Maximization of Neumann eigenvalues under Diameter constraint

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Shape Optimization, Geometric Inequalities, and Related Topics

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Neumann eigenvalues

Let $\Omega \subset \mathbb{R}^d$ be a connected and bounded domain sucht that the embedding $H^1(\Omega) \to L^2(\Omega)$ is compact (ex. Lipschitz domains), we consider Neumann eigenvalues

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

that we denote by

$$0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \cdots \to +\infty.$$

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that we denote by

$$0 = \mu_0(\Omega) < \mu_1(\Omega) \le \mu_2(\Omega) \le \cdots \to +\infty.$$

These eigenvalues can also be characterized by

$$\mu_k(\Omega) = \inf_{E_k} \sup_{0 \neq u \in E_k} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx},$$

where the infimum is taken over all k-dimensional subspaces of the Sobolev space $H^1(\Omega)$ which are L^2-orthogonal to constants on Ω.

Bounds under Diameter constraint

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This problem was already studied:

- S. Y. Cheng (1975), gives general upper bounds involving the diameter for smooth and complete Riemannian manifolds. The given bound is sharp for μ_1 in dimension $d = 2$
- R. Banuelos and K. Burdzy (1999) proved (via different method) sharp and explicit upper bound for μ_1 in the plane. They also characterize the maximizing sequence.
- P. Kröger (1999) prove sharp upper bounds for convex domains in all dimensions.
- L. Brasco, C. Nitsch and C. Trombetti (2016) proved sharp upper bounds for the first eigenvalue of the p-laplacian.

We present sharp upper bounds for an "optimal" class of domains.

Definition

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Let $\Omega \subset \mathbb{R}^d$ be a domain, the **profile function** g associated to Ω is the function defined in the following way:

$$g(x_1) = \mathcal{H}^{d-1}(\{x' \in \mathbb{R}^{d-1} \mid (x_1, x') \in \Omega,\ x_1 \in [0, D(\Omega)]\}).$$

Main Theorem

Theorem

Let $\Omega \subset \mathbb{R}^d$ be a domain, let g be the profile function associated to Ω. If the function g is a optimal $\frac{1}{\alpha}$-concave function with $\alpha \geq 1$, then the following bounds hold:

- *let $\alpha < 2$ then:* $D(\Omega)^2 \mu_k(\Omega) \leq (2 j_{\frac{\alpha-1}{2},1} + (k-1)\pi)^2$
- *let $\alpha = 2$ then:* $D(\Omega)^2 \mu_k(\Omega) \leq ((k+1)\pi)^2$
- *let $\alpha > 2$ then:*
  - *if k is odd then* $D(\Omega)^2 \mu_k(\Omega) \leq 4 j^2_{\frac{\alpha-1}{2},\frac{k+1}{2}}$
  - *if k is even then* $D(\Omega)^2 \mu_k(\Omega) \leq (j_{\frac{\alpha-1}{2},\frac{k}{2}} + j_{\frac{\alpha-1}{2},\frac{k+2}{2}})^2$

where $j_{\nu,m}$ is the $m-$th zero of the Bessel function J_ν. Moreover all the inequality above are optimal in the sense that they are saturated by sequence of collapsing domains.

Convex case

The convex case is a particular case indeed if $\Omega \subset \mathbb{R}^d$ is a convex set then, by an easy application of Brunn-Minkowski inequality, the profile function g is $\frac{1}{d-1}-$concave.

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Corollary

Let $\Omega \subset \mathbb{R}^d$ be a convex domain, then the following bounds hold:

- *let $d = 2$ then $D(\Omega)^2 \mu_k(\Omega) \leq (2j_{0,1} + (k-1)\pi)^2$*
- *let $d = 3$ then: $D(\Omega)^2 \mu_k(\Omega) \leq ((k+1)\pi)^2$*
- *let $d \geq 4$ then:*
    - *if k is odd then $D(\Omega)^2 \mu_k(\Omega) \leq 4 j^2_{\frac{d-2}{2},\frac{k+1}{2}}$*
    - *if k is even then $D(\Omega)^2 \mu_k(\Omega) \leq (j_{\frac{d-2}{2},\frac{k}{2}} + j_{\frac{d-2}{2},\frac{k+2}{2}})^2$*

Some examples in $\mathbb{R}^2$

If $\Omega \subset \mathbb{R}^2$ is convex, then the profile function is concave. But there are plenty of domains associated to a given profile functions. Given a concave function h there are domains Ω with profile function h but they are not convex:

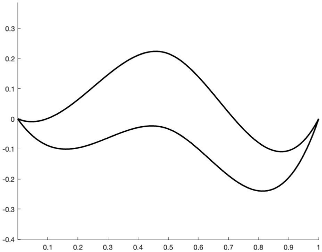


FIGURE: A plane domain $\mathcal{D}_1$ with profile function $x(1-x)$

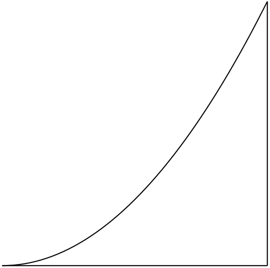
So we have $\mu_k(\mathcal{D}_1) \leq (2j_{0,1} + (k-1)\pi)^2$

Some examples in $\mathbb{R}^2$

The set given by:

$$\mathcal{D}_2 = \{(x, y) \in \mathbb{R}^2 | \, 0 < x < 1, 0 < y < x^2\}$$

has a profile function that is optimal $\frac{1}{2}-$concave but is not convex.



In this case $\mu_k(\mathcal{D}_2) \leq ((k+1)\pi)^2$.

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We have that in the class of domains $\Omega \subset \mathbb{R}^d$ we have sharp upper bounds.
But we can construct domains with fixed diameter with arbitrary large Neumann eigenvalues. Indeed let M be a positive number, then there exists a number α such that $M < 4j_{\frac{\alpha-1}{2},1}$. From the main theorem we can conclude that there exists a domain $\Omega \subset \mathbb{R}^d$ with profile function that is $\frac{1}{\alpha}-$concave such that:

$$M < D(\Omega)^2 \mu_1(\Omega).$$

Relaxed Strum-Liouville eigenvalues

Definition (Relaxed Sturm-Liouville eigenvalues)

Let $h \in L^\infty(0,1)$ be a non negative function, then we define the following quantity

$$\mu_k(h) = \inf_{E_k} \sup_{0 \neq u \in E_k} \frac{\int_0^1 (u')^2 h dx}{\int_0^1 u^2 h dx},$$

where the infimum is taken over all k-dimensional subspaces of the Sobolev space $H^1([0,1])$ which are L^2-orthogonal to h on $[0,1]$.

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We called it relaxed eigenvalues because, in general, we cannot assume the existence of egienfunctions.

Links Between $\mu_k(\Omega)$ and $\mu_k(h)$

Using 1−dimensional test functions it is easy to see that, if Ω is a domain with profile function h, then

$$D(\Omega)^2\mu_k(\Omega) \leq \mu_k(h).$$

Theorem

Let $h \in L^\infty(0,1)$ be a non negative function (not identically zero), let $\Omega_{\epsilon,h}$ be a domain with profile function equal to $\epsilon^{d-1}h$ then

$$\lim_{\epsilon \to 0} D(\Omega_{\epsilon,h})^2\mu_k(\Omega_{\epsilon,h}) = \mu_k(h).$$

A general way of proving optimal upper bounds in a given class of domains $\mathcal{S}$ in $\mathbb{R}^d$ is to solve the following maximization problem:

$$\sup\{D(\Omega)^2 \mu_k(\Omega), \Omega \in \mathcal{S}\}$$

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where $\mathcal{B}$ is a given space of functions.
In order to prove our theorem we need to study the following maximization problem:

$$\max\{\mu_k(h^\alpha), h : [0,1] \to \mathbb{R}_+, h \text{ concave}\}$$

with $\alpha \geq 1$

In order to simplify the notation we introduce the following space of functions

$$\mathcal{L} := \{h : [0,1] \to [0,1], h \text{ concave }, \max h = 1\}.$$

Theorem

Let $\alpha \geq 1$ then there exists a solution $\overline{h}_{\alpha,k}$ of the following problem

$$\sup\{\mu_k(h^\alpha), h \in \mathcal{L}\},$$

Moreover we have an explicit expression for $\mu_k(\overline{h}^\alpha_{\alpha,k})$ and for the functions $\overline{h}_{\alpha,k}$

It is not difficult to prove that a maximizer $\overline{h}$ exists, a difficult part is to prove that there exists an eigenfunction associated to $\mu_k(\overline{h})$.

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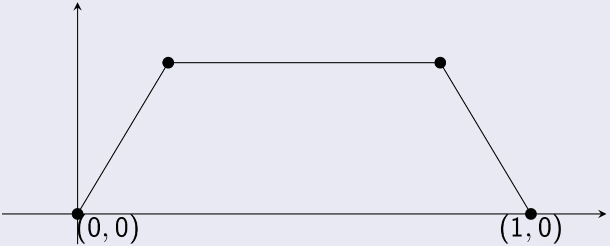
Theorem

For any $k \geq 1$, the maximizer h_k^* satisfies $\max\{\mu_k(h), h \text{ concave }\} = \mu_k(h_k^*) = (2j_{0,1} + (k-1)\pi)^2$ and h_k^* has the following shape in general:



Sketch of the proof

We recall that h_k^* is the maximizer for the Sturm-Liouville problem among concave functions. We will denote it by h in the following and u the eigenfunction.

- The first (and main) step is to prove that $suppt(h'')$ is a discrete set with at most $k+1$ points.
- Then, refining our analysis, we prove that $suppt(h'')$ has only 1 or 2 points inside.
- Finally, using the explicit form of the eigenfunction we are able to conclude.

For that purpose, we write the optimality conditions using Lagrange multipliers in particular, for the concavity constraint.

Optimality conditions

The derivative of the eigenvalue is given by

$$< d\mu_k(h), v >= \int_0^1 [u'^2 - \mu_k u^2] v dx$$

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Due to the constraints $h'' \leq 0$ and $h(0) \geq 0$, $h(1) \geq 0$ we infer the existence of

- a function $\xi \in H^1(0,1)$, $\xi \geq 0$, $\xi = 0$ on the support S of the measure h''
- two non-negative numbers λ_0, λ_1 with $\lambda_0 = 0$ (resp $\lambda_1 = 0$) if $h(0) > 0$ (resp $h(1) > 0$)

such that, for any $v \in H^1(0,1)$:

$$< d\mu_k(h), v >= \int_0^1 [u'^2 - \mu_k u^2] v dx = - < \xi'', v > + \lambda_0 v(0) + \lambda_1 v(1).$$

Analysis of he optimality condition

Let us introduce the function $f := {u'}^2 - \mu_k u^2$. From the optimality condition, we infer $-\xi'' = f$ (in the sense of distributions) on the interval $(0, 1)$. We deduce that ξ is actually a C^1 function.

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We have $-\xi'' = f$ on I, $\xi(\alpha) = \xi(\beta) = 0$.
Moreover, since $\xi \geq 0$ and is C^1, we must have $\xi'(\alpha) = \xi'(\beta) = 0$.
We deduce

$$\int_\alpha^\beta f(x)dx = \int_\alpha^\beta xf(x)dx = 0$$

and also that **f must vanish at least twice on (α, β)**.

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and also that f **must vanish at least twice on** (α, β).
If we are able to control the zeros of the function f we conclude

Sketch of the proof (3)

On the other hand, we look at $f := u'^2 - \mu_k u^2$ on the nodal intervals of u. By a precise analysis, using the ODE

$$u'' + \frac{h'}{h} u' + \mu_k u = 0$$

we are able to prove that **f vanishes exactly twice on each internal nodal interval**. (and exactly once on each boundary interval).

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Since u has exactly $k+1$ nodal intervals, it follows that f has $k+2$ zeros and so there are at most $k+1$ intervals in the complement of the support. Therefore the support of h'' is discrete and has at most $k+2$ points (including the extremities).

In other words h is a polygonal line composed of at most $k+1$ segment.

Last step in the proof

Let us denote by $0 = x_0 < x_1 < x_2 \ldots < x_{m+1} = 1$, $m \leq k$ the points in S, extremities of the segments defining h.

We prove four other qualitative results (using optimality and the ODE), namely:

1. We have $h(0) = h(1) = 0$.
2. For any $i, 0 \leq i \leq m+1$, $u(x_i)u'(x_i) = 0$.

Last step in the proof (2)

3. On an increasing segment $[x_i, x_{i+1}]$, (resp. decreasing), $u'(x_i) = 0$ (resp. $u'(x_{i+1}) = 0$).
4. At any point x_i, such that $1 \leq i \leq m$ we have $u(x_i) = 0$ (therefore, the intervals $[x_i, x_{i+1}]$ coincide with the nodal intervals)

We deduce from 3. and 4. that there are only two or three segments in the optimal h.

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We deduce from 3. and 4. that there are only two or three segments in the optimal h.
Moreover, the nodes x_1 or x_1, x_2 are related to zeros of the Bessel function J_0 since the eigenfunction is expressed in terms of J_0 on the first and the last interval. A simple analysis, using properties of the zeros of Bessel functions and counting the number of nodal intervals, gives the final result.

We proved that for every $\Omega \subset \mathbb{R}^d$ with concave profile function (a particular case is $\Omega \subset \mathbb{R}^2$ convex)

$$D(\Omega)^2 \mu_k(\Omega) \leq \mu_k(\overline{h}_k) = (2j_{0,1} + (k-1)\pi)^2,$$

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Moreover let $\Omega_{\epsilon,\overline{h}_k}$ be a domain with profile function given by $\epsilon^{d-1}\overline{h}_k$ then

$$D(\Omega_{\epsilon,\overline{h}_k})^2 \mu_k(\Omega_{\epsilon,\overline{h}_k}) \to (2j_{0,1} + (k-1)\pi)^2.$$

The case $\alpha > 1$

Let $h \in \mathcal{L}$ it is not clear if there exists an eigenfuntion corresponding to the relaxed eigenvalue $\mu_k(h^\alpha)$. Let $g(h; x, y)$ be the Green Kernel associated to the Strum-Liouville ODE, then

$$g(h; x, y) \in L^2(0,1) \times L^2(0,1)$$

, but in general

$$g(h^\alpha; x, y) \notin L^2(0,1) \times L^2(0,1).$$

The techniques we develop forthe case $\alpha = 1$ are not directly available.

The case $\alpha > 1$

We introduce a new maximization problem:

$$\sup\{\mu_k(h^\alpha), h \in \mathcal{L}, h \geq \epsilon\},$$

we denote $\overline{h}_\epsilon$ the maximizer for this problem:

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- We prove that $\overline{h}_\epsilon$ is a maximizing sequence
- $\overline{h}_\epsilon \overset{*}{\rightharpoonup} \overline{h}$ we give a precise formula for $\overline{h}$ and $\mu_k(\overline{h}^\alpha)$.

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Moreover let $\Omega_{\epsilon,\overline{h}^{\alpha}_{\alpha,k}}$ be a domain with profile function given by $\epsilon^{d-1}\overline{h}^{\alpha}_{\alpha,k}$ then

$$D(\Omega_{\epsilon,\overline{h}^{\alpha}_{\alpha,k}})^2\mu_k(\Omega_{\epsilon,\overline{h}^{\alpha}_{\alpha,k}}) \to \mu_k(\overline{h}^{\alpha}_{\alpha,k}).$$

This conclude th proof of the main theorem

Thank you for your attention!