On the critical $p-$Laplace equation

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Politecnico di Milano

Shape Optimization, Geometric Inequalities, and Related Topics

Two days workshop for young researchers in Naples

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The generalized Lane-Emden equation I

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In this seminar we consider the following quasilinear equation:

$$\Delta_p u + |u|^{q-1} u = 0 \quad \text{in } \mathbb{R}^n \,, \tag{1}$$

where $q > 1$, $1 < p < n$ and Δ_p is the $p-$**Laplace operator**

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- from the study of problems in conformal geometry, like prescribed scalar curvature problem[2].

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[1] S. Chandrasekhar. *An Introduction to the Study of Stellar Structure*, 1957.
[2] M. Struwe. *Variational Methods. Applications to Nonlinear PDEs and Hamiltonian Systems*, 1990.

The generalized Lane-Emden equation II

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$$q = p^* - 1\,, \qquad \text{where} \qquad p^* := \frac{np}{n-p}\,,$$

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Two cases:

- $q < p^* - 1$, **subcritical regime**;
- $q = p^* - 1$, **critical regime.**

The case: $p = 2$ and $q < 2^* - 1$

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Theorem [Gidas-Spruck (1981)]
Let $u \in C^2(\mathbb{R}^n)$ be a solution of

$$\begin{cases} \Delta u + u^q = 0 & \text{in } \mathbb{R}^n \\ u \geq 0\,, \end{cases}$$

with

$$1 \leq q < 2^* - 1 = \frac{n+2}{n-2}\,,$$

then $u \equiv 0$.

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- Proof based on a **test functions argument** and on **integral identities.**
- The same result holds in complete noncompact **Riemannian manifolds** (M^n, g) with **nonnegative Ricci curvature.**

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An **explicit family of solutions** to

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is given by the **Talentiane** or **Aubin-Talenti bubbles**

$$\mathcal{U}_{\lambda,x_0}(x) := \left(\frac{\sqrt{n(n-2)}\lambda}{1 + \lambda^2 |x - x_0|^2} \right)^{\frac{n-2}{2}}, \quad \text{where } \lambda > 0 \text{ and } x_0 \in \mathbb{R}^n\,. \tag{3}$$

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These functions have been constructed by *Aubin (1976)* and *Talenti (1976)* as **minimizers of the Sobolev constant**:

$$S := \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 \, dx}{\left(\int_{\mathbb{R}^n} u^{2^*} \, dx \right)^{2/2^*}}\,,$$

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Problem (2) is also related to the **Yamabe problem.**

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Theorem [Yamabe ('60), Trudinger ('68), Aubin ('76), Schoen ('84)]. Let (M, g_0) be a compact Riemannian manifold of dimension $n \geq 3$. Then there exists a metric g on M which is conformal to g_0 and has constant scalar curvature.

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If we write,

$$g = u^{\frac{4}{n-2}} g_0$$

for some positive function u. Then u solves

$$\frac{4(n-1)}{n-2} \Delta_{g_0} u - R_{g_0} u + R_g u^{\frac{n+2}{n-2}} = 0\,,$$

where R_{g_0}, R_g denotes the scalar curvature of g_0, g respectively.

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When (M, g_0) is the round sphere (**Nirenberg problem**), by stereographic projection we get

$$\frac{4(n-1)}{n-2} \Delta u + R_g u^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad \mathbb{R}^n \,,$$

and hence

$$\Delta u + u^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n.$$

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Theorem [Obata (1971) and Gidas-Ni-Nirenberg (1981)]
Let $u \in C^2(\mathbb{R}^n)$ be a solution of

$$\begin{cases} \Delta u + u^{2^*-1} = 0 & \text{in } \mathbb{R}^n \\ u > 0\,, \end{cases} \tag{4}$$

such that

$$u(x) = O(|x|^{2-n}) \quad \text{for } x \text{ large.}$$

Then $u(x) = \mathcal{U}_{\lambda,x_0}(x)$.

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- *Chen-Li (1991)* and *Li (1996)* shorter proof.

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Weak solutions: *a weak solution u to*

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is a function

$$u \in W^{1,p}_{loc}(\mathbb{R}^n) \cap L^{\infty}_{loc}(\mathbb{R}^n) ,$$

such that

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where $W^{1,p}_c(\mathbb{R}^n)$ denotes the space of compactly supported functions in $W^{1,p}(\mathbb{R}^n)$.

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In general, solutions to quasilinear equations **are not smooth.**

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satisfies:

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for some $\alpha \in (0,1)$ and where $\mathcal{Z} := \{x \in \mathbb{R}^n \,:\, \nabla u(x) = 0\}$.

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$$|\nabla u|^{p-2}\nabla u \in W^{1,2}_{loc}(\mathbb{R}^n)\,,$$

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$$|\nabla u|^{p-2}\nabla^2 u \in \begin{cases} L^2_{loc}(\mathbb{R}^n) & \text{for } 1 < p \leq 2\,, \\ L^2_{loc}(\mathbb{R}^n \setminus \mathcal{Z}) & \text{for } 2 < p < n\,. \end{cases}$$

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- What about Riemannian manifolds?

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General p and $q = p^* - 1$

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These functions have been constructed by *Aubin (1976)* and *Talenti (1976)* as **minimizers of the Sobolev constant**:

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Question: are the Talentiane (7) the only solutions to (6)?

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Theorem [Damascelli-Merchán-Montoro-Sciunzi (2014)]
Let $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ be a weak solution to

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- *Ciraolo-Figalli-R. (2021)* alternative proof.

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- For $p = 2$: it is possible to construct "many" **sign-changing solutions** to
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which are **not radial!**

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- For $1 < p < n$, with $n \geq 4$, it is possible to construct "many" **sign-changing solutions** to

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Theorem [Caffarelli-Gidas-Spruck (1989)]

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Question: is it possible to remove (or weaken) the assumption $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$?

A big difference between $p = 2$ and $p \neq 2$: the finite energy assumption

It is well-known that the energy associated to

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This functional is also interesting from the point of view of the **calculus of variations.** Since the embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$ is **not compact**, the classical tools of the calculus of variations (e.g. the Mountain Pass Lemma or the direct method) do not apply!

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$$u \in \mathcal{D}^{1,p}(\mathbb{R}^n) \quad \Rightarrow \quad \mathcal{E}(u) < \infty\,.$$

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The hypothesis $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ is called the **finite energy assumption.**

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Let u be a weak solution to

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- **True** if

$$n = 2 \text{ and } 1 < p < 2 \quad \text{or} \quad n = 3 \text{ and } \frac{3}{2} < p < 2\,,$$

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- Proof based on **integral identities** and inspired by *Gidas, Spruck (1981), Serrin, Zou (2002), Ciraolo, Figalli, R. (2021), Catino, Monticelli (2022).*
- For general n and p additional **assumptions on the energy**:

$$\mathcal{E}_{\mathbb{R}^n}(u) := \frac{1}{p}\int_{\mathbb{R}^n} |\nabla u|^p \, dx - \frac{1}{p^*}\int_{\mathbb{R}^n} u^{p^*} \, dx$$

  or on the **behaviour at infinity of the solution**:

$$u(x) \leq C|x|^{\alpha}\,, \quad \text{as } |x| \to \infty,$$

  are much **weaker** than $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$.

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- Arguing as *Serrin-Zou (2002)* we obtain:

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where

$$\mathrm{V} := \begin{cases} \nabla \mathrm{v} & \text{in } \mathcal{Z}^c \\ 0 & \text{in } \mathcal{Z} \end{cases} \quad \text{with} \quad \mathrm{v} := u^{-\frac{n(p-1)}{n-p}} |\nabla u|^{p-2} \nabla u \quad \text{and} \quad \mathring{\mathrm{V}} := \mathrm{V} - \frac{\operatorname{tr} \mathrm{V}}{n} \, \mathrm{Id}_n \;.$$

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- Using Cauchy-Schwarz and Young's inequalities and the definition of v:

$$\int_{\mathbb{R}^n} u^{\frac{(n-1)p}{n-p}} |\mathring{\mathrm{V}}|^2 \eta^2 \, dx \le C \int_{\mathbb{R}^n} u^{\frac{(2-p)n-p}{n-p}} |\nabla u|^{2(p-1)} |\nabla \eta|^2 \, dx \,,$$

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while, from Holder's inequality

$$\int_{\mathbb{R}^n} u^{\frac{(n-1)p}{n-p}} |\mathring{\mathrm{V}}|^2 \eta^2 \, dx \le C \left(\int_{\mathrm{supp}|\nabla \eta|} u^{\frac{(n-1)p}{n-p}} |\mathring{\mathrm{V}}|^2 \eta^2 \, dx \right)^{\frac{1}{2}} \times$$

$$\left(\int_{\mathbb{R}^n} u^{\frac{(2-p)n-p}{n-p}} |\nabla u|^{2(p-1)} |\nabla \eta|^2 \, dx \right)^{\frac{1}{2}} \,, \qquad \text{for all } 0 \le \eta \in C_c^\infty(\mathbb{R}^n) \,.$$

Idea of the proof for $n = 2$ and $1 < p < 2$

- Take η such that $\eta \equiv 1$ in B_R, $\eta \equiv 0$ in B_{2R}^c, $0 \le \eta \le 1$ and

$$|\nabla \eta|^2 \le \frac{C}{R^2} \quad \text{in } B_{2R} \setminus B_R\,.$$

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&\le \frac{C}{R^2} \left(\int_{B_{2R} \setminus B_R} u^{-\frac{p}{2-p}} |\nabla u|^p \, dx \right)^{\frac{2(p-1)}{p}} \left(\int_{B_{2R} \setminus B_R} u^{\frac{p}{2-p}} \, dx \right)^{\frac{2-p}{p}}
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&\le \frac{C}{R^2} \int_{B_{2R}\setminus B_R} u \left(u^{-\frac{p}{2-p}} |\nabla u|^p \right)^{\frac{2(p-1)}{p}} dx \\
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&\le \frac{C}{R^2} \left(\int_{B_{2R}\setminus B_R} u^{-\frac{p}{2-p}} |\nabla u|^p\,dx + \int_{B_{2R}\setminus B_R} u^{\frac{p}{2-p}}\,dx \right)
\end{aligned}$$

Idea of the proof for $n = 2$ and $1 < p < 2$

- Take η such that $\eta \equiv 1$ in B_R, $\eta \equiv 0$ in B_{2R}^c, $0 \leq \eta \leq 1$ and

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thanks to a **weak energy estimate** on balls.

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It is classical that

$$\Delta_p u \leq 0 \text{ in } \mathbb{R}^n\setminus K \quad\Rightarrow\quad u(x)\geq C|x|^{-\frac{n-p}{p-1}} \text{ for } |x|\geq\rho\,;$$

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Choose ε small enough and reorder terms.

Our general results

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Theorem [Catino-Monticelli-R. (2022)]

Let u be a weak solution to

$$\begin{cases} \Delta_p u + u^{p^*-1} = 0 & \text{in } \mathbb{R}^n \\ u > 0\,, \end{cases} \tag{9}$$

Then $u(x) = \mathcal{U}_{\lambda,x_0}(x)$, if one of the following holds:

- $\mathcal{E}_{B_{2R}\setminus B_R}(u) = O(R^\theta)$, *for some suitable* $\theta = \theta(n,p) > 0$,
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Remark [Vétois (2016)]

$$u \in \mathcal{D}^{1,p}(\mathbb{R}^n) \quad \Rightarrow \begin{cases} u \text{ bounded} \\ u(x) \le \frac{C}{1+|x|^{\frac{n-p}{p-1}}} \\ |\nabla u(x)| \le \frac{C}{1+|x|^{\frac{n-1}{p-1}}} \end{cases}$$

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Remark (weaker assumptions) From Young's and Holder's inequalities

$$\mathcal{E}_{B_{2R}\setminus B_R}(u) = O(R^\theta) \quad \Leftrightarrow \quad \mathcal{E}^{pot}_{B_{2R}\setminus B_R}(u) = O(R^\theta) \quad \Leftrightarrow \quad \mathcal{E}^{kin}_{B_{2R}\setminus B_R}(u) = O(R^\theta)$$

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Corollary Let u be a bounded weak solution to (9) with

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Let (M^n, g), be a complete noncompact Riemannian manifold with $\mathrm{Ric} \geq 0$ and let $u \in C^2(M)$ be a solution of

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- On Cartan-Hadamard manifolds by *Muratori, Soave (2022)*.

Final remarks: anisotropic setting

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Let $H : \mathbb{R}^n \to \mathbb{R}$ an **anisotropic norm**, i.e. H is convex, H is positive 1-homogeneous and H is positive such that H is uniformly convex and $H^2 \in C^2(\mathbb{R}^n \setminus \{\mathcal{O}\}) \cap C^{1,1}(\mathbb{R}^n)$.

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- Same picture in convex cones of $\mathbb{R}^n$ (see *Lions, Pacella, Tricarico (1988), Ciraolo, Figalli, R. (2021)*).

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- **Partial results** if $n \geq 5$: *de Figueiredo,-Felmer (1994), Lin (1998), Busca-Manásevich (2002), Reichel-Zou (2000), Souplet (2009), Li-Zhang (2019)*.

GRAZIE DELL'ATTENZIONE!