Sharp stability of higher order Dirichlet eigenvalues

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Plan

1. Presentation of the results
2. Proof of the linear bound
3. Some applications and open questions

Presentation of the results

Notations

$n \geq 2$ is fixed, B is the unit ball of $\mathbb{R}^n$, $\mathcal{A} = \{\Omega \subset \mathbb{R}^n$ open s.t. $|\Omega| = |B|\}$.

$$\lambda_k(\Omega) = \inf \left\{ \sup_{v \in V} \frac{\int_\Omega |\nabla v|^2}{\int_\Omega v^2},\ V \subset H_0^1(\Omega) : \dim(V) = k \right\}$$

We write $(u_k)_{k \in \mathbb{N}^*}$ the L^2-normalized eigenfunctions: they verify

$$\begin{cases} -\Delta u_k = \lambda_k(\Omega) u_k & (\Omega) \\ u_k = 0 & (\partial\Omega) \end{cases}$$

Faber-Krahn inequality:

$$\lambda_1(\Omega) \geq \lambda_1(B)$$

with equality if and only if $\Omega = B$.

Structure of $(\lambda_k(B))$

When $n = 2$:

- $j_{m,p}$: p-th positive zero of the m-th Bessel function J_m.
- $\{\lambda_k(B), k \in \mathbb{N}^*\} = \{j_{m,p}^2,\ m \in \mathbb{N}, p \in \mathbb{N}^*\}$
- $u_{m,p}(re^{i\theta}) = \begin{cases} J_0(j_{0,p} r) & \text{if } m = 0 \\ J_m(j_{m,p} r)\cos(m\theta + \phi) & \text{if } m \geq 1 \end{cases}.$

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In general:

- $\{\lambda_k(B), k \in \mathbb{N}^*\} = \{j_{m+\frac{n-2}{2},p}^2,\ m \in \mathbb{N}, p \in \mathbb{N}^*\}$
- $\mathbb{H}_m[X_1, \dots, X_n]$: homogeneous harmonic polynomials of degree m.
- $u_{m,p}(x) = \dfrac{J_{m+\frac{n-2}{2}}\left(j_{m+\frac{n-2}{2},p}x\right)}{|x|^{\frac{n-2}{2}}} P\left(\frac{x}{|x|}\right),\ P \in \mathbb{H}_m[X_1, \dots, X_n].$

Torsional rigidity

$$T(\Omega) = \sup_{v \in H_0^1(\Omega)} \int_\Omega \left(2v - |\nabla v|^2\right)$$

We write w the associated torsion function that verifies

$$\begin{cases} -\Delta w = 1 & (\Omega) \\ w = 0 & (\partial\Omega) \end{cases}$$

We also have

$$T(\Omega)^{-1} = \inf_{v \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla v|^2}{\left(\int_\Omega v\right)^2}$$

Saint-Venant inequality:

$$T(\Omega) \leq T(B)$$

with equality if and only if $\Omega = B$.

Initial question

Suppose $\lambda_1(\Omega)$ is close to $\lambda_1(B)$, can we say $\lambda_k(\Omega)$ is close to $\lambda_k(B)$?

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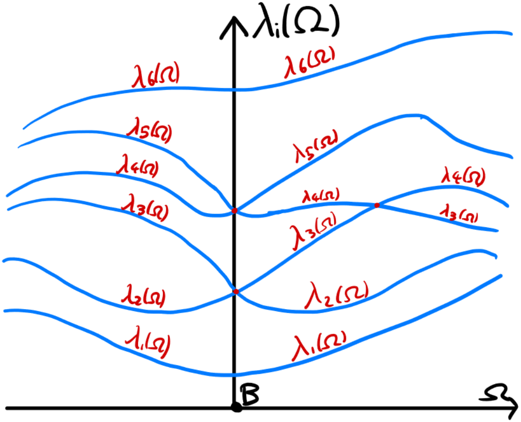
Let $\Omega = B_{t\zeta} := (I + t\zeta)(B)$ for some small $\zeta \in \mathcal{C}_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$, then:



Initial question

We have directional derivatives:

$$\lambda_k'(B)\cdot\zeta := \frac{d}{dt}\Big|_{t=0^+}\lambda_k(B_{t\zeta})$$

considered for any $\zeta \in \mathcal{C}^\infty$ with $\int_{\partial B}\zeta\cdot\nu_B = 0$.

- If $\lambda_k(B)$ is simple, then u_k is radial and

$$\lambda_k'(B)\cdot\zeta = -\int_{\partial B}(\zeta\cdot\nu_B)|\nabla u_k|^2 = 0$$

- If $\lambda_k(B)$ is multiple, $\lambda_k'(B)\neq 0$.

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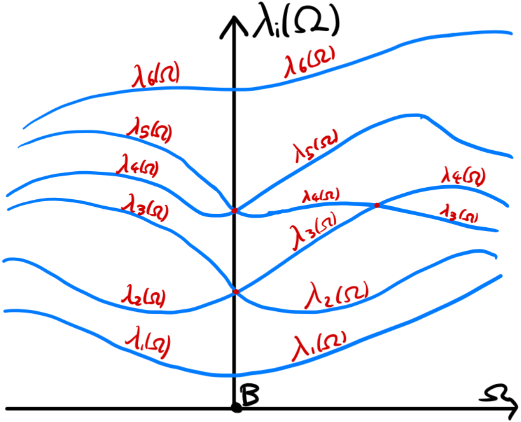
- If $\lambda_k(B)$ is multiple, $\lambda_k'(B)\neq 0$.
- If ζ is normal to ∂B, $|B_\zeta| = |B|$, $\mathrm{bar}(B_\zeta) = 0$, $\|\zeta\|_{\mathcal{C}^3(\partial B)} \ll 1$, then

$$\lambda_1(B_\zeta) - \lambda_1(B) \geq c_n\|\zeta\cdot\nu_B\|^2_{H^{1/2}(\partial B)}$$

(Brasco, De Philippis, Velichkov - 2015)



$$|\lambda_k(\Omega) - \lambda_k(B)| \leq \begin{cases} C_{n,k}(\lambda_1(\Omega) - \lambda_1(B)) & \text{if } \lambda_k(B) \text{ is simple} \\ C_{n,k}(\lambda_1(\Omega) - \lambda_1(B))^{\frac{1}{2}} & \text{if } \lambda_k(B) \text{ is degenerate} \end{cases}$$

Some known results

- (Bertrand, Colbois - 2005) When $\lambda_1(\Omega)$ is close to $\lambda_1(B)$:

$$|\lambda_k(\Omega) - \lambda_k(B)| \lesssim (\lambda_1(\Omega) - \lambda_1(B))^{\frac{1}{80n}}.$$

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- (Mazzoleni, Pratelli - 2019)
  $n = 2$:

$$-(\lambda_1(\Omega) - \lambda_1(B))^{\frac{1}{4} - o(1)} \lesssim \lambda_k(\Omega) - \lambda_k(B) \lesssim (\lambda_1(\Omega) - \lambda_1(B))^{\frac{1}{8} - o(1)}$$

  $n = 3$:

$$-(\lambda_1(\Omega) - \lambda_1(B))^{\frac{1}{6} - o(1)} \lesssim \lambda_k(\Omega) - \lambda_k(B) \lesssim (\lambda_1(\Omega) - \lambda_1(B))^{\frac{1}{12} - o(1)}$$

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- (Brasco, De Phillipis, Velichkov - 2015)

$$\inf_{x\in\mathbb{R}^n} |\Omega\Delta(B+x)| \lesssim \sqrt{\lambda_1(\Omega) - \lambda_1(B)}.$$

Sharp bound for degenerate eigenvalues

Theorem

There exists $C_n > 0$ such that for any $\Omega \in \mathcal{A}$,

$$|\lambda_k(\Omega) - \lambda_k(B)| \leq C_n k^{2+\frac{4}{n}} \lambda_1(\Omega)^{\frac{1}{2}} \sqrt{\lambda_1(\Omega) - \lambda_1(B)}$$

Kohler-Jobin inequality: $\Omega(\in \mathcal{A}) \mapsto T(\Omega)^{\frac{2}{n+2}} \lambda_1(\Omega)$ is minimal on the ball. As a consequence,

$$T(\Omega)^{-1} - T(B)^{-1} \leq C_n(\lambda_1(\Omega) - \lambda_1(B)).$$

In the theorem we actually prove:

$$|\lambda_k(\Omega) - \lambda_k(B)| \leq C_n k^{2+\frac{4}{n}} \lambda_1(\Omega)^{\frac{1}{2}} \sqrt{T(\Omega)^{-1} - T(B)^{-1}}$$

then apply Kohler-Jobin.

Sharp bound for simple eigenvalues

Theorem

Let k be such that $\lambda_k(B)$ is simple, there exists $C_{n,k} > 0$ such that for any $\Omega \in \mathcal{A}$,

$$|\lambda_k(\Omega) - \lambda_k(B)| \leq C_{n,k}(\lambda_1(\Omega) - \lambda_1(B))$$

We define the spectral gap

$$g(k) = \operatorname{dist}(\lambda_k(B), \{\lambda_i(B), i \in \mathbb{N}^*\} \setminus \{\lambda_k(B)\})$$

We can take

$$C_{n,k} = C_n \frac{k^{4+\frac{8}{n}}}{g(k)}.$$

In dimension 2 the valid choices of k are

$$k = (1), 6, 15, 30, 51, 74, 105, 140, 175, 222, 269, 326, 383, 446, 517, 588, \ldots$$

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Sharp bound for a group of eigenvalues

Theorem

For every $k \leq l$, such that

$$\lambda_{k-1}(B) < \lambda_k(B) = \lambda_l(B) < \lambda_{l+1}(B),$$

then there exists $C_{n,k} > 0$ such that for any open set $\Omega \in \mathcal{A}$,

$$\left|\sum_{i=k}^{l}\left[\lambda_i(\Omega) - \lambda_i(B)\right]\right| \leq C_{n,k}\left(\lambda_1(\Omega) - \lambda_1(B)\right)$$

We can take $C_{n,k} = C_n \frac{k^{6+\frac{8}{n}}}{g(k)}$. Example in $2d$:

$$\left|\frac{\lambda_2(\Omega) + \lambda_3(\Omega)}{2} - \lambda_2(B)\right| \leq C(\lambda_1(\Omega) - \lambda_1(B))$$

so $\begin{cases} \lambda_2(\Omega) - \lambda_2(B) \\ \lambda_3(B) - \lambda_3(\Omega) \end{cases} \leq C(\lambda_1(\Omega) - \lambda_1(B)).$

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Proof of the linear bound

An equivalent formulation

Let $k \leq l$ be such that $\lambda_{k-1}(B) < \lambda_k(B) = \lambda_l(B) < \lambda_{l+1}(B)$, then

$$\forall \Omega \in \mathcal{A}, \ \left| \sum_{i=k}^{l} \left[\lambda_i(\Omega) - \lambda_i(B) \right] \right| \leq C_{n,k}\left(T(\Omega)^{-1} - T(B)^{-1}\right)$$

$$\Leftrightarrow \forall \Omega \in \mathcal{A}, \ T(\Omega)^{-1} \pm \frac{1}{C_{n,k}} \sum_{i=k}^{l} \lambda_i(\Omega) \geq T(B)^{-1} \pm \frac{1}{C_{n,k}} \sum_{i=k}^{l} \lambda_i(B)$$

Theorem

There exists $\delta_{n,k} > 0$ such that for any $|\delta| \leq \delta_{n,k}$,

$$\Omega \in \mathcal{A} \mapsto \frac{1}{T(\Omega)} + \delta \sum_{i=k}^{l} \lambda_i(\Omega)$$

is minimized by the ball.

Plan of proof

When δ is small enough, then:

1) There exists a minimizer Ω among quasi-open sets.

2) Ω is open and $\sup |\nabla w_\Omega| \leq C_n$.

3) $\Omega = \phi(B)$ where $\|\phi - \mathrm{Id}\|_{\mathcal{C}^3} \ll 1$.

4) $T^{-1} + \delta \sum_{i=k}^{l} \lambda_i$ is minimal on the ball in a small $\mathcal{C}^3$ neighbourhood of the ball.

Conclusion: there exists a minimizer Ω, and Ω is the ball.

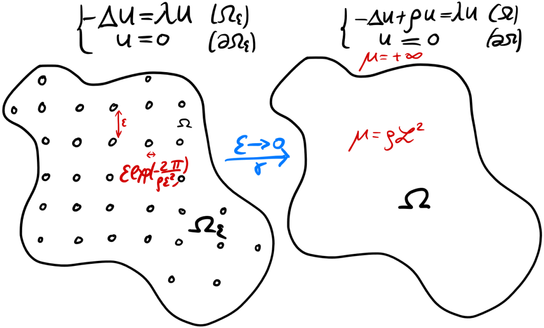
First existence result

Lemma

If δ is small enough, then $\Omega \in \mathcal{A} \mapsto \frac{1}{T(\Omega)} + \delta \sum_{i=k}^{l} \lambda_i(\Omega)$ has a minimizer Ω. Moreover, w_Ω is Lipschitz and

$$B_{1-o_{\delta\to 0}(1)} \subset \Omega \subset B_{1+o_{\delta\to 0}(1)}$$

When $\delta < 0$, it must be small enough to avoid this kind of phenomena



Overdetermined equation on $(w, u_k, \ldots, u_l)$

Suppose everything is completely smooth, then a shape derivative would give the following equation on $(w, u_k, \ldots, u_l)$:

$$\begin{cases} -\Delta w = 1, \ -\Delta u_i = \lambda_i u_i & (\Omega) \\ w = u_i = 0 & (\partial\Omega) \\ (\partial_\nu w)^2 + \delta T(\Omega)^2 \sum_{i=k}^{l} (\partial_\nu u_i)^2 = Q & (\partial\Omega) \end{cases}$$

where ∂_ν is the inward normal derivative and $Q := \frac{1}{n^2} + \mathcal{O}(|\delta|)$.

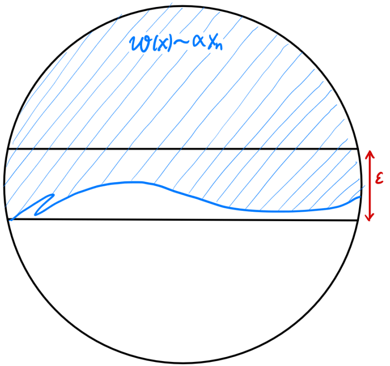
Serrin's theorem: if $\partial_\nu w_\Omega = \frac{1}{n}$, then $\Omega = B$.

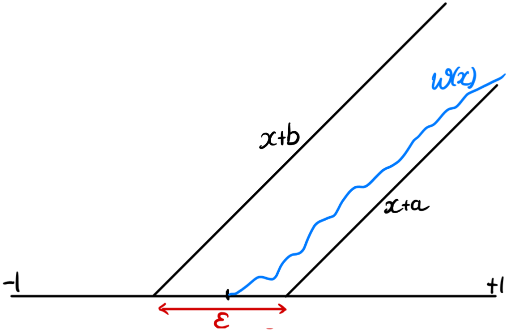
Here we have $\partial_\nu w_\Omega = \frac{1}{n} + \mathcal{O}(|\delta|)$.

Flat solution

Let $e \in \mathbb{S}^{n-1}$, $\epsilon > 0$, a function $w \in H^1(B_r, \mathbb{R}_+)$ is ϵ, e-flat in B_r if

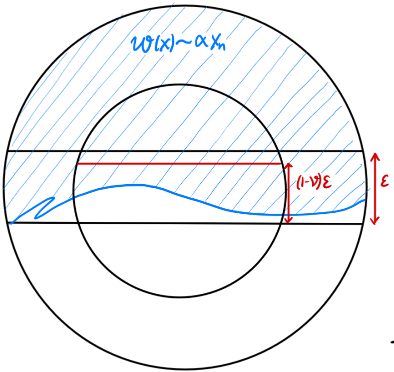
- $0 \in \partial\{w > 0\}$.
- $\alpha(x \cdot e + a)_+ \leq w(x) \leq \alpha(x \cdot e + b)_+$ where $\alpha > 0$, $b - a \leq \epsilon r$.
- $|\Delta w| \leq \alpha\epsilon^2$ in $B_r \cap \{w > 0\}$.

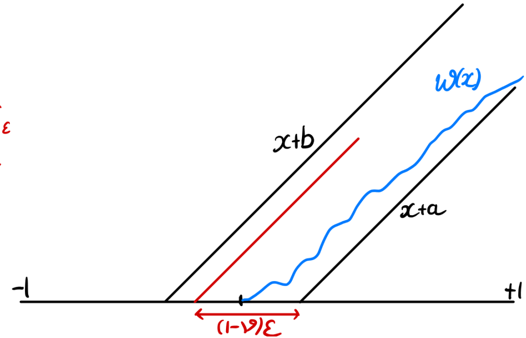




Partial Harnack inequality

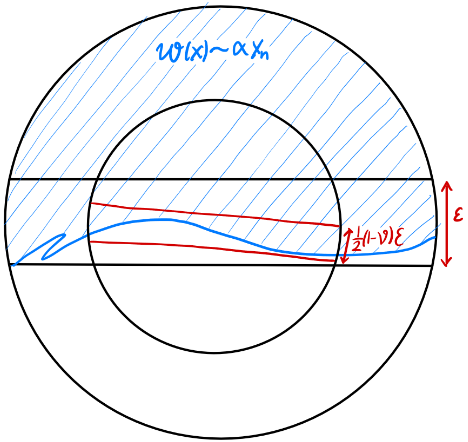
Let Ω be a minimizer, if $(w_\Omega \pm c\sqrt{|\delta|}u_{i,\Omega})_{i=k\dots,l}$ are ϵ, e-flat in B_r, where $r, \epsilon \ll 1$, then $(w_\Omega \pm c\sqrt{|\delta|}u_{i,\Omega})_{i=k\dots,l}$ are $(2(1-\nu)\epsilon, e)$-flat in $B_{\frac{1}{2}r}$.





Improvement of flatness

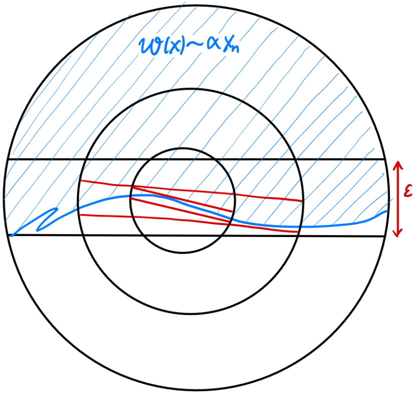
Let Ω be a minimizer, if $(w_\Omega \pm c\sqrt{|\delta|}u_{i,\Omega})_{i=k\dots,l}$ are ϵ, e-flat in B_r, where $r,\epsilon \ll 1$, then $(w_\Omega \pm c\sqrt{|\delta|}u_{i,\Omega})_{i=k\dots,l}$ are $((1-\nu)\epsilon, e')$-flat in $B_{\frac{1}{2}r}$ for some $e' \in \mathbb{S}^{n-1}$.



Improvement of flatness (II)

We can iterate this result: if $(w_\Omega \pm c\sqrt{|\delta|}u_{i,\Omega})_{i=k\dots,l}$ are ϵ, e-flat in B_r, then they are $((1-\nu)^p\epsilon, e^p)$-flat in $B_{2^{-p}r}$ for some sequence $(e^p)_p$.

Let Ω be a minimizer, if $(w_\Omega \pm c\sqrt{|\delta|}u_{i,\Omega})_{i=k\dots,l}$ are ϵ, e-flat in B_r, where $r, \epsilon \ll 1$, then $\partial\Omega \cap B_{\frac{1}{2}r}$ is a $\mathcal{C}^{1,\gamma}$ graph. Here $\gamma := \log_2 \frac{1}{1-\nu}$.



Statement of the regularity result

Lemma

For any minimizer Ω of $T^{-1} + \delta \sum_i \lambda_i$ in $\mathcal{A}$, up to a translation of Ω we have

$$\Omega = \phi(B) \text{ where } \|\phi - Id\|_{\mathcal{C}^3(B)} = o_{\delta \to 0}(1).$$

- First step: $B_{1-o(1)} \subset \Omega \subset B_{1+o(1)}$.
- Then, $\Omega = \phi(B)$ where $\|\phi - 1\|_{\mathcal{C}^{1,\gamma}} = o(1)$ by improvement of flatness.
- Finally, $\|\phi - 1\|_{\mathcal{C}^3} = o(1)$ by hodograph transform.

Conclusion

Lemma

Suppose $\partial\Omega = \{(1+h(x))x,\ x \in \partial B\}$ *where* $\|h\|_{\mathcal{C}^3(\partial B)} \ll 1$, $|\Omega| = |B|$, $bar(\Omega) = 0$, *then*

$$T(\Omega)^{-1} - T(B)^{-1} \geq c_n \|h\|^2_{H^{1/2}(\partial B)}$$

$$\left| \sum_{i=k}^{l} \left[\lambda_i(\Omega) - \lambda_i(B) \right] \right| \leq C_{n,k} \|h\|^2_{H^{1/2}(\partial B)}$$

Consequence: let Ω be as in the lemma, then

$$\begin{aligned} T(\Omega)^{-1} + \delta\lambda_k(\Omega) &\geq T(B)^{-1} + \delta\lambda_k(B) + (c_n - C_{n,k}|\delta|)\, \|h\|^2_{H^{1/2}(\partial B)} \\ &\geq T(B)^{-1} + \delta\lambda_k(B) \text{ when } |\delta| \leq \frac{c_n}{C_{n,k}} \end{aligned}$$

Some applications and open questions

Application: reverse Kohler-Jobin

Kohler-Jobin inequality: $\Omega(\in \mathcal{A}) \mapsto T(\Omega)^{\frac{2}{n+2}}\lambda_1(\Omega)$ is minimal on the ball.

We get an opposite inequality : there exists $\delta > 0$ small enough such that $\Omega \in \mathcal{A} \mapsto T(\Omega)^{-1} - \delta\lambda_1(\Omega)$ is minimal on the ball.

Corollary

There exists $c_n, C_n > 0$ such that for any $\Omega \in \mathcal{A}$:

$$c_n \leq \frac{T(\Omega)^{-1} - T(B)^{-1}}{\lambda_1(\Omega) - \lambda_1(B)} (\leq C_n).$$

Corollary

There exists $p_n > 1$ such that for any $p > p_n$,

$$\Omega \in \mathcal{A} \mapsto T(\Omega)^p \lambda_1(\Omega)$$

is maximal on the ball.

What about more general functions ?

Is this still true for

$$Z_\Omega(t) = \sum_{k\geq 1} e^{-t\lambda_k(\Omega)},\ \zeta_\Omega(s) = \sum_{k\geq 1} \lambda_k(\Omega)^{-s}\ ?$$

We remind that

$$|\lambda_k(\Omega) - \lambda_k(B)| \leq C_n k^{2+\frac{4}{n}} \lambda_1(\Omega)^{\frac{1}{2}} \sqrt{\lambda_1(\Omega) - \lambda_1(B)} \text{ for any } k$$

$$\left| \sum_{i=k}^{l} \left[\lambda_i(\Omega) - \lambda_i(B) \right] \right| \leq C_n \frac{k^{6+\frac{8}{n}}}{g(k)} \left(\lambda_1(\Omega) - \lambda_1(B) \right) \text{ for a cluster } [k,l]$$

where

$$g(k) = \text{dist}(\lambda_k(B), \{\lambda_i(B), i \in \mathbb{N}^*\} \setminus \{\lambda_k(B)\})$$

Issue: we know no reasonable lower bound on $g(k)$!

Open question about $(\lambda_k(B))_{k\in\mathbb{N}^*}$

Is there some $C, a > 0$ such that $g(k) > Ck^{-a}$?

- If we replace B with $[0,1]\times[0,L]$, this is true when L is algebraic. It is false when
$$L = \sum_{i\geq 1} 10^{-i!}.$$
- The eigenvalues are given by $(j^2_{p+\frac{n}{2}-1,k})_{p\in\mathbb{N},k\in\mathbb{N}^*}$ where $j_{\mu,k}$ is the k-th zero of the μ-th Bessel function J_μ.

  Bourget's hypothesis (proved by Siegel): if $\mu,\nu\in\mathbb{N}/2$, $\mu-\nu\in\mathbb{N}^*$, then for any $k,l\in\mathbb{N}^*$:
$$j_{\mu,k}\neq j_{\nu,l}$$

  Our (?) hypothesis: $\exists C, a > 0$ s.t. $|j_{\mu,k}-j_{\nu,l}| > Cj_{\mu,k}^{-a}$.

Thank you for your attention !