the probability that the drift demand *D* exceeds a specific value *d*. In order to minimize the calculation efforts, we'll make use of the drift demand hazard that was derived in the previous section assuming that there was no variability due to uncertainty. The drift hazard or the annual frequency that the drift demand exceeds a specific value was derived from Equation 2-25 as:

$$H_D(d) = \nu \cdot P[D > d] = k_0 \cdot \left(s_a^d\right)^{-k} \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2_{RD}} = H_{S_a}(s_a^d) \cdot e^{\frac{1}{2}\cdot\frac{k^2}{b^2}\cdot\beta^2_{RD}}$$

The spectral acceleration hazard for a given value of deviation in its evaluation, ε_{UH}, can be found from Equation 2-54 as:

$$H_{S_a|\varepsilon_{UH}}(x) = \widehat{H}_{S_a}(x) \cdot \varepsilon_{UH} = k_0 \cdot x^{-k} \cdot \varepsilon_{UH}$$

Replacing the above value for spectral acceleration hazard in Equation 2-25, we obtain the drift demand hazard for a given value of deviation in spectral acceleration hazard ε_{UD}:

$$H_{D|\varepsilon_{UH}}(d) = \nu \cdot P[D > d] = \widetilde{H}_{S_a|\varepsilon_{UH}}(s_a^d) \cdot e^{\frac{1}{2}\cdot\frac{k^2}{b^2}\cdot\beta^2_{RD}} = \hat{H}_{S_a}(s_a^d) \cdot \varepsilon_{UH} \cdot e^{\frac{1}{2}\cdot\frac{k^2}{b^2}\cdot\beta^2_{RD}}$$

In the next step, we derive the drift hazard function for a given value of deviation in spectral acceleration hazard, ε_{UH}, and the given value of deviation of the median drift demand, ε_{UD}:

$$H_{D|\varepsilon_{UH},\varepsilon_{UD}}(d) = \nu \cdot P[D > d \mid \varepsilon_{UD}, \varepsilon_{UH}] = \hat{H}_{S_a|\varepsilon_{UD}}(s_a^d) \cdot \varepsilon_{UH} \cdot e^{\frac{1}{2}\cdot\frac{k^2}{b^2}\cdot\beta^2_{RD}} \qquad (2\text{-}61)$$

The term $\hat{H}_{S_a|\varepsilon_{UD}}(s_a^d)$ must be interpreted as the median spectral acceleration hazard for the spectral acceleration that corresponds to drift *d* for a <u>given value</u> of deviation ε_{UD} in the median drift evaluation:

$$\hat{H}_{S_a|\varepsilon_{UD}}(s_a^d) = \hat{H}_{S_a}(s_a^{d|\varepsilon_{UD}}) \qquad (2\text{-}62)$$

In order to be able to calculate the above value, we need to find $s_a^{d|\varepsilon_{UD}}$ or the spectral acceleration corresponding to drift *d* for a given value of deviation ε_{UD} in drift evaluation. The median drift demand for a given value of deviation ε_{UD} can be found from Equation 2-57 as:

$$\eta_D(x) = \hat{\eta}_D(x) \cdot \varepsilon_{UD}$$

At this stage we assume that $\hat{\eta}_D(x)$ has the same functional form as the one $\eta_D(x)$ had in the previous part, stated in Equation 2-4, namely:

$$\hat{\eta}_D(x) = a \cdot x^b \qquad (2\text{-}63)$$

Substituting the value for $\hat{\eta}_D(x)$ from Equation 2-63 in Equation 2-57, the median drift demand can be evaluated as:

$$\eta_D(x) = a \cdot x^b \cdot \varepsilon_{UD} \qquad (2-64)$$

$s_a^{d|\varepsilon_{UD}}$ or the spectral acceleration corresponding to drift *d* for a given value of deviation ε_{UD} in drift evaluation can be evaluated by setting $\eta_D(x)$ in Equation 2-64 equal to *d* and solving for $s_a^{d|\varepsilon_{UD}}$. Hence, we can define $s_a^{d|\varepsilon_{UD}}$ as[4]:

$$s_a^{d|\varepsilon_{UD}} = \left(\frac{d}{a \cdot \varepsilon_{UD}}\right)^{\frac{1}{b}} = \left(\frac{d / \varepsilon_{UD}}{a}\right)^{\frac{1}{b}} = s_a^{d/\varepsilon_{UD}} \qquad (2-65)$$

The graphic interpretation of $s_a^{d|\varepsilon_{UD}}$ can be seen from Figure 2-11. In simple words, this means that we find the corresponding S_a value from the median curve $a \cdot x^b \cdot \varepsilon_{UD}$:

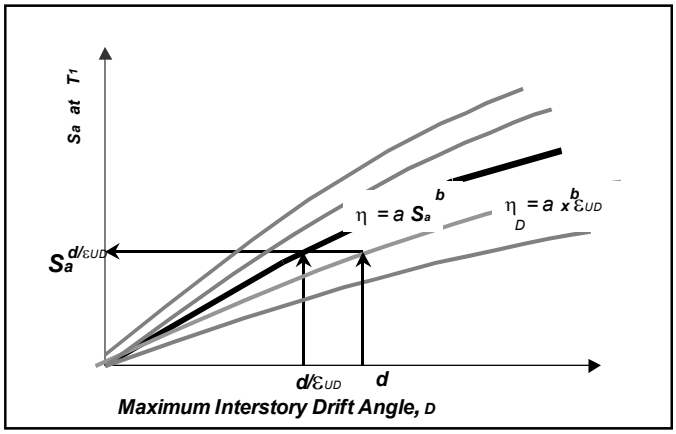


Figure 2-11 - Spectral acceleration corresponding to the inter-story drift ratio value *d* for a given value of deviation of median drift, ε_{UD}

Replacing the value for $s_a^{d|\varepsilon_{UD}}$ from Equation 2-65 in Equation 2-61:

$$H_{D|\varepsilon_{UH},\varepsilon_{UD}}(d) = \hat{H}_{S_a|\varepsilon_{UD}}(s_a^d) \cdot \varepsilon_{UH} \cdot e^{\frac{1}{2}\cdot\frac{k^2}{b^2}\cdot\beta^2_{RD}} = \hat{H}_{S_a}(s_a^{d/\varepsilon_{UD}}) \cdot e^{\frac{1}{2}\cdot\frac{k^2}{b^2}\cdot\beta^2_{RD}} \cdot \varepsilon_{UH} \qquad (2-66)$$

$\hat{H}(s_a^{d/\varepsilon_{UH}})$ can be calculated from Equations 2-65 and 2-50 as follows:

$$\hat{H}_{S_a}(s_a^{d/\varepsilon_{UD}}) = k_0 \cdot \left(\frac{d}{a \cdot \varepsilon_{UD}}\right)^{\frac{-k}{b}} \qquad (2-67)$$

[4] In fact $s_a^{d|\varepsilon_{UD}}$ is nothing but the spectral acceleration corresponding to drift demand *d* which is being calculated from the median curve $a \cdot x^b \cdot \varepsilon_{UD}$ instead of $a \cdot x^b$.

Substituting the value of $\hat{H}(s_a^{d/\varepsilon_{UH}})$ from Equation 2-67 in Equation 2-66 results in:

$$H_{D|\varepsilon_{UH},\varepsilon_{UD}}(d) = \hat{H}_{S_a}(s_a^{d/\varepsilon_{UD}}) \cdot e^{\frac{1}{2}\cdot\frac{k^2}{b^2}\cdot\beta^2_{D|S_a}} \cdot \varepsilon_{UH} = k_0 \cdot \left(\frac{d}{a\cdot\varepsilon_{UD}}\right)^{\frac{-k}{b}} \cdot \varepsilon_{UH} \cdot e^{\frac{1}{2}\cdot\frac{k^2}{b^2}\cdot\beta^2_{D|S_a}}$$

$$= k_0 \cdot \left(\frac{d}{a}\right)^{\frac{-k}{b}} \cdot e^{\frac{1}{2}\cdot\frac{k^2}{b^2}\cdot\beta^2_{D|S_a}} \cdot \varepsilon_{UH} \cdot \varepsilon_{UD}^{\frac{k}{b}} \qquad (2-68)$$

In short, we have an expression for the drift hazard conditioned on the spectral acceleration hazard and variables representing the uncertainty in drift, which is a simple analytical function of ε_{UH} and ε_{UD} the random variable representation of those two uncertainties. Recalling from the last section where the spectral acceleration hazard could be interpreted as an uncertain variable,

$$\widetilde{H}_{S_a}(s_a^d) = k_0 \cdot \left(\frac{d}{a}\right)^{\frac{-k}{b}} \cdot \varepsilon_{UH}$$

we can interpret the drift hazard itself as an uncertain (random) variable $\widetilde{H}_D(d)$, which is a function of the uncertain spectral acceleration hazard $\widetilde{H}_{S_a}(s_a)$ and uncertainty in drift prediction, ε_{UD}:

$$\widetilde{H}_D(d) = \nu \cdot P[D > d] = k_0 \cdot \left(\frac{d}{a}\right)^{\frac{-k}{b}} \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2_{RD}} \cdot \widetilde{\varepsilon}_{UH} \cdot \widetilde{\varepsilon}^{\frac{k}{b}}_{UD} = \widetilde{H}_{S_a}(s_a^d) \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2_{RD}} \cdot \widetilde{\varepsilon}^{\frac{k}{b}}_{UD}$$

$$(2-69)$$

The product of independent lognormal random variables raised to powers, such as k/b, is again a lognormal random variable (Benjamin and Cornell, 1970). Therefore, we can conclude that the drift hazard is also a log-normal random variable whose distribution parameters can be calculated based on the information about the distribution characteristics of ε_{UH} and ε_{UD} from Equations 2-52 and 2-60:

$$median(\widetilde{H}_D(d)) = \hat{H}_D(d) = median(\hat{H}_{S_a}(s_a^d) \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2_{RD}} \varepsilon_{UH} \cdot \varepsilon^{\frac{k}{b}}_{UD}) = \hat{H}_{S_a}(s_a^d) \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2_{RD}}$$

$$\beta_{UH_D} = \sqrt{\beta_{UH}{}^2 + \frac{k^2}{b^2}\beta_{UD}{}^2}$$

Therefore, the drift hazard $\widetilde{H}_D(d)$ is an uncertain quantity with median:

$$\hat{H}_D(d) = \hat{H}_{S_a}(s_a^d) \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2{}_{RD}} \qquad (2-70)$$

and fractional standard deviation:

$$\beta_{UH_D} = \sqrt{\beta_{UH}{}^2 + \frac{k^2}{b^2}\beta_{UD}{}^2} \qquad (2-71)$$

and, also mean value $\overline{H}_D(d)$ is equal to:

$$\overline{H}_D(d) = \hat{H}_D(d) \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2{}_{UH_D}} = \hat{H}_{S_a}(s_a^d) \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2{}_{RD}} \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2{}_{UH_D}} \qquad (2\text{-}72)$$

After substituting β_{UH_D} from Equation 2-69:

$$\overline{H}_D(d) = \hat{H}_{S_a}(s_a^d) \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2{}_{RD}} \cdot e^{\frac{1}{2}\beta^2{}_{UH}} \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2{}_{UD}} = \overline{H}_{S_a}(s_a^d) \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2{}_{RD}} \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2{}_{UD}} \qquad (2\text{-}73)$$

Note that the uncertainty in the hazard curve can be dealt with simply by using the mean estimate of the hazard curve. Similar to Figure 2-3, Figure 2-12 illustrates a graphical presentation of basic components for drift hazard evaluation, but in this case there is uncertainty both in the estimation of median drift curve $\tilde{\eta}_D$ and spectral acceleration hazard $\tilde{H}_{S_a}(s_a)$. In Figure 2-12, we have plotted, the median estimate, $\hat{\eta}_D$, of the uncertain median drift curve, the probability density reflecting the uncertainty in $\hat{\eta}_D$ about that estimate, with a fractional standard deviation equal to, β_{UD}, the median drift curve for a given value of uncertainty ε_{UD}, a realization of median drift curve $\hat{\eta}_D \cdot \varepsilon_{UD}$ and the fractional standard deviation β_{RD} due to record to record variability in the results of dynamic analyses around it. The median estimate for spectral acceleration hazard $\hat{H}_{S_a}(s_a)$ and the spectral acceleration hazard for a given value of uncertainty in hazard evaluation $\hat{H}_{S_a}(s_a) \cdot \varepsilon_{UH}$ are shown in the same manner as in Figure 2-10.

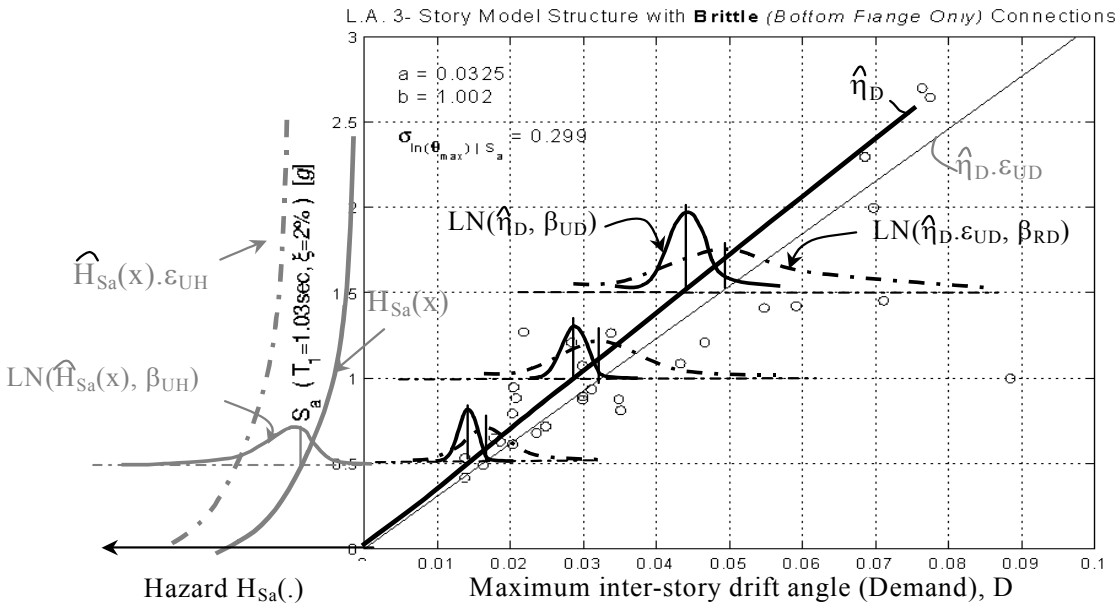


Figure 2-12 - Basic components for the evaluation of drift hazard with uncertainty in the estimation of spectral acceleration hazard $\widetilde{H}_{S_a}(s_a)$ and median drift $\widetilde{\eta}_D$

2.5.8.2.1 Numerical Example

Returning to our numerical example, we calculate the mean estimate of the drift hazard in the case where there is uncertainty in the evaluation of the drift demand. We have the maximum inter-story drift values resulting from 30 different nonlinear time history runs plotted in Figure 2-2. Fitting a line in log-log space to the data points gives us the following information about the median inter-story drift and the dispersion around it:

$$\eta_D(x) = 0.0325 s_a^{1.002}$$
$$\beta_{RD} = \beta_{D|S_a} = 0.299 \approx 0.3$$

But strictly this is just the median estimate, $\hat{\eta}_D(x)$ of the median drift curve. The uncertainty in the median estimate for the inter-story drift can be due to modeling errors and other approximations involved in the analysis procedure. Here we limit the consideration to the statistical uncertainty in the median due to the finite sample size ($n_{sample}=30$). The statistical properties of the median inter-story drift can be estimated as (see Rice 1995 for the statistical parameters for the mean estimate):

$$\hat{\eta}_D(x) = 0.0325 \cdot x^{1.002}$$

$$\beta_{UD} = \frac{\beta_{D|S_a}}{\sqrt{\mathrm{n}_{\text{sample}}}} \approx \frac{0.3}{5.48} = 0.055$$

Analogous to the previous section, we would like to evaluate the mean estimate of the probability

that the maximum inter-story drift angle exceeds a specific value, say 2%, $\overline{H}_D(0.02)$. If we substitute 0.02 for *d* in Equation 2-73:

$$\overline{H}_D(0.02) = \hat{H}_D(0.02) \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2_{UH_D}} = \overline{H}_{S_a}(s_a^{0.02}) \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2_{RD}} \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2_{UD}}$$

Recall that s_a^d is equal to $s_a^d = \left(\frac{d}{a}\right)^{\frac{1}{b}}$ per definition:

$$s_a^{0.02} = \left(\frac{0.02}{0.0325}\right)^1 = 0.615 \ \ [\mathrm{g}]$$

The mean estimate for the spectral acceleration hazard can be evaluated as:

$$\overline{H}_{S_a}(s_a) = \hat{H}_{S_a}(s_a) \cdot e^{\frac{1}{2}\beta^2_{UH}} \ .$$

β_{UH} here is assumed to be equal 0.50. With this assumption we can look up the value for $\hat{H}_{S_a}(0.615)$ from the spectral acceleration hazard curve in Figure 2-5, which is equal to *0.007*. Hence,

$$\overline{H}_{S_a}(0.615) = \hat{H}_{S_a}(0.615) \cdot e^{\frac{1}{2}(0.5)^2} = 0.007 \times 1.13 = 0.079$$

$\overline{H}_D(0.02)$ can be derived as:

$$\overline{H}_D(0.02) = \nu \cdot P[D > 0.02] = 0.0079 \cdot e^{\frac{1}{2}x\frac{3^2}{1^2}x0.3^2} \cdot e^{\frac{1}{2}x\frac{3^2}{1^2}x0.055^2} = 0.0079 \times 1.5 \times 1.014 = 0.012$$

Note that in the previous section where there was no uncertainty involved in the estimation of inter-story drift demand $H_D(0.02)$ was equal to 0.0105. The net uncertainty here in the estimation of $H_D(0.02)$ is equal to:

$$\beta_{UH_D} = \sqrt{(0.5)^2 + \frac{3^2}{1^2} 0.055} = 0.526$$

The primary contribution is from the PSHA, but this can change if β_{UD} increases to as large as 0.15. If non-linear dynamic modeling errors are considered, this value is likely to be considerably larger than 0.15.

2.5.8.3 Annual Probability of Exceeding a Limit State

We would like to derive the probability that the drift demand exceeds the drift capacity or the limit state probability, or simply the limit state frequency, in the face of epistemic uncertainty. In the last section we derived the probability that the drift demand exceeds a specified value of drift demand also known as drift hazard. The uncertainties in the derivation of the spectral acceleration hazard and drift demand were considered in this derivation. Now we are interested in calculating the probability that the drift demand exceeds a drift capacity, which is an uncertain quantity. In the previous part the capacity was assumed to be a uncertain (random) variable whose variability was caused by aleatory elements, e.g., connection-to-connection variability in a to-be-built design, or record-to-record variability observed , as will be observed in Chapter 4 in global instability capacity.

The drift capacity variable was introduced above as a median capacity value times a random variable ε_C representing the variation around the median value. We assumed that ε_C has a lognormal distribution:

$$C = \eta_C \cdot \varepsilon_C \qquad (2-74)$$

In general, the median capacity is also an uncertain (random) variable. The uncertainty in the median capacity is caused by the limited knowledge and data about the for example, untested connection designs or the nonlinear structural modeling and/or analysis underlying global stability prediction. The median capacity variable can be expressed as the product of its median value, $\hat{\eta}_C$ and a random variable ε_{UC} (*UC* stands for the uncertainty in evaluation of capacity, C) representing the uncertainty involved in the evaluation of η_C :

$$\eta_C = \hat{\eta}_C \cdot \varepsilon_{UC} \qquad (2-75)$$

Finally, we subscript ε with *RD,* standing for the randomness in drift demand evaluation:

$$C = \eta_C \cdot \varepsilon_{UC} = \hat{\eta}_C \cdot \varepsilon_{UC} \cdot \varepsilon_{RC} \qquad (2-76)$$

ε_{RC} and ε_{RC} are assumed to be independent and to have log-normal distributions with following characteristics:

$$\begin{aligned} &median(\varepsilon_{RC}) = median(\varepsilon_{UC}) = e^{mean(\ln(\varepsilon))} = 1 \\ &\sigma_{\ln(\varepsilon_{RC})} = \beta_{RC} \\ &\sigma_{\ln(\varepsilon_{UC})} = \beta_{UC} \\ &(2-77) \end{aligned}$$

Our objective in this section is to derive the probability that the drift demand *D* exceeds the drift capacity *C*, recognizing all these uncertainty elements. Based on the expression derived for the limit state frequency in previous part, Equation 2-38:

$$H_{LS} = \nu \cdot P[D > C] = H_D(\eta_C) \cdot e^{\frac{1}{2} \cdot \frac{k^2}{b^2} \cdot \beta_{RC}^2} \qquad (2\text{-}78)$$

We are going to use the above equation and combine it with our findings from the previous section for the drift hazard in order to derive the limit state frequency, H_{LS}. If capacity is a specific deterministic value, *c*, (i.e., there is no randomness nor uncertainty in the evaluation of capacity), the drift hazard function associated with exceeding drift level *c* (for a given value of uncertainty in spectral acceleration hazard, ε_{UH}, and uncertainty in drift demand, ε_{UD}) can be derived based on the results of previous section from Equation 2-61, substituting *c* for *d*:

$$H_{D|\varepsilon_{UH},\varepsilon_{UD}}(c) = P[D > c \mid \varepsilon_{UH}, \varepsilon_{UD}] = k_0 \cdot \left(\frac{c}{a}\right)^{\frac{-k}{b}} \cdot \varepsilon_{UH} \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2{}_{RD}} \cdot \varepsilon_{UD}^{\frac{k}{b}} \qquad (2-79)$$

We would like to first find the limit state frequency conditioned on the uncertainties in spectral acceleration hazard, drift demand and drift capacity. The median capacity for a given deviation, ε_{UC}, of drift capacity median from its median estimator can be written as below based on Equation 2-75:

$$\eta_{C|\varepsilon_{UC}} = \hat{\eta}_C \cdot \varepsilon_{UC} \qquad (2-80)$$

If we substitute the median capacity associated with this given deviation, $\eta_{C|\varepsilon_{UC}}$, from Equation 2-80 for c in Equation 2-79, the drift hazard conditioned on the uncertainties in spectral acceleration hazard, drift demand and drift capacity will be found as:

$$H_{D|\varepsilon_{UH},\varepsilon_{UD},\varepsilon_{UC}}(\eta_{C|\varepsilon_{UC}}) = k_0 \cdot \left(\frac{\eta_{C|\varepsilon_{UC}}}{a}\right)^{\frac{-k}{b}} \cdot \varepsilon_{UH} \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2_{RD}} \cdot \varepsilon^{\frac{k}{b}}{}_{UD} \cdot \qquad (2\text{-}81)$$

Substituting the conditional drift hazard term at $\eta_{C|\varepsilon_{UC}}$ in the above equation into Equation 2-78, the limit state frequency for a given value of uncertainty in spectral acceleration hazard, drift demand and drift capacity can be found:

$$H_{LS|\varepsilon_{UH},\varepsilon_{UD,\varepsilon_{UC}}} = \nu \cdot P[D > C \mid \varepsilon_{UH}, \varepsilon_{UD}, \varepsilon_{UC}] = H_{D|\varepsilon_{UH},\varepsilon_{UD},\varepsilon_{UC}}(\eta_{C|\varepsilon_{UC}}) \cdot e^{\frac{1}{2}\cdot\frac{k^2}{b^2}\cdot\beta^2_{RC}}$$

$$= k_0 \cdot \left(\frac{\hat{\eta}_C \cdot \varepsilon_{UC}}{a}\right)^{\frac{-k}{b}} \cdot \varepsilon_{UH} \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2{}_{RD}} \cdot \varepsilon^{\frac{k}{b}}{}_{UD} \cdot e^{\frac{1}{2}\cdot\frac{k^2}{b^2}\cdot\beta^2_{RC}}$$

$$\Rightarrow H_{LS|\varepsilon_{UH},\varepsilon_{UD,\varepsilon_{UC}}} = k_0 \cdot \left(\frac{\hat{\eta}_C}{a}\right)^{\frac{-k}{b}} \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2_{RD}} \cdot e^{\frac{1}{2}\cdot\frac{k^2}{b^2}\cdot\beta^2_{RC}} \cdot \varepsilon_{UH} \cdot \varepsilon_{UD}^{\frac{k}{b}} \cdot \varepsilon_{UC}^{\frac{-k}{b}} \qquad (2\text{-}82)$$

The above expression gives the limit state frequency conditioned on the spectral acceleration hazard, drift demand and drift capacity as an analytical function of $\varepsilon_{UH}, \varepsilon_{UD}$ and ε_{UC} the random variables representing the above-mentioned uncertainties. Similar to what we did in the previous section for the drift hazard, here we can interpret the limit state frequency itself as an uncertain (random) variable $\widetilde{H}_{LS}$. Recalling from Equation 2-54, the term,

$$k_0 \cdot \left(\frac{\hat{\eta}_C}{a}\right)^{\frac{-k}{b}} \cdot \varepsilon_{UH}$$

is equal to $\widetilde{H}_{S_A}(S_a^{\hat{\eta}_C})$. Hence, the limit state frequency can be introduced as a random quantity that is a function of the uncertain hazard $\widetilde{H}_{S_a}(\hat{\eta}_C)$, the uncertainty in drift demand prediction, ε_{UD} and the uncertainty in drift capacity prediction, ε_{UC}:

$$\widetilde{H}_{LS} = \widetilde{H}_{S_A}(S_a^{\hat{\eta}_C}) \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2{}_{RD}} \cdot e^{\frac{1}{2}\cdot\frac{k^2}{b^2}\cdot\beta_{RC}} \cdot \varepsilon^{\frac{k}{b}}{}_{UD} \cdot \varepsilon^{\frac{-k}{b}}{}_{UC} \qquad (2\text{-}83)$$

It can be observed that the limit state frequency is also a log-normal random variable whose distribution parameters can be calculated based on the information about the distribution characteristics of ε_{UH}, ε_{UD} and ε_{UC} from Equations 2-52, 2-60, and 2-77:

$$median(\widetilde{H}_{LS}) = \hat{H}_{LS} = median(\widetilde{H}_{S_A}(s_a^{\hat{\eta}_C}) \cdot \varepsilon^{\frac{k}{b}}{}_{UD} \cdot \varepsilon^{\frac{-k}{b}}{}_{UC} \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2{}_{RD}} \cdot e^{\frac{1}{2}\cdot\frac{k^2}{b^2}\cdot\beta_C^2}) = \hat{H}_{S_a}(\hat{\eta}_C) \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2{}_{RD}} \cdot e^{\frac{1}{2}\cdot\frac{k^2}{b^2}\cdot\beta_C^2}$$

$$\beta_{\widetilde{H}_{LS}} = \sqrt{\beta_{UH}{}^2 + \frac{k^2}{b^2}\beta_{UD}{}^2 + \frac{k^2}{b^2}\beta_{UC}{}^2}$$

Therefore, the uncertain limit state frequency $\widetilde{H}_{LS}$ is an uncertain quantity with median,

$$\hat{H}_{LS} = \hat{H}_{S_a}(s_a^{\hat{\eta}_C}) \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2{}_{RD}} \cdot e^{\frac{1}{2}\cdot\frac{k^2}{b^2}\cdot\beta_C^2} \qquad (2-84)$$

and fractional standard deviation equal to:

$$\beta_{\widetilde{H}_{LS}} = \sqrt{\beta_{UH}{}^2 + \frac{k^2}{b^2}\beta_{UD}{}^2 + \frac{k^2}{b^2}\beta_{UC}{}^2} \qquad (2-85)$$

and, also the mean limit state probability $\overline{H}_{LS}$ is equal to:

$$\overline{H}_{LS} = \hat{H}_{LS} \cdot e^{\frac{1}{2}\beta^2{}_{PLS}} = \hat{H}_{LS} \cdot e^{\frac{1}{2}\beta^2{}_{UH}} \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2{}_{UD}} \cdot e^{\frac{1}{2}\cdot\frac{k^2}{b^2}\beta^2{}_{UC}} = \overline{H}_{S_a}(s_a^{\hat{\eta}_C}) \cdot e^{\frac{1}{2}\frac{k^2}{b^2}(\beta^2{}_{RD}+\beta^2{}_{UD})} \cdot e^{\frac{1}{2}\cdot\frac{k^2}{b^2}\cdot(\beta^2{}_{RC}+\beta^2{}_{UC})}$$
$$(2-86)$$

Note that in this final form, the mean estimate of $\widetilde{H}_{LS}$ looks like H_{LS} without uncertainty (Equation 2-38) but now based on the mean estimate of $\widetilde{H}_{S_a}(s_a)$, and with increased β^2 in the capacity and demand (given S_a) exponential terms. Figure 2-13 illustrates a graphical presentation of basic components for evaluation of the limit state probability; there is uncertainty in the estimation of spectral acceleration hazard $\widetilde{H}_{S_a}(s_a)$, median drift demand curve $\widetilde{\eta}_D$ and median drift capacity $\widetilde{\eta}_C$. In Figure 2-13, we plot together the median estimate, $\hat{\eta}_C$, of the uncertain median drift capacity, the probability density reflecting the uncertainty in η_C about that estimate, with dispersion, β_{UC}, median drift capacity for a given deviation, ε_{UC}, in the estimation of median drift capacity $\hat{\eta}_C \cdot \varepsilon_{UC}$, the probability density reflecting the randomness type of uncertainty (for example, specimen-to-specimen variability the estimation of capacity) in capacity C about the median drift capacity $\hat{\eta}_C \cdot \varepsilon_{UC}$, with dispersion, β_{RC}.

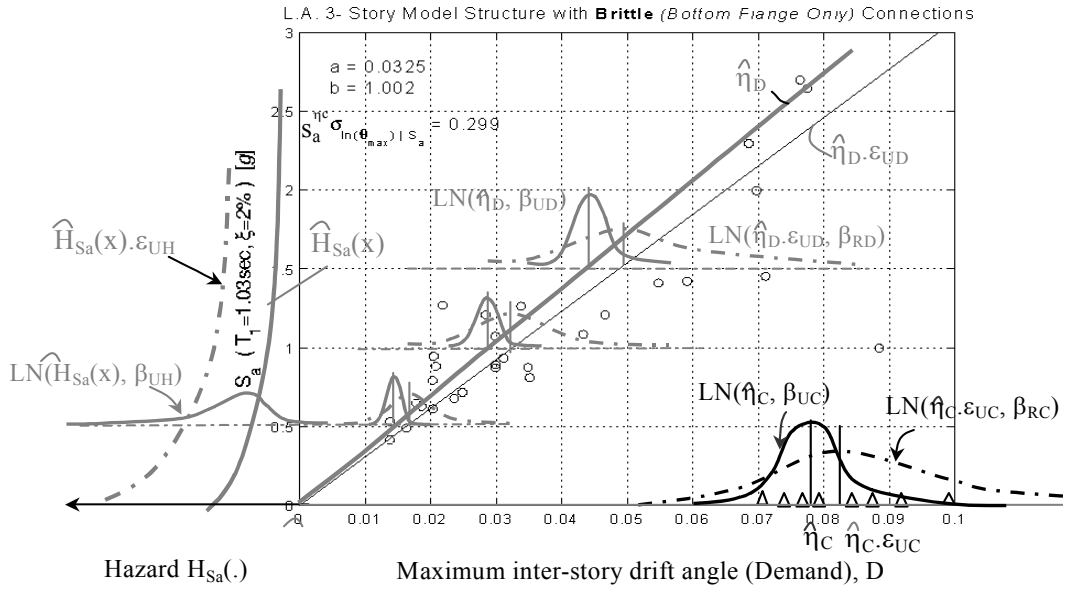


Figure 2-13 - Basic components for the derivation of the limit state probability when there is uncertainty in the estimation of the spectral acceleration hazard $\widetilde{H}_{S_a}(s_a)$, median drift demand η_D, and median drift capacity η_C.

2.5.8.3.1 Numerical Example

For our 3-story frame numerical example, we would like to calculate the mean limit state probability, when there is uncertainty both in the estimation of median drift demand and median drift capacity. Recalling from last section, the median drift and the dispersion of drift for a given level of spectral acceleration was estimated (by fitting a line in the log-log space to the data points obtained by performing 30 different nonlinear time history analyses) as:

$$\eta_D(x) = 0.0325 \cdot s_a^{\,1.002}$$
$$\beta_{RD} = \beta_{D|S_a} = 0.299 \approx 0.3$$

We have also estimated the statistical properties of the uncertain median drift demand as:

$$\hat{\eta}_D(x) = \eta_D(x) = 0.0325 \cdot x^{1.002}$$
$$\beta_{UD} \cong \frac{\beta_{D|S_a}}{\sqrt{\mathrm{n}_{\mathrm{sample}}}} = \frac{0.3}{5.48} = 0.055$$

The median and dispersion for drift capacity were given before:

$$\eta_C = 0.07$$
$$\beta_{RC} = \sigma_{\ln(C)} = \beta_C = 0.20$$

Note that the dispersion parameter β_C is due to the randomness type of uncertainty in drift capacity. The statistical properties for the now uncertain median drift capacity we have estimated here as (for an assumed sample of size 4 as the number of tests upon which the estimate of the median connection capacity is based):

$$\hat{\eta}_C \cong 0.07$$
$$\beta_{UC} \cong \frac{\beta_C}{\sqrt{4}} = \frac{0.2}{2} = 0.1$$

Equation 2-86 gives the mean limit state probability as:

$$\overline{H}_{LS} = \hat{H}_{LS} \cdot e^{\frac{1}{2}\beta^2_{UH}} \cdot e^{\frac{1}{2}\frac{k^2}{b^2}\beta^2_{UD}} \cdot e^{\frac{1}{2}\cdot\frac{k^2}{b^2}\beta^2_{UC}} = \overline{H}_{S_a}(s_a^{\hat{\eta}_C}) \cdot e^{\frac{1}{2}\frac{k^2}{b^2}(\beta^2_{RD}+\beta^2_{UD})} \cdot e^{\frac{1}{2}\cdot\frac{k^2}{b^2}\cdot(\beta^2_{RC}+\beta^2_{UC})}$$

We have defined s_a^d equal to $\left(\frac{d}{a}\right)^{\frac{1}{b}}$, therefore $s_a^{\hat{\eta}_C}$ can be calculated as $\left(\frac{\hat{\eta}_C}{a}\right)^{\frac{1}{b}}$:

$$s_a^{\hat{\eta}_C} = s_a^{0.07} = \left(\frac{0.07}{0.0325}\right)^1 = 2.15 \quad [\text{g}]$$

Our next step is to calculate the median spectral acceleration hazard at a spectral acceleration corresponding to median drift capacity:

$$\overline{H}_{S_a}(s_a^{\hat{\eta}_C}) = \hat{H}_{S_a}(s_a^{\hat{\eta}_C}) \cdot e^{\frac{1}{2}\beta^2_{UH}} = k0 \cdot \left(s_a^{\hat{\eta}_C}\right)^{-k} \cdot e^{\frac{1}{2}\beta^2_{UH}} = 0.00124 \times 2.15^{-3.0} \times e^{\frac{1}{2}(0.5^2)}$$

$$= 0.000124 \times 1.13 = 0.00014$$

Assuming $\beta_{UH} = 0.5$, same as in the previous sections. We can also look up the value for $\hat{H}_{S_a}(2.15)$ from the spectral acceleration hazard curve from Figure 2-5, which is equal to 0.00012. Hence, the mean estimate of the limit state probability $\overline{H}_{LS}$ can be derived as:

$$\overline{H}_{LS} = 0.00014 \cdot \mathrm{e}^{\frac{1}{2}(\frac{3^2}{1^2})(0.3^2)} \cdot \mathrm{e}^{\frac{1}{2}(\frac{3^2}{1^2})(0.2^2)} \cdot \mathrm{e}^{\frac{1}{2}(\frac{3^2}{1^2})(0.1^2)} \cdot \mathrm{e}^{\frac{1}{2}(\frac{3^2}{1^2})(0.055^2)}$$
$$= 0.00014 \times 1.5 \times 1.2 \times 1.014 \times 1.05 = 2.68 \times 10^{-4}$$

Note that in the first part of the document we calculated the limit state probability when the epistemic uncertainty in the estimation of median demand and capacity was not taken into account. The limit state probability in that case was equal to 2.2×10^{-4}, whereas the mean estimate of the limit state probability calculated in the presence of uncertainty in the estimation of hazard, demand and capacity is 2.68×10^{-4}

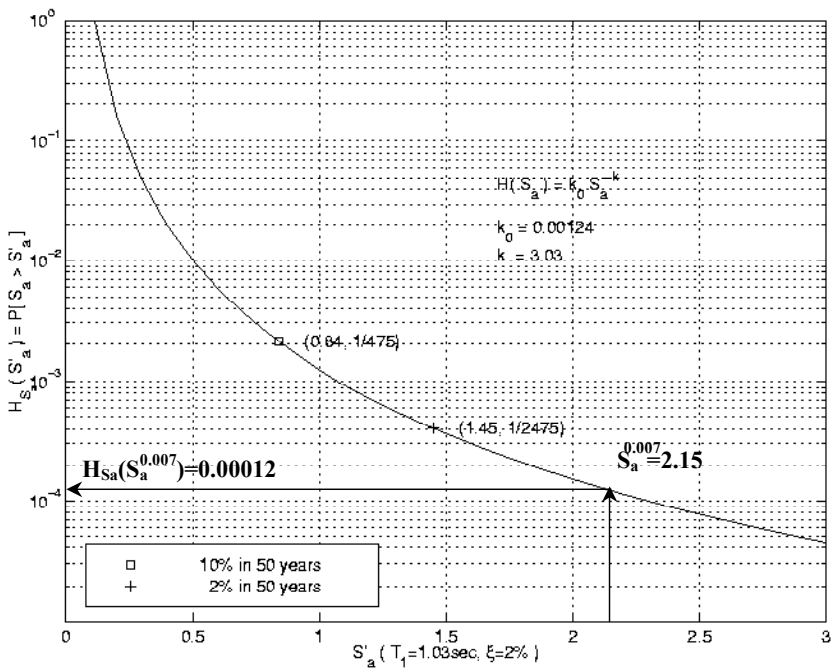


Figure 2-14 - The median estimate for the spectral acceleration hazard curve. The hazard value for a spectral acceleration equal to 2.15 is shown on the figure.

2.5.8.4 Annual probability of exceeding a limit state: the IM-based approach

The annual probability of exceeding a limit state following the IM-based was derived in Section 2.5.7.5, considering only the aleatory uncertainty (due to record-to-record variability) in demand and capacity. In this section we are going to follow the same approach in order to derive the limit state probability considering also the epistemic uncertainty. Similar to the previous sections, we assume that the median capacity variable can be expressed as the product of its median value, $\hat{\eta}_{S_{a,C}}$ and a random variable $\varepsilon_{US_{a,C}}$ representing the uncertainty involved in the evaluation of $\eta_{S_{a,C}}$:

$$\eta_{S_{a,C}} = \hat{\eta}_{S_{a,C}} \cdot \varepsilon_{US_{a,C}} \qquad (2-87)$$

Therefore, similar to the previous sections, we can represent the spectral acceleration capacity as:

$$S_{a,C} = \eta_{S_{a,C}} \cdot \varepsilon_{RS_{a,C}} = \hat{\eta}_{S_{a,C}} \cdot \varepsilon_{RS_{a,C}} \cdot \varepsilon_{US_{a,C}} \qquad (2-88)$$

where $\varepsilon_{RS_{a,C}}$ and $\varepsilon_{US_{a,C}}$ are assumed to be independent and to have log-normal distributions with following characteristics:

$$\begin{aligned} &median(\varepsilon_{RS_{a,C}}) = median(\varepsilon_{US_{a,C}}) = 1 \\ &\sigma_{\ln(\varepsilon_{RS_{a,C}})} = \beta_{RS_{a,C}} \\ &\sigma_{\ln(\varepsilon_{US_{a,C}})} = \beta_{US_{a,C}} \\ &(2-89) \end{aligned}$$

where $\beta_{RS_{a,C}}$ and $\beta_{US_{a,C}}$ are fraction standard deviations representing the randomness and uncertainty in the spectral acceleration capacity respectively. It can be shown (the procedure is similar to the one described for the displacement-based approach in detail) that the limit state frequency $\widetilde{H}_{LS}$ is an uncertain quantity with median,

$$\hat{H}_{LS} = \hat{H}_{S_a}(\hat{\eta}_{S_{a,C}}) \cdot \exp(\frac{1}{2} k^2 \beta_{RS_{a,C}}{}^2) \qquad (2-90)$$

and fractional standard deviation equal to:

$$\beta_{\widetilde{H}_{LS}} = \sqrt{\beta_{UH}{}^2 + k^2 \cdot \beta_{US_{a,C}}{}^2} \qquad (2-91)$$

and, also the mean limit state probability $\overline{H}_{LS}$ is equal to:

$$\overline{H}_{LS} = \hat{H}_{LS} \cdot e^{\frac{1}{2}\beta^2{}_{PLS}} = \hat{H}_{LS} \cdot e^{\frac{1}{2}\beta^2{}_{UH}} \cdot e^{\frac{1}{2}k^2\beta^2{}_{US_{a,C}}} = \overline{H}_{S_a}(s_a^{\hat{\eta}_C}) \cdot e^{\frac{1}{2}\frac{k^2}{b^2}(\beta^2{}_{RS_{a,C}} + \beta^2{}_{US_{a,C}})} \qquad (2-92)$$

As with the limit state probability derived following the displacement-based case, the mean estimate of $\widetilde{H}_{LS}$ looks like H_{LS} without uncertainty (Equation 2-48) but based on the mean estimate of $\widetilde{H}_{S_a}(s_a)$, and with increased β^2 in the spectral acceleration capacity due to the consideration of epistemic uncertainty.

2.6 Summary

A technical foundation for probabilistic seismic assessment of structures has been developed, taking into account the randomness (aleatory uncertainty) and uncertainty (epistemic uncertainty) in the seismic hazard, demand and capacity parameters. This foundation is based on a closed-form analytical expression for the mean annual frequency of exceeding a limit state (limit state frequency in short). Two different approaches were presented for deriving the limit state frequency, namely, the displacement-based and the IM-based. Both approaches are based on simplifying assumptions regarding the shape of the hazard curve and the probabilistic models representing demand and capacity. This technical foundation forms an analytic basis upon which alternative design and assessment formats can be developed. These formats are discussed in the next chapter which is the second part to this report.

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