Maximisation of Neumann eigenvalues

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LAMA / LJK

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Problem setting

Introduction

Spectrum of the Laplacian with Neumann b.c.

Let $N \geq 1$ and $\Omega \subseteq \mathbb{R}^N$ be an bounded open set with Lipschitz boundary.

We consider the problem : find $u \in H^1(\Omega) \setminus \{0\}$, $\mu \in \mathbb{R}$ such that

$$\begin{cases} -\Delta u & = \mu u \text{ dans } \Omega \\ \frac{\partial u}{\partial n} & = 0 \text{ sur } \partial\Omega \end{cases}.$$

This problem has a discrete sequence of eigenvalues going to infinity:

$$0 = \mu_0(\Omega) \leq \mu_1(\Omega) \leq \mu_2(\Omega) \leq \ldots \to +\infty.$$

For $m > 0$, we consider the following problem :

Problem

$$\max\left\{\mu_k(\Omega) : \Omega \subseteq \mathbb{R}^N, \Omega \text{ open bounded Lipschitz }, |\Omega| = m\right\}.$$

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By scale-invariance, we can consider :

Equivalent problem

$$\max \left\{ |\Omega|^{\frac{2}{N}} \mu_k(\Omega) : \Omega \subseteq \mathbb{R}^N, \Omega \text{ open bounded Lipschitz } \right\}.$$

Remark

This question is related to the famous Pólya conjecture :

$$\mu_k(\Omega) \leq \frac{4\pi^2 k^{\frac{2}{N}}}{(\omega_N|\Omega|)^{\frac{2}{N}}}.$$

with ω_N the volume of the unit ball of $\mathbb{R}^N$.

State of the art

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- For $k = 2$, the disjoint union of two same balls is optimal : proved by Girouard, Nadirashvili and Polterovich (2009) for simply connected domains in $\mathbb{R}^2$ and by Bucur and Henrot in general (2016);

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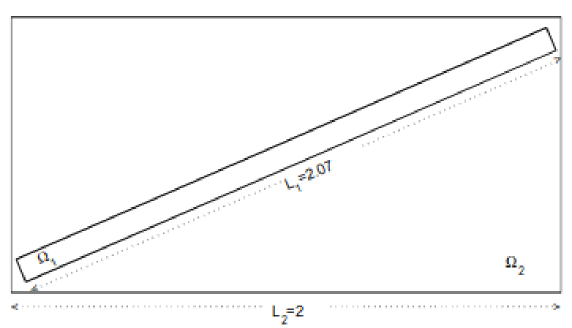
Why ?

Non-monotonicity

In the case of Dirichlet b.c. we have

$$\Omega_1 \subseteq \Omega_2 \implies \mu_k(\Omega_1) \geq \mu_k(\Omega_2),$$

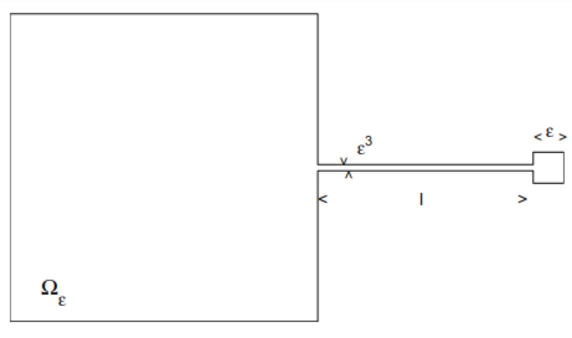
which isn't the case for Neumann b.c. :



Here $\Omega_1 \subseteq \Omega_2$ but $\mu_1(\Omega_1) < \mu_1(\Omega_2)$.

Instability

Let $\Omega = (0,1)^2$ and Ω_ε shown on the following figure. Then $\mu_1(\Omega_\varepsilon)$ **does not** converges to $\mu_1(\Omega)$.



Here $\mu_1(\Omega_\varepsilon) \to 0$ and $\mu_1(\Omega) = \pi^2$ (Courant-Hilbert, 1953).

Relaxation in a class of densities

Relaxation

Theorem (Courant-Hilbert)

For all $k \geq 1$,

$$\mu_k(\Omega) = \min_{S \in \mathcal{S}_{k+1}} \max_{u \in S \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega u^2 \, dx},$$

where $\mathcal{S}_k$ is the set of subspaces of dimension k in $H^1(\Omega)$.

Definition

Let $\rho : \mathbb{R}^N \to [0,1]$ such that $0 < \int_{\mathbb{R}^N} \rho dx < +\infty$. We consider the following degenerate problem : for $k \geq 0$

$$\mu_k(\rho) := \inf_{S \in \mathcal{S}_{k+1}} \max_{u \in S} \frac{\int_{\mathbb{R}^N} \rho |\nabla u|^2 \, dx}{\int_{\mathbb{R}^N} \rho u^2 \, dx},$$

with $\mathcal{S}_{k+1}$ the set of subspaces of dimension $k+1$ in

$$\{u \cdot 1_{\{\rho(x)>0\}} : u \in C_c^\infty(\mathbb{R}^N)\}.$$

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Remark

If ρ is regular enough it is the spectrum of an operator (ex : $\rho = \mathbf{1}_\Omega$).

Relaxation

We now focus on the new problem

$$\max\left\{\mu_k(\rho) : \rho : \mathbb{R}^N \to [0,1], \int_{\mathbb{R}^N} \rho dx = m\right\}.$$

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Remark

For $k = 1, 2$, Bucur and Henrot have shown that this problem is equivalent to the shape optimization problem.

Relaxation

Questions :

1. Does the optimal density exists for every dimension N and every eigenvalue k ?

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Relaxation

Questions :

1. Does the optimal density exists for every dimension N and every eigenvalue k ?
2. Is the relaxed problem equivalent to the original one for $k \geq 3$?
3. Can we extrapolate Pòlya's conjecture in the class of densities ?
4. What does the optimal densities looks like for $k \geq 3$?

Existence of a collection of densities

Theorem (D.Bucur, E.M, E.Oudet)

$$\max\left\{\mu_k(\rho) : \rho : \mathbb{R}^N \to [0,1], \int_{\mathbb{R}^N} \rho dx = m\right\}.$$

is attained. More precisely, there exists $j \in \mathbb{N}$*,* $j \leq k$*,* $\rho_1, \ldots, \rho_j : \mathbb{R}^N \to [0,1]$ *and* $n_1, \ldots, n_j \in \mathbb{N}$ *with* $n_1 + \cdots + n_j = k + 1 - j$ *such that*

$$\sum_{i=1}^{j} \int_{\mathbb{R}^N} \rho_i dx = m \quad \textit{et} \quad \mu_k^* = \mu_{n_1}(\rho_1) = \cdots = \mu_{n_j}(\rho_j).$$

Solution to the Pòlya's in dimension 1

Theorem (D.Bucur, E.M.,E.Oudet)

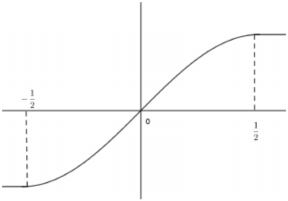
Let $\rho : \mathbb{R} \to [0,1]$, $\int_{\mathbb{R}} \rho = m$. *Then*

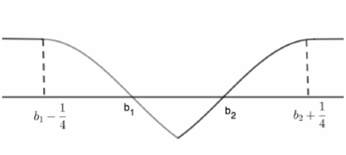
$$\forall k \in \mathbb{N},\ \mu_k(\rho) \leq \frac{\pi^2 k^2}{m^2}.$$

The equality is realized for a density ρ *being the characteristic function of* k *disjoint segments of length* m/k.

Solution to the Pòlya's in dimension 1

The proof consists in the **construction a "good" test function** which relies on a **topological degree** argument and on the properties of the eigenfunctions associated to **non-degenerate** densities.





Test functions used for the proof.

Kröger-type inequalities

Kröger (1992) showed that for a domain Ω

$$\mu_k(\Omega) \leq 4\pi^2 \left(\frac{(N+2)k}{2\omega_N|\Omega|} \right)^{2/N}.$$

This result translates into the density framework :

Theorem (D.Bucur, E.M., E.Oudet)

Let $\rho : \mathbb{R}^N \to [0,1]$, $0 < \int_{\mathbb{R}^N} \rho < \infty$. *Then*

$$\mu_k(\rho) \leq 4\pi^2 \left(\frac{(N+2)k}{2\omega_N} \frac{||\rho||_\infty}{||\rho||_1} \right)^{2/N}.$$

Simulations

An approximation result

Question

Can the degenerated eigenvalues be approximated by eigenvalues of well-posed problems ?

An approximation result

Let $D = (0,1)^2$.

Definition

Let $\rho : D \to [0,1]$ and $\varepsilon > 0$ be small. Define

$$\mu_k^\varepsilon(\rho) := \min_{\substack{\dim(S) = k+1 \\ S \subset \mathbf{H}^1(D)}} \max_{u \in S \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (\rho + \varepsilon)|\nabla u|^2 \, dx}{\int_{\mathbb{R}^N} (\rho + \varepsilon^2) u^2 \, dx}. \tag{1}$$

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Remark

Those are the eigenvalues of the elliptic problem

$$\begin{cases} -div\,[(\rho + \epsilon)\nabla u] = \mu_k^\epsilon(\rho)(\rho + \epsilon^2) u \text{ in } D \\ \partial_n u = 0 \text{ on } \partial D \end{cases}$$

An approximation result

Lemma (D.Bucur, E.M., E.Oudet)

Under the previous notations,

$$\mu_k^\varepsilon(\rho) \xrightarrow[\varepsilon \to 0]{} \mu_k(\rho).$$

Theorem (D.Bucur, E.M., E.Oudet)

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$$\max_\rho \mu_k^\varepsilon(\rho) \xrightarrow[\varepsilon \to 0]{} \max_\rho \mu_k(\rho).$$

Implementation

Suppose that D is meshed by a set of triangles $(T_p)_p$.

The set of densities $\rho : D \to [0,1]$ is approximated by a **finite element** space V_h and $\mathbf{H}^1(D)$ is approximated by a finite element space U_h.

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The set of densities $\rho : D \to [0,1]$ is approximated by a **finite element** space V_h and $\mathbf{H}^1(D)$ is approximated by a finite element space U_h.

The following problem is then solved

$$\begin{aligned} \max_{\rho \in V_h} \quad & \mu_k^{\varepsilon}(\rho) \\ \text{s.t.} \quad & \|\rho\|_1 = m, \\ & 0 \leq \rho \leq 1 \end{aligned} \tag{2}$$

Results : μ_1

Play

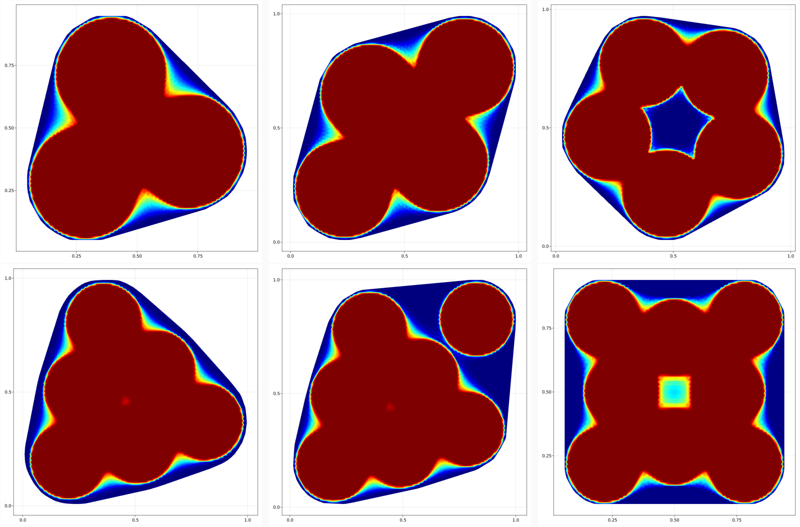
Results : μ_2

Play

Results : μ_3

Play

Results



Approximation of μ_k for $k = 3, .., 8$

Neumann problem on the sphere

Problem setting

Let $\mathbf{S}^N$ be the unit sphere of $\mathbb{R}^{N+1}$ and $\Omega \subseteq \mathbf{S}^N$ be a Lipschitz domain on $\mathbf{S}^N$. Let

$$0 = \mu_0(\Omega) \leq \mu_1(\Omega) \leq \mu_2(\Omega) \leq \ldots \to +\infty.$$

be the eigenvalues of the problem

$$\begin{cases} -\Delta_\Gamma u & = \mu_k(\Omega) u \text{ in } \Omega \\ \frac{\partial u}{\partial n} & = 0 \text{ on } \partial\Omega \end{cases}$$

with $u \in H^1(\Omega) \setminus \{0\}$.

Problem setting

For $0 < m \leq |\mathbf{S}^N|$ we consider the same problem as previously

$$\max \left\{ \mu_k(\Omega) \text{ s.t. } \Omega \subseteq \mathbf{S}^N, \Omega \text{ open, Lipschitz }, |\Omega| = m \right\}.$$

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Remark

We don't have scale invariance of the eigenvalues on the sphere !
This will lead to different behaviours when making m vary.

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- On $\mathbf{S}^2$, Laugesen and Langford (2022) showed that the geodesic ball is optimal for μ_1 and $0 < m < 0.94|\mathbf{S}^2|$ among simply connected domains;
- In the whole sphere and for other eigenvalues, we don't know.

State of the art

Questions

- Does the geodesic ball maximizes μ_1 in all $\mathbf{S}^N$ with the additionnal constraint that $m < \frac{|\mathbf{S}^N|}{2}$? What about $m > \frac{|\mathbf{S}^N|}{2}$?

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- What about μ_2, μ_3 ... ?

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- Does the geodesic ball maximizes μ_1 in all $\mathbf{S}^N$ with the additionnal constraint that $m < \frac{|\mathbf{S}^N|}{2}$? What about $m > \frac{|\mathbf{S}^N|}{2}$?
- What about μ_2, μ_3 … ?
- What happens in the density framework ?

Existence result in the class of densities

Just like in $\mathbb{R}^N$, we can define the degenerate eigenvalues of a density $\rho : \mathbf{S}^N \to [0,1]$

$$\mu_k(\rho) := \inf_{S \in \mathcal{S}_{k+1}} \max_{u \in S} \frac{\int_{\mathbf{S}^N} \rho |\nabla_\Gamma u|^2 \, dx}{\int_{\mathbf{S}^N} \rho u^2 \, dx}, \tag{3}$$

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Theorem (E.M.)

Let $0 < m < |\mathbf{S}^N|$. For all $k \in \mathbb{N}$, there exists $\bar{\rho}$ such that

$$\mu_k(\bar{\rho}) = \max \left\{ \mu_k(\rho) \text{ s.t. } \rho : \mathbf{S}^N \to [0,1], \int_{\mathbf{S}^N} \rho = m \right\}$$

Numerical explorations

Two numerical methods

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- Shape optimization :

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- Density optimization :

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- Density optimization :

$$\max\left\{\mu_k(\rho) \text{ s.t. } \rho : \mathbf{S}^N \to [0,1], \int_{\mathbf{S}^N} \rho = m\right\}$$

$\implies$ This will lead to two different optimization techniques !

Density optimization

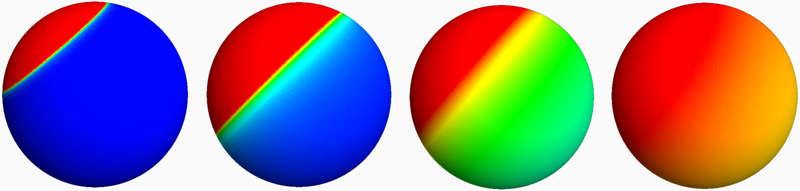
Same technique as in the plane !

To compare, let UB_k^m be the union of k disjoint geodesic balls of total measure m. For $k > 0$ and

$$0 < m < |\mathbf{S}^2| \approx 12.56,$$

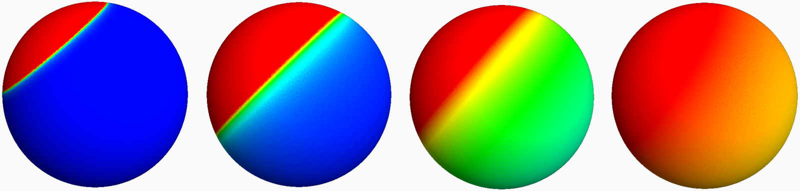
we will compute $\mu_k(UB_k^m)$ and compare it to $\mu_k(\bar{\rho})$.

Density optimization : results for μ_1



Examples of optimal densities for μ_1 and $m \in \{2.0, 4.98, 8.05, 11.2\}$.

Density optimization : results for μ_1



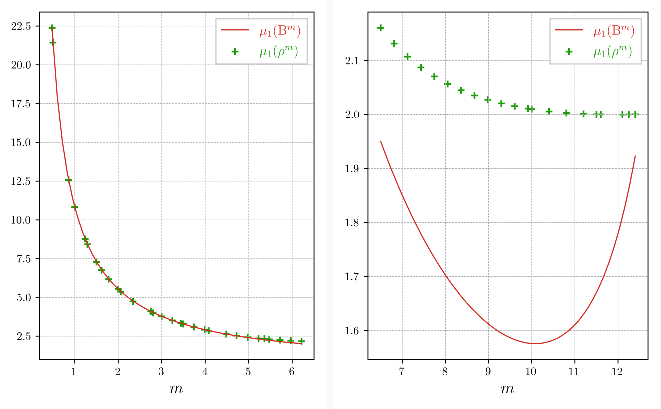
Examples of optimal densities for μ_1 and $m \in \{2.0, 4.98, 8.05, 11.2\}$.

Remark

The optimal density isn't always *"bang-bang"*. An important consequence is that even in the geodesic ball is optimal among the domains, **it will be impossible to prove it by a *"Weinberger-type"* argument**.

Density optimization : results for μ_1

We display the optimal values of $\mu_1(\bar{\rho})$ along with the values of $\mu_1(B^m)$:



Optimal values for μ_1 as function of m obtained by the density method.

Density optimization : results for μ_1

Conjecture

Let $m \in (0, |\mathbf{S}^N|)$. The optimal density of the problem

$$\max\left\{\mu_1(\rho) : \rho : \mathbf{S}^n \to [0,1], \int_{\mathbf{S}^n} \rho dx = m\right\}.$$

is axially symmetric.

Density optimization : results for μ_1

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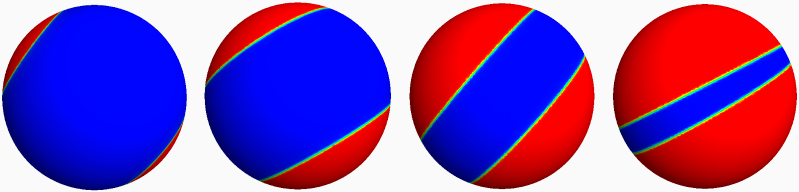
is axially symmetric.

Conjecture

There exists $\delta > 0$ such that for all $m \in (0, \delta)$ the optimal density is the characteristic function of a geodesic ball.

Density optimization : results for μ_2

Here are some optimal densities for μ_2 for different values of m:



Examples of optimal densities for μ_2 and $m \in \{2.31, 5.46, 8.23, 11.01\}$.

Density optimization : results for μ_2

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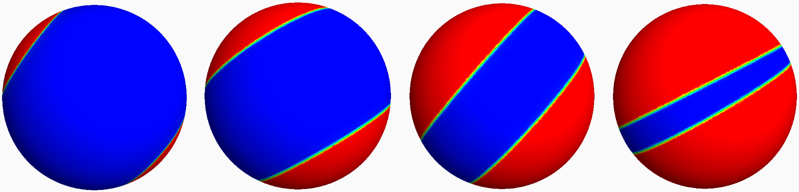


Examples of optimal densities for μ_2 and $m \in \{2.31, 5.46, 8.23, 11.01\}$.

Remark

In opposition to μ_1, the optimal density seems to always be the one of two geodesic balls.

Density optimization : results for μ_2

Theorem (D. Bucur, E.M., M. Nahon)

Let $0 < m < |\mathbf{S}^N|$. The density for which μ_2 is maximal is the characteristic function of two disjoint balls of equal measure.

Density optimization : results for μ_2

Theorem (D. Bucur, E.M., M. Nahon)

Let $0 < m < |\mathbf{S}^N|$. The density for which μ_2 is maximal is the characteristic function of two disjoint balls of equal measure.

Surprisingly, this result generalizes the one of Ashbaugh and Beguria for μ_1 :

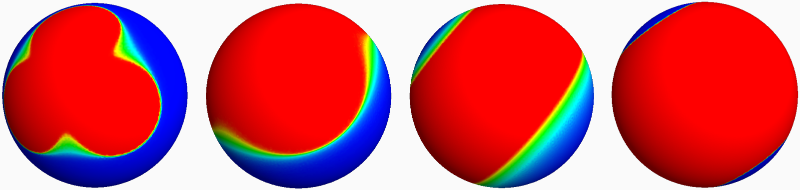
Theorem (D. Bucur, E.M., M. Nahon)

Let $0 < m < |\mathbf{S}^N|$ and $\rho : \mathbf{S}^N \setminus B^m \to [0,1]$ with $\int_{\mathbf{S}^N \setminus B^m} \rho = m$. Then

$$\mu_1(\rho) \leq \mu_1(B^m).$$

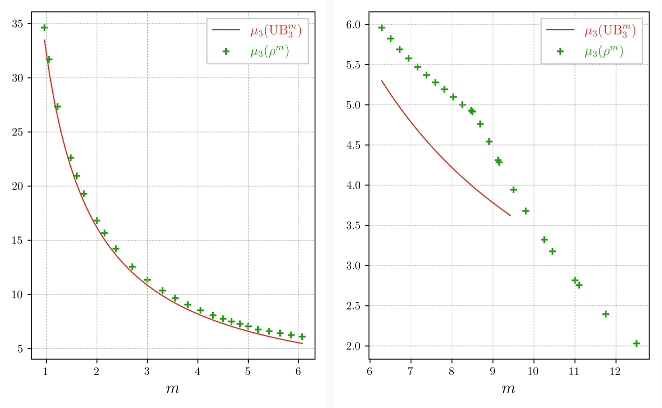
Density optimization : results for μ_3

The case of μ_3 shows a wide varietey of optima :



Examples of optimal densities for μ_3 and $m \in \{2.0, 5.0, 8.03, 11.0\}$.

Density optimization : results for μ_3

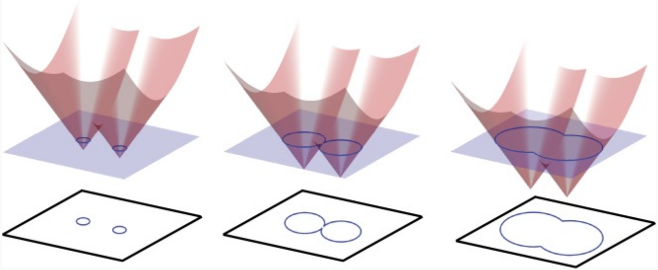


Optimal values for μ_3 as function of m obtained by the density method.

Shape optimization : the level set method in 2 minutes

Let $\Omega(t) \subset \mathbf{S}^N$ be a domain moving depending on $t \in [0, T]$. We can represent the domain $\Omega(t)$ by a **level set** function $\phi : [0, T] \times \mathbf{S}^N \to \mathbb{R}$ such that

$$\forall x \in \mathbf{S}^N, \forall t \in [0, T], \begin{cases} \phi(t, x) < 0 & \text{if } x \in \Omega(t) \\ \phi(t, x) = 0 & \text{if } x \in \partial\Omega(t) \\ \phi(t, x) > 0 & \text{if } x \in^c \Omega(t) \end{cases}.$$



Shape optimization : the level set method in 2 minutes

Let us suppose that $\Omega(t)$ evolves according to a velocity field $V : \mathbf{S}^N \to \mathbf{TS}^N$. More precisely, if $\Omega_0 \subset \mathbf{S}^N$ and χ is the flow of V,

$$\Omega(t) := \chi(\Omega_0, t).$$

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THe motion of $\Omega(t)$ is equivalent to the advection of its level set

$$\partial_t \phi(t,x) + V(x) \cdot \nabla \phi(t,x) = 0 \text{ on } (0,T) \times \mathbf{S}^N.$$

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If we suppose that $V(x) = v(x) n_{\Omega(t)}(x)$ then :

$$\partial_t \phi(t,x) + v(x) |\nabla \phi(t,x)| = 0 \text{ on } (0, T) \times \mathbf{S}^N.$$

Shape optimization : the level set method in 2 minutes

Let us suppose that $\Omega(t)$ evolves according to a velocity field $V : \mathbf{S}^N \to \mathbf{TS}^N$. More precisely, if $\Omega_0 \subset \mathbf{S}^N$ and χ is the flow of V,

$$\Omega(t) := \chi(\Omega_0, t).$$

THe motion of $\Omega(t)$ is equivalent to the advection of its level set

$$\partial_t \phi(t,x) + V(x) \cdot \nabla \phi(t,x) = 0 \text{ on } (0,T) \times \mathbf{S}^N.$$

If we suppose that $V(x) = v(x) n_{\Omega(t)}(x)$ then :

$$\partial_t \phi(t,x) + v(x) |\nabla \phi(t,x)| = 0 \text{ on } (0,T) \times \mathbf{S}^N.$$

We now have to determine v thanks to the **shape derivative** of μ_k.

Theorem (Zanger (2001))

Define

$$\mu_k'(\Omega_0, V) := \lim_{t\to 0^+} \frac{\mu_k(\Omega(t)) - \mu_k(\Omega_0)}{t}.$$

Under some assumptions on Ω_0, this limit exists and is equal to

$$\mu_k'(\Omega_0, V) = \int_{\partial\Omega_0} \left(|\nabla u|^2 - \mu_k(\Omega_0)u^2\right)(V.n)\,d\sigma$$

where u is a unitary eigenfunction associated to $\mu_k(\Omega_0)$.

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One candidate for a gradient-type algorithm could be

$$v = |\nabla u|^2 - \mu_k(\Omega_0) u^2.$$

Shape optimization : the level set method in 2 minutes

Algorithm

1. Initialization of the level set function

Shape optimization : the level set method in 2 minutes

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Shape optimization : the level set method in 2 minutes

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1. Initialization of the level set function
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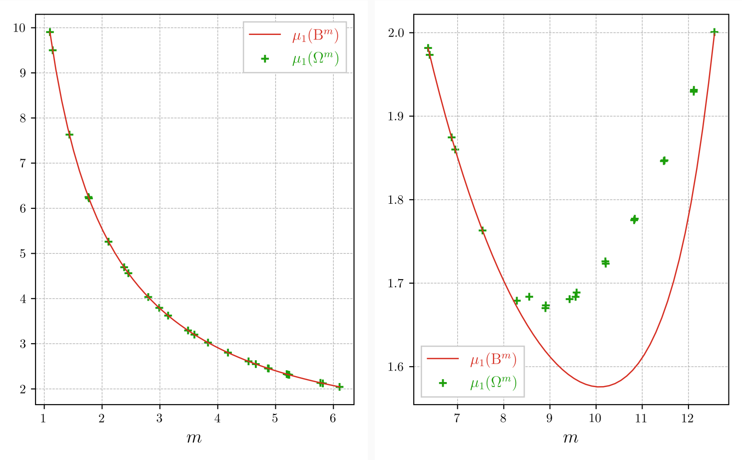
   - 2.1 Compute the eigen elements
   - 2.2 Compute the shape derivative
   - 2.3 Advect the level set function according to the field v given by the shape derivative

Levelset : Results for μ_1.



Optimal values for μ_1 obtained by the level set method.

Levelset : Results for μ_1

Play Play Play Play

Examples of optimal domains for μ_1 and $m \in \{2.03, 5.1, 8.0, 10.85\}$.

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Remark

For a large enough mass, the geodesic ball isn't optimal.

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Remark

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Remark

For a large enough mass, the method seems to try to "homogenize".

Levelset : Results for μ_2

The results are compliant with the theorem :

Play Play Play Play

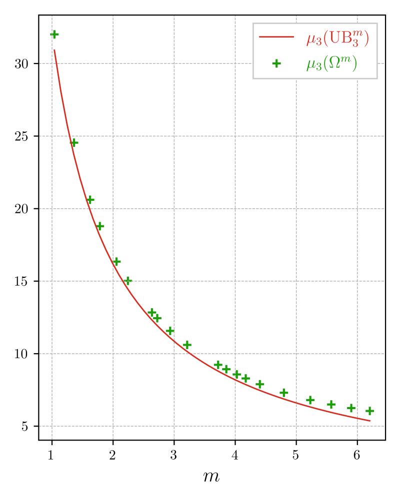
Examples of optimal domains for μ_2 and $m \in \{2.12, 5.1, 8.13, 11.17\}$.

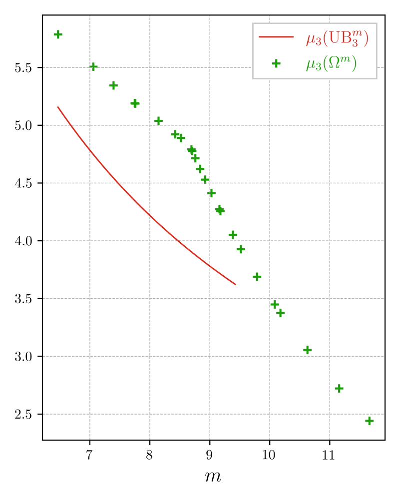
Levelset : Results for μ_3

Play Play Play Play

Examples of optimal domains for μ_3 and $m \in \{2.0, 5.22, 8.0, 11.04\}$.

Levelset : Results for μ_3





Optimal values for μ_3 obtained by the level set method.

What next ?

- Study in greater details the homogenization phenomenon μ_1;

- Study in greater details the homogenization phenomenon μ_1;
- Provide a theoretical proof of the non-optimality of the geodesic ball for m large enough (work in progress with D.Bucur, R.Laugesen and M.Nahon).

Resources

- Preprint for the optimization in $\mathbb{R}^n$: *https://arxiv.org/abs/2204.11472*
- Preprint for the optimization in $\mathbf{S}^n$: *https://arxiv.org/abs/2208.11413*
- Optimal domains and densities on the sphere (MEDIT format): *https://github.com/EloiMartinet/Neumann_Sphere/*

Thanks for your attention.

The ersatz material approach

The problem is approximated by the following one, posed on the whole sphere

$$-\operatorname{div}\left[(\mathbf{1}_\Omega + \varepsilon)\nabla u\right] = \mu_k^\varepsilon(\mathbf{1}_\Omega)(\mathbf{1}_\Omega + \varepsilon^2)u.$$

This allows to always keep the same mesh.

The two different implementations of the level set method

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Remeshing approach

At each iteration, we remesh the domain $\Omega(t) = \{\phi < 0\}$ and solve the original problem on Ω

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We then need to define the velocity field v on all $\mathbf{S}^2$ by **extension-regularisation**.

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- More precise.

Handling area constraint

Let $b, m' > 0$. To handle the area constraint we add a penality term. However, the functionnal

$$\Omega \mapsto \mu_k(\Omega) - b(|\Omega| - m')^2$$

has maximal equal to $+\infty$ (take the sequence $\mathbf{B}^\varepsilon$, $\varepsilon \to 0$).

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