Correlation of the Fields Scattered by a Fractal Surface at Two Closely Spaced Receivers

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***Abstract*— Natural surfaces show scale invariance statistical properties over a wide range of scales, and exhibit power-law spectra over a wide range of spatial frequencies: these properties are well modelled by fractional Brownian motion (fBm) two-dimensional processes. We here present a closed form expression of the correlation coefficient of the fields scattered by a fBm surface and measured at two closely spaced positions. The obtained formulation shows that the correlation coefficient in the near-specular scattering case depends on a parameter that is related to the rms surface slope measured at the electromagnetic wavelength scale: when this parameter decreases, the correlation coefficient smoothly increases from the value obtained by the roughness-independent expression already available in literature to a value close to unity.**

I. Introduction

Evaluation of the correlation coefficient of the fields scattered by a natural surface and measured at two closely spaced points is of great interest in monostatic and bistatic Synthetic Aperture Radar (SAR) interferometry [1] and in Global Navigation Satellite Systems Reflectometry (GNSS-R) [2]. In the far-from-specular scattering direction case, the correlation coefficient dependence on surface roughness is negligible, and the classical expression of [1-2], only dependent on radar parameters and observation geometry, can be used [3]. However, it has been recently shown [3-5] that in the near specular case, of interest for GNSS-R applications (in this case, the two points are the positions occupied by the moving receiver at two slightly different times), the correlation coefficient also depends on surface roughness, and it increases from the value obtained with the classical expression for very rough surfaces to a value close to unity for gently undulating surfaces. In [3-5], the randomly rough scattering surface is described in terms of its rms height standard deviation σ and correlation length L, and its autocorrelation function is assumed Gaussian. In [5], non-Gaussian autocorrelation is also accounted for, provided that it is regular with its derivatives in the origin, in which case the surface is described in terms of its rms slope. However, experimental data show that, for soil surfaces, measured values of σ and L increase with the length of the considered height profile (see, e.g., [6]); and it is well known that measured rms slope of a sea surface increases with the maximum spatial frequency that is considered to estimate it. Accordingly, σ, L and rms slope are not well suitable to characterize the roughness of natural surfaces. This is related to the fact that soil and sea surfaces exhibit power-law spectra over a wide range of spatial frequencies and show scale invariance statistical properties over a wide range of scales. Both features are well modeled by using fractal geometry, and in particular by using fractional Brownian motion (fBm) two-dimensional processes [7]. Therefore, in this work we extend the result of [5] to the case of fBm surfaces.

II. Surface model

A two-dimensional (2D) fBm is a random process $z(x,y)$ whose increments $z(x,y) - z(x',y')$ over a fixed horizontal distance $\Delta = \sqrt{(x-x')^2 + (y-y')^2}$ are zero-mean Gaussian random variables with variance $Q(\Delta) = s^2\Delta^{2H}$ (so that $Q(\Delta)$ is the fBm structure function), where s is a parameter measured in m^{1-H}, and H is the Hurst coefficient, with $0<H<1$ [7-8]. Realizations of the 2D fBm process are fractal surfaces with fractal dimension $D=3-H$.

The fBm process is statistically non-stationary, with stationary increments, and has infinite variance [7-8]. However, real natural surfaces only obey the fBm definition up to an outer scale l that may be their linear size or, for a sea surface, the dominant wavelength. Such *physical* (or *bandlimited*) fBm random surfaces are statistically stationary [7-8] and have finite variance $\sigma^2 = \frac{1}{2}s^2 l^{2H}$, so that we can write, for $\Delta \le l$,

$$Q(\Delta) = 2\sigma^2[1 - C(\Delta)] = s^2\Delta^{2H} \quad , \tag{1}$$

where $C(\Delta)$ is the surface normalized autocorrelation function. We want to compute the correlation of GNSS-R signals for such surfaces. It is clear from (1) that $C(\Delta)$ is not Gaussian, and that its second derivative $C''(\Delta)$ diverges in the origin, so that results of [3-5] cannot be used. The needed new, original evaluation is described in the next Section.

III. Evaluation of the Correlation Coefficient

The geometry of the problem is illustrated in Fig.1, where T is the position of the transmitter, R_1 and R_2 are the positions of the receiver at times t and $t+\Delta t$, respectively, the origin O is the specular point at time t and all other symbols are defined in [5]. The correlation coefficient of the fields is defined as

$$\rho(\Delta t) = \frac{|\mathrm{cov}[E(\mathrm{R}_1)E(\mathrm{R}_2)]|}{\sqrt{\mathrm{var}[E(\mathrm{R}_1)]\mathrm{var}[E(\mathrm{R}_2)]}} \quad , \tag{2}$$

where $E(\mathrm{R}_1)$ and $E(\mathrm{R}_2)$ are the generic component of the electric fields at R_1 and R_2. By following the same procedure as in [5] we get, see eqs. (10) and (12) of [5],

$$\mathrm{cov}[E(\mathrm{R}_1)E(\mathrm{R}_2)] \cong$$
$$\cong \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\exp\{-jk[R_T(x,y)+R_{R1}(x,y)-R_T(x',y')-R_{R2}(x',y')]\}}{R_{R1}(x,y)R_{R2}(x',y')} \cdot \tag{3}$$
$$F(x,y)w(x,y)F^*(x',y')w(x',y')f(x,y,x',y')dxdydx'dy'$$

where $$w(x,y) = \exp\left(-\frac{x^2}{2A_x^2} - \frac{y^2}{2A_y^2}\right) \tag{4}$$

is the sensor illumination function, peaked around the origin, so that A_x and A_y are the x and y sensor resolutions,

$$f(x,y,x',y') = \exp\left\{-\frac{k^2\sigma^2}{2}[u_{z1}(x,y) - u_{z2}(x',y')]^2\right\}$$
$$\left[\exp\left\{-\frac{k^2 u_{z1}(x,y) u_{z2}(x',y')}{2} Q(\Delta)\right\} - \exp\{-k^2\sigma^2 u_{z1}(x,y) u_{z2}(x',y')\}\right] \cong \quad (5)$$
$$\cong \exp\left\{-\frac{k^2\sigma^2}{2}[u_{z1}(x,y) - u_{z2}(x',y')]^2\right\} \exp\left\{-\frac{k^2 s^2 \Delta^{2H} u_{z1}(x,y) u_{z2}(x',y')}{2}\right\}$$

and all other symbols are defined in [5]. In (5) we have used (1) and have assumed $k^2\sigma^2 = \frac{1}{2}k^2 s^2 l^{2H} \gg 1$ (i.e., σ larger than the electromagnetic wavelength λ, that is about 20 cm for GNSS-R). Proceeding again as in [5] we get

$$\mathrm{cov}[E(\mathrm{R}_1)E(\mathrm{R}_2)] \cong \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|F(x,y)|^2 w^2(x,y) \exp\{-jk[R_{R1}(x,y) - R_{R2}(x,y)]\}}{R_{R1}(x,y) R_{R2}(x,y)}$$
$$\exp\left\{-\frac{k^2\sigma^2}{2}[u_{z1}(x,y) - u_{z2}(x,y)]^2\right\} \quad (6)$$
$$2\pi \int_0^\infty \mathrm{J}_0[k u_{xy}(x,y)\Delta] \exp\left\{-\frac{k^2 s^2 \Delta^{2H} u_z^2(x,y)}{2}\right\} \Delta\, d\Delta\, dx dy \quad ,$$

where $u_{xy} = \sqrt{\left[\frac{\partial(R_T + R_{R1})}{\partial x}\right]^2 + \left[\frac{\partial(R_T + R_{R1})}{\partial y}\right]^2}$, $u_z = \sqrt{u_{z1} u_{z2}}$, and J_0 is the zero-order Bessel function. The integral over Δ in (6) can be expressed via a series expansion around u_{xy}=0 (which is the value of u_{xy} at the specular point, i.e., in the origin) [8]. By arresting the expansion at the second order, we get

$$\int_0^\infty \mathrm{J}_0[k u_{xy}(x,y)\Delta] \exp\left\{-\frac{k^2 s^2 \Delta^{2H} u_z^2(x,y)}{2}\right\} \Delta\, d\Delta \cong$$
$$\frac{\Gamma(1/H)}{2H\left(\frac{k^2 s^2 u_z^2(x,y)}{2}\right)^{1/H}} \left(1 - \frac{\Gamma(2/H)\, k^2 u_{xy}^2(x,y)}{4\,\Gamma(1/H)\left(\frac{k^2 s^2 u_z^2(x,y)}{2}\right)^{1/H}}\right) \cong$$
$$\frac{\Gamma(1/H)}{2H\left(\frac{k^2 s^2 u_z^2(x,y)}{2}\right)^{1/H}} \exp\left(-\frac{\Gamma(2/H)\, k^2 u_{xy}^2(x,y)}{4\,\Gamma(1/H)\left(\frac{k^2 s^2 u_z^2(x,y)}{2}\right)^{1/H}}\right) \quad , \quad (7)$$

where Γ is the gamma function. The exponential function in (7) is peaked around the origin as $w^2(x,y)$, and its width, to be compared with the sensor resolution, can be evaluated by expanding the exponent around the origin: by using the same approach as in [5], we get

$$\exp\left(-\frac{\Gamma(2/H)\, k^2 u_{xy}^2(x,y)}{4\,\Gamma(1/H)\left(\frac{k^2 s^2 u_z^2(x,y)}{2}\right)^{1/H}}\right) \cong \exp\left(-\frac{x^2}{G_x^2} - \frac{y^2}{G_y^2}\right) \quad , \quad (8)$$

where
$$G_x = \sqrt{\frac{\Gamma(1/H)}{\Gamma(2/H)}} \left(\sqrt{2} k^{1-H} s \cos^{1-2H}\vartheta_0\right)^{1/H} 2 r_{R1}$$
$$G_y = \sqrt{\frac{\Gamma(1/H)}{\Gamma(2/H)}} \left(\sqrt{2} k^{1-H} s \cos\vartheta_0\right)^{1/H} 2 r_{R1} \quad , \quad (9)$$

with r_{R1} being the receiver distance, see Fig. 1. It can be verified that $k^{1-H}s \sim s\lambda^H/\lambda$, which is the rms slope at the wavelength scale. From this point on, results of [5] apply, in which (9) replaces (23) of [5]. Accordingly, we find

$$\rho(\Delta t) \cong \exp\left\{-\frac{k^2 \Delta t^2 [W_x^2 \cos^4\vartheta_0 v_x^2 + W_y^2 v_y^2]}{4 r_{R1}^2}\right\} \quad (10)$$

with $W_{x,y} = A_{x,y} G_{x,y} / \sqrt{A_{x,y}^2 + G_{x,y}^2}$ and v_x,v_y being the x and y components of the receiver's velocity v.
For $A_{x,y} \ll G_{x,y}$, we get $W_{x,y} \cong A_{x,y}$ and

$$\rho(\Delta t) \cong \exp\left\{-\frac{\pi^2 \Delta t^2 [A_x^2 \cos^4\vartheta_0 v_x^2 + A_y^2 v_y^2]}{\lambda^2 r_{R1}^2}\right\} \quad , \quad (11)$$

which is the classical solution of [2]. Conversely, for $G_{x,y} \ll A_{x,y}$, we have $W_{x,y} \cong G_{x,y}$ and, using (9) in (10),

$$\rho(\Delta t) \cong \exp\left\{-\frac{\Gamma(1/H)}{\Gamma(2/H)} (2k^2 s^2 \cos^2\vartheta_0)^{1/H} 4\Delta t^2 v^2\right\} \quad . \quad (12)$$

Eq.(12) reduces to the result of [5] for $H\to 1$, with s assuming the meaning of rms slope.

By fitting the rms height-vs-patch size data for the Z4 site of [6] (flat soil) we get H=0.93, s=0.06: for the high-altitude airborne system of [5] and $\vartheta_0 = 30°$ this corresponds to $G_x = 1975$ m and $G_y = 1711$ m, which are larger than resolution, so that (11) applies; for $\Delta t = 7$ ms we get ρ=0.33.

For the Z1 site of [6] (very flat soil) we get H=0.83, s=0.01: for the high-altitude airborne system of [5] and $\vartheta_0 = 30°$ this corresponds to $G_x = 230$ m and $G_y = 199$ m, which are smaller than resolution, so that in this case (12) applies; for $\Delta t = 7$ ms we get ρ=0.41.

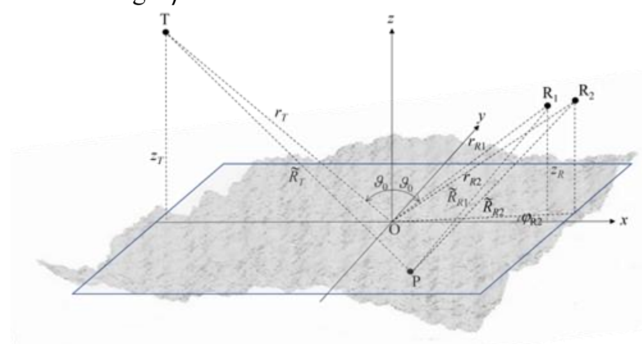

Figure 1. Geometry of the problem. We assume $r_T \gg r_{R1,2}$

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