Appendix A

The expected value of Y^{α} where Y is a log normal random variable:

Assume lnY is a normal random variable (i.e., Y is lognormal) with mean *m* and standard deviation σ. One can always write the following relationship for Y raised to a power, α:

$$Y^{\alpha} = e^{\alpha \ln Y}$$

lnY can be transformed into a standard normal variable *U*,

$$U = \frac{\ln Y - m}{\sigma} \qquad \text{(A - I)}$$

for which, the standard normal probability density function (PDF) at *U*=*u* is equal to:

$$\phi(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} \qquad \text{(A - II)}$$

Based on the linear relation between lnY and U (Equation A-I), and the standard normal PDF for U (Equation A-II), the PDF for normal random variable lnY at lnY=*x* can be obtained as:

$$f_{\ln Y}(x) = \frac{1}{\sigma}\phi(u) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}u^2} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{\ln Y - m}{\sigma})^2} \qquad \text{(A - III)}$$

where f(.) denotes the PDF function. The expected value for a function g(.) of a continuous random variable Z can calculated as:

$$E[g(Z)] = \int_{-\infty}^{\infty} g(z) \cdot f_Z(z) \cdot dz \qquad \text{(A - IV)}$$

Therefore, the expected value of $g(Y) = Y^{\alpha}$ can be written as (using Equations A-II, A-III and A-IV):

$$E[Y^{\alpha}] = E[e^{\alpha \ln Y}] = \int_{-\infty}^{+\infty} e^{\alpha x} \cdot f_{\ln Y}(x) \cdot dx = \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{\alpha x} \cdot e^{-\frac{1}{2}(\frac{x-m}{\sigma})^2} \cdot dx$$

After some algebraic operations, which involves adding and subtracting some(necessary) square terms the following equation is obtained:

$$E(e^{\alpha \ln Y}) = \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{\alpha x} \cdot e^{\frac{(x-m)^2}{2\sigma^2}} dx = e^{m\alpha} \cdot \mathrm{e}^{\frac{1}{2}\alpha^2\sigma^2} \cdot \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-m-\alpha\sigma^2)^2}{2\sigma^2}} \cdot dx$$

We can recognize that the term inside the integral is nothing but the PDF for a normal variable with a mean equal to $m + \alpha \cdot \sigma^2$, and a standard deviation equal to, σ. Therefore, the resulting integral (from $-\infty$ to ∞) is equal to unity. Hence, the expected value of Y^{α} is simplified to the product of the following two terms:

$$E[Y^{\alpha}] = E[e^{\alpha \ln Y}] = e^{m\alpha}.\mathrm{e}^{\frac{1}{2}\alpha^2\sigma^2}$$

For a lognormal random variable, the mean of the logarithm of the variable is equal to the logarithm of the median of the variable (see Benjamin and Cornell, 1970):

$$\ln \eta_{\mathrm{Y}} = E[\ln Y]$$

where $\eta(.)$ denotes the median. Hence, for normal random variable lnY with mean *m* and standard deviation σ, the expected values of Y^{α} can be written as:

$$E[Y^{\alpha}] = E[e^{\alpha \ln Y}] = e^{m\alpha} \cdot \mathrm{e}^{\frac{1}{2}\alpha^2\sigma^2} = e^{\alpha \ln \eta_Y} \cdot \mathrm{e}^{\frac{1}{2}\alpha^2\sigma^2} = (\eta_{\mathrm{Y}})^{\alpha} \cdot \mathrm{e}^{\frac{1}{2}\alpha^2\sigma^2} \qquad \text{(A - V)}$$

Thus, the expected value of a lognormal random variable raised to a power α can be expressed as the product of the median value raised to the power times a *magnification factor*, which is an exponential function of the variance of lnY times $\frac{1}{2}\alpha^2$.