Shape Optimization, Geometric Inequalities, and Related Topics

Two days workshop for young researchers in Naples,
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Abstracts

Some new bounds for a *mean-to-max* problem involving the torsion function

Luca Briani

Università di Pisa

Given a domain $\Omega \subset \mathbb{R}^d$ with finite measure the torsion function of Ω is defined to be the unique solution $w_{p,\Omega}$ to the following boundary value problem:

$$-\Delta_p w = 1, \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega.$$

In this seminar I will discuss the optimization problems for the *mean-to-max* shape functional

$$\Omega \mapsto \|w_{p,\Omega}\|^{-1}_{L^\infty(\Omega)} \frac{\int_\Omega w_{p,\Omega}(x)dx}{|\Omega|}.$$

The latter is known in the literature as the *efficiency* of the torsion function.

Most of the known results for the linear case extend without difficulties to the case when $1 < p \leq d$ and I will focus on the case $p > d$ where the results abruptly change.

In the extremal case $p = +\infty$ I will show the maximality of an honeycomb-type structure.

At last I will describe some applications to the problems of comparing the torsional rigidity $T_p(\Omega)$ with the first eigenvalue $\lambda_p(\Omega)$ of the $p-$Laplace operator.

Comparison results for a nonlocal singular elliptic problem

Ida de Bonis

Università degli studi di Roma La Sapienza

We provide symmetrization results in the form of mass concentration comparisons for fractional singular elliptic equations in bounded domains, coupled with homogeneous external Dirichlet conditions. We are referring to the following problem

$$(2) \quad \begin{cases} (-\Delta)^s u = \frac{f(x)}{u^\gamma} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where Ω is a bounded open set in $\mathbb{R}^N$ $(N \geq 2)$, $\gamma > 0$ and f is a positive summable function. The operator which appears in the left hand side is the fractional Laplacian operator.
Two types of comparison results are presented, depending on the summability of the right-hand side of the equation. The maximum principle arguments employed in the core of the proofs of the main results offer a nonstandard, flexible alternative to the ones described in Theorem 3.1 of [2]. Some interesting consequences are L^p regularity results and nonlocal energy estimates for solutions.
These results are contained in the recent paper [1].

Bibliography

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Capacities, outward minimizing sets and curvature flows

Mattia Fogagnolo

Università degli studi di Padova

In this talk, I will discuss some results concerning the p-capacity of a bounded set and the area of its minimal envelope. Huisken-Ilmanen's notion of weak Inverse Mean Curvature Flow will naturally emerge. I will present some geometric applications, and highlight the strength of these notions in Riemannian geometry.

Rearrangement of Gradient

Andrea Gentile

Scuola Superiore Meridionale

Symmetrization is a well-known technique in mathematical analysis: starting from a measurable function u defined on an open set $\Omega \subset \mathbb{R}^n$ of finite measure, this procedure allows to construct a radially symmetric function, $u^\sharp$, defined on the ball $\Omega^\sharp$ having same measure as Ω. This function $u^\sharp$ is called the *Schwarz rearrangement* of u.

Here we want to present a symmetrization technique that involves the gradient of the function. In particular we extend the work [2] in which the authors are able to prove some comparison results between the L^p norm of a function u and a function $u^\star$ obtained suitably symmetrizing the gradient of the function u.

We generalize this result firstly to non zero trace functions [1] and then to BV functions (in preparation).

This is a joint work with Vincenzo Amato, Carlo Nitsch and Cristina Trombetti.

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Two variational problems coming from thermal insulation

Domenico Angelo La Manna

Università degli Studi di Napoli Federico II

We present two variational models related to thermal insulation problem, which are the result of two papers in collaboration with Bozhidar Velichkov and Serena Guarino Lo Bianco. In the first part of the talk we discuss some results obtained in [1]. We study the minimization problem

$$\inf_{(u\Omega)\in\mathcal{V}\times\mathcal{E}}\left\{\int_D |\nabla u|^2\,dx+\beta\int_{\partial^*\Omega} u^2\,d\mathcal{H}^{d-1}\right\}, \tag{1}$$

where $\beta>0$ and

$$\mathcal{V}=\left\{u\in H^1_{loc}(\mathbb{R}^d)\ :\ u-v\in H^1_0(D)\right\},$$
$$\mathcal{E}=\left\{\Omega\subset\mathbb{R}^d\ :\ \mathrm{Per}(\Omega)<+\infty\ \text{ and }\ \Omega=E\ \text{ in }\ \mathbb{R}^d\setminus D\right\}.$$

We show that the problem (1) admits a solution and we also discuss about the regularity of solutions. In the second part of the talk we present some new results contained in [2]. We analize a variational problem which models the thermal insulation of two disjoint rooms. To be more precise, we fix two smooth sets E_1, E_2 with positive distance and a function $g\in H^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ be a non-negative function such that $g\equiv 1 \quad \text{on} \quad E_1\cup E_2$. We provide existence and regularity for solutions of the problem

$$\inf_{(u,\Omega_1,\Omega_2)\in\mathcal{A}}\left\{\int_D |\nabla u|^2\,dx+\beta\int_{\partial\Omega_1\cap\partial\Omega_2} u^2\,d\mathcal{H}^1+\Lambda|\{u>0\}\cap D|\right\} \tag{2}$$

where $(u,\Omega_1,\Omega_2)\in\mathcal{A}$ if and only if

- $u-g\in H^1_0(D)$ with $D:=\mathbb{R}^2\setminus\left(\overline{E}_1\cup\overline{E}_2\right)$;
- Ω_i is a set of finite perimeter and $\Omega_i=E_i$ in $\mathbb{R}^2\setminus D$ for $i=1,2$.
- $\{u>0\}\subset\Omega_1\cup\Omega_2$.

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Numerical optimization of domain on the sphere

Eloi Martinet
Université Savoie Mont Blanc

We consider the numerical optimization of the first lowest eigenvalues of the Laplace-Beltrami operator of domains on the sphere with Neumann boundary conditions. We address two approaches : one is a shape optimization procedure via the level-set method and the other one is a relaxation of the initial problem leading to a density method. These computations give some strong insight on the optimal shapes of those eigenvalue problems and show a rich variety of shapes regarding the proportion of the surface area of the sphere occupied by the domain.

Rigidity results for the Robin $p-$Laplacian

Alba Lia Masiello
Università degli Studi di Napoli Federico II

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded, open and Lipschitz set and let f be a positive function. We consider the following problem

$$\begin{cases} -\Delta_p u := -\text{div}(|\nabla u|^{p-2}\nabla u) = f & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta |u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where ν is the unit exterior normal to $\partial\Omega$ and $\beta > 0$, and its symmetrized version

$$\begin{cases} -\Delta_p v = f^\sharp & \text{in } \Omega^\sharp \\ |\nabla v|^{p-2}\dfrac{\partial v}{\partial \nu} + \beta |v|^{p-2}v = 0 & \text{on } \partial\Omega^\sharp, \end{cases} \tag{4}$$

where $\Omega^\sharp$ is the ball centered at the origin with the same measure of Ω.

In $[1, 2]$ the authors prove a comparison á la Talenti between the solutions to equations (3) and (4). In particular, they prove

$$||u||_{L^{pk,p}(\Omega)} \leq ||v||_{L^{pk,p}(\Omega^\sharp)}, \qquad \forall\, 0 < k \leq \frac{n(p-1)}{(n-2)p+n}, \tag{5}$$

and in the case $f \equiv 1$, they prove

$$||u||_{L^{pk,p}(\Omega)} \leq ||v||_{L^{pk,p}(\Omega^\sharp)}, \qquad \forall\, 0 < k \leq \frac{n(p-1)}{(n-2)p+n}, \quad \forall p > 1, \tag{6}$$

where $||\cdot||_{k,q}$ is the Lorentz norm of a measurable function.

In this seminar, we are interested in characterizing the equality cases in (5) and (6), proving that these estimates are rigid, i.e. the equality case can occur only in the symmetric setting.

This is a joint work with Gloria Paoli.

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Optimal bounds for Neumann eigenvalues in terms of the diameter

Marco Michetti

Université Paris-Saclay

In this talk we study the maximization problem of the Neumann eigenvalues under diameter constraint in an "optimal" class of domains.

We define the profile function f associated to a domain $\Omega \subset \mathbb{R}^d$ (defined as the $d-1$ dimensional measure of the slices orthogonal to a diameter), assuming that this function is β-concave we will give sharp upper bounds of the quantity $D(\Omega)^2\mu_k(\Omega)$ in terms of β. These optimal bounds will go to infinity when β goes to zero giving in this way a geometric characterization of domains for which the diameter is fixed, but the Neumann eigenvalues are arbitrarily large. This includes the case of convex domains in $\mathbb{R}^d$, containing and generalizing previous results by P. Kröger. The proof of these results are based on a maximization problem for relaxed Sturm-Liouville eigenvalues.

This talk is based on a joint work with Antoine Henrot

Debonding models: the wave equation on time-dependent domains and related coupled problems

Riccardo Molinarolo

Università degli studi di Firenze

In this talk we revisit some issues about existence and regularity for the wave equation in non-cylindrical domains. Using a method of diffeomorphisms we show, through increasing regularity assumptions, how the existence of weak solutions, their improved regularity and an energy balance can be derived. Possible generalizations of the pde considered will be also mentioned.

In the second part of the talk, as an application, we give a rigorous definition of dynamic energy release rate density for some debonding models, and we formulate a proper notion of solutions for such problems. We will compare the consistence of such a formulation with that of previous ones, given in the literature for some particular cases.

This talk is based on two works in collaborations with G. Lazzaroni, F. Riva, and F. Solombrino (cf. [1, 2]).

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Sharp stability of higher order Dirichlet eigenvalues

Mickaël Nahon

Max Planck Institute Leipzig

Let $\Omega \subset \mathbb{R}^n$ be an open set with same area as the unit ball B and call $\lambda_k(\Omega)$ the k-th eigenvalue of the Laplacian with Dirichlet condition on Ω. Suppose $\lambda_1(\Omega) - \lambda_1(B)$ is small, how large can $|\lambda_k(\Omega) - \lambda_k(B)|$ be? We establish bounds with sharp exponents depending on the multiplicity of $\lambda_k(B)$ through the study of a perturbed shape optimization problem. This is a joint work with Dorin Bucur, Jimmy Lamboley and Raphaël Prunier.

Stability in shape optimization under convexity constraint

Raphael Prunier

Sorbonne Universitè

This talk is concerned with shape optimization, which is the study of minimization problems for which the optimization runs among a certain class $\mathcal{A}$ of subsets of $\mathbb{R}^n$. The additionnal convexity constraint means that we are interested in problems where $\mathcal{A} \subset \{\text{convex subsets of } \mathbb{R}^n\}$. When a ball $B \subset \mathbb{R}^n$ is a minimizer of a functionnal $\Omega \in \mathcal{A} \mapsto J(\Omega)$ (e.g. when J is the perimeter functionnal), one can wonder about its stability in the following way: if one perturbates J into $J + \varepsilon R$ where R is another functionnal and $\varepsilon \to 0$, is B still a (the?) solution of the minimization of $J + \varepsilon R$? In this talk we present a general method – the so-called *selection principle* strategy – very efficient for proving stability of the ball by 1) proving stability for smooth perturbations and 2) proving stability in general using a regularizing procedure. The second step relies on a regularity theory for J, which is where the convexity constraint comes into play since it allows (in certain cases) to obtain more regularity for minimizers. We will present the specific case of the functionnal $P - \varepsilon\lambda_1$ where λ_1 is the first Dirichlet eigenvalue, for which we obtained sharp stability of the ball. This is based on a joint work with J. Lamboley.

On the critical $p-$Laplace equation

Alberto Roncoroni
Politecnico di Milano

The starting point of the seminar is the well-known generalized Lane-Emden equation

$$\Delta_p u + |u|^{q-1}u = 0 \quad \text{in } \mathbb{R}^n\,, \tag{7}$$

where Δ_p is the usual $p-$Laplace operator with $1 < p < n$ and $q > 1$. I will discuss several non-existence and classification results for positive solutions of (7) in the subcritical ($q < p^* - 1$) and in the critical case ($q = p^* - 1$). In the critical case, it has been recently shown, exploiting the moving planes method, that positive solutions to the critical $p-$Laplace equation (i.e. (7) with $q = p^* - 1$) and with finite energy, i.e. such that $u \in L^{p^*}(\mathbb{R}^n)$ and $\nabla u \in L^p(\mathbb{R}^n)$, can be completely classified. In this talk, I will present some recent classification results for positive solutions to the critical $p-$Laplace equation with (possibly) infinite energy satisfying suitable conditions at infinity. Moreover, if time permits I will discuss analogue results in the anisotropic, conical and Riemannian settings.

This is based on a recent joint work with G. Catino and D. Monticelli.

Existence and multiplicity results for some classes of nonlinear differential problems

Angela Sciammetta
Università degli studi di Palermo

The present talk is devoted to the existence and multiplicity of solutions for some classes of nonlinear differential problems.

First of all we analyze the existence of two nontrivial solutions for the following nonlinear elliptic equations driven by an anisotropic Laplacian operator

$$\begin{cases} -\Delta_{\vec{p}} u = \lambda f(x,u) & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_1}$$

where Ω is a nonempty bounded open set of the real Euclidean space $\mathbb{R}^N$, $N \geq 2$, with a boundary of class C^1, $\Delta_{\vec{p}} u = \displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)$ is the anisotropic $p-$Laplacian operator, $\lambda \in \,]0,+\infty[$ and $f : \Omega \times R \to R$ is an L^1-Carathéodory function.

Our main tool is a two critical points theorem established in [3]. Such critical point result is an appropriate combination of the local minimum theorem obtained in [2], with the classical and seminal Ambrosetti-Rabinowitz theorem (see [1]). The functional framework involves the anisotropic Sobolev spaces.

The second part of the talk is dedicated to the existence and location of solutions to the following Neumann problem

$$
\begin{cases} -\mathrm{div}(A(x,\nabla u)) + \alpha(x)|u|^{p-2}u = f(x,u,\nabla u) & \text{in } \Omega, \\ \\ A(x,\nabla u)\cdot \nu(x) = 0 & \text{su } \partial\Omega. \end{cases} \tag{P_2}
$$

Here $1 < p < \infty$, $\Omega \subset \mathbb{R}^N$ is a nonempty bounded domain with boundary $\partial\Omega$ of class $C^{1,\gamma}$, $\gamma \in]0,1[$ and $\nu(x)$ designates the unit outward normal vector to $\partial\Omega$ at each point $x \in \partial\Omega$. In the equation (P_2) we assume that $\alpha \in L^\infty(\Omega)$ with $\alpha \geq 0$, $\alpha \not\equiv 0$ and $\mathrm{div}(A(\cdot,\nabla u))$ denotes the divergence of the composition $A(\cdot,\nabla u)$ corresponding to a continuous map $A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$. It is not required that the divergence term $\mathrm{div}(A(\cdot,\nabla u))$ comes from a potential operator. The right-hand side of the equation (P_2) is in the form of a Nemytskii operator, determined by a Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. We emphasize that function f depends not only on the solution u, but also on its gradient ∇u. Consequently, problem (P_2) does not have a variational structure, which makes any variational method inapplicable. Thereby, we develop a non-variational approach, based on sub and supersolution method (see Theorem 2.99 [5]).

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Non-local BV functions and a denoising model with L^1 fidelity

Giorgio Stefani

SISSA

We consider a total variation denoising model with weighted L^1 fidelity and with regularizing term given by a non-local variation induced by a suitable (non-integrable) kernel K. Our approach relies on the analysis of non-local BV functions with finite total K-variation, with special emphasis on compactness, Lusin-type estimates, Sobolev embeddings and isoperimetric and monotonicity properties. We also link the study of the fidelity in our model with the theory of Cheeger sets in this non-local setting. This is a joint work with K. Bessas.

About maximal distance minimizers

Yana Teplitskaya
Universiteit Leiden

Consider a compact $M \subset \mathbb{R}^d$ and $l > 0$. A maximal distance minimizers problem is to find a connected compact set Σ of the length (one-dimensional Hausdorff measure $\mathcal{H}^1$) at most l that minimizes $\max_{y \in M} \operatorname{dist}(y; \Sigma)$ where dist stands for the Euclidean distance. I will provide a survey on the results (and open problems) concerning the maximal distance minimizers and some related subjects.