Capacities, outward minimizing sets and geometric flows

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Least area problem with obstacle and outward minimizing sets

Given $\Omega \subset \mathbb{R}^n$ (for the time being) bounded with finite perimeter, we are interested in the existence of a bounded set Ω^* such that

$$P(\Omega^*) = \inf\{P(E)\,\Omega \subseteq E\,\text{bounded}\}.$$

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We call it minimizing hull of Ω.

If such a set is Ω itself, we say that it is outward minimizing: We say that Ω is outward minimizing if for any E with $\Omega \subseteq E$ we have $P(\Omega) \leq P(E)$.

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RMK: In a noncompact Riemannian manifold, this works everytime we have a bounded outward minimizing set containing Ω, and in particular if we have a outward minimizing exhaustion.

Construction of minimizing hulls

Let $\Omega \subset M$ be a bounded subset with finite perimeter. Bassanezi-Tamanini (Ann. Univ. Ferrara, 1984) showed that, with

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the desired solution to the least area problem with obstacle Ω is given by

$$\Omega^* = \mathrm{Int}\left(\bigcap_{E \in \mathrm{OM}(\Omega)} \mathrm{Int}\,(E) \right).$$

Technical minor issue: the intersection should be thought in a measure theoretic sense.

The strategy of the proof is the most natural one: they show that <u>countable</u> intersections of outward minimizing sets (containing Ω) are still outward minimizing,

$$\Rightarrow \Omega^* = \mathrm{Int}\left(\bigcap_{E\in\mathrm{OM}(\Omega)} \mathrm{Int}\,(E)\right)$$

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is outward minimizing.

If there were a solution $E \supset \Omega$ with $P(E) < P(\Omega^*)$, it would be outward minimizing $\Rightarrow$ it would be part of the intersection $\Rightarrow \Omega^* \subseteq E \Rightarrow P(\Omega^*) \leq P(E)$ contradiction.

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<u>RMK:</u> Bassanezi-Tamanini's proof actually gives you a bounded minimizing hull if you have an <u>exhaustion of outward minimizing sets.</u>

Existence in case of positive isoperimetric constant

We show that

Theorem (F., Mazzieri (JFA 2022)) Assume (M, g) is a complete noncompact Riemannian manifold with a positive Isoperimetric/Sobolev constant $C_{\mathrm{iso}} > 0$, that is

$$\frac{|\partial E|^n}{|E|^{n-1}} \geq \mathrm{C}_{\mathrm{iso}} > 0.$$

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Then, any bounded $\Omega \subset M$ with finite perimeter admits a bounded minimizing hull Ω^*.

RMK: Relevant classes of manifolds satisfy the above assumptions: nonnegative Ricci curvature with Euclidean volume growth, Cartan-Hadamard (simply connected with nonpositive sectional)...

PROOF: Assume that any (bounded) minimizing sequence gives rise to an unbounded set with finite perimeter F. Then, $|F|^{(n-1)/n} \leq CP(F) \leq CP(\Omega)$ and in particular of finite volume , that in particular satisfies $m(r) = |F \setminus B(O,r)| > 0$ for any r big enough.

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This yields

$$m(r)^{\frac{n-1}{n}} \leq -Cm'(r).$$

Integrating it, we get

$$(r_2 - r_1) \leq C \left[m(r_1)^{\frac{1}{n}} - m(r_2)^{\frac{1}{n}} \right]$$

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for any $r_2 > r_1$ big enough. Letting $r_2 \to +\infty$ contradiction.

Capacitary interpretation

It is straightforward to show (Maz'ya) that

$$\inf\{P(E)\,|\,\Omega\subseteq E, E\,\text{smooth}\} = \mathrm{Cap}_1(\Omega) = \inf\left\{\int_M |\nabla f|\,\mathrm{d}\mu, \quad C_c^\infty(M)\ni f\geq \chi_\Omega\right\}$$

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In particular,

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when the latter exists. Direct proof:

$$\int_M |\nabla f|\,\mathrm{d}\mu \geq \int_0^1 |\{f = t\}|\,\mathrm{d}t \geq \inf\left\{P(E)\,\middle|\,\Omega \subset E,\right\} = P(\Omega^*).$$

Asymptotic relation with p-capacities

Recall that, for $p \geq 1$,

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Theorem (Agostiniani, F., Mazzieri- F. Mazzieri; 2022)

Let (M, g) admit a positive isoperimetric constant, and let Ω with $C^{1,\alpha}$-boundary. Then

$$\lim_{p \to 1^+} \mathrm{Cap}_p(\Omega) = \mathrm{Cap}_1(\Omega) = P(\Omega^*).$$

Main step I We apply the L^p Sobolev inequality for $p > 1$. With a careful choice of exponents (Xu) we get

$$\mathrm{Cap}_1(\Omega) \leq \; q_p \; \mathrm{C}_{n,p}{}^{(p-1)/p} \left(\int_M |\nabla f|^p \, \mathrm{d}\mu \right)^{(n-1)/(n-p)} ,$$

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Main step 2 We claim that

$$\mathrm{Cap}_1(\Omega) = P(\Omega^*).$$

It suffices to show "$\leq$"; the inequality "$\geq$" was already known: recall that

$$\inf\{P(E) \,|\, \Omega \subseteq E, E \,\text{smooth}\} = \mathrm{Cap}_1(\Omega) = \inf \left\{ \int_M |\nabla f| \, \mathrm{d}\mu, \quad C_c^\infty(M) \ni f \geq \chi_\Omega \right\}$$

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True if Ω is $C^{1,\alpha}$, in this case Ω^* is $C^{1,\beta}$ except for a set Σ with $H^{n-8} = 0$.

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Related to the regularity of Ω^*. Does an outward minimizing set satisfy some kind of regularity allowing to be approximated from the outside?

Relation with the Inverse Mean Curvature Flow

The level sets of a function w defined on a Riemannian manifold (M, g) evolves by Inverse Mean Curvature Flow if w satisfies

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$$P(\{w \leq t\}) \leq P(F) - \int_{F \setminus \{w \leq t\}} |\nabla w|$$

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for any F containing $\{w \leq t\}$, with $w \in \mathrm{Lip}$.
If the sublevel sets of w are compact and exhaust M, this is an outward minimizing exhaustion $\Rightarrow$ the construction of Ω^* works "like in $\mathbb{R}^n$", even in absence of a positive isoperimetric constant!

p-capacitary potentials and IMCF

Recall that $\mathrm{Cap}_p(\Omega)$ is realized by u_p satisfying

$$\Delta_p u_p = 0, \quad u_p = 1 \text{ on } \partial\Omega, \quad u_p \to 0 \text{ at infinity.}$$

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$$w_p = -(p-1)\log u_p \to w,$$

solution of the weak IMCF described above (Moser, Kotschwar-Ni, Mari-Rigoli-Setti).

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Real deal: In a manifold where you can build the envelopes Ω^*, does the weak IMCF with compact sublevel exists?

Example of application to the Minkowski Inequality

In joint papers with Agostiniani, Benatti, Mazzieri and Pinamonti we got the L^p-Minkowski inequality (sharp with equality only on spheres)

$$C_{n,p}\,\mathrm{Cap}_p(\Omega)^{\frac{n-p-1}{n-p}} \leq \int_{\partial\Omega} \frac{|H|^p}{n-1}\,\mathrm{d}\sigma,$$

through monotonicity formulas along the p-harmonic potentials (also on Riemannian manifolds).

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We can pass to the limit as $p \to 1^+$ and obtain the Minkowski Inequality in terms of Ω^*

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Similar version in the anisotropic case (Xia-Yin).

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Applications to the Penrose Inequality in GR (Agostiniani-Mantegazza-Mazzieri-Oronzio, Hirsh-Miao-Tam...).

Thank you!