A NOVEL MODEL FOR THE IMAGES OF FRACTAL PROFILES

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ABSTRACT

In this paper we investigate the characteristics of the images relevant to fractal profiles: in particular, we show that the signal backscattered from a fractal profile modeled as a fractional Brownian motion (fBm) stochastic process is strictly related to the associated fractional Gaussian noise (fGn) process. Moreover, we compute in closed form the structure function and the spectrum of the image, highlighting their key properties and asymptotic behavior.
An experimental validation of the above mentioned results is also provided.

1. INTRODUCTION

A new generation of sensors is providing a great amount of data that potentially could give the possibility to extract very valuable information. In particular, as far as geophysical applications are under concern, retrieving from remote sensing data significant parameters relevant to an observed surface is of open issue of key importance. The lack of adequate mathematical models supporting the retrieving is often one of the main problems when facing the development of inversion techniques.
In this paper, we focus on natural landscapes, which are well described by fractal geometry [1]. In particular, an fBm ([1], [2]) model for the surface is used here to describe the observed surface. In the following we deal with the problem of the imaging of fractal profiles, i.e. on the evaluation in closed analytical form of some key properties of the image relevant to a fractal profile. In particular, we will show the need for the introduction of a smoothed process to circumvent the fbm lack of derivatives [1].
In literature the works coping with this issue are sparse and not always so accurate. Most of them refer to the early works by Pentland ([3], [4]), who studied the optical imaging of a fractal surface assuming a lambertian scattering behavior. A first evident limitation of the work by Pentland is the choice of this particular scattering behavior that is surely not adequate in every situation: for example, has been demonstrated that at microwave frequencies the scattering from natural surfaces is definitely not Lambertian-like [2]. Then, in this paper we present a more general approach to study the imaging of fractal profiles; this approach do not rely on any particular scattering model: results here presented apply to any scattering model provided that the scattering behavior, in the case of interest, can be considered to be linear as a function of the derivative of the profile. Hence, in Section 2 we briefly describe Pentland model.
In this paper we focus on the (Euclidean) one-dimensional problem. In fact, the extension to the two-dimensional case is not straightforward, involving isotropy and depolarization issues. Hence, in Section 3 we briefly describe the fractal models used throughout the paper.
In Section 4, assuming the validity of the Pentland model, we present some new relevant results on the characteristics of the radar image relevant to a fractal (fBm) profile. In particular, we show how the image can be related to the fractional Gaussian noise (fGn) process [5] associated with the fBm process used to describe the profile of interest. Main implications on the extraction techniques of the profile fractal parameters from the image are also stressed.
In Section 5 we provide relevant simulation results, obtained by using the elaboration chain recently presented in literature by the authors [6]; the simulations fully support the analytical conclusions drawn in the previous section.

2. PENTLAND MODEL

In Refs. [3] and [4], Pentland copes with the problem of the imaging of fBm surfaces. His approach is based on a linear approximation of the radar cross section as a function of the partial derivatives of the surface. His work consists in the evaluation of the spectrum of the image, assuming a particular irradiation diagram (i.e., the Lambertian one) for the considered surface.
We already noted that that the considered scattering behavior is not always adequate to describe the electromagnetic scattering from the considered surface: for instance, theoretical and experimental results ([2], [7]) show that at microwave frequencies the scattering from natural surfaces is definitely not Lambertian-like.
However, the main weakness of Pentland analysis is in the fact that he formally works with the partial derivatives of the surface, despite the non-differentiability of the mathematical fBm. As a matter of fact, the features of the scene relevant to spatial scales much smaller than the wavelength one do not contribute to the scattering phenomenon: in this sense, the electromagnetic incident field acts as a low-pass filter on the surface. Furthermore, scales smaller and greater than the resolution one contribute in different ways to the formation

of the image: anyway, this crucial issue deserve the maximum attention and is then fully clarified in the following sections. Note that this issue is strongly underestimated in all the existing literature on the subject [8], [9].

3. FRACTAL MODELS

In this paper we use an fBm stochastic process to model the surface. The fBm is defined in terms of the probability density function of its height increments. A stochastic process $z(x)$ is an fBm profile if, for every x and x' it satisfies the following relation:

$$\Pr\left\{z(x)-z(x')<\bar{\zeta}\right\}=\frac{1}{\sqrt{2\pi}s\tau^H}\int_{-\infty}^{\bar{\zeta}}\exp\left(-\frac{\zeta^2}{2s^2\tau^{2H}}\right)d\zeta \quad (3.1)$$

where $\tau = x - x'$, H is the *Hurst coefficient* ($0 < H < 1$), related to the fractal dimension D by means of the relation $D = 2 - H$, and s [$\mathrm{m}^{(1-H)}$] is the standard deviation of the profile increments at unitary distance, related to a characteristic length of the surface, the topothesy T, by means of the relation $s = T^{(1-H)}$.

The structure function (whose plot is termed the variogram), $V(\tau)$, is defined as the mean square increment of elevation points placed at distance τ and for an fBm profile can be evaluated in terms of the parameters H and s as:

$$V(\tau)=s^2\tau^{2H} \quad (3.2)$$

It exhibits in a log – log plane a linear behavior with slope $2H$, and intercept with the vertical axis in $2\log s$ [1], [2].

It has been demonstrated ([1], [2]) that the power density spectrum $S(k)$ of an isotropic fBm one-dimensional process exhibits appropriate power-law behaviors provided by:

$$S(k)=S_o k^{-(1+2H)} \quad (3.3)$$

wherein:

$$S_0 = s^2\frac{\pi H}{\cos(\pi H)}\frac{1}{\Gamma(1-2H)} \quad (3.4)$$

$\Gamma(\cdot)$ being the Gamma function. Note that also the fBm spectral dependence (3.5) provides a linear plot in a $\log(S)$ – $\log(k)$ plane.

In the previous section we have shown that to employ Pentland approach we need an expression for the derivative of the surface. The fractional Gaussian noise (fGn) is defined as the derivative process of the fBm: the mathematical fBm is strictly non-differentiable, implying that its derivative process has to be handled with care. The most elementary way to deal with the fBm lack of derivative is to smooth the original process, discarding the high frequency effects responsible for the non-differentiability of the fBm [5]. Hence, starting from the standard fBm process $z(x)$, we build the random function $z_\varphi(x)$:

$$z_\varphi(x)=\int_{-\infty}^{\infty} z(x')\varphi(x-x')dx' \quad (3.5)$$

with

$$\varphi(x)=\begin{cases}\dfrac{1}{\varepsilon} & \text{if } x\in[0,\varepsilon]\\ 0 & \text{otherwise}\end{cases} \quad (3.6)$$

in which $\varepsilon > 0$.

Thanks to our particular choice we obtain a simple expression for the derivative of the smoothed process and we can easily evaluate its autocorrelation function:

$$R_{z'}(\tau;\varepsilon)=\frac{1}{2}s^2\varepsilon^{2H-2}\left[\left(\frac{|\tau|}{\varepsilon}+1\right)^{2H}-2\left|\frac{\tau}{\varepsilon}\right|^{2H}+\left|\frac{|\tau|}{\varepsilon}-1\right|^{2H}\right] \quad (3.7)$$

In the limit of $\tau >> \varepsilon$ we obtain:

$$R_{z'}(\tau)\cong s^2H(2H-1)|\tau|^{2H-2} \quad (3.8)$$

We can easily evaluate also the structure function of this process, being it a stationary one:

$$V_{z'}(\tau;\varepsilon)=s^2\varepsilon^{2H-2}\left[2-\left(\frac{|\tau|}{\varepsilon}+1\right)^{2H}+2\left|\frac{\tau}{\varepsilon}\right|^{2H}-\left|\frac{|\tau|}{\varepsilon}-1\right|^{2H}\right] \quad (3.9)$$

And in the limit of $\tau >> \varepsilon$:

$$V_{z'}(\tau;\varepsilon)=2s^2\left[\varepsilon^{2H-2}-H(2H-1)|\tau|^{2H-2}\right] \quad (3.10)$$

Being this process stationary, we can evaluate its spectrum by means of the Wiener-Kintchine theorem and express it as:

$$S_{z'}(k;\varepsilon)=2s^2\varepsilon^{-2}\Gamma(1+2H)\sin(\pi H)\left(1-\cos(k\varepsilon)\right)\frac{1}{|k|^{1+2H}}$$

(3.11)

In the limit of $k\varepsilon << 1$, Eq. (3.11) takes the relevant form:

$$S_{z'}(k)=s^2\Gamma(1+2H)\sin(\pi H)\frac{1}{|k|^{2H-1}} \quad (3.12)$$

From Eq. (3.13) follows that the fGn process is not a fractal process, because $H \notin [0,1]$; but this process presents the interesting property of being an asymptotically power law process with exponent $1 - 2H$.
It is worth noting that our asymptotic results are equal to those obtained by Pentland, due to the fact that he worked with $\varepsilon = 0$.

4. FRACTAL MODEL FOR THE IMAGE

The imaging process can be seen as a system whose input is the surface profile, and whose output is the radar image which is proportional, to the first order, to the derivative of the surface profile, as we have seen in Section 2. Hence, in the following we assume a first order series expansion of the image for small values of the derivative of the considered profile. Combining the results achieved in the previous section and setting $\varepsilon = \Delta x$, where Δx is the system resolution, we can compute the spectrum and structure function of the image as:

$$S_I(k;\varepsilon) = 2a_1^2 s^2 \varepsilon^{-2} \Gamma(1+2H)\sin(\pi H)\left(1-\cos(k\varepsilon)\right)\frac{1}{|k|^{1+2H}}$$

(4.1)

$$V_I(\tau;\varepsilon) = a_1^2 s^2 \varepsilon^{2H-2}\left[2-\left(\left|\frac{\tau}{\varepsilon}+1\right|^{2H}+2\left(\frac{\tau}{\varepsilon}\right)^{2H}-\left|\frac{\tau}{\varepsilon}-1\right|^{2H}\right)\right]$$

(4.2)

where a_1 is the coefficient relevant to the first order term in the above mentioned image series expansion.
It is evident that fractal parameters retrieval based on a linear regression on the log-log variogram is no longer possible. Conversely, in the asymptotic limit it is still possible to use the image to estimate these parameters by a linear regression on the log- log plot of the spectrum.

5. EXPERIMENTAL RESULTS

In this Section we provide an experimental validation of our analytical results. This validation is obtained availing of the processing chain recently presented by the authors [6].
Accordingly, we generated a fractal profile of known fractal dimension and imaged it through an SPM scattering model [2]. Hence we evaluated the power spectrum of the received signal, comparing it to the theoretical behavior (4.1). We had to pay particular care in estimating the spectra both of the surface and of its image: we used a Capon filter to circumvent leakage and high variance problems [10]. The obtained results are presented in Fig. 1.

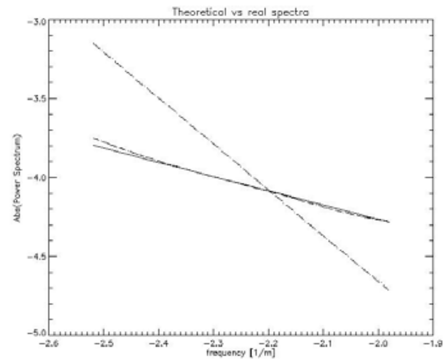

Fig. 1 Comparison between theoretical spectra of the surface (long dashed) and of the image (solid) and experimental ones (dotted and dash dot dot, respectively): H=0.95; s=0.01 m^{1-H}; Δx=5 m.

7. REFERENCES

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