# Some new bounds for the mean-to-max ratio of the torsion function.

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Joint work with D. Bucur

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# Outline

# Basic ingredients and motivations

Fix $1 < p < \infty$ and $\Omega \subseteq \mathbb{R}^d$ a **nonempty open set with finite measure** (a *domain*). We call **torsion function of** $\Omega$ the weak solution to:

$$\begin{cases} -\Delta_p w = \operatorname{div}(|\nabla u|^{p-2}\nabla w) = 1 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

**Some properties:**

- By the maximum principle $w_{p,\Omega} > 0$.
- In general we cannot expect more regularity than $C^{1,\alpha}(\Omega)$.
- Exact expressions are available only for $\Omega = B(x_0, r)$.
- By setting $w_{p,\Omega} = 0$ in $\Omega^c$, it holds $w_{p,\Omega} \in W^{1,p}(\mathbb{R}^d)$ and $-\Delta_p w_{p,\Omega} \leq 1$ weakly.

- **Theorem** [Bhattacharya, DiBenedetto, Manfredi '89; Kawohl '90]:

$$w_{p,\Omega} \to d(\cdot, \Omega^c) \text{ uniformly in } \Omega \text{ as } p \to \infty.$$

for every domain $\Omega \subset \mathbb{R}^d$.

Above for $C \subset \mathbb{R}^d$, we denote by $d(\cdot, C)$ the usual **distance function**:

$$d(x, C) = \inf\{|x - y| : \ y \in C\}.$$

Thus we set: $w_{\infty,\Omega}(x) = d(x, \Omega^c)$.

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Thus we set: $w_{\infty,\Omega}(x) = d(x, \Omega^c)$.

- **Elasticity theory**: The torsion function appears in the computation of the angular change when a beam of a given length and a given modulus of rigidity is exposed to a twisting moment.
- **Probability**: The value $w_{2,\Omega}(x)$ equals the expected lifetime of a Brownian motion in $\Omega$ starting at $x \in \Omega$.
- **Heat conduction**: Given an object $\Omega \subset \mathbb{R}^d$, whose boundary is constantly kept at zero temperature, whose initial temperature and mass are fixed, $w_{2,\Omega}(x)$ equals the temperature at $x \in \Omega$ averaged in the whole thermal process.
- **Potential theory**: The geometry of the torsion function is studied to understand localization properties of high order eigenfunctions of different elliptic operators.

For $1 < p \leq \infty$ we define the **mean-to-max ratio**:

$$\Phi_p(\Omega) = \frac{1}{\|w_{p,\Omega}\|_\infty} \fint_\Omega w_{p,\Omega}(x)dx.$$

**GOAL**

Compute or give estimates to the following quantities:

$$\inf / \sup\{\Phi_p(\Omega) : \ \Omega \subset \mathbb{R}^d \text{ domain}\},$$

**Remark**: $\Omega$ varies among *all* the possible domains.

Playing with disjoint balls with decreasing radii, it is easy to construct a sequence $\Omega_n$ for which

$$\Phi_p(\Omega_n) \to 0, \ \text{ as } n \to \infty.$$

Hence we only look at the maximization problem.

Objective

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**Triviality:**

$$\Phi_p(\Omega) = \frac{1}{\|w_{p,\Omega}\|_\infty} \fint_\Omega w_{p,\Omega}(x)dx \leq 1.$$

**Questions:**

- **Q1.** Is the value 1 attained by some sequence of domains?
- **Q2.** Can we characterize the geometry of the maximizing sequences?

A physical motivation

- In many thermo-energetic processes the average temperature of the generator $\Omega$ and the power produced are proportional. A high mean-to-max ratio of the torsion function of $\Omega$ will help to provide adequate average power without exceeding the maximum temperature which can be dictated for instance by the material (e.g. metallurgical considerations). To optimizing this process we can either modify the material or design a generator having an optimal shape.
- The optimization problems for the mean-to-max ratio of other relevant functions (eigenfunctions with different boundary conditions) has been also considered ( Payne-Stakgold, Berg-Bucur, Berg-Bucur-Keppeller, . . . ).

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# Known results

1. Henrot, Lucardesi, Philippin[1] proved the following.

**Theorem (Henrot, Lucardesi, Philippin '18)**

*We have*

$$\sup\{\Phi_2(\Omega) : \ \Omega \subseteq \mathbb{R}^d \ \textit{domain}\} = 1.$$

The proof exploits the relaxation of the class of domains to the set of capacitary measure. In particular, it happens that **any** periodical structures à la Cioranescu-Murat that $\gamma$-converge to a constant multiple of the Lebesgue measure lead to the supremum value 1.

With some difficulties due to the non-linearity of the operator $-\Delta_p$ the proof extends to the case $1 < p \leq d$. Hence we can state the following:

**Proposition**

$$1 < p \leq d \Longrightarrow \sup\{\Phi_p(\Omega) : \Omega \subset \mathbb{R}^d \textit{domain} \} = 1.$$

*Moreover the the value 1 is attained by any periodical structure à la Cioranescu-Murat.*

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2. It is worth mentioning that in [HLP '18] it is also proved that

$$\frac{1}{(d+1)^2} \le \Phi_2(\Omega) \le \frac{2}{3}, \quad \text{for every convex domain.}$$

-Della Pietra, Gavitone,Lo Bianco [2] extended the latter inequality to the nonlinear case.

**Theorem (Della Pietra, Gavitone, Lo Bianco '18)**

*For every convex domain $\Omega \subset \mathbb{R}^d$ it holds*

$$\frac{p'}{d^{p'-1}(d+p')} \le \Phi_p(\Omega) \le \frac{p'}{p'+1}.$$

*The right-hand side of the previous inequality is sharp.*

→The sharp constant for the left hand side is still unknown.

3. Taking the limit for $p \to \infty$ in the inequality above we get

$$\frac{1}{d+1} \le \Phi_\infty(\Omega) \le \frac{1}{2}.$$

for every $\Omega$ convex set. The bounds above are both sharp. The upper bound is still optimal if we consider simply connected domains or even for 1-connected domains ([B., Buttazzo, Prinari 21']).

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# Theorem 1: refining Jensen inequality for the torsion function

**Recall**:

$$\Phi_p(\Omega) = \frac{1}{\|w_{p,\Omega}\|_\infty} \fint_\Omega w_{p,\Omega}(x)dx, \quad \text{for every } p \in (1, \infty].$$

Theorem (L.B., D. Bucur)

*For every $d < p \leq \infty$, there exists a constant $c_{p,d}$ depending on the dimension $d$ and on $p$ such that $c_{p,d} < 1$ and*

$$\Psi_p(\Omega) := \frac{1}{\left(\fint w_{p,\Omega}^p dx\right)^{1/p}} \fint w_{p,\Omega} dx \leq c_{p,d}$$

*for every $\Omega$ domains. In particular, since $\Phi_p(\Omega) \leq \Psi_p(\Omega)$, it holds*

$$\sup\{\Phi_p(\Omega) : \ \Omega \subset \mathbb{R}^d \ \textit{domain}\} < 1.$$

The key Lemma to prove the theorem above is the following one.

**Lemma**

*There exists a constant $C_{d,p}$, depending only on $p$ and $d$, such that for every domain $\Omega \subset \mathbb{R}^d$, if we set*

$$d_0 = \fint_\Omega w_{p,\Omega} dx, \quad E_1 = \{x \in \Omega \ : \ w_{p,\Omega}(x) < d_0/2\}, \quad E_2 = \{x \in \Omega : \ w_{p,\Omega}(x) > 3d_0/2\},$$

*then at least one of the following cases occur:*

$$|E_1| \geq \frac{|\Omega|}{C_{d,p}}, \quad |E_2| \geq \frac{|\Omega|}{C_{d,p}}.$$

**proof of the Lemma, case $p = \infty$:**
Recall that $w_{\infty,\Omega}(x) = d(x, \Omega^c)$.
Apply Vitali covering Lemma to $(B(x, d(x, \Omega^c)))_{x\in\Omega}$, to get

$$\mathcal{F} = \{B_i = B(x_i, r_i) : i \in I\}, \text{ with } r_i = d(x_i, \Omega^c) \text{ such that}$$
$$B_i \cap B_j = \emptyset, \text{ if } i \neq j, \quad \text{ and } \Omega \subset \bigcup_{i\in I} 5B_i.$$

Let $\mathcal{F}_L$ the family of large balls defined through

$$\mathcal{F}_L = \{B_j \in \mathcal{F} \ : r_j > Kd_0\}, \quad d_0 = \fint_\Omega d(x, \Omega^c)dx,$$

where $K > 0$ is a large constant.
**Claim:**

$$\sum_{B_j\in\mathcal{F}_L} |B(x_j, r_j)| \leq \frac{|\Omega|}{5^d 2}, \tag{3.1}$$

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If this is not the case

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\begin{aligned}
d_0|\Omega| &\geq \sum_{B_j \in \mathcal{F}_L} \left( \int_{B(x_j, r_j)} d(x, \Omega^c) dx \right) \\
&\geq \sum_{B_j \in \mathcal{F}_L} \left( \int_{B(x_i, r_i)} (r_i - |x - x_i|) dx \right) \\
&= \sum_{B_j \in \mathcal{F}_L} \frac{\omega_d r_j^{d+1}}{d+1} \geq \frac{K d_0}{d+1} \sum_{B_j \in \mathcal{F}_L} \omega_d r_j^d \\
&= \frac{K d_0}{(d+1)} \sum_{B_j \in \mathcal{F}_L} |B(x_j, r_j)| > \frac{K d_0}{5^d 2(d+1)} |\Omega|.
\end{aligned}
$$

**A contradiction if K is large enough!** Hence the Claim is true. Equivalently

$$
\sum_{B_i \in \mathcal{F} \setminus \mathcal{F}_L} |B(x_i, r_i)| \geq \frac{|\Omega|}{5^d 2}.
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To conclude notice that we can compare

$$\sum_{B_i\in\mathcal{F}\setminus\mathcal{F}_L}|B(x_i,r_i)|\sim\sum_{B_i\in\mathcal{F}\setminus\mathcal{F}_L}|B(x_i,r_i)\cap E_1|,$$

being the radius of the balls in $\mathcal{F}\setminus\mathcal{F}_L$ not too large. The latter, implies the lemma. $\square$

**Proof of theorem.** Fix $\Omega$ a domain and denote $w = w_{p,\Omega}$ and

$$d_0 = \frac{1}{|\Omega|}\int_\Omega w(x)dx, \quad E = \{x \in \Omega \ : \ w(x) < d_0/2\}.$$

The elementary inequality

$$⨍_A f^p(x)dx \geq \left(⨍_A f(x)dx\right)^p + \frac{1}{2^{p-1}-1}⨍_A \left|f(x) - ⨍_A f(y)dy\right|^p dx,$$

applied to the torsion function gives

$$\begin{aligned}&⨍_\Omega w^p(x)dx \geq d_0^p + \frac{1}{2^{p-1}-1}⨍_\Omega |w(x)-d_0|^p\,dx \\ &\geq d_0^p + \frac{1}{|\Omega|(2^{p-1}-1)}\int_E |w(x)-d_0|^p\,dx \geq d_0^p\left(1 + \frac{|E|}{|\Omega|}\frac{1}{2^p(2^{p-1}-1)}\right)\end{aligned}$$

Same if $E = \{x \in \Omega \ : \ w(x) > 3d_0/2\}$. $\square$

# Theorem 2: honeycomb structure

To answer **Q2.** we consider now the case $p = +\infty$, $d = 2$. Recall that

$$\Phi_\infty(\Omega) = \frac{1}{\|d(\cdot, \Omega^c)\|_\infty} \fint_\Omega d(x, \Omega^c) dx.$$

**An easy computation.**
Let $H \subset \mathbb{R}^2$ be the hexagon centered at the origin with unitary side. For every $\varepsilon > 0$ we fix $C_\varepsilon \subset \mathbb{R}^2$ to be a set of points such that the family $\{\varepsilon H + x\}_{x \in C_\varepsilon}$ produces an hexagonal tiling for $\mathbb{R}^2$. We then define $\Omega_\varepsilon$ as

$$\Omega_\varepsilon = B(0,1) \setminus \{x \in C_\varepsilon : \varepsilon H + x \Subset B(0,1)\}.$$



Figure: *Disk perforated with the centers of a fine hexagonal tiling.*

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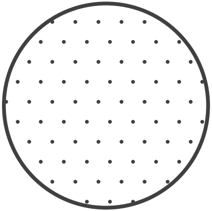

Figure: *Disk perforated with the centers of a fine hexagonal tiling.*

As $\varepsilon \to 0$, the effects of the boundary of $B(0,1)$ become negligible and we have

$$\lim_{\varepsilon \to 0} \Phi_\infty(\Omega_\varepsilon) = \frac{1}{\varepsilon} \left( \frac{|\Omega|}{|\varepsilon H|} \right) \frac{\int_{\varepsilon H} |x| dx}{|\Omega|} = \fint_H |x| dx = \frac{1}{3} + \frac{\ln(3)}{4}.$$

In particular

$$\lim_{\varepsilon \to 0} \Phi(\Omega_\varepsilon) > \sup\{\Phi_\infty(\Omega) \ : \Omega \subset \mathbb{R}^d, \ \textbf{convex domain} \ \} = \frac{1}{2}.$$

Another easy computations shows that if we start with a **square** tiling and we produce the related sequence of perforated domains $\widetilde{\Omega}_\varepsilon$ we get

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**Question:** Is the hexagonal configuration the best one?

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**Theorem (L.B., D. Bucur)**

*We have*

$$\sup\{\Phi_\infty(\Omega) : \Omega \subset \mathbb{R}^2 \mathit{domain}\ \} = \frac{1}{3} + \frac{\ln(3)}{4}.$$

**Remark:** The maximization problem for the efficiency is somehow related to the *optimal compliance conjecture* [G. Buttazzo, F. Santambrogio, N. Varchon '06]:

- Given a smooth bounded domain $\Omega$ and set $r = c\sqrt{n}$ determine the solution to

$$\min\left\{\int_\Omega w_{2,\Omega\setminus\overline{B}_r(x_i)}dx : \quad x_1,\ldots,x_n \in \mathbb{R}^d\right\}.$$

- The optimal configurations of points is conjectured to be asymptotically given by a tiling of the set $\Omega$ with regular hexagons.

The problem above generalize to $p \neq 2$. The case $p = \infty$ lead to the classical *Allocation Problem*. **This is the only known case.**
**Allocation Problem:** given a uniform distribution of consumers in a domain $\Omega$ with unitary measure, the goal is to build $n$ markets $C = \{x_1,\ldots,x_n\} \subset \Omega$, in such a way that the average distance of the consumers from the nearest centers is minimal. Precisely one looks at

$$\min\left\{\int_\Omega d(x,C)dx : \ C \subset \Omega, \#C = n\right\}.$$

When $n$ increases the best configuration is asymptotically given by building the markets in the centers of an hexagonal tiling $\Omega$ as proved by Morgan and Bolton.[3]

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**Remark:** The maximization problem for the efficiency is somehow related to the *optimal compliance conjecture* [G. Buttazzo, F. Santambrogio, N. Varchon '06]:

- Given a smooth bounded domain $\Omega$ and set $r = c\sqrt{n}$ determine the solution to

$$\min\left\{\int_\Omega w_{2,\Omega\setminus\overline{B}_r(x_i)}dx : \quad x_1,\dots,x_n \in \mathbb{R}^d\right\}.$$

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The problem above generalize to $p \neq 2$. The case $p = \infty$ lead to the classical *Allocation Problem*. **This is the only known case.**
**Allocation Problem:** given a uniform distribution of consumers in a domain $\Omega$ with unitary measure, the goal is to build $n$ markets $C = \{x_1,\dots,x_n\} \subset \Omega$, in such a way that the average distance of the consumers from the nearest centers is minimal. Precisely one looks at

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**Morgan and Bolton's proof key point:**

- Given a family of points $\mathcal{S}$, the **Voronoi cell** of a point $p \in \mathcal{S}$ is defined by the planar convex region:
$$V(p) = \{x \in \mathbb{R}^2 \;:\; |x-p| \leq |x-q|, \text{ for every } q \in \mathcal{S}\}.$$
- A distribution of points divide the unit square into polygonal convex regions: the Vornoi cells of the points.
- Regular cells are better: *For every $m \in \mathbb{N}$ $m \geq 2$, among planar m-gons $R \subset \mathbb{R}^d$ of unit area, the average distance from the origin*
$$R \mapsto \fint_R |x|dx$$
*is minimized uniquely by the regular m-gon centered at the origin.*

In our case the term $\|d(\cdot, \Omega^c)\|_\infty$ prevent us to write down good optimality condition for the "Voronoi cells" in any useful way.

## Proof of Theorem 2

**Step 1: Discretization.**
Denote by $\mathcal{Q}_\varepsilon$ the family of all the finite union of closed squares, of size $\varepsilon$ and vertexes in $\varepsilon\mathbb{Z}^2$.
For $Q \in \mathcal{Q}_\varepsilon$ we define its *$\varepsilon$-discrete boundary* as $\partial_{d,\varepsilon} Q = \partial Q \cap \varepsilon\mathbb{Z}^2$.
Let $\Omega_\varepsilon$ the "best" interior approximation of $\Omega$ by elements in $\mathcal{Q}_\varepsilon$.
Define the following quantity:

$$\Phi_{d,\infty}(\Omega_\varepsilon) = \frac{\int_{\Omega_\varepsilon} d(x, \partial_{d,\varepsilon}\Omega_\varepsilon) dx}{|\Omega_\varepsilon| \|d(\cdot, \partial_{d,\varepsilon}\Omega_\varepsilon)\|_{L^\infty(\Omega_\varepsilon)}}.$$

$\rightarrow$ Is enough to prove that, for $\varepsilon$ small enough it holds

$$\Phi_{d,\infty}(\Omega_\varepsilon) \leq \frac{1}{3} + \frac{\ln(4)}{3}.$$

**Step 2: Triangulation** Connecting points of $\partial_{d,\varepsilon}\Omega_\varepsilon$ with a triangulation (denoting by $\mathcal{F}$ the family of the triangles contained in $\Omega_\varepsilon$) we have

$$\Phi_{d,\infty}(\Omega_\varepsilon) = \frac{\sum_{\Delta\in\mathcal{F}} \int_\Delta d(x, \partial_{d,\varepsilon}\Omega_\varepsilon)dx}{\left(\sum_{\Delta\in\mathcal{F}} |\Delta|\right) \|d(\cdot, \partial_{d,\varepsilon}\Omega_\varepsilon)\|_{L^\infty(\Omega_\varepsilon)}}$$

Now, as in the proof of Morgan and Bolton we look at an isoperimetric-type inequality for triangles.

Given a triangle $\Delta \subset \mathbb{R}^2$, denote by $r(\Delta)$ the **circumradius** and by $V(\Delta)$ the set of vertexes. Define the following quantity:

$$E(\Delta) = \frac{\int_\Delta d(x, V(\Delta))dx}{|\Delta| r(\Delta)}.$$

**Lemma**

*For every triangle $\Delta \subset \mathbb{R}^2$ we have*

$$E(\Delta) \leq E(\Delta_{eq}) = \frac{1}{3} + \frac{\ln(3)}{4}, \tag{4.1}$$

*where $\Delta_{eq} \subset \mathbb{R}^2$ is any equilateral triangle.*

**Step 3: Conclusion**

$$\Phi_{d,\infty}(\Omega_\varepsilon) = \frac{\sum_{\Delta\in\mathcal{F}}\int_\Delta d(x,\partial_{d,\varepsilon}\Omega_\varepsilon)dx}{\left(\sum_{\Delta\in\mathcal{F}}|\Delta|\right)\|d(\cdot,\partial_{d,\varepsilon}\Omega_\varepsilon)\|_{L^\infty(\Omega_\varepsilon)}}$$
$$\leq \frac{\sum_{\Delta\in\mathcal{F}}\int_\Delta d(x,V(\Delta))dx}{\left(\sum_{\Delta\in\mathcal{F}}|\Delta|\right)\|d(\cdot,\partial_{d,\varepsilon}\Omega_\varepsilon)\|_{L^\infty(\Omega_\varepsilon)}}$$

Using the lemma above the thesis follows if for every $\Delta \in \mathcal{F}$ it holds:

$$r(\Delta) \leq \|d(\cdot,\partial_{d,\varepsilon}\Omega_\varepsilon)\|_{L^\infty(\Omega_\varepsilon)}.$$

Is there a lucky triangulation?.

**Voronoi cells and Delauney triangulations** Starting from the Voronoi diagram of $\mathcal{S}$ we can consider its *dual structure*: the Delaunay tessellation. This is the straight-line graph with vertex set $\mathcal{S}$ determined by saying that a segment connecting two points of $\mathcal{S}$ belongs to the graph if and only if the Voronoi regions $V(p)$ and $V(q)$ are edge-adjacent. In general the faces determined by the Delaunay tessellation can be polygons other than triangles (consider again the case of four co-circular points), however we can always add to the graph new edges to obtain a new graph which has only triangular faces.

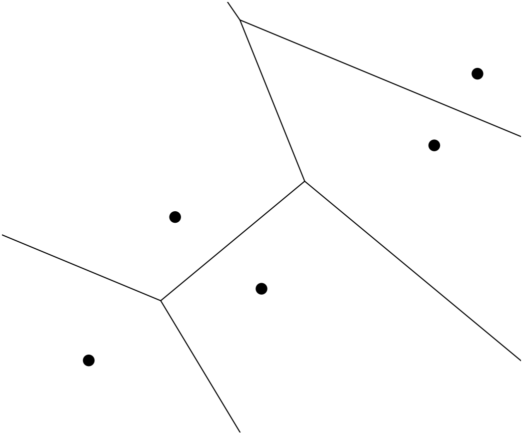

Figure: *An example of Voronoi diagram spanned by five points*

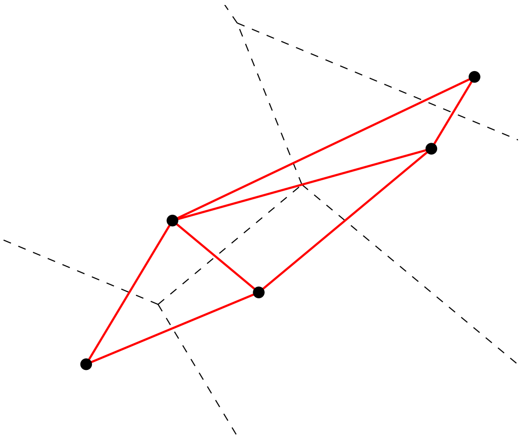

Figure: *Delauney triangulation of five points, the dashed lines represent the Voronoi diagram.*

**empty-circle property**: the circle that circumscribes any triangle does not contain, in its interior, any other point of $\mathcal{S}$.
Applying the property above we can prove the inequality

$$r(\Delta) \leq \|d(\cdot, \partial_{d,\varepsilon}\Omega_\varepsilon)\|_{L^\infty(\Omega_\varepsilon)},$$

for every $\Delta \in \mathcal{F}$. $\square$

# Conclusions

An intuitive conclusion of our results is not only that the supremum values of the efficiency of the torsion function are of different nature depending on $p$ (below or above $d$) but also the asymptotical maximization structures might behave differently, being much more rigid for $p > d$ than for $p \leq d$.

### Open problem

Is it true that, in the two dimensional case, for $p \geq 2$ the only maximizing sequences for the shape functional $\Phi_p(\Omega)$ are the hexagonal ones? What about higher dimensions?

# An application

Recently the problem of comparing the first eigenvalue $\lambda(\Omega)$ of the Dirichlet Laplacian of $\Omega$ with the torsional rigidity $T(\Omega) = \|w_{2,\Omega}\|_1$, has been widely studied.
A classical inequality by G. Polya asserts that

$$\lambda(\Omega)T(\Omega) \leq |\Omega|, \qquad \text{for every domain } \Omega.$$

Berg, Ferone, Nitsch, Trombetti[4] proved that such an inequality is sharp, showing that

$$\sup\{\lambda(\Omega)T(\Omega),\ \Omega \text{ domain with } |\Omega| = 1\} = 1, \text{and the supremum is not attained.}$$

Generalizing the problem to the $p$-Laplacian, toghether with G. Buttazzo and F. Prinari[5] we proved that if $1 < p \leq d$ it holds

$$\sup_{\Omega}\{\lambda_p(\Omega)T_p(\Omega),\ \Omega \text{ domain with } |\Omega| = 1\} = 1.$$

where

$$\lambda_p(\Omega) = \inf_{u\in W_0^{1,p}(\Omega), u\neq 0} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx}, \quad T_p(\Omega) = \left(\int_\Omega w_{p,\Omega}\right)^{p-1}$$

[4] van den Berg, M.; Ferone, V.; Nitsch, C.; Trombetti, C. Int. Equations Operator Theory '16
[5] B., L.; Buttazzo, G.; Prinari, F.; Calc. Var. Partial Differential Equations '22

For capacitary reason the proof fails in the case $p > d$.

**Pólya proof:**

$$\lambda_p(\Omega) \leq \frac{\int_\Omega |\nabla w_{p,\Omega}|^p dx}{\int_\Omega |w_{p,\Omega}|^p dx} = \frac{\int_\Omega w_{p,\Omega} dx}{\int_\Omega |w_{p,\Omega}|^p dx} \leq \frac{\int_\Omega w_{p,\Omega} dx}{\left(\int_\Omega |w_{p,\Omega}| dx\right)^p} = \frac{1}{T_p(\Omega)}.$$

By Theorem 1, we have

$$\sup\left\{ \fint_\Omega w_{p,\Omega} dx \left( \fint_\Omega w_{p,\Omega}^p dx \right)^{-1/p} : \Omega \subset \mathbb{R}^d \text{domain} \right\} < 1.$$

Hence using the latter to improve the red inequality we get

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Thank you for your attention