Stability in shape optimization under convexity constraint

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Shape Optimization, Geometric Inequalities, and Related Topics,
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1. Introduction
    - Shape optimization under convexity constraint
    - Stability
2. The *selection principle* method
    - General strategy
    - General strategy applied to the QII
3. Stability of the ball for $P - \varepsilon\lambda_1$ under c.c.
    - Regularity theory of the perimeter under c.c.
    - (IT) property

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Example. Isoperimetric problem:

$$\min \{P(A),\ A \subset \mathbb{R}^n,\ |A| = 1\} = P(B)$$

$P(A)$: Perimeter of A.
B: ball of measure $|B| = 1$.

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Definition (Stability)

B is said to be stable for $J + \varepsilon R$ if there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ the problem

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$$\forall A \in \mathcal{A} \text{ with } |A \Delta B| \ll 1, \ J(A) + \varepsilon R(A) \geq J(B) + \varepsilon R(B) \quad (S_\varepsilon)$$

with equality only if (up to translation) $A = B$.

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Examples

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1) **Quantitative iso. ineq. [Fusco, Maggi, Pratelli, '08].**

$$\text{If } |A| = 1, \ \delta_F(A) := \inf\{|A\Delta(B+x)|, \ x \in \mathbb{R}^n\}$$

The ball is stable for $P - \varepsilon\delta_F^2$: there exists $\varepsilon_n > 0$ s.t. for all $A \subset \mathbb{R}^n$ with $|A| = 1$

$$P(A) - P(B) \geq \varepsilon_n \delta_F(A)^2 \qquad \text{(QII)}$$

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The ball is stable in 2d for $P - \varepsilon\lambda_1$ among simply connected sets: there exists $\varepsilon > 0$ s.t. for all $A \subset \mathbb{R}^2$ open sply co. with $|A| = 1$

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$$\exists A_j \subset \mathbb{R}^n \text{ open, } |A_j| = 1 \text{ and } \begin{cases} (P - \varepsilon\lambda_1)(A_j) < (P - \varepsilon\lambda_1)(B) \\ |A_j \Delta B| \to 0 \text{ avec } j \to \infty \end{cases}$$

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$\rightsquigarrow$ Geometric constraint to ensure stability in dimension ≥ 3 ?

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Step 2 : Stability for all sets thanks to a regularity theory.

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Method: differential calculus with repect to $\xi \in X$:

$$\left.\frac{d}{d\xi}\right|_{\xi=0} (J + \varepsilon R)(B_\xi) = 0, \quad \left.\frac{d^2}{d\xi^2}\right|_{\xi=0} (J + \varepsilon R)(B_\xi) \text{ is coercive}$$

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2. We build $\widetilde{A}_j \in \mathcal{A}$ s.t. (1) is still verified for $(\widetilde{A}_j)$ and

$$\exists \xi_j \in X, \begin{cases} \widetilde{A}_j = (\mathrm{Id} + \xi_j)(B) \\ \|\xi_j\|_X \to 0 \end{cases}$$

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3. We conclude using Step 1.

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$$\forall\big(\|\phi\|_{W^{1,\infty}} \ll 1, |B_\phi| = 1\big),\ P(B_\phi) - P(B) \geq c\|\phi\|_{H^1}^2$$
$$\big(\gtrsim \delta_F(B_\phi)^2\big)$$

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Idea :

$$P(B_\phi) = \int_{\partial B} f\left(\phi, \nabla_\tau \phi\right) d\mathcal{H}^{n-1}$$

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$\rightsquigarrow$ Expansion of f at second order.

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Definition (q.m.p)

$A \subset \mathbb{R}^n$ is a (Λ, r_0)-quasi-minimizer of the perimeter (q.m.p.) if for all $0 < r < r_0$ and $F \Delta A \Subset B_r(x)$

$$P(A) \leq P(F) + \Lambda r^n$$

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Theorem (Regularity and cvg. of the q.m.p., Tamanini '88...)

If (A_j) is a sequence of (Λ, r_0)-q.m.p such that $|A_j \Delta B| \to 0$, then there exists $\phi_j \in C^{1,1/2}(\partial B)$ with

$$A_j = (Id + \phi_j \nu_B)(B) \text{ and } \|\phi_j\|_{C^{1,\alpha}(\partial B)} \to 0 \text{ for } \alpha \in (0, 1/2)$$

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1. There exists $|A_j| = 1$ s.t.

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3. We conclude thanks to the Thm. of cvg. of the q.m.p. and Step 1.

Goal: Stability for $P - \varepsilon\lambda_1$ under convexity constraint thanks to the *selection principle*.

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Theorem (Sharp stability of the ball; Lamboley, P.)

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$$\forall K \in \mathcal{K}_1^n \text{ with } |K \Delta B| \ll 1,\ P(K) - \varepsilon\lambda_1(K) \geq P(B) - \varepsilon\lambda_1(B)$$

with equality only if (up to translation) $K = B$.

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– *For* $\varepsilon \in (\varepsilon_n, \infty)$, *the ball is not stable for* $P - \varepsilon\lambda_1$ *under c.c.*

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– Non sharp stability is already known in any dimensions:

In 2d : Payne-Weinberger for the simply connected sets.

In dimensions ≥ 3 : Brandolini, Nitsch, Trombetti ('10) for the convex sets.

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(A q.m.p.c.c. is not a q.m.p. !)

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Let (K_j) be a sequence of (Λ,η)-q.m.p.c.c. s.t. $|K_j\Delta B|\to 0$. Then there exists $\phi_j\in C^{1,1}(\partial B)$ s.t. $K_j=(Id+\phi_j\nu_B)(B)$ and

$$\forall\alpha\in(0,1),\ \|\phi_j\|_{C^{1,\alpha}(\partial B)}\to 0$$

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$$\forall\alpha\in(0,1),\ \|\phi_j\|_{C^{1,\alpha}(\partial B)}\to 0$$

Optimal in general: there exists a q.m.p.c.c. $\mathcal{C}^{1,1}$ which is not $\mathcal{C}^2$.

$$\forall\Big(\widetilde{K}\in\mathcal{K}^n,\ |K\Delta\widetilde{K}|\leq\eta\Big),\ P(K)\leq P(\widetilde{K})+\Lambda|K\Delta\widetilde{K}| \qquad \text{(q.m.p.c.c.)}$$

Step 2

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$\rightsquigarrow$ The space X aimed for is $\mathcal{C}^{1,\alpha}(\partial B)$.

1. Introduction
   - Shape optimization under convexity constraint
   - Stability
2. The *selection principle* method
   - General strategy
   - General strategy applied to the QII
3. Stability of the ball for $P - \varepsilon\lambda_1$ under c.c.
   - Regularity theory of the perimeter under c.c.
   - (IT) property

Step 1

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$\rightsquigarrow$ Expand $P(B_\phi) - \varepsilon\lambda_1(B_\phi)$ at second order ((IT) property).

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P verifies $(IT)_{H^1, W^{1,\infty}}$: there exists a m.o.c. $\omega : \mathbb{R}^+ \to \mathbb{R}$ such that if $\phi \in W^{1,\infty}(\partial B)$,

$$P(B_\phi) = P(B) + P'(B) \cdot \phi + \frac{1}{2} P''(B) \cdot (\phi, \phi) + \omega\left(\|\phi\|_{W^{1,\infty}}\right) \|\phi\|_{H^1}^2$$

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(IT) 1st Dirichlet egv (Lamboley, P.)

There exists $\alpha \in (0,1)$ such that λ_1 verifies $(IT)_{H^1,\mathcal{C}^{1,\alpha}}$: for $\phi \in \mathcal{C}^{1,\alpha}(\partial B)$,

$$\lambda_1(B_\phi) = \lambda_1(B) + \lambda_1'(B)\cdot\phi + \frac{1}{2}\lambda_1''(B)\cdot(\phi,\phi) + \omega\left(\|\phi\|_{\mathcal{C}^{1,\alpha}}\right)\|\phi\|_{H^1}^2$$

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There exists $\alpha \in (0,1)$ *such that* $P_\varepsilon := P - \varepsilon\lambda_1$ *verifies* $(IT)_{H^1, \mathcal{C}^{1,\alpha}}$*: there exists a m.o.c.* $\omega : \mathbb{R}^+ \to \mathbb{R}$ *such that for* $\phi \in \mathcal{C}^{1,\alpha}(\partial B)$*,*

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Moreover, for all $\varepsilon \in (0, \varepsilon_n)$,

$$P''_\varepsilon(B)\cdot(\phi,\phi) \gtrsim \|\phi\|^2_{H^1}$$

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$\Longrightarrow$ Stability for $\mathcal{C}^{1,\alpha}$ perturbations.
$\Longrightarrow$ Stability under c.c. thanks to the regularity theory.

Thank you for your attention !