Deformation of Kählerian Lie groups

(joint with P. Bieliavsky, P. Bonneau, V. Gayral, Y. Maeda, Y. Voglaire)

Francesco D’Andrea

International School for Advanced Studies (SISSA)

Via Beirut 2-4, Trieste, Italy

09/09/2010

Workshop on Quantum Groups and Physics, Caen, 6–10 September 2010.
Outline

1 Introduction
   Hopf algebras and compact quantum groups
   Drinfeld twist/cocycle quantization
   The operator algebra approach

2 From formal to non-formal deformations: Moyal
   The Moyal plane
   Moyal-Weyl quantization

3 Deformation of Kählerian Lie groups
   Definition and decomposition
   Formal deformations
   Multiplicative unitaries
Motivating example: if \( G \) is a compact topological group, the group structure is encoded in the algebra \( A := \mathcal{R}(G) \) of complex-valued representative functions.

Dually to the group operations, one can define co-operations

\[
\Delta : A \to A \otimes A \quad \quad \epsilon : A \to \mathbb{C} \quad \quad S : A \to A
\]

\[
\Delta(f)(g_1, g_2) := f(g_1 g_2) \quad \quad \epsilon(f) := f(1_G) \quad \quad S(f)(g) := f(g^{-1})
\]

satisfying the axioms of a commutative Hopf algebra (HA):

\[
(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta \quad \quad \text{(coassociativity)}
\]

\[
(\epsilon \otimes \text{id}) \circ \Delta = (\text{id} \otimes \epsilon) \circ \Delta = \text{id} \quad \quad \text{(counity)}
\]

\[
(S \otimes \text{id}) \circ \Delta = (\text{id} \otimes S) \circ \Delta \quad \quad \text{(coinverse)}
\]

\[
\Delta(ab) = \Delta(a) \Delta(b) \quad \quad \epsilon(ab) = \epsilon(a) \epsilon(b)
\]

Tannaka-Krein duality: \( G \) can be reconstructed from its irreps., i.e. from \( \mathcal{R}(G) \).

Then [Drinfeld-Jimbo, Majid, \(~'80s\):]

quantum groups := non-commutative non-cocommutative Hopf algebras.
Quantization of Poisson-Lie groups

Let $G := $ is both a compact group and a Poisson manifold and $\mathcal{A} := \mathcal{R}(G)$. It is natural to look for formal deformations of $\mathcal{A}$, i.e. associative products $*_{\hbar}$ on the $\mathbb{C}[[\hbar]]$-module $\mathcal{A}[[\hbar]]$ of the form

$$f_1 *_{\hbar} f_2 = f_1 \cdot f_2 + \frac{i\hbar}{2} \{f_1, f_2\} + O(\hbar^2),$$

such that $(\mathcal{A}[[\hbar]], *_{\hbar}, \Delta, \ldots)$ is still an Hopf algebra. The necessary condition

$$\Delta(f_1 *_{\hbar} f_2) = \Delta(f_1) (*_{\hbar} \otimes *_{\hbar}) \Delta(f_2)$$

at the leading order in $\hbar$ gives

$$(*) \quad \Delta\{a, b\} = \{a_{(1)}, b_{(1)}\} \otimes a_{(2)} b_{(2)} + a_{(1)} b_{(1)} \otimes \{a_{(2)}, b_{(2)}\},$$

for all $a, b \in \mathcal{A}$, and with $\Delta(u) = u_{(1)} \otimes u_{(2)}$ the Sweedler notation.

A Lie group with a Poisson structure satisfying $(*)$ is called Poisson-Lie group.

Most concrete examples of quantum groups are of this type: deformations $(\mathcal{R}(G)[[\hbar]], *_{\hbar})$ of Poisson-Lie groups with undeformed coproduct.
Drinfeld twist/cocycle quantization

A twist based on a bialgebra $\mathcal{U}$ (e.g. $\mathcal{U} = U(g)[[\hbar]]$) is an invertible $F \in \mathcal{U} \otimes \mathcal{U}$ s.t.

$$(\Delta \otimes \text{id})(F) \cdot (F \otimes 1) = (\text{id} \otimes \Delta)(F) \cdot (1 \otimes F),$$

$$(\epsilon \otimes \text{id})(F) = (\text{id} \otimes \epsilon)(F) = 1 \otimes 1 .$$

$\Leftrightarrow F^{-1}$ is a counital Hopf 2-cocycle (cf. e.g. [Majid, 1995]). With $F$ we can define:

- a new bialgebra $\mathcal{U}_F = (\mathcal{U}, \Delta_F)$ by replacing $\Delta$ of $\mathcal{U}$ with

$$\Delta_F(X) := F^{-1} \Delta(X)F , \quad \forall X \in \mathcal{U} .$$

- for any left $\mathcal{U}$-module algebra $\mathcal{A}$ with product $m(a \otimes b) = ab$ (e.g. $\mathcal{R}(G)[[\hbar]]$),

a left $\mathcal{U}_F$-module algebra $\mathcal{A}_F = (\mathcal{A}, *_F)$ with product

$$a *_F b = m \circ F(a \otimes b) , \quad \forall a, b \in \mathcal{A} .$$

- for any bialgebra $\mathcal{O}$ dual to $\mathcal{U}$ (e.g. $\mathcal{R}(G)[[\hbar]]$), a $\mathcal{O}_F = (\mathcal{O}, *_F)$ dual to $\mathcal{U}_F$ by

$$a *_F b := m(\triangleright (a \otimes b) \triangleleft F^{-1}) .$$

Here $\triangleright$ and $\triangleleft$ are the left/right canonical actions. Rem.: the coproduct is undeformed!
The operator algebra approach

- Gelfand-Naǐmark thm: the top. space $G$ can be reconstructed from $Q := C_0(G)$.

- A compact quantum group (CQG) is a pair $(Q, \Delta)$ given by a complex unital $C^*$-algebra $Q$ and a coassociative unital $C^*$-algebra morphism $\Delta : Q \to Q \hat{\otimes} Q$ satisfying certain density properties [Woronowicz, ~‘80s].

  **Theorem:** commutative compact quantum groups $=$ compact topological groups.

- Other possible definitions/generalizations to the non-compact case:

  - **Hopf $C^*$-algebras** [Vaes & Van Daele, 2001]: similar to HAs but $\text{Im}(\Delta) \subset Q \hat{\otimes} Q$.
  
  - **Multiplier Hopf algebras** [Van Daele, 1994]: HA with no 1, $\text{Im}(\Delta) \subset M(Q \hat{\otimes} Q)$.
  
  - **LCQG** [Kusterman & Vaes, 2000]: $(Q, \Delta, \varphi, \psi)$ with $\Delta : Q \to M(Q \hat{\otimes} Q)$ and $\varphi$ (resp. $\psi$) left (resp. right) invariant faithful KMS weight. Modelled on $C_0(G)$.

  - **Bornological QGs** [Voigt, 2005]: $(\mathcal{A}^\infty, \Delta, \varphi)$, modelled on $\mathcal{A}^\infty := C_c(G)$, uses bornological algebras.

  - A pair $(Q, \Delta)$ can be constructed from a **multiplicative unitary**!
Quantum groups via multiplicative unitaries

[Baaj-Skandalis ~ ‘90s, later Woronowicz & Sołtan]

All informations about a locally compact group $G$ are encoded in the Kac-Takesaki operator, that is the op. $W \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ — $\mathcal{H} = L^2(G, d^R \mu)$ — closure of

$$(W \psi)(g, g') = \psi(gg', g').$$

$W$ is unitary . . . and multiplicative:

$$(\dagger) \quad W_{12}W_{13}W_{23} = W_{23}W_{12}.$$

From $W$ one can reconstruct $C_0(G)$ (then the topological space $G$) as

$$C_0(G) = \{(\omega \otimes \text{id})(W) \mid \omega \in \mathcal{B}(\mathcal{H})_*\}^\text{norm cl.}$$

and the group structure from the property

$$W^*(1 \otimes T)W = T \otimes T \iff T = R_g \text{ for some } g \in G.$$ 

Idea: define a LCQG via a multiplicative unitary, i.e. a unitary $W$ on some $\mathcal{H} \otimes \mathcal{H}$ satisfying the Pentagon equation $(\dagger)$. 
Outline

1 Introduction
   Hopf algebras and compact quantum groups
   Drinfeld twist/cocycle quantization
   The operator algebra approach

2 From formal to non-formal deformations: Moyal
   The Moyal plane
   Moyal-Weyl quantization

3 Deformation of Kählerian Lie groups
   Definition and decomposition
   Formal deformations
   Multiplicative unitaries
The Moyal plane

Let \( \{ P_1, P_2 \} := \text{basis of } \mathbb{R}^2 \) and \( A := \mathcal{U}(\mathbb{R}^2) \)-module algebra with multiplication \( m \).

An associative product on \( A[[\hbar]] \) is given by

\[
a \ast_{\hbar}^{M} b := m \circ e^{i\hbar P_1 \wedge P_2} (a \otimes b).
\]

Example: if \( A = C^\infty(\mathbb{R}^2) \) and \( P_i = \partial_i \) we get a deformation quantization of \( \mathbb{R}^2 \).

Memorandum

A deformation quantization of \((M, \{ , \})\) is an ass. product \( \ast_{\hbar} \) on \( C^\infty(M)[[\hbar]] \) given by:

\[
a \ast_{\hbar} b := ab + \frac{i\hbar}{2}\{a, b\} + \sum_{n \geq 2} \hbar^n C_n(a, b),
\]

where \( C_n \) are bidifferential operators.

From deformation quantizations to twists and back

Left invariant \( \ast_{\hbar} \)'s on \( G \) are in bijection with twisting elements (Hopf 2-cocycles) \( F_{\hbar} \) based on \( \mathcal{U}(g)[[\hbar]] \) via the formula

\[
a \ast_{\hbar} b = m \circ F_{\hbar}(a \otimes b).
\]
Moyal-Weyl quantization

Non-formal deformations: if \( f_1, f_2 \in S(\mathbb{R}^2) \) then \( f_1 \ast_{\hbar}^M f_2 \) is convergent (to a Schwartz function) for any \( \hbar \in \mathbb{R}^\times \), and can be rewritten as an oscillatory integral:

\[
(f_1 \ast_{\hbar}^W f_2)(x) := \frac{1}{(\pi \hbar)^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{\frac{i}{\hbar} \{\omega(x,y) + \omega(y,z) + \omega(z,x)\}} \ f_1(y) f_2(z) \, d^2y \, d^2z,
\]

with \( \omega := \) standard symplectic form. Remark:

- \( \ast_{\hbar}^W \) can be extended to larger classes of functions: C*- / Hilbert algebras, . . .
  [Gracia-Bondía & Várilly, 1988]
- Weyl: formulation of quantum mechanics on phase-space.
- Rieffel: strict deformation quantization for actions of \( \mathbb{R}^d \).
- Examples: \( SU_q(2) \) [Sheu, 1991], \( G_\theta \) for any compact \( G \) with rank \( \geq 2 \) [Wang, 1996], Connes-Landi spheres, . . .

Using the integral kernel of \( \ast_{\hbar}^W \) one can deform the Kac-Takesaki operator of any locally compact group \( G \) with rank \( \geq 2 \) [Vaes et al., Lecture Notes, 2001].
Outline

1 Introduction
   - Hopf algebras and compact quantum groups
   - Drinfeld twist/cocycle quantization
   - The operator algebra approach

2 From formal to non-formal deformations: Moyal
   - The Moyal plane
   - Moyal-Weyl quantization

3 Deformation of Kählerian Lie groups
   - Definition and decomposition
   - Formal deformations
   - Multiplicative unitaries
j-algebras were introduced in the study bounded homogeneous domains in $\mathbb{C}^n$.

**Definition [Piatetski-Shapiro]**

A normal j-algebra $(\mathfrak{b}, \alpha, j)$ is given by

1. a solvable Lie algebra $\mathfrak{b}$ split over $\mathbb{R}$ (i.e. $\text{Ad}_X$ has only real eigenvalues $\forall X \in \mathfrak{b}$);
2. $j \in \text{End}(\mathfrak{b})$ s.t. $j^2 = -1$ and $j[X, Y] = [jX, Y] + [X, jY] + j[jX, jY]$ for all $X, Y \in \mathfrak{b}$;
3. a linear form $\alpha : \mathfrak{b} \to \mathbb{R}$ s.t. $\alpha([jX, X]) > 0 \forall X \neq 0$ and $\alpha([jX, jY]) = \alpha([X, Y])$.

- One can associate a $(\mathfrak{b}, \alpha, j)$ to any bounded homogeneous domain.
- The Lie group $\mathbb{B}$ is a Kähler manifold with an invariant Kähler structure ($j$ gives the complex structure and the Chevalley coboundary $d\alpha$ gives the Kähler form). We call this a Kählerian Lie group.
- To each semisimple $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ of Hermitian type (i.e. rank $\mathfrak{k} = 1$) one can attach a normal j-algebra, with $\mathfrak{b} := \mathfrak{a} \oplus \mathfrak{n}$. These are called **elementary**.
Decomposition and formal extension lemma

- Elementary normal $j$-algebras are building blocks for normal $j$-algebras.

**Theorem [Piatetski-Shapiro]**

For any $(b, \alpha, j)$ there exists a split exact sequence

$$0 \to b_0 \to b \to b_1 \to 0$$

where $b_0$ is an elementary normal $j$-ideal.

**Corollary**

Any $b$ can be constructed by iteration as semidirect product of elementary normal $j$-algebras and possibly an abelian factor.

**Formal extension lemma**

Let $b = b_0 \rtimes b_1$ and $F_i$ twists based on $\mathcal{U}(b_i)[[\hbar]]$. If $F_0 \in \mathcal{U}(b_0)^{b_1} \otimes \mathcal{U}(b_0)^{b_1}[[\hbar]]$, then

$$F := F_0 \otimes F_1$$

is a twist based on $\mathcal{U}(b)[[\hbar]]$. 
Elementary normal j-algebras

Definition

Fix \( n \in \mathbb{N} \). Then

i) \( b \simeq \mathbb{R} \times \mathbb{R}^{2n} \times \mathbb{R} \) has basis \( H, \{X_i\}_{i=1,\ldots,2n}, E \) and relations

\[
[X_i, X_j] = (\delta_{i+n,j} - \delta_{i-n,j})E, \quad \quad \quad \quad [E, X_i] = 0, \\
[H, E] = 2E, \quad \quad \quad \quad [H, X_i] = X_i.
\]

ii) The map

\[
b \to \mathbb{B}, \quad \quad x := (a, v, t) \mapsto \exp(aH)\exp(\sum_i v_i X_i + tE)
\]

is a global Darboux diffeomorphism.

iii) Within these coordinates the group law is

\[
x \cdot x' = (a + a', e^{-a'}v + v', e^{-2a'}t + t' + \frac{1}{2}e^{-a'}\Omega(v, v'))
\]

with \( \Omega(v, v') = \sum_{i=1}^{n} (v_i v'_{i+n} - v_{i+n} v'_i) \) standard symplectic structure on \( \mathbb{R}^{2n} \).

Remark 1: \( \mathbb{B} \) is a 1D extension of the Heisenberg group \( \mathbb{H}_n := \{x = (a, v, t) \in \mathbb{B} : a = 0\} \).

Remark 2: there is extra structure on \( \mathbb{B} \), namely it is a symplectic symmetric space.
A star product on \( \mathbb{B} \)

Within the theory of symplectic symmetric spaces [Bieliavsky-Cahen-Gutt, 1997] it is possible to define an associative product (on suitable test functions) in the form of an oscillatory integral. For \( \hbar \in \mathbb{R}^\times \) and \( \tau : \mathbb{R} \to \mathbb{C} \) satisfying some technical conditions, we define

\[
(f_1 \ast_{\hbar, \tau} f_2)(x_0) = \int_{\mathbb{B} \times \mathbb{B}} K_{\hbar, \tau}(x_1, x_2) f_1(x_0 x_1) f_2(x_0 x_2) dx_1 dx_2,
\]

where \( dx \) is the left-invariant Haar,

\[
K_{\hbar, \tau}(x_1, x_2) := 2^{-2n} (\pi \hbar)^{-2(n+1)} A_{\text{can}}(x_1, x_2) e^{i S_{\text{can}}(x_1, x_2)} e^{\tau(a_1) + \tau(-a_2) - \tau(a_1 - a_2)},
\]

the canonical amplitude and phase are

\[
A_{\text{can}}(x_1, x_2) := (\cosh a_1 \cosh a_2 \cosh(a_1 - a_2))^n \sqrt{\cosh 2a_1 \cosh 2a_2 \cosh 2(a_1 - a_2)},
\]

\[
S_{\text{can}}(x_1, x_2) := \sinh(2a_1) t_2 - \sinh(2a_2) t_1 + \cosh a_1 \cosh a_2 \Omega(v_1, v_2).
\]
The formal twist
— n = 0, \tau = \tau_0 —

Define \( \tau_0(a) := \frac{1}{2} \log \cosh 2a \).

Proposition
For \( n = 0 \) and \( \tau = \tau_0 \) the formal twist is given by

\[
\mathcal{F}_h = \sum_{n \geq 0} \left( \frac{i \hbar}{2} \right)^n \mathcal{F}_n, \quad \mathcal{F}_n := \sum_{j, k, l, m \geq 0 \atop j + l = n} (-1)^l B_{l, m} B_{j, k} H^k E^l \otimes H^m E^j,
\]

where

\[
B_{j, k} := \sum_{n_1 + n_2 + \ldots + n_j = k \atop n_1 + 2n_2 + \ldots + jn_j = j} \frac{\beta_1^{n_1} \beta_2^{n_2} \ldots \beta_j^{n_j}}{n_1! n_2! \ldots n_j!},
\]

and \( \beta_n \) are the Taylor coefficients of \( \text{arcsinh} \).

Remark 1: since \( B_{j, k} = 0 \) for all \( j < k \), \( \mathcal{F}_n \) is a finite sum (i.e. \( \mathcal{F}_n \in \text{UEA} \)).

Remark 2: a completely explicit formula can be given for arbitrary \( \tau \).
The formal twist
— $n$ and $\tau$ arbitrary —

**Proposition**

Denoting by $\mathcal{F}_{h, \tau}\big|_{n=0}$ the twist in the $n = 0$ case, we have:

$$\mathcal{F}_{h, \tau} = \mathcal{F}_{h, \tau}\big|_{n=0} \prod_{i=0}^{n} G_{i}^i$$

where the $G_{i}^i$’s are mutually commuting (so that their order in the product doesn’t matter) and are given by

$$G_{i}^i := \sum_{k \geq 0} \frac{(-i\hbar c_{h}(E))^{k}}{k!} \sum_{\sigma \in \{\pm\}^{k}} (-1)^{||\sigma||} X^{\sigma_{1}} \ldots X^{\sigma_{k}} \otimes X^{-\sigma_{1}} \ldots X^{-\sigma_{k}},$$

with the shorthand notation $X^{-} = X_i$ and $X^{+} = X_{n+i}$, with $||\sigma||$ the number of $+$ signs in $\sigma$ and with

$$c_{h}(E) := (1 - \hbar^{2}E^{2} \otimes 1)^{-\frac{1}{2}} (1-1 \otimes \hbar^{2}E^{2})^{-\frac{1}{2}}.$$
Deforming the Kac-Takesaki operator of $\mathbb{B}$

The Kac-Takesaki operator of $\mathbb{B}$ is given on 2-point functions by

$$W = (\text{id} \otimes m)(\Delta \otimes \text{id})$$

where $m(\psi)(g) = \psi(g, g)$ extends the multiplication map and $\Delta(\psi)(g, g') = \psi(gg')$.

- The pentagon equation follows from $\Delta(f_1 f_2) = \Delta(f_1)\Delta(f_2)$.
- Idea: deform the product $m$ into $m_\times$ such that $\Delta$ is still an algebra morphism.

$\Rightarrow$ Formally this is obtained by twisting the product.

In the framework of bounded operators we proceed as follows... We define

$$F_{\hbar, \tau} = \int dx_1 dx_2 K_{\hbar, \tau}(x_1, x_2) R_{x_1}^* \otimes R_{x_2}^* \quad \tilde{F}_{\hbar, \tau} = \int dx_1 dx_2 K_{\hbar, \tau}(x_1, x_2)^* L_{x_1}^* \otimes L_{x_2}^*$$

where $(R_{x}^* f)(g) := f(gx)$ and $(L_{x}^* f)(g) := f(xg)$. The operators $F_{\hbar, \tau}$ and $\tilde{F}_{\hbar, \tau}$ are the ‘non-formal analogue’ of $\mathcal{F}_{\hbar, \tau} \triangleright$ and $\triangleleft \mathcal{F}_{\hbar, \tau}^{-1}$, with $\triangleright/\triangleleft$ the left/right canonical actions.

Proposition

$F_{\hbar, \tau}$ and $\tilde{F}_{\hbar, \tau}$ are unitary operators on $L^2(\mathbb{B}, d^R\chi)$ and $L^2(\mathbb{B}, d^L\chi)$ respectively.
A common domain for the operators $F$

From now on assume $\tau(a)^* = -\tau(a) = \tau(-a)$. One of the main technical points:

**Proposition [P. Bieliavsky, V. Gayral]**

Let $T : (a, v, t) \mapsto (\sinh 2a, (\cosh a)^{-1}v, t)$. The Fréchet space $\mathcal{D} := T(S(\mathbb{R}^{2n+2}))$ is stable under left/right translations and $F_{\hbar, \tau}$ and $\tilde{F}_{\hbar, \tau}$ are continuous operators on $\mathcal{D} \hat{\otimes} \mathcal{D}$.

As a corollary, the **doubly twisted product**

$$m_* : \mathcal{D} \hat{\otimes} \mathcal{D} \to \mathcal{D}, \quad m_* = m \circ F_{\theta, \tau} \circ \tilde{F}_{\theta, \sigma(\tau)},$$

is a well defined associative product on $\mathcal{D}$. It satisfies $\Delta(f_1 \star_\hbar f_2) = \Delta(f_1)(\star_\hbar \otimes \star_\hbar)\Delta(f_2)$ where $\Delta : \mathcal{D} \to \mathcal{M}(\mathcal{D} \hat{\otimes} \mathcal{D})$ (bornological quantum group??). The operator

$$W_* := (id \otimes m_*)(\Delta \otimes id)$$

with domain $\mathcal{D} \hat{\otimes} \mathcal{D}$, satisfies the pentagon equation.

Is it bounded? (unitary?) \(\leadsto\) not on $\mathcal{H} = L^2(\mathbb{B}, d^R \chi)$!
On the unitarity of $W_*$

Lemma

The operator $W_*$ can be rewritten as

\[
W_* = (1 \otimes \chi^{1/2}) Y_{h, \tau}^\dagger (1 \otimes \chi^{-1/2}) W F_{h, \tau},
\]

where $W$ is the (undeformed) Kac-Takesaki operator of $\mathbb{B}$, $\chi$ is the modular function and

\[
Y_{h, \tau} := \int dx_1 dx_2 K_{h, \tau}(x_1, x_2) R_{x_2}^* \otimes \chi(x_1)^{1/2} L_{x_1}^*.
\]

Remarks:

- both $R_x^*$ and $x \mapsto \chi(x)^{1/2} L_x^*$ are unitary left actions of $\mathbb{B}$ on $L^2(\mathbb{B}, d^2 x)$;
- $W$, $F_{h, \tau}$ and $Y_{h, \tau}$ in (*) are unitary on $L^2(\mathbb{B}, d^2 x)$ but $\chi$ is not.
- a minimal modification of $W_*$ in order to get a unitary operator is $U_* := Y_{h, \tau}^\dagger W F_{h, \tau}$
  \[\sim \Rightarrow\] $U_*$ does not satisfy the pentagon equation.

$\Rightarrow$ change Hilbert space!
Modifying the Hilbert space

Definition/Proposition

Let $\langle \cdot, \cdot \rangle_\star : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ be the sesquilinear form:

$$\langle \psi, \varphi \rangle_\star = \int m_\star (\overline{\psi}, \varphi)(x) \, d^R x ,$$

$\langle \psi, \varphi \rangle = \int \overline{\psi}(x) \varphi(x) d^R x$ the $L^2$-inner product and $P_\hbar$ the pseudo-differential operator

$$P_\hbar := (D + \sqrt{1 + D^2})^{(n+1)/2}$$

with $D := -i\hbar \partial / \partial t$. Then

i) $\langle \psi, \varphi \rangle_\star = \langle P_\hbar \psi, P_\hbar \varphi \rangle$;

ii) $\langle \cdot, \cdot \rangle_\star$ is an inner product ($P_\hbar$ is positive and invertible);

iii) $W_\star$ extends to a unitary operator on $\mathcal{H}_\star \otimes \mathcal{H}_\star$, where $\mathcal{H}_\star$ is the Hilbert space completion of $\mathcal{D}$ w.r.t. the inner product $\langle \cdot, \cdot \rangle_\star$.

Remark: $\mathcal{H}_\star = L^2(\mathbb{R}^{2n+1}, e^{2(n+1)\alpha} \, d\alpha d^2 n \nu) \otimes H^{(n+1)/2}(\mathbb{R})$, where $H^s(\mathbb{R})$ is the $s$-th Sobolev space in the variable $t$. 
Thank you for your attention.