Rearrangement of Gradient

Shape Optimization, Geometric Inequalities, and Related Topics
Two days workshop for young researchers in Naples.

January 30, 2023

Andrea Gentile

Mathematical and Physical Sciences
for Advanced Materials and Technologies

Scuola Superiore Meridionale

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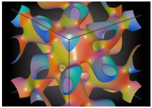

Introduction

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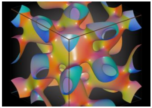

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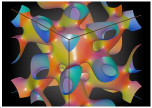
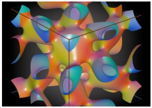


Introduction

Rearrangements (1)



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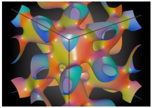

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The radially increasing and decreasing rearrangement of u are respectively defined as

$$u^\sharp(x) = u^*(\omega_n |x|^n) \qquad u_\sharp(x) = u_*(\omega_n |x|^n).$$

where ω_n is the measure of the n-dimensional ball.

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Rearrangements (2)

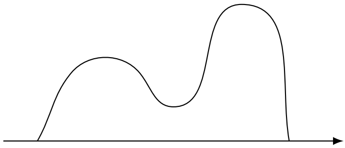

Figure: The decreasing rearrangement $u^\sharp$.

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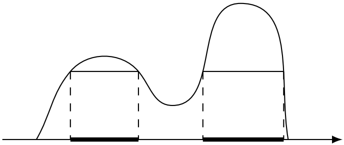

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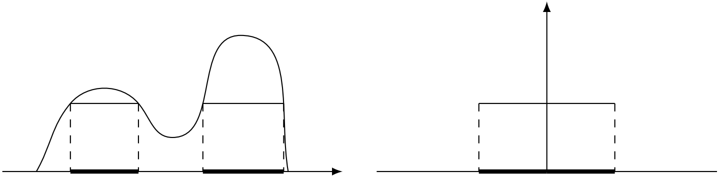

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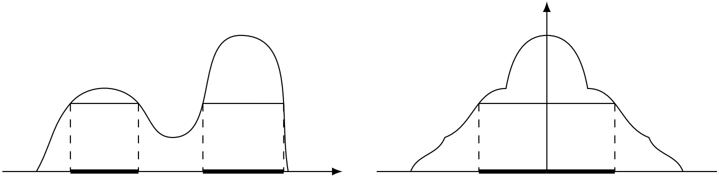

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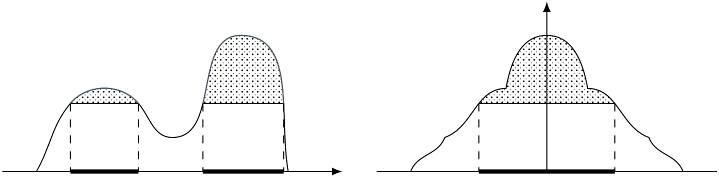

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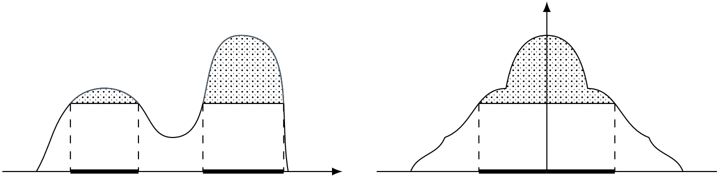

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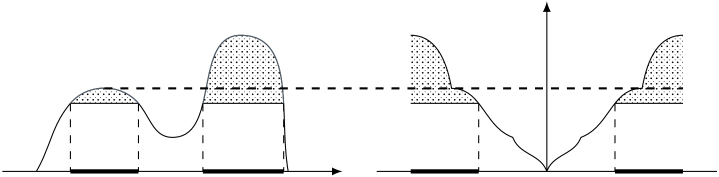
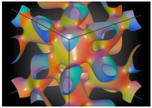


Figure: The increasing rearrangement $u_\sharp$.



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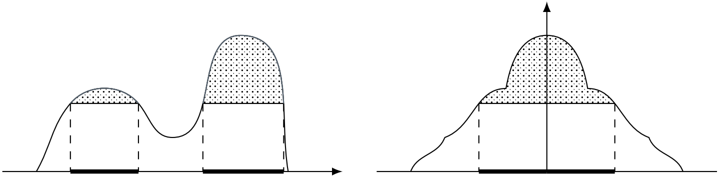

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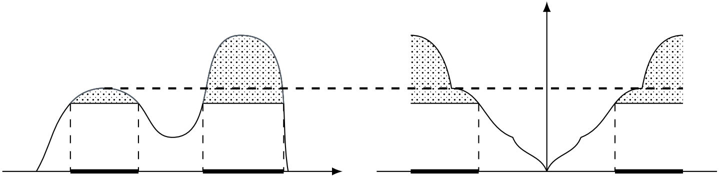

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By Cavalieri's principle, the L^p norms are equal for every p.

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Some literature

Rearrangements are very useful in order to obtain comparison result.

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- Polya-Szegö: if $u \in W^{1,p}(\mathbb{R}^n)$ then $u^\sharp \in W^{1,p}(\mathbb{R}^n)$ and it holds:

$$\int_{\mathbb{R}^n} |\nabla u^\sharp|^p \, dx \le \int_{\mathbb{R}^n} |\nabla u|^p \, dx$$

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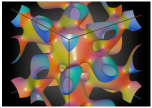

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- Talenti Comparison results: let $f \in L^{\frac{n}{n+2}}$ be a positive function, denoting with u, v respectively the solution to

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} -\Delta v = f^\sharp & \text{in } \Omega^\sharp \\ v = 0 & \text{on } \partial\Omega^\sharp \end{cases},$$

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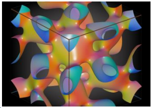

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and therefore

$$\|u^\sharp\|_{L^p} \leq \|v\|_{L^p} \qquad \text{for every } p.$$

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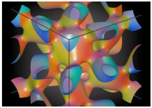
An Hamilton-Jacobi comparison



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Assume $f : \mathbb{R}^n \to \mathbb{R}$ is a non-negative function. Denoting with u and v respectively the solutions to

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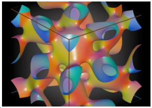

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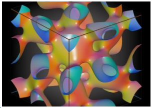

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They also proved a L^∞ comparison replacing the increasing rearrangement with the decreasing rearrangement of f.

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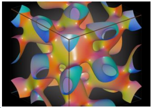

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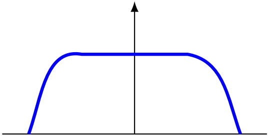

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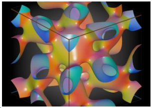

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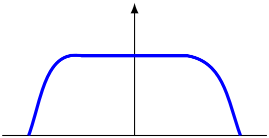

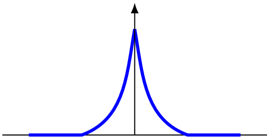

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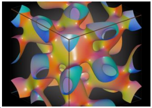
L^q comparison (1)

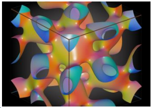


Let $1 < p < \infty$, let Ω be a bounded open set in $\mathbb{R}^n$, let $\varphi = \varphi^* \in L^p(0, |\Omega|)$ and let q be such that

- $1 \leq q \leq \frac{np}{n-p}$ if $p < n$
- $1 \leq q < +\infty$ if $p = n$
- $1 \leq q \leq +\infty$ if $p > n$.

Let us define

$$I(\Omega) := \sup \left\{ \|u\|_{L^q} \;\middle|\; \right.$$



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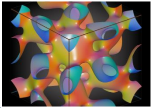

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Questions:

- Does $I(\Omega)$ achieve maximum?

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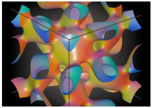

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Questions:

- Does $I(\Omega)$ achieve maximum?
- What is the optimal shape?

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L^q comparison (2)

Theorem (Alvino, P.L. Lions, G. Trombetti, 1989)

Let $\Omega^\sharp$ be the ball centered at the origin with same measure as Ω and R its radius. Then there exists v, g spherically symmetric on $\Omega^\sharp$ such that $g^* = \varphi$, $I(\Omega^\sharp) = \|v\|_{L^q}$,

$$v(x) = \int_{|x|}^{R} g(s)\, ds$$

and thus

$$|\nabla v| = g \qquad a.e. \ in \ \Omega^\sharp, \ v \in W_0^{1,p}(\Omega^\sharp), \ v \geq 0 \ in \ \Omega^\sharp$$

Furthermore $I(\Omega^\sharp) \geq I(\Omega)$ for all open sets Ω in $\mathbb{R}^n$ with $|\Omega^\sharp| = |\Omega|$.

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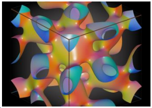

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In [Cianchi, 1996] the author proved a representation formula for g.

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Sobolev case

Main result

Theorem (Amato, G. - to appear on Rendiconti Lincei)

Let $\Omega \subset \mathbb{R}^n$ be a bounded open and Lipschitz set and let $u \in W^{1,p}(\Omega)$ be a non-negative function. Then there exists a non-negative radial function $v \in W^{1,p}(\Omega^\sharp)$ that satisfies

$$\begin{cases} |\nabla v|(x) = |\nabla u|_\sharp(x) & \text{a.e. in } \Omega^\sharp \\ v = \dfrac{\displaystyle\int_{\partial\Omega} u \, d\mathcal{H}^{n-1}}{\operatorname{Per}(\Omega^\sharp)} & \text{on } \partial\Omega^\sharp. \end{cases}$$

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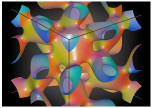

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and verifies

$$\|u\|_{L^1(\Omega)} \le \|v\|_{L^1(\Omega^\sharp)},$$

$$\mathrm{Per}(\Omega^\sharp)^{p-1} \int_{\partial\Omega^\sharp} v^p \, dx \le \mathrm{Per}(\Omega)^{p-1} \int_{\partial\Omega} u^p \, dx \qquad \forall p \ge 1.$$

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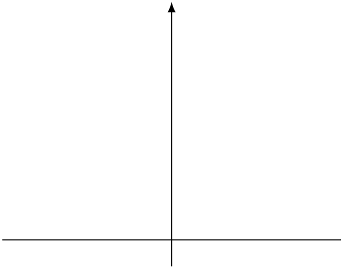
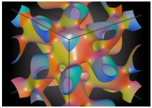
Idea of the proof

Suppose that u and Ω are smooth.





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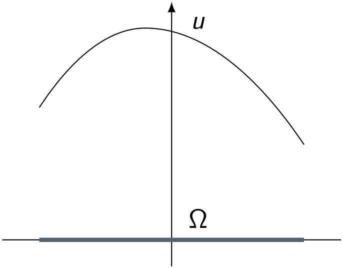

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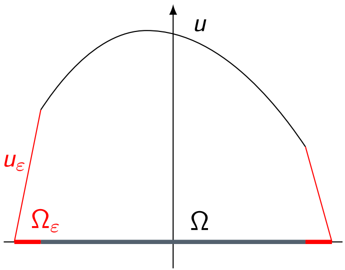

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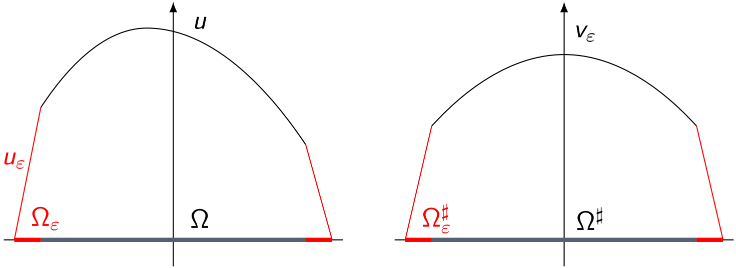

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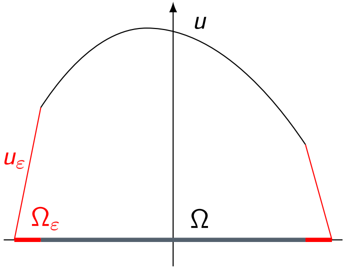
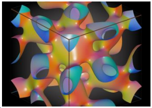


So we can apply Giarrusso-Nunziante comparison.



BV case

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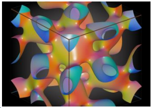
BV function



We say that $u \in L^1(\Omega)$ is a BV function if

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i}\, dx = \int_\Omega \varphi \, d(D_i u) \qquad \forall \varphi \in C_C^\infty(\Omega).$$

and Du is a Radon measure.

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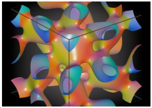

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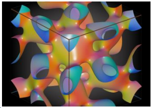

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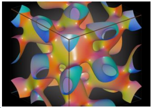

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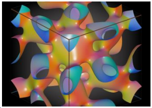

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Moreover for BV functions it holds the Fleming-Rishel formula:

$$|Du|(\Omega) = \int_{-\infty}^{+\infty} \mathrm{Per}(u > t) \, dt.$$

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BV rearrangement

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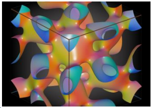

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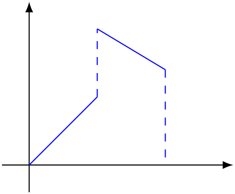

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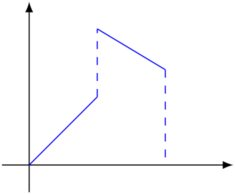

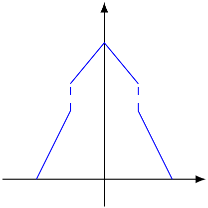

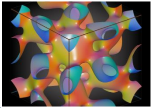

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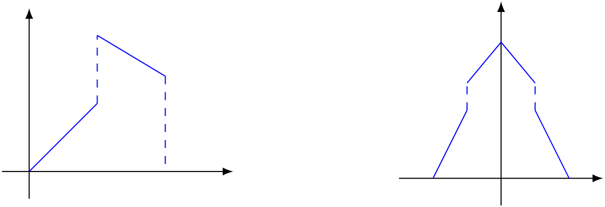

Regular and singular part can mix!

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Main Theorem

Let us define

$$BV_0(\Omega) := \{u \in BV(\mathbb{R}^n) : \ u \equiv 0 \text{ in } \mathbb{R}^n \setminus \Omega\} .$$

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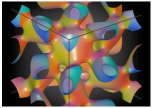

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Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $\Omega^\sharp$ be the centered ball and let $u \in \mathrm{BV}_0(\Omega)$ be a non-negative function. Then there exists a non-negative function $v \in W^{1,1}(\Omega^\sharp) \cap \mathrm{BV}_0(\Omega^\sharp) \cap L^\infty(\Omega^\sharp)$

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$$\begin{cases} |\nabla v|(x) = |\nabla^{\mathrm{a}} u|_\sharp(x) & \text{a.e. in } \Omega^\sharp \\ v(x) = \dfrac{1}{\mathrm{Per}(\Omega)} |D^{\mathrm{s}} u|(\mathbb{R}^n) & \text{on } \partial\Omega^\sharp \end{cases},$$

such that

$$\|u\|_{L^1(\Omega)} \leq \|v\|_{L^1(\Omega^\sharp)}.$$

Idea of the proof (1)

For every $s \in [0, |\Omega|]$ we can define

$$G(s) = \left|D\big(u - u^*(s)\big)\right|(\mathbb{R}^n)$$
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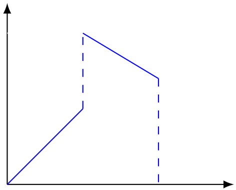

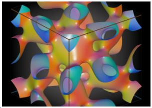

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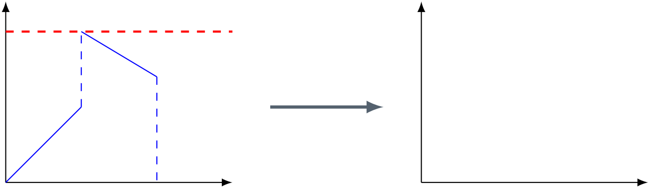

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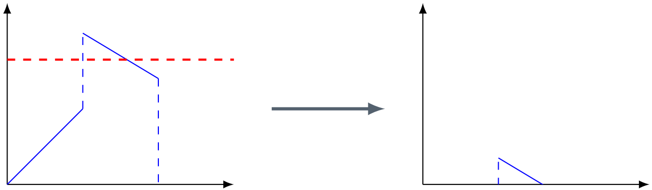

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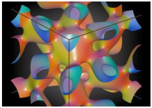

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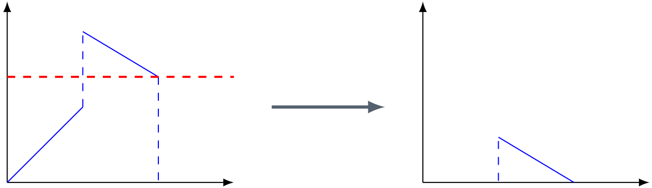

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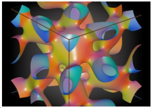

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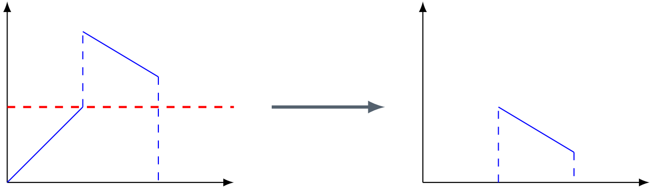

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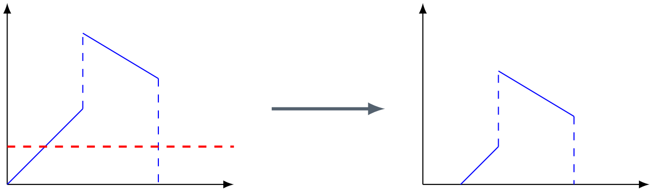

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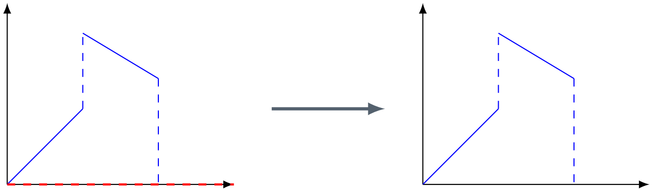

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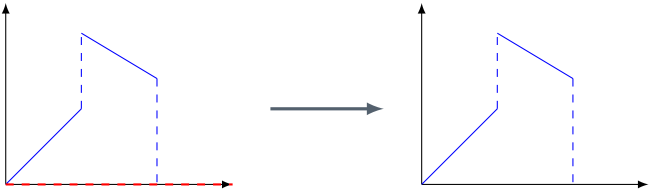
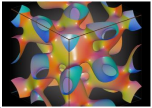


Every G_i is increasing, so they are BV functions and we have

$$G(s) = \int_0^s dF_1(s) + \int_0^s dF_2(s)$$



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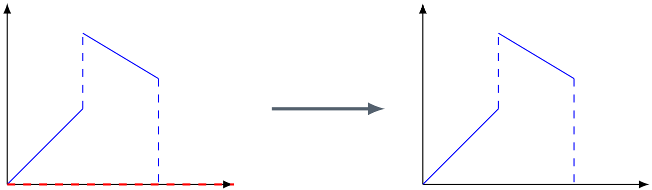

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Idea of the proof (2)

So we can define

$$z(s) := \int_s^{+\infty} \frac{1}{n\omega_n^{\frac{1}{n}} \tau^{1-\frac{1}{n}}} \, dF(\tau) \qquad \forall s \in [0, +\infty),$$

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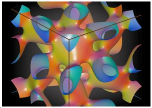
Lemma

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and suppose that u is a non-negative $\mathrm{BV}_0(\Omega)$ function. Then

$$u^*(s) \leq z(s) \qquad a.e. \;\; s \in [0, +\infty).$$



Idea of the proof (3)

Integrating from 0 to $+\infty$ we get

$$\|u\|_{L^1(\Omega)} = \int_0^{+\infty} u^*(s)\, ds \leq \int_0^{+\infty} z(s)\, ds$$

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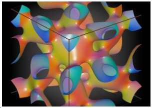

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Idea of the proof (3)

Integrating from 0 to $+\infty$ we get

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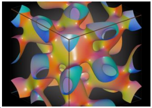
Some remarks



We recall that v has the following explicit expression

$$v(x) = \int_{\omega_n|x|^n}^{+\infty} \frac{|\nabla^{\mathrm{a}} u|_*(t)}{n\omega_n^{\frac{1}{n}} t^{1-\frac{1}{n}}}\, dt + \frac{1}{\mathrm{Per}(\Omega^\sharp)} |D^s u|(\mathbb{R}^n)\chi_{[0,|\Omega|]}(\omega_n|x^n|),$$

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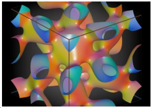

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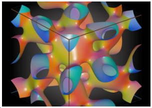
Applications

Robin Torsional Rigidity (1)



Let $\beta > 0$, $\Omega \subset \mathbb{R}^n$ a bounded open set with Lipschitz boundary and let us consider the functional

$$\mathcal{F}_\beta(\Omega, w) = \frac{\displaystyle\int_\Omega |\nabla w|^2\, dx + \beta\, \mathrm{Per}(\Omega) \int_{\partial\Omega} w^2\, d\mathcal{H}^{n-1}}{\left(\displaystyle\int_\Omega w\, dx\right)^2} \qquad w \in H^1(\Omega)$$

and the associate minimum problem

$$\frac{1}{T(\Omega, \beta)} = \min_{w \in H^1(\Omega)} \mathcal{F}_\beta(\Omega, w)$$

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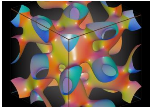
Robin Torsional Rigidity (1)



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and the associate minimum problem

$$\frac{1}{T(\Omega, \beta)} = \min_{w \in H^1(\Omega)} \mathcal{F}_\beta(\Omega, w)$$

The minimum is a weak solution to

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} + \beta\ \mathrm{Per}(\Omega)\, u = 0 & \text{on } \partial\Omega \end{cases}$$

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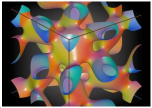
Robin Torsional Rigidity (2)

Corollary (Amato, G. - to appear on Rendiconti Lincei)

Let $\beta > 0$, let $\Omega \subset \mathbb{R}^n$ be a bounded open and Lipschitz set. If we denote with $\Omega^\sharp$ the ball centered at the origin with same measure as Ω, it holds

$$T(\Omega, \beta) \leq T(\Omega^\sharp, \beta)$$

Weightded L^1 comparison



Moreover we generalize a result by Talenti (1994).

Theorem (Amato, G. - to appear on Rendiconti Lincei)

Let $\Omega \subset \mathbb{R}^n$ be a bounded open and Lipschitz set and $u \in W^{1,p}(\Omega)$. Let f be in $L^\infty(\Omega)$ a function such that

$$f^*(t) \geq \left(1 - \frac{1}{n}\right)\frac{1}{t}\int_0^t f^*(s)\,ds \qquad \forall t \in [0, |\Omega|]. \qquad (\star)$$

Then it holds

$$\int_\Omega f(x)u(x)\,dx \leq \int_{\Omega^\sharp} f^\sharp(x)v(x)\,dx.$$

where v is the radially symmetric function such that

$$\begin{cases} |\nabla v|(x) = |\nabla u|_\sharp(x) & \text{a.e. in } \Omega^\sharp \\ v = \dfrac{\displaystyle\int_{\partial\Omega} u\,d\mathcal{H}^{n-1}}{\mathrm{Per}(\Omega^\sharp)} & \text{on } \partial\Omega^\sharp. \end{cases}$$

Lorentz comparison

Let $\Omega \subseteq \mathbb{R}^n$ a measurable set, $0 < p < +\infty$ and $0 < q < +\infty$. Then a function w belongs to the Lorentz space $L^{p,q}(\Omega)$ if

$$|||w|||_{L^{p,q}(\Omega)} = \left(\int_0^{+\infty} \left[t^{\frac{1}{p}} w^*(t) \right]^q \frac{dt}{t} \right)^{\frac{1}{q}} < +\infty$$

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Corollary (Amato, G. - to appear on Rendiconti Lincei)

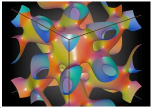
Let $1 \leq p \leq \frac{n}{n-1}$, let $\Omega \subset \mathbb{R}^n$ be a bounded open and Lipschitz set and $u \in W^{1,p}(\Omega)$ a non-negative function. Then it holds

$$\|u\|_{L^{p,1}(\Omega)} \leq \|v\|_{L^{p,1}(\Omega^\sharp)}$$

where $u^\star$ is the function

$$\begin{cases} |\nabla v|(x) = |\nabla u|_\sharp(x) & \text{a.e. in } \Omega^\sharp \\ v = \dfrac{\displaystyle\int_{\partial\Omega} u \, d\mathcal{H}^{n-1}}{\mathrm{Per}(\Omega^\sharp)} & \text{on } \partial\Omega^\sharp. \end{cases}$$

An insulating problem (1)



Now we deal with the functional

$$\mathcal{G}(\psi) := \frac{\displaystyle\int_\Omega |\nabla\psi|^2\,dx - \frac{1}{m}\left(\int_{\partial\Omega}|\psi|\,d\mathcal{H}^{n-1}\right)^2}{\left(\displaystyle\int_\Omega |\psi|\,dx\right)^2} \qquad \psi \in H^1(\Omega)\setminus\{\,0\,\},$$

with $m > 0$ and the associate minimum problem

$$\frac{1}{T_{\mathcal{G}}(\Omega)} := \min_{\psi\in H^1(\Omega)} \mathcal{G}(\psi).$$

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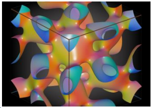

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Why this functional?

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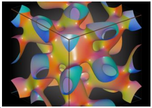
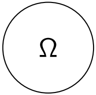
An insulating problem (2)



The functional is linked to the problem of optimal insulation of a given domain $\Omega \subset \mathbb{R}^n$.



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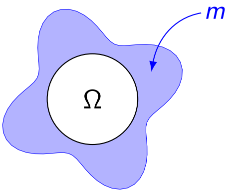

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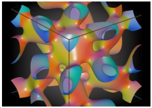

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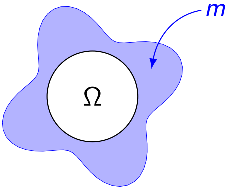

Corollary (Amato, G. - to appear on Rendiconti Lincei)

Let $\Omega \subset \mathbb{R}^n$ be a bounded and open set, let $\Omega^\sharp$ be the centered ball with same measure as Ω and let $m > 0$, then

$$T_{\mathcal{G}}(\Omega) \leq T_{\mathcal{G}}(\Omega^\sharp).$$

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Thanks for your attention!

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