Debonding models: the wave equation on time-dependent domains and related coupled problems

Shape Optimization, Geometric Inequalities and Related Topics

R. Molinarolo

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UNIVERSITÀ DEGLI STUDI FIRENZE

Introduction: dynamic debonding model

In this talk we present a notion of weak solution for the wave equation on a time-dependent domain with homogeneous Dirichlet boundary value data and standard initial conditions. We will analyse existence, regularity and energy balance. As an application of that we will propose a notion of solution for dynamic debonding models.

What is a debonding model?

Consider a flexible, inextensible, thin film, initially attached to a planar rigid substrate. The film is progressively peeled off by applying a tension and an opening to its edge.

The free part of the film, debonded region, is parameterized in the reference configuration by a time-dependent domain where the displacement satisfies the wave equation.

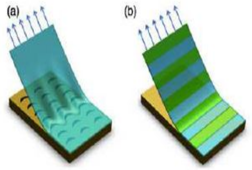
The part of the film still attached to the substrate is called bonded region.

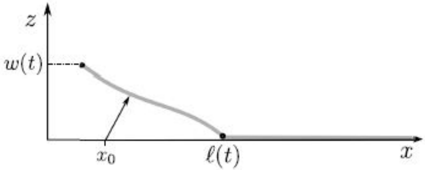
The interface between the two parts is called debonding front.

When we prescribe the evolution of the debonding front, we will have to dealt with a wave equation on a moving domain.

When the evolution of the debonding front is unknown we will see that it is governed by energetic criteria, called Griffith's criterion. In that case the problem becomes coupled and the aim is to determine the evolution of the debonding front and of the displacement.

Literature: 1-dim debonding model





If one assumes that the model only depends on one spatial variable, i.e. 1-dim debonding process, case (b) in figure, then a lot of results are available in literature.

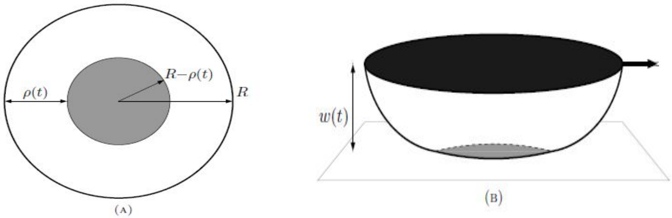
We just mention, e.g., the works of G. Dal Maso, G. Lazzaroni and L. Nardini[1], G. Lazzaroni, and L. Nardini[2,3] and F. Riva, and L. Nardini[4].

[1] G. Dal Maso, G. Lazzaroni and L. Nardini, *Existence and uniqueness of dynamic evolutions for a peeling test in dimension one*, J. Differential Equations, 261 (2016), pp. 4897–4923.

[2] G. Lazzaroni, and L. Nardini, *Analysis of a dynamic peeling test with speed-dependent toughness*, SIAM J. Appl. Math., 78 (2018), pp. 1206-1227.

[3] G. Lazzaroni, and L. Nardini, *On the 1d wave equation in time-dependent domains and the problem of debond initiation*, ESAIM:COCV, 25 (2019), 80.

[4] F. Riva, and L. Nardini, *Existence and uniqueness of dynamic evolutions for a one-dimensional debonding model with damping*, J. Evol. Equ., 21 (2021), pp. 63–106.



If one assumes that the model depends on two spatial variable, i.e. 2-dim debonding process, case (a) and (b) in figure, but with a prescribe radial structure, the problem was analysed and solved by G. Lazzaroni, R. Molinarolo and F. Solombrino[5]

With minor modification, the aforementioned results extends to the N-dim radial case.

The full generalisation, from a geometrical point of view, was considered by G. Lazzaroni, R. Molinarolo, F. Riva and F. Solombrino[6]

[5] G. Lazzaroni, R. Molinarolo and F. Solombrino, *Radial solutions for a dynamic debonding model in dimension two*, Nonlinear Anal., 219 (2022), 112822.

[6] G. Lazzaroni, R. Molinarolo, F. Riva and F. Solombrino, *On the wave equation on moving domains: regularity, energy balance and application to dynamic debonding*, Interfaces Free Bound. (2022).

Mathematical formulation of the problem

Fix $T > 0$ and let $\{\Omega_t\}_{t\in[0,T]}$ be a family of domains such that:

1. for every $t \in [0, T]$, the set $\Omega_t \subset \mathbb{R}^N$ is nonempty, open, bounded and Lipschitz;
2. for every $s, t \in [0, T]$, with $s \le t$, one has $\Omega_s \subset \Omega_t$.

Define

$$\Omega_t^c := \mathbb{R}^N \setminus \overline{\Omega_t}, \quad \mathcal{O} := \bigcup_{t\in(0,T)} \{t\} \times \Omega_t, \quad \Gamma := \bigcup_{t\in(0,T)} \{t\} \times \partial\Omega_t.$$

Consider the formal problem for a function $u : \mathcal{O} \to \mathbb{R}$:

$$\begin{cases} \ddot{u}(t,x) - \Delta u(t,x) = f(t,x) & \text{for } (t,x) \in \mathcal{O}, \\ u(t,x) = 0 & \text{for } (t,x) \in \Gamma, \\ u(0,x) = u_0(x) & \text{for } x \in \Omega_0, \\ \dot{u}(0,x) = u_1(x) & \text{for } x \in \Omega_0. \end{cases} \tag{1}$$

with forcing term

$$f \in L^2(\mathcal{O}),$$

complemented with initial conditions

$$u_0 \in H_0^1(\Omega_0) \text{ and } u_1 \in L^2(\Omega_0).$$

Existence of weak solutions for problem (1)

Definition 1 (Weak solutions of problem (1))

We say that $u\colon \overline{\mathcal{O}} \to \mathbb{R}$ *is a weak solution to problem* (1) *if*

(i) $u \in L^2(0,T;H^1_0(\Omega_t))$ *and* $\dot u \in L^2(0,T;L^2(\Omega_t))$;

(ii) $u(0) = u_0$ *in* $C^0([0,T];L^2(\Omega_0))$, $\dot u(0) = u_1$ *in* $C^0([0,T];H^{-1}(\Omega_0))$;

(iii) u *satisfies*

$$-\int_0^T \langle \dot u(t), \dot\eta(t)\rangle_{L^2(\Omega_t)}\,\mathrm{d}t + \int_0^T \langle \nabla u(t), \nabla\eta(t)\rangle_{L^2(\Omega_t)}\,\mathrm{d}t = \int_0^T \langle f(t), \eta(t)\rangle_{L^2(\Omega_t)}\,\mathrm{d}t, \quad (2)$$

for every $\eta \in L^2(0,T;H^1_0(\Omega_t))$ *with* $\dot\eta \in L^2(0,T;L^2(\Omega_t))$ *and* $\eta(T) = \eta(0) = 0$.

Theorem 1 (Existence theorem for problem (1))

There exists a weak solution u *of problem* (1) *in the sense of Definition 1. Moreover*

1. $u \in L^\infty(0,T;H^1_0(\Omega_t)) \cap C^0_w([0,T];H^1_0(\Omega_T))$;
2. $\dot u \in L^\infty(0,T;L^2(\Omega_t)) \cap C^0_w([\bar t,T];L^2(\Omega_{\bar t}))$ *for every* $t \in [0,T)$;
3. *the energy inequality: for every* $t \in [0,T]$

$$\frac{1}{2}\,\|\dot u(t)\|^2_{L^2(\Omega_t)} + \frac{1}{2}\,\|\nabla u(t)\|^2_{L^2(\Omega_t)} \le \frac{1}{2}\,\|u_1\|^2_{L^2(\Omega_0)} + \frac{1}{2}\,\|\nabla u_0\|^2_{L^2(\Omega_0)} + \int_0^t \langle f(s), \dot u(s)\rangle_{L^2(\Omega_s)}\,\mathrm{d}s.$$

Observations

Few observations on the proof of Theorem 1:

1. Our proof of Theorem 1 is based on a time-discretisation argument as done in the parabolic setting by J. Calvo, M. Novaga, and G. Orlandi.[7]
2. Alternative proofs: one can proceed by minimizing movements as in U. Gianazza, and G. Savaré[8], or by a Galerkin method with penalization as in J. P. Zolésio.[9]
3. More regularity of the solution is required in order to obtain energy balance.
4. The proof is flexible and therefore can be adopted to other frameworks, like anisotropic setting (Finsler laplacian) or fractional setting (Fractional laplacian): it just required the analysis of the solution of the model problem on a time slice, i.e. on the cylindrical domain.

[7] J. Calvo, M. Novaga, and G. Orlandi, *Parabolic equations in time-dependent domains*, J. Evol. Equ., 17 (2017), pp. 781–804.

[8] U. Gianazza, and G. Savaré, *Abstract evolution equations on variable domains: an approach by minimizing movements*, Ann. Sc. Norm. Sup. Pisa Cl. Sci., 23 (1996), pp. 149–178.

[9] J. P. Zolésio, *Galerkin approximation for wave equation in moving domain*, Lecture Notes in Control and Information Sciences, Stabilization of Flexible Structures, Third Working Conference, Montpellier, France, Springer–Verlag Berlin Heidelberg New York 153 (1989), pp. 191–225.

First energy balance

Theorem 2 (First energy balance)

Assume that $\mathcal{O}$ is open with Lipschitz boundary and that

$$\partial\mathcal{O} = \Gamma \cup (\{T\} \times \overline{\Omega}_T) \cup (\{0\} \times \overline{\Omega}_0).$$

Let u be a weak solution of problem (1) *in the sense of Definition 1, satisfying the following regularity property:*

$$u \in L^2(0,T;H^2(\Omega_t) \cap H^1_0(\Omega_t)), \quad \dot{u} \in L^2(0,T;H^1(\Omega_t)), \quad \ddot{u} \in L^2(0,T;L^2(\Omega_t)).$$

Then, for every $t \in [0,T]$, the following energy balance holds true:

$$\frac{1}{2}\|\dot{u}(t)\|^2_{L^2(\Omega_t)} + \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega_t)} - \int_{\Gamma_t} \frac{\nu^t_{\mathcal{O}}}{2}\left[1 - \left(\frac{\nu^t_{\mathcal{O}}}{|\nu^x_{\mathcal{O}}|}\right)^2\right] |\nabla u|^2 \,\mathrm{d}\mathcal{H}^N$$
$$= \frac{1}{2}\|u_1\|^2_{L^2(\Omega_0)} + \frac{1}{2}\|\nabla u_0|^2_{L^2(\Omega_0)} + \int_0^t \langle f(s), \dot{u}(s)\rangle_{L^2(\Omega_s)} \,\mathrm{d}s,$$

with $\Gamma_t := \{(s,x) \in \Gamma : s \in (0,t)\}$ and $\nu_{\mathcal{O}} := (\nu^t_{\mathcal{O}}, \nu^x_{\mathcal{O}})$ is the outside unit normal of Γ.

Proof.

By the regularity assumption, $u \in H^2(\mathcal{O})$ and satisfies

$$\ddot{u}(t,x) - \operatorname{div}(\nabla u(t,x)) = f(t,x), \qquad \text{for a.e. } (t,x) \in \mathcal{O}.$$

Multiplying by a function $\varphi \in L^2(0,T;H^1(\Omega_t)) \cap H^1(0,T;L^2(\Omega_t))$, integrating by parts in $\mathcal{O}$, for all $t \in [0,T]$ we obtain:

$$\langle \dot{u}(t), \varphi(t)\rangle_{L^2(\Omega_t)} - \langle u_1, \varphi(0)\rangle_{L^2(\Omega_0)} - \int_0^t \langle \dot{u}(s), \dot{\varphi}(s)\rangle_{L^2(\Omega_s)}\, \mathrm{d}s + \int_0^t \langle \nabla u(s), \nabla \varphi(s)\rangle_{L^2(\Omega_s)}\, \mathrm{d}s$$
$$= \int_0^t \langle f(s), \varphi(s)\rangle_{L^2(\Omega_s)}\, \mathrm{d}s - \int_{\Gamma_t} (\dot{u}\,\nu^t_{\mathcal{O}} - \nabla u \cdot \nu^x_{\mathcal{O}})\, \varphi \, \mathrm{d}\mathcal{H}^N.$$

Choosing as test function $\varphi = \dot{u}$, we obtain

$$\|\dot{u}(t)\|^2_{L^2(\Omega_t)} - \|u_1\|^2_{L^2(\Omega_0)} - \int_0^t \langle \dot{u}(s), \ddot{u}(s)\rangle_{L^2(\Omega_s)}\, \mathrm{d}s + \int_0^t \langle \nabla u(s), \nabla \dot{u}(s)\rangle_{L^2(\Omega_s)}\, \mathrm{d}s$$
$$= \int_0^t \langle f(s), \dot{u}(s)\rangle_{L^2(\Omega_s)}\, \mathrm{d}s - \int_{\Gamma_t} (\dot{u})^2 \nu^t_{\mathcal{O}}\, \mathrm{d}\mathcal{H}^N + \int_{\Gamma_t} (\nabla u \cdot \nu^x_{\mathcal{O}})\, \dot{u}\, \mathrm{d}\mathcal{H}^N.$$

Again integration by parts yields to

$$\int_0^t \langle \dot{u}(s), \ddot{u}(s)\rangle_{L^2(\Omega_s)}\, \mathrm{d}s = \frac{1}{2}\|\dot{u}(t)\|^2_{L^2(\Omega_t)} - \frac{1}{2}\|u_1\|^2_{L^2(\Omega_0)} + \frac{1}{2}\int_{\Gamma_t} (\dot{u})^2 \nu^t_{\mathcal{O}}\, \mathrm{d}\mathcal{H}^N,$$
$$\int_0^t \langle \nabla u(s), \nabla \dot{u}(s)\rangle_{L^2(\Omega_s)}\, \mathrm{d}s = \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega_t)} - \frac{1}{2}\|\nabla u_0\|^2_{L^2(\Omega_0)} + \frac{1}{2}\int_{\Gamma_t} |\nabla u|^2 \nu^t_{\mathcal{O}}\, \mathrm{d}\mathcal{H}^N.$$

Since $u \equiv 0$ on Γ, then

$$\dot{u}\,\nu_{\mathcal{O}}^{x} = \nu_{\mathcal{O}}^{t} \nabla u, \quad \mathcal{H}^N\text{-a.e. on } \Gamma,$$

which in particular implies the relations

$$\begin{aligned} \dot{u}(\nabla u \cdot \nu_{\mathcal{O}}^{x}) &= \nu_{\mathcal{O}}^{t} |\nabla u|^2, \qquad && \mathcal{H}^N\text{-a.e. on } \Gamma, \\ (\dot{u})^2 |\nu_{\mathcal{O}}^{x}|^2 &= (\nu_{\mathcal{O}}^{t})^2 |\nabla u|^2, \qquad && \mathcal{H}^N\text{-a.e. on } \Gamma. \end{aligned}$$

We conclude that

$$\begin{aligned} &\frac{1}{2}\,\|\dot{u}(t)\|^2_{L^2(\Omega_t)} - \frac{1}{2}\,\|u_1\|^2_{L^2(\Omega_0)} + \frac{1}{2}\,\|\nabla u(t)\|^2_{L^2(\Omega_t)} - \frac{1}{2}\,\|\nabla u_0\|^2_{L^2(\Omega_0)} \\ &= \int_0^t \langle f(s), \dot{u}(s)\rangle_{L^2(\Omega_s)}\,\mathrm{d}s + \int_{\Gamma_t} \frac{\nu_{\mathcal{O}}^{t}}{2} \left[1 - \left(\frac{\nu_{\mathcal{O}}^{t}}{|\nu_{\mathcal{O}}^{x}|} \right)^2 \right] |\nabla u|^2 \,\mathrm{d}\mathcal{H}^N, \end{aligned}$$

thus the statement is proved. □

Remark 3

If the monotonicity condition on the domains is in force one has $\nu_{\mathcal{O}}^{t} \leq 0$; moreover if the growth of the domains is subsonic, i.e. $|\nu_{\mathcal{O}}^{t}| \leq |\nu_{\mathcal{O}}^{x}|$ (and the set $\mathcal{O}$ is usually called time-like). In applications to debonding models, the integral over Γ_t can be thus interpreted as energy dissipated in the debonding process.

Change of variables from Ω_t into Ω_0

We now recast problem (1) into a hyperbolic problem in a fixed domain. To this end we adapt a method based on diffeomorphism. Essentially, we build up suitable change of variable in order to map, for every $t \in [0, T]$, the time dependent domain Ω_t into the fixed domain Ω_0.

We thus assume the existence of two functions

$$\Phi\colon [0, T] \times \overline{\Omega}_0 \to \mathbb{R}^N, \qquad \Psi\colon \overline{\mathcal{O}} \to \overline{\Omega}_0,$$

satisfying

$$\Phi(t, \Omega_0) = \Omega_t \text{ and } \Psi(t, \Omega_t) = \Omega_0 \text{ for all } t \in [0, T], \tag{5a}$$

$$\Phi(t, \Psi(t, x)) = x, \quad \text{for all } (t, x) \in \overline{\mathcal{O}}, \tag{5b}$$

$$\Psi(t, \Phi(t, y)) = y, \quad \text{for all } (t, y) \in [0, T] \times \overline{\Omega}_0, \tag{5c}$$

$$\Phi(0, y) = y, \quad \text{for all } y \in \overline{\Omega}_0. \tag{5d}$$

We also assume that they fulfil the following assumptions:

(H_1) Φ, Ψ are of class $C^{1,1}$ on their domains of definition;

(H_2) $|\dot{\Phi}(t, y)| < 1$ for every $(t, y) \in [0, T] \times \overline{\Omega}_0$.

Last condition ensures that the growth speed of the sets Ω_t is always strictly less than the speed of the travelling waves of problem (1); it is crucial in order to guarantee that the transformed problem is still hyperbolic.

The auxiliary function v

Given a weak solution u of problem (1), we now consider the auxiliary function

$$v(t,y) := u(t,\Phi(t,y)), \quad \text{for all } (t,x) \in [0,T] \times \overline{\Omega}_0.$$

Equivalently,

$$u(t,x) = v(t,\Psi(t,x)) \quad \text{for all } (t,x) \in \overline{\mathcal{O}}.$$

This change of variables yields to the following problem with fixed domain:

$$\begin{cases} \ddot{v} - \operatorname{div}(B\nabla v) + a\cdot\nabla v - 2b\cdot\nabla\dot{v} = g, & \text{in } (0,T)\times\Omega_0, \\ v = 0, & \text{in } (0,T)\times\partial\Omega_0, \\ v(0) = v_0, & \text{in } \Omega_0, \\ \dot{v}(0) = v_1, & \text{in } \Omega_0, \end{cases} \tag{7}$$

where the coefficients are given by

$$\begin{aligned} B(t,y) &:= D\Psi(t,\Phi(t,y))D\Psi(t,\Phi(t,y))^T - \dot{\Psi}(t,\Phi(t,y)) \otimes \dot{\Psi}(t,\Phi(t,y)), \\ a(t,y) &:= -\left\{B(t,y)\nabla \det D\Phi(t,y) + \partial_t\left[b(t,y)\det D\Phi(t,y)\right]\right\} \det D\Psi(t,\Phi(t,y)), \\ b(t,y) &:= -\dot{\Psi}(t,\Phi(t,y)), \\ g(t,y) &:= f(t,\Phi(t,y)). \end{aligned}$$

and the initial data are given by

$$v_0 := u_0 \in H^1_0(\Omega_0), \quad v_1 := u_1 + \dot{\Phi}(0,\cdot)\cdot\nabla u \in L^2(\Omega_0).$$

Scheme of the Galerkin method on fixed domain

Let $\{w_k\}_{k\in\mathbb{N}} \subseteq H^2(\Omega_0) \cap H_0^1(\Omega_0)$ be the set of eigenfunctions of $-\Delta$ in $H_0^1(\Omega_0)$ normalized in $L^2(\Omega_0)$. It is a standard fact that they form an orthogonal basis of $H_0^1(\Omega_0)$ and an orthonormal basis of $L^2(\Omega_0)$. Furthermore, for every $k \in \mathbb{N}$

$$\langle \phi, w_k \rangle_{L^2(\Omega_0)} = \frac{\langle \nabla\phi, \nabla w_k \rangle_{L^2(\Omega_0)}}{\|\nabla w_k\|^2_{L^2(\Omega_0)}}, \quad \text{for all } \phi \in H_0^1(\Omega_0). \tag{8}$$

For every $m \in \mathbb{N}$, we seek functions $d_k^m \in H^2(0,T)$ such that the function defined by

$$v^m(t) := \sum_{k=1}^{m} d_k^m(t) w_k \in H^2(0,T; H^2(\Omega_0) \cap H_0^1(\Omega_0)) \tag{9}$$

satisfies for every $k = 1, \dots, m$ and for almost every $t \in [0,T]$ the finite-dimensional version of problem (7), namely

$$\begin{aligned}\langle \ddot{v}^m(t), w_k \rangle_{H_0^1(\Omega_0)} + \langle B(t)\nabla v^m(t), \nabla w_k \rangle_{L^2(\Omega_0)} + \langle a(t)\cdot\nabla v^m(t), w_k \rangle_{L^2(\Omega_0)} \\ -2\langle b(t)\cdot\nabla \dot{v}^m(t), w_k \rangle_{L^2(\Omega_0)} = \langle g(t), w_k \rangle_{L^2(\Omega_0)},\end{aligned} \tag{10}$$

with initial conditions

$$d_k^m(0) = \langle v_0, w_k \rangle_{L^2(\Omega_0)}, \tag{11a}$$

$$\dot{d}_k^m(0) = \langle v_1, w_k \rangle_{L^2(\Omega_0)}. \tag{11b}$$

Higher regularity

Assume that

$$\Omega_0 \text{ is convex or of class } C^2. \tag{12}$$

and the initial data are

$$v_0 \in H^2(\Omega_0) \cap H_0^1(\Omega_0), \quad \text{and} \quad v_1 \in H_0^1(\Omega_0). \tag{13}$$

Under these assumptions, we have

$$v^m \in H^3(0, T; H^2(\Omega_0) \cap H_0^1(\Omega_0)).$$

We are now in a position to deduce higher uniform estimates for the functions v^m.

Proposition 4 (Higher a priori estimates)

There exists a constant $D > 0$ (independent of $m \in \mathbb{N}$) such that

$$\sup_{0 \le t \le T} \left(\|\ddot{v}^m(t)\|^2_{L^2(\Omega_0)} + \|\dot{v}^m(t)\|^2_{H_0^1(\Omega_0)} + \|v^m(t)\|^2_{H^2(\Omega_0)} \right) \le D.$$

Theorem 5 (Higher regularity of strong weak solution of problem (7))

There exists a unique strong-weak solution v of problem (7) which satisfies:

$$\begin{aligned} &v \in L^\infty(0, T; H^2(\Omega_0) \cap H_0^1(\Omega_0)), \\ &\dot{v} \in L^\infty(0, T; H_0^1(\Omega_0)), \\ &\ddot{v} \in L^\infty(0, T; L^2(\Omega_0)). \end{aligned} \tag{14}$$

Proposition 6

Let Φ, Ψ be as in (5) and satisfy (H_1). Then $\mathcal{O}$ be open with Lipschitz boundary with $\partial\mathcal{O} = \Gamma \cup (\{T\} \times \overline{\Omega}_T) \cup (\{0\} \times \overline{\Omega}_0)$. Furthermore

$$\nu_{\mathcal{O}}(t,x) = (\nu_{\mathcal{O}}^t(t,x), \nu_{\mathcal{O}}^x(t,x)) = \frac{(-\omega(t,x), \nu_{\Omega_t}(x))}{\sqrt{1+\omega(t,x)^2}}, \tag{15}$$

for all $t \in [0,T]$ and $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega_t$, where we introduce the scalar normal velocity

$$\omega(t,x) := \dot{\Phi}(t, \Psi(t,x)) \cdot \nu_{\Omega_t}(x), \qquad \text{for all } t \in [0,T] \text{ and } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega_t. \tag{16}$$

Moreover the following identity holds true for every $\mathfrak{h} \in L^1(\Gamma)$:

$$\int_\Gamma \mathfrak{h}\, \nu_{\mathcal{O}} \, \mathrm{d}\mathcal{H}^N = \int_0^T \int_{\partial\Omega_t} \mathfrak{h}(t,x) \begin{pmatrix} -\omega(t,x) \\ \nu_{\Omega_t}(x) \end{pmatrix} \mathrm{d}\mathcal{H}^{N-1}(x)\, \mathrm{d}t. \tag{17}$$

Remark 7

We point out that actually the scalar normal velocity ω does not depend on the choice of the diffeomorphisms Φ and Ψ, but it is intrinsically related to the set $\mathcal{O}$ (and so to the family $\{\Omega_t\}_{t\in[0,T]}$). Indeed by (15) we have

$$\omega(t,x) = -\frac{\nu_{\mathcal{O}}^t(t,x)}{|\nu_{\mathcal{O}}^x(t,x)|}, \quad \text{for all } t \in [0,T] \text{ and } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega_t. \tag{18}$$

Higher regularity result for problem (1)

Combining the results on fixed domain and exploiting the previous corollary we can now state the following theorem, which rigorously extends Theorem 2:

Theorem 8 (Higher regularity and energy balance theorem)

Let the forcing term f be in $H^1(\mathcal{O})$ and assume the initial data satisfy

$$u_0 \in H^2(\Omega_0) \cap H^1_0(\Omega_0), \quad \textit{and} \quad u_1 + \dot{\Phi}(0,\cdot)\cdot\nabla u_0 \in H^1_0(\Omega_0). \tag{19}$$

Then there exists a unique weak solution u of problem (1) in the sense of Definition 1, which satisfies

$$\begin{aligned} &u \in L^\infty(0,T;H^2(\Omega_t)\cap H^1_0(\Omega_t)),\\ &\dot{u} \in L^\infty(0,T;H^1(\Omega_t)),\\ &\ddot{u} \in L^\infty(0,T;L^2(\Omega_t)). \end{aligned}$$

Moreover for every $t \in [0,T]$ the following energy balance holds true:

$$\begin{aligned} \frac{1}{2}\|\dot{u}(t)\|^2_{L^2(\Omega_t)} + \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega_t)} + \int_0^t \int_{\partial\Omega_s} \frac{\omega(s,x)}{2}(1-\omega(s,x)^2)\left(\frac{\partial u}{\partial \nu_{\Omega_s}}(s,x)\right)^2 \mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}s \\ = \frac{1}{2}\|u_1\|^2_{L^2(\Omega_0)} + \frac{1}{2}\|\nabla u_0\|^2_{L^2(\Omega_0)} + \int_0^t \langle f(s),\dot{u}(s)\rangle_{L^2(\Omega_s)}\,\mathrm{d}s, \end{aligned} \tag{20}$$

where ω is the scalar normal velocity.

Application to dynamic debonding

Differently from previously, now also the evolution of the sets $t \mapsto \Omega_t$ is unknown, and it has to be recovered by means of energetic considerations which involve the solution u of problem (1) in an implicit and complex way.

We propose a rigorous definition of the dynamic energy release rate in a general framework, i.e. without any ansatz on the shape of the domains. This allows to state the energetic principle governing the evolution, called Griffith criterion.

We assume that the energy needed to debond a portion of film parametrized on a (measurable) set $E \subseteq \mathbb{R}^N$ is given by

$$\int_E \kappa(x)\,\mathrm{d}x,$$

where $\kappa \in C^0(\mathbb{R}^N)$ is a positive function, representing the toughness of the glue between the film and the substrate. Notice that for every $t \in [0, T]$ there holds

$$\int_{\Omega_t} \kappa(x)\,\mathrm{d}x = \int_{\Omega_0} \kappa(x)\,\mathrm{d}x + \int_0^t \int_{\partial\Omega_s} \omega(s,x)\kappa(x)\,\mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}s.$$

In particular we have

$$|\Omega_t| = |\Omega_0| + \int_0^t \int_{\partial\Omega_s} \omega(s,x)\,\mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}s.$$

Dynamic energy release rate

Let u be the weak solution found in Theorem 8 for a given nondecreasing family Ω_t.

The dynamic energy release rate is the opposite of the (infinitesimal) energy variation due to the change in time of the domain.

Recalling the energy balance (20), we thus consider the internal energy (kinetic and potential) subtracted with the work of external forces:

$$\mathcal{E}(t) := \frac{1}{2}\|\dot{u}(t)\|^2_{L^2(\Omega_t)} + \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega_t)} - \int_0^t \langle f(s), \dot{u}(s)\rangle_{L^2(\Omega_s)}\,\mathrm{d}s. \tag{22}$$

We now provide the definition of dynamic energy release rate.

Definition 2 (Dynamic energy release rate)

For $t \in [0, T]$, we define the dynamic energy release rate of the debonding model as

$$\mathcal{G}(t) := \lim_{h\to 0^+} -\frac{\mathcal{E}(t+h) - \mathcal{E}(t)}{|\Omega_{t+h} \setminus \Omega_t|},$$

whenever such limit exists.

Due to the energy balance (20), we infer that the dynamic energy release rate can be computed as follows:

$$\mathcal{G}(t) = -\frac{\dot{\mathcal{E}}(t)}{\frac{\mathrm{d}}{\mathrm{d}t}|\Omega_{(\cdot)}|(t)} = \frac{\displaystyle\int_{\partial\Omega_t} \frac{\omega(t,x)}{2}(1-\omega(t,x)^2)\left(\frac{\partial u}{\partial \nu_{\Omega_t}}(t,x)\right)^2 \mathrm{d}\mathcal{H}^{N-1}(x)}{\displaystyle\int_{\partial\Omega_t} \omega(t,x)\,\mathrm{d}\mathcal{H}^{N-1}(x)},$$

if

$$\int_{\partial\Omega_t} \omega(t,x)\,\mathrm{d}\mathcal{H}^{N-1}(x) > 0.$$

Definition 3

Given $t \in [0,T]$ and $x \in \partial\Omega_t$ for which $\alpha := \omega(t,x) > 0$, the dynamic energy release rate density at the point (t,x) with speed $\alpha \in (0,1)$ is defined by

$$G_\alpha(t,x) := \lim_{r\to 0^+} \frac{\displaystyle\int_{\partial\Omega_t\cap B_r(x)} \frac{\omega(t)}{2}(1-\omega(t)^2)\left(\frac{\partial u}{\partial \nu_{\Omega_t}}(t)\right)^2 \mathrm{d}\mathcal{H}^{N-1}}{\displaystyle\int_{\partial\Omega_t\cap B_r(x)} \omega(t)\,\mathrm{d}\mathcal{H}^{N-1}} = \frac{1}{2}(1-\alpha^2)\left(\frac{\partial u}{\partial \nu_{\Omega_t}}(t,x)\right)^2 \tag{23}$$

If $\alpha = 0$, the dynamic energy release rate density is extended by continuity, setting:

$$G_0(t,x) := \frac{1}{2}\left(\frac{\partial u}{\partial \nu_{\Omega_t}}(t,x)\right)^2.$$

Remark 9

We notice that the dynamic energy release rate density can be written in an equivalent way by using the relation

$$\dot{u}(t,x) + \omega(t,x)\frac{\partial u}{\partial \nu_{\Omega_t}}(t,x) = 0, \qquad \text{for a.e. } t \in [0,T] \text{ and } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega_t,$$

which follows since $u \equiv 0$ on Γ. Indeed, from the above equality we deduce

$$\begin{aligned} G_{\omega(t,x)}(t,x) &= \frac{1}{2}(1-\omega(t,x)^2)\left(\frac{\partial u}{\partial \nu_{\Omega_t}}(t,x)\right)^2 = \frac{1}{2}\frac{1-\omega(t,x)}{1+\omega(t,x)}\left[(1+\omega(t,x))\frac{\partial u}{\partial \nu_{\Omega_t}}(t,x)\right]^2 \\ &= \frac{1}{2}\frac{1-\omega(t,x)}{1+\omega(t,x)}\left[\frac{\partial u}{\partial \nu_{\Omega_t}}(t,x) - \dot{u}(t,x)\right]^2. \end{aligned} \tag{24}$$

Given a positive toughness $\kappa \in C^0(\mathbb{R}^N)$, we now postulate that during the evolution process the following energy balance is satisfied:

$$\mathcal{E}(t) + \int_{\Omega_t \setminus \Omega_0} \kappa(x)\,\mathrm{d}x = \mathcal{E}(0), \qquad \text{for every } t \in [0,T]. \tag{25}$$

We observe that the energy is conserved if one requires

$$\omega(t,x)\kappa(x) = \omega(t,x)G_{\omega(t,x)}(t,x), \quad \text{for a.e. } t \in [0,T] \text{ and for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega_t.$$

However, the above condition is not sufficient to determine a proper evolution of the sets Ω_t, indeed $\omega \equiv 0$ (i.e. $\Omega_t \equiv \Omega_0$) is always an admissible choice.

A stronger requirement is the following local maximum dissipation principle, which essentially says that Ω_t grows whenever it is possible, while preserving the energy balance:

$$\omega(t,x) = \max\{\alpha \in [0,1) : \alpha\kappa(x) = \alpha G_\alpha(t,x)\}, \tag{26}$$

for a.e. $t \in [0,T]$ and for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega_t$. Then we have:

Proposition 10

The following three conditions are equivalent:

- *the local maximum dissipation principle* (26) *holds true;*
- *the local dynamic Griffith criterion holds true, namely*

$$\begin{cases} 0 \le \omega(t,x) < 1, \\ G_{\omega(t,x)}(t,x) \le \kappa(x), \\ \omega(t,x)\left[G_{\omega(t,x)}(t,x) - \kappa(x)\right] = 0, \end{cases} \quad \text{for a.e. } t \in [0,T] \text{ and for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega_t; \tag{27}$$

- *for a.e.* $t \in [0,T]$ *and for* $\mathcal{H}^{N-1}$*-a.e.* $x \in \partial\Omega_t$ *the scalar normal velocity* ω *is given by*

$$\omega(t,x) = \max\left\{ \frac{\left[\frac{\partial u}{\partial \nu_{\Omega_t}}(t,x) - \dot{u}(t,x)\right]^2 - 2\kappa(x)}{\left[\frac{\partial u}{\partial \nu_{\Omega_t}}(t,x) - \dot{u}(t,x)\right]^2 + 2\kappa(x)},\ 0 \right\}. \tag{28a}$$

We are now in the position to provide a proper formulation of a dynamic debonding model. We point out that the resulting system features a strong coupling: indeed, the evolution of the domain of the wave equation is governed by (28a), which in turn depends on the solution u to the wave equation itself.

Given the following data:

- $\Omega_0 \subseteq \mathbb{R}^N$ satisfying of class C^2 (or convex),
- $\kappa \in C^0(\mathbb{R}^N)$ satisfying $\kappa(x) > 0$ for all $x \in \mathbb{R}^N$,
- $f \in H^1(0,T; L^2_{\mathrm{loc}}(\mathbb{R}^N)) \cap L^2(0,T; H^1_{\mathrm{loc}}(\mathbb{R}^N))$,
- $u_0 \in H^2(\Omega_0) \cap H^1_0(\Omega_0)$ and $u_1 \in H^1(\Omega_0)$ satisfying

$$\text{either} \quad u_1(x) = 0 \quad \text{and} \quad \left(\frac{\partial u_0}{\partial \nu_{\Omega_0}}(x)\right)^2 \leq 2\kappa(x),$$

$$\text{or} \quad u_1(x) \neq 0, \quad \left(\frac{\partial u_0}{\partial \nu_{\Omega_0}}(x)\right)^2 - u_1(x)^2 = 2\kappa(x) \quad \text{and} \quad \frac{\frac{\partial u_0}{\partial \nu_{\Omega_0}}(x)}{u_1(x)} < -1,$$

for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega_0$,

Coupled problem (1)&(26)

Definition 4 (Weak solution of the coupled problem)

We say that an evolution $[0,T] \ni t \mapsto (u(t), \Omega_t)$ *is a weak solution of the coupled problem* (1)&(26) *if the following conditions are satisfied:*

1. *there exists a map* $\Phi\colon [0,T] \times \overline{\Omega}_0 \to \mathbb{R}^N$ *with "space-inverse"* $\Psi(t,\cdot)$ *satisfying* (5), (H_1') *and* (H_2), *for which*
$$\Omega_t = \Phi(t, \Omega_0), \quad \text{for every } t \in [0,T];$$
2. *u is the weak solution to problem* (1) *with forcing term f and initial data* u_0, u_1;
3. *the local maximum dissipation principle* (26) *is satisfied, or equivalently the scalar normal velocity* $\omega(t,x) = \dot{\Phi}(t, \Psi(t,x)) \cdot \nu_{\Omega_t}(x)$ *fulfils* (28a) *for a.e.* $t \in [0,T]$ *and for* $\mathcal{H}^{N-1}$*-a.e.* $x \in \partial\Omega_t$.

We conclude by showing how Definition 4 covers the particular cases of the 1-dimensional and radial models already analysed in literature.

In fact, in those cases the notion of solution to the coupled problem is given in a slightly different form, and the existence is obtained by exploiting d'Alembert's formula. We prove that, if the initial data are well-prepared, the solution found in the above mentioned works fulfils Definition 4, at least for short times.

Theorem 11 (1-dim coupled problem)

Let $\ell_0 > 0$ and let $\kappa \in C^{1,1}_{\mathrm{loc}}([\ell_0, +\infty))$ satisfy $\kappa(x) > 0$ for all $x \in [\ell_0, +\infty)$. Assume that $f \in C^{0,1}([0,T] \times [0,+\infty))$, $u_0 \in C^{2,1}([0,\ell_0])$ and $u_1 \in C^{1,1}([0,\ell_0])$ satisfy

$$u_0(0) = 0, \quad u_0(\ell_0) = 0 \quad \text{and} \quad u_1(0) = 0,$$

$$u_1(\ell_0) \neq 0, \quad u_0'(\ell_0)^2 - u_1(\ell_0)^2 = 2\kappa(\ell_0), \quad \frac{u_0'(\ell_0)}{u_1(\ell_0)} < -1.$$

Then, there exist $T^* \in (0,T]$ and a unique weak solution $t \mapsto (u(t), (0, \ell(t)))$ to the coupled problem (1)&(26) in $[0, T^*]$ in the sense of Definition 4.

Theorem 12 (2-dim radial coupled problem)

Let $R > \rho_0 > 0$ and let $\kappa \in C^{1,1}_{\mathrm{rad}}(\overline{B_R(0)})$ satisfy $\kappa(x) > 0$ for all $x \in \overline{B_R(0)}$. Setting $\Omega_0 := \{x \in \mathbb{R}^2 : R - \rho_0 < |x| < R\}$, assume that $f \in C^{0,1}([0,T]; C^{0,1}_{\mathrm{rad}}(\overline{B_R(0)}))$, $u_0 \in C^{2,1}_{\mathrm{rad}}(\overline{\Omega_0})$ and $u_1 \in C^{1,1}_{\mathrm{rad}}(\overline{\Omega_0})$ satisfy

$$\begin{aligned} u_0(x) &= 0 \quad \text{if } |x| = R \text{ or } |x| = R - \rho_0, \\ u_1(x) &= 0 \quad \text{if } |x| = R, \end{aligned}$$

$$u_1(x) \neq 0, \quad \left(\frac{\partial u_0}{\partial \nu_{\Omega_0}}(x)\right)^2 - u_1(x)^2 = 2\kappa(x) \quad \text{and} \quad \frac{\frac{\partial u_0}{\partial \nu_{\Omega_0}}(x)}{u_1(x)} < -1 \quad \text{if } |x| = R - \rho_0.$$

Then, there exist $T^* \in (0,T]$ and a unique weak solution $t \mapsto (u(t), \Omega_t)$ to the coupled problem (1)&(26) in $[0, T^*]$ in the sense of Definition 4, where

$$\Omega_t := \{x \in \mathbb{R}^2 : R - \rho(t) < |x| < R\}, \quad \text{for a suitable } \rho \in C^{2,1}([0, T^*]).$$

Thanks for your attention!