Existence and multiplicity results for some classes of nonlinear differential problems

Shape Optimization, Geometric Inequalities and Related Topics

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G. Bonanno, G. D'Aguì, A. Sciammetta, *Existence of two positive solutions for anisotropic nonlinear elliptic equations*, Advances in Differential Equations, vol. **26** (2021), 229-258.

D. Motreanu, A. Sciammetta, E. Tornatore, *A sub-super solutions approach for Neumann boundary value problems with gradient dependence*, Nonlinear Anal. Real World Appl. **54** (2020) 1–12.

First part

Anisotropic nonlinear elliptic equations via variational methods

G. Bonanno, G. D'Aguì, A. Sciammetta, *Existence of two positive solutions for anisotropic nonlinear elliptic equations*, Advances in Differential Equations, vol. **26** (2021), 229-258.

$$\begin{cases} -\Delta_{\vec{p}} u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (D_\lambda^{\vec{p}})$$

- $\Omega \subset \mathbb{R}^N$ with a boundary of class C^1 and with $N \geq 2$;
- $\vec{p} = (p_1, p_2, \ldots, p_N), \vec{p} \in \mathbb{R}^N$;
- $p^- = \min\{p_1, p_2 \ldots, p_N\} > N$;
- $p^+ = \max\{p_1, p_2 \ldots, p_N\}$;
- $\lambda > 0$;
- $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1−Carathéodory function, that is:
  1. $x \mapsto f(x,\xi)$ is measurable for every $\xi \in \mathbb{R}$;
  2. $\xi \mapsto f(x,\xi)$ is continuous for almost every $x \in \Omega$;
  3. for every $s > 0$ there is a function $l_s \in L^1(\Omega)$ such that

$$\sup_{|\xi| \leq s} |f(x,\xi)| \leq l_s(x), \qquad \text{for a.e.} \quad x \in \Omega.$$

Anisotropic $p-$Laplacian operator

$$\Delta_{\vec{p}} u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i - 2} \frac{\partial u}{\partial x_i} \right)$$

If $p_i = p$ for all $i = 1, \ldots, N$

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) = \tilde{\Delta}_p u, \quad \text{pseudo}-p-\text{Laplacian operator.}$$

If $p_i = 2$ for all $i = 1, \ldots, N$

$$\sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2} = \Delta u, \qquad \text{Laplacian operator.}$$

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[1] M. Belloni, B. Kawohl, *The pseudo-p-Laplace eigenvalue problem and viscosity solutions as $p \to \infty$*, ESAIM Control Optim. Calc. Var. **10** (2004), 28–52.

[2] L. Brasco, G. Franzina, *An anisotropic eigenvalue problem of Stekloff type and weighted Wulff inequalities*, Nonlinear Differ. Equ. Appl. **20** (2013), 1795–1830.

Some references

[1] S.M. Nikol'skii, *An imbedding theorem for functions with partial derivatives considered in different metrics*, Izv. Akad. Nauk SSSR Ser. Mat. **22** (1958), 321–336.

[2] J. Rákosník, *Some remarks to anisotropic Sobolev spaces I*, Beiträge zur Analysis **13** (1979) 55–68.

[3] J. Rákosník, *Some remarks to anisotropic Sobolev spaces II*, Beiträge zur Analysis **15** (1981), 127–140.

[4] M. Troisi, *Teoremi di inclusione per spazi di Sobolev non isotropi*, Ricerche Mat. **18** (1969), 3–24.

Let $\alpha \in \mathbb{N}^N$ be multiindices such that $\alpha = (\alpha_1, \ldots, \alpha_N)$. The length of α is $|\alpha| = \alpha_1 + \ldots + \alpha_N$.

$$D^\alpha u := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_N^{\alpha_N}}, \tag{1}$$

$$D^0 u := u.$$

$E = \left\{\alpha \in \mathbb{N}_0^N : |\alpha| \leq 1\right\}$ and $\vec{p} = (p_0, p_1, \ldots, p_N)$ with $p_0 \geq p_i \geq 1$ for $i = 1, \ldots, N$.

$$W^{E,\vec{p}}(\Omega) = \{u = u(x) : D^\alpha u \in L^{p_\alpha}(\Omega), \text{ for } \alpha \in E\}, \tag{2}$$

is a reflexive Banach space if it is equipped with the norm

$$\|u\|_{W^{E,\vec{p}}(\Omega)} := \sum_{\alpha \in E} \|D^\alpha u\|_{L^{p_\alpha}(\Omega)}. \tag{3}$$

We denote by $W_0^{E,\vec{p}}(\Omega)$ as closure of $C_0^\infty(\Omega)$ in the topology of $W^{E,\vec{p}}(\Omega)$.

Anisotripic Sobolev spaces

Consider the following $N+1$ multiindices of $N-$tuple

$$E = \{(0,0,\dots,0),(1,0,\dots,0),(0,1,\dots,0),\dots(0,0,\dots,1)\},$$

and consiter $\vec{p} = (p_0, p_1, p_2, \dots, p_N)$ with $p_i \geq 1$ for all $i = 1, \dots N$.
Then, the set (2) becomes

$$W^{1,\vec{p}}(\Omega) = \left\{ u \in L^{p_0}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega),\ \text{for } i = 1, \dots, N \right\}, \tag{4}$$

in which we consider the norm

$$\|u\|_{W^{1,\vec{p}}(\Omega)} = \|u\|_{L^{p_0}(\Omega)} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}. \tag{5}$$

We define $W_0^{1,\vec{p}}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (5). On $W_0^{1,\vec{p}}(\Omega)$ we can also define the following norm

$$\|u\|_{W_0^{1,\vec{p}}(\Omega)} := \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}. \tag{6}$$

Remark

We observe also that if $\vec{p}$ is constant (that is $p_i = p$ for all $i = 0, 1, \dots, N$) we get

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{\partial u}{\partial x_i} \in L^p(\Omega) \right\}.$$

Other references

[1] S.N. Antontsev, S. Shmarev, *Energy methods for free boundary problems: applications to nonlinear PDEs and fluid mechanics*, Progress in Nonlinear Differential Equations and Their Applications, Vol 48, Birkhauser Boston, Boston, MA, 2002.

[2] M. Bendahmane, M. Langlais, M. Saad, *On some anisotropic reaction-diffusion systems with L^1-data modeling the propagation of an epidemic disease*, Nonlinear Anal., (4) 54 (2003), 617–636.

[3] A. Cianchi, *Local boundedness of minimizers of anisotropic functionals*, Ann. Inst. H. Poincarè ANL **17** (2000), 147–168.

[4] R. Di Nardo, F. Feo, *Existence and uniqueness for nonlinear anisotropic elliptic equations*, Arch. Math. **102** (2014), 141–153.

[5] A. El Hamidi, J.M. Rakotoson, *Extremal functions for the anisotropic Sobolev inequalities*, Ann. Inst. H. Poincaré Anal. Non Linéaire **24** (2007), 741–75.

[6] G.M. Figueiredo, J.R.S. Silva, *A critical anisotropic problem with discontinuous nonlinearities*, Nonlinear Analysis: Real world and applications **47** (2019), 364–374.

[7] I. Fragalà, F. Gazzola, B. Kawohl, *Existence and nonexistence results for anisotropic quasilinear elliptic equations*,Ann. Inst. H. Poincaré Anal. Non Linéaire **21**,715–734.

[8] N. Fusco, C. Sbordone, *Some remarks on the regularity of minima of anisotropic integrals*, Commun. in Partial Differential Equations **18** (993), 153–167.

[9] S.M. Nikol'skii, *An imbedding theorem for functions with partial derivatives considered in different metrics*, Izv. Akad. Nauk SSSR Ser. Mat. **22** (1958), 321–336. English translation: Amer. Math.

Main tool

Theorem (G. Bonanno and G. D'Aguì)

Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two functionals of class C^1 such that $\inf_X \Phi(u) = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that

$$\frac{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}, \tag{7}$$

and, for each

$$\lambda \in \Lambda = \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \right[,$$

the functional $I_\lambda = \Phi - \lambda \Psi$ satisfies the $(PS)-$condition and it is unbounded from below.
Then, for each $\lambda \in \Lambda$, the functional I_λ admits at least two non-zero critical points $u_{\lambda,1}, u_{\lambda,2} \in X$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

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[1] A. Ambrosetti, P.H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973) 349–381.

[2] G. Bonanno, G. D'Aguì, *Two non-zero solutions for elliptic Dirichlet problems*, Z. Anal. Anwend. **35** (2016), 449–464.

Variational approach

$\Phi, \Psi : W_0^{1,\vec{p}}(\Omega) \to \mathbb{R}$, $F(x,t) = \int_0^t f(x,\xi)d\xi$ for all $(x,t) \in \Omega \times \mathbb{R}$.

$$\underbrace{I_\lambda(u) = \underbrace{\sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx}_{\Phi(u)} - \lambda \underbrace{\int_\Omega F(x,u(x))dx}_{\Psi(u)}}_{\text{Energy functional}} .$$

Definition

A function $u : \Omega \to \mathbb{R}$ is a weak solution of problem $(D_\lambda^{\vec{p}})$ if $u \in X$ satisfies the following condition for all $v \in X$

$$\underbrace{\sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx}_{\Phi'(u)(v)} = \lambda \underbrace{\int_\Omega f(x,u(x))v(x)dx}_{\Psi'(u)(v)} .$$

(AR) *There exist constants $\mu > p^+$ and $M > 0$ such that, $0 < \mu F(x,t) \leq tf(x,t)$ for all $x \in \Omega$ and for all $|t| \geq M$.*

Lemma 1

Assume that the $(AR)-$condition holds. Then I_λ satisfies the $(PS)-$condition and it is unbounded from below.

Preliminary results

$\left(W_0^{1,\vec{p}}(\Omega), \|\cdot\|_{W_0^{1,\vec{p}}(\Omega)}\right)$ is a Banach space, where $W_0^{1,\vec{p}}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with

$$\|u\|_{W_0^{1,\vec{p}}(\Omega)} := \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)} .$$

Proposition

$W_0^{1,\vec{p}}(\Omega)$ is compactly embedded in $C^0(\bar{\Omega})$ and for each $u \in W_0^{1,\vec{p}}(\Omega)$

$$\|u\|_{C^0(\bar{\Omega})} \leq \underbrace{2^{\frac{(N-1)(p^- -1)}{p^-}} m_{p^-} \max_{1\leq i\leq N} \{|\Omega|^{\frac{p_i - p^-}{p_i p^-}}\}}_{=T_0} \|u\|_{W_0^{1,\vec{p}}(\Omega)}$$

Proof: $p^- > N$, $W_0^{1,p^-}(\Omega)$ is continuously embedded in $C^0(\bar{\Omega})$, the embedding is compact and

$$\|u\|_{C^0(\bar{\Omega})} \leq m_{p^-} \|u\|_{W_0^{1,p^-}(\Omega)} \leq 2^{\frac{(N-1)(p^- -1)}{p^-}} m_{p^-} \max_{1\leq i\leq N} \{|\Omega|^{\frac{p_i - p^-}{p_i p^-}}\} \|u\|_{W_0^{1,\vec{p}}(\Omega)} .$$

$$m_{p^-} = \frac{N^{-\frac{1}{p^-}}}{\sqrt{\pi}} \left[\Gamma\left(1 + \frac{N}{2}\right)\right]^{\frac{1}{N}} \left(\frac{p^- - 1}{p^- - N}\right)^{1-\frac{1}{p^-}} |\Omega|^{\frac{1}{N} - \frac{1}{p^-}}$$

[1] M. Troisi, *Teoremi di inclusione per spazi di Sobolev non isotropi*, Ricerche Mat. **18** (1969), 3–24.

Preliminary results

Proposition

Fix $r > 0$. Then for each $u \in W_0^{1,\vec{p}}(\Omega)$ such that

$$\sum_{i=1}^{N} \frac{1}{p_i} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}^{p_i} < r,$$

one has

$$\|u\|_{C^0(\bar{\Omega})} < T \max\{r^{1/p^-}; r^{1/p^+}\},$$

where $T = T_0 \sum_{i=1}^{N} {p_i}^{1/p_i}$.

The sign of solutions

$$f^+(x,t) = \begin{cases} f(x,0), & \text{if } \quad t < 0, \\ f(x,t), & \text{if } \quad t \geq 0, \end{cases} \tag{8}$$

for all $(x,t) \in \Omega \times \mathbb{R}$ and

$$\begin{cases} -\Delta_{\vec{p}} u = \lambda f^+(x,u) & \text{in } \ \Omega, \\ u = 0 & \text{on } \ \partial\Omega. \end{cases} \qquad (D^{\vec{p}}_{\lambda,f^+})$$

Lemma 2

Assume that

$$f(x,0) \geq 0 \quad \textit{for a.e. } x \in \Omega.$$

Then, any weak solution of $(D^{\vec{p}}_{\lambda,f^+})$ is nonnegative and it is also a weak solution of $(D^{\vec{p}}_{\lambda})$.

Lemma 3

Assume that

$$f(x,t) \geq 0 \quad \textit{for a.e. } x \in \Omega, \quad \textit{for all } t \geq 0.$$

Then, any non-zero weak solution of $(D^{\vec{p}}_{\lambda,f^+})$ is positive and it is also a weak solution of $(D^{\vec{p}}_{\lambda})$.

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[1] A. Di Castro, E. Montefusco, *Nonlinear eigenvalues for anisotropic quasilinear degenerate elliptic equations*, Nonlinear Anal. **70** (2009), 4093–4105.

Main result

$$R := \sup_{x \in \Omega} \operatorname{dist}(x, \partial\Omega) \ \Rightarrow \ \exists x_0 \in \Omega \ \text{ such that } \ B(x_0, R) \subseteq \Omega$$

$$\omega_R := |B(x_0, R)| = \frac{\pi^{\frac{N}{2}}}{\Gamma(1 + \frac{N}{2})} R^N, \quad \mathcal{K} = \frac{1}{\left[\sum_{i=1}^{N} \frac{1}{p_i} \left(\frac{2}{R}\right)^{p_i}\right] \omega_R \left(\frac{2^N - 1}{2^N}\right) \max\left\{T^{p^-}; T^{p^+}\right\}}$$

Theorem

Assume that the (AR)-condition holds and $\exists c, d > 0$, with $\max\left\{d^{p^-}; d^{p^+}\right\} < \min\left\{c^{p^-}; c^{p^+}\right\}$, s.t.

$$F(x,t) \geq 0, \quad \text{for all} \quad (x,t) \in \Omega \times [0, d], \tag{9}$$

$$\frac{\int_\Omega \max_{|\xi| \leq c} F(x, \xi) dx}{\min\left\{c^{p^-}; c^{p^+}\right\}} < \mathcal{K} \ \frac{\int_{B(x_0, \frac{R}{2})} F(x, d)\, dx}{\max\left\{d^{p^-}; d^{p^+}\right\}}. \tag{10}$$

Then, for each

$$\lambda \in \tilde{\Lambda} := \left]\frac{1}{\max\left\{T^{p^-}; T^{p^+}\right\}} \frac{1}{\mathcal{K}} \frac{\max\left\{d^{p^-}; d^{p^+}\right\}}{\int_{B(x_0, \frac{R}{2})} F(x, d)\, dx}, \frac{1}{\max\left\{T^{p^-}; T^{p^+}\right\}} \frac{\min\left\{c^{p^-}; c^{p^+}\right\}}{\int_\Omega \max_{|\xi| \leq c} F(x, \xi) dx}\right[,$$

problem $(D_\lambda^{\vec{p}})$ has at least two non-zero weak solutions.

Sketch of Proof

- $X = W_0^{1,\vec{p}}(\Omega)$ and $\lambda \in \tilde{\Lambda}$.
- $I_\lambda = \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx - \lambda \int_\Omega F(x,u(x))dx = \Phi(u) - \lambda \Psi(u).$
- from $(AR)-$condition $\overset{\text{Lemma 1}}{\Rightarrow}$ I_λ satisfies the (PS)-condition
  I_λ is unbounded from below.
- Put $r = \min\{\left(\frac{c}{T}\right)^{p^-}; \left(\frac{c}{T}\right)^{p^+}\}$ and

$$\tilde{u}(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0,R), \\ \frac{2d}{R}(R - |x - x_0|) & \text{if } x \in B(x_0,R) \setminus B\left(x_0, \frac{R}{2}\right), \\ d & \text{if } x \in B\left(x_0, \frac{R}{2}\right). \end{cases}$$

Clearly, $\tilde{u} \in W_0^{1,\vec{p}}(\Omega)$. From $\max\left\{d^{p^-}; d^{p^+}\right\} < \min\left\{c^{p^-}; c^{p^+}\right\}$ + (10) $\Rightarrow 0 < \Phi(\tilde{u}) < r$

$$\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \geq \max\left\{T^{p^-}; T^{p^+}\right\} \mathcal{K} \frac{\int_{B(x_0,\frac{R}{2})} F(x,d)dx}{\max\left\{d^{p^-}; d^{p^+}\right\}} > \max\left\{T^{p^-}; T^{p^+}\right\} \frac{\int_\Omega \max_{|\xi| \leq c} F(x,\xi)dx}{\min\left\{c^{p^-}; c^{p^+}\right\}} \geq \frac{\sup_{u \in \Phi^{-1}(]-\infty,r])} \Psi(u)}{r}$$

- $\lambda \in \tilde{\Lambda} \subseteq \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty,r])} \Psi(u)} \right[$

Some consequences

Theorem

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x,t) \geq 0$ for a.e. $x \in \Omega$ and for all $t \geq 0$. Assume that

$$(AR^+)\ \exists\ \mu > p^+ \text{ and } M > 0 \text{ such that } 0 < \mu F(x,t) \leq tf(x,t)\ \forall\, x \in \Omega \text{ and } \forall\, t \geq M.$$

Moreover, assume that there are two positive constants c and d, with $d < 1 \leq c$, such that

$$\frac{\int_\Omega F(x,c)dx}{c^{p^-}} < \mathcal{K} \frac{\int_{B\left(x_0,\frac{R}{2}\right)} F(x,d)\,dx}{d^{p^-}}.$$

Then, for each $\lambda \in \left] \frac{1}{\max\left\{T^{p^-};T^{p^+}\right\}} \frac{1}{\mathcal{K}} \frac{d^{p^-}}{\int_{B\left(x_0,\frac{R}{2}\right)} F(x,d)\,dx}, \frac{1}{\max\left\{T^{p^-};T^{p^+}\right\}} \frac{c^{p^-}}{\int_\Omega F(x,c)dx} \right[$, problem $(D_\lambda^{\vec{p}})$ has at least two positive weak solutions.

Sketch of Proof

- from $(AR^+)-$condition $\overset{\text{Lemma 1}}{\Rightarrow}$ $I_\lambda^+ := \Phi - \lambda\Psi^+$ satisfies the $(PS)-$condition
  I_λ^+ is unbounded from below
- From Lemma 3, any non-zero weak solution of $(D_{\lambda,f^+}^{\vec{p}})$ is a positive weak solution of $(D_\lambda^{\vec{p}})$.

Some consequences

Theorem

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x,t) \geq 0$ for a.e. $x \in \Omega$ and for all $t \geq 0$. Assume that

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Moreover, assume that there are two positive constants c and d, with $d < c \leq 1$, such that

$$\frac{\displaystyle\int_\Omega F(x,c)dx}{c^{p^+}} < \mathcal{K}\frac{\displaystyle\int_{B\left(x_0,\frac{R}{2}\right)} F(x,d)\,dx}{d^{p^-}}.$$

Then, for each $\lambda \in \left]\frac{1}{\max\left\{T^{p^-};T^{p^+}\right\}}\frac{1}{\mathcal{K}}\frac{d^{p^-}}{\displaystyle\int_{B\left(x_0,\frac{R}{2}\right)} F(x,d)\,dx}, \frac{1}{\max\left\{T^{p^-};T^{p^+}\right\}}\frac{c^{p^+}}{\displaystyle\int_\Omega F(x,c)dx}\right[$, problem $(D_\lambda^{\vec{p}})$ has at least two positive weak solutions.

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- From Lemma 3, any non-zero weak solution of $(D^{\vec{p}}_{\lambda,f^+})$ is a positive weak solution of $(D^{\vec{p}}_\lambda)$.

Some consequences

Theorem

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x,t) \geq 0$ for a.e. $x \in \Omega$ and for all $t \geq 0$. Assume that

$$(AR^+)\ \exists\ \mu > p^+ \text{ and } M > 0 \text{ such that } 0 < \mu F(x,t) \leq tf(x,t)\ \forall\, x \in \Omega \text{ and } \forall\, t \geq M.$$

Moreover, assume that there are two positive constants c and d, with $1 \leq d < c$, such that

$$\frac{\int_\Omega F(x,c)dx}{c^{p^-}} < \mathcal{K} \frac{\int_{B\left(x_0,\frac{R}{2}\right)} F(x,d)\,dx}{d^{p^+}}.$$

Then, for each $\lambda \in \left] \frac{1}{\max\left\{T^{p^-};T^{p^+}\right\}} \frac{1}{\mathcal{K}} \frac{d^{p^+}}{\int_{B\left(x_0,\frac{R}{2}\right)} F(x,d)\,dx}, \frac{1}{\max\left\{T^{p^-};T^{p^+}\right\}} \frac{c^{p^-}}{\int_\Omega F(x,c)dx} \right[$, problem $(D_\lambda^{\vec{p}})$ has at least two positive weak solutions.

Sketch of Proof

- from $(AR^+)-$condition $\overset{\text{Lemma 1}}{\Rightarrow}$ $I_\lambda^+ := \Phi - \lambda\Psi^+$ satisfies the $(PS)-$condition; I_λ^+ is unbounded from below
- From Lemma 3, any non-zero weak solution of $(D_{\lambda,f^+}^{\vec{p}})$ is a positive weak solution of $(D_\lambda^{\vec{p}})$.

Example 1: $N = 3, \Omega = B(0,2), p_1 = 4, p_2 = 5, p_3 = 6, c = 1$ and $d = 10^{-14}$

$$\begin{cases} -\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) = 10^{-12}(x^2+y^2+z^2)u^8 + 10^{-12}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{11}$$

$$f(x,y,z,t) = (x^2+y^2+z^2)t^8 + t^2 \;\Rightarrow\; F(x,y,z,t) = (x^2+y^2+z^2)\frac{t^9}{9} + \frac{t^3}{3}.$$

We have that $(AR^+)-$condition holds and

$$m_{p^-} = \sqrt[4]{\frac{3^3}{2\pi}}, \quad T_0 = \sqrt[3]{\frac{2^5\cdot 3^2}{\sqrt{\pi}}}, \quad T = (\sqrt{2}+\sqrt[5]{5}+\sqrt[6]{6})\sqrt[3]{\frac{2^5\cdot 3^2}{\sqrt{\pi}}},$$

$$\max\left\{T^{p^-}; T^{p^+}\right\} = T^6 = (\sqrt{2}+\sqrt[5]{5}+\sqrt[6]{6})^6\frac{(2^5\cdot 3^2)^2}{\pi}, \quad \mathcal{K} = \frac{5}{2^{10}\cdot 3^2\cdot 7\cdot 37(\sqrt{2}+\sqrt[5]{5}+\sqrt[6]{6})^6}.$$

$$\frac{1}{\max\left\{T^{p^-}; T^{p^+}\right\}}\frac{1}{\mathcal{K}}\frac{d^{p^-}}{\int_{B\left(x_0,\frac{R}{2}\right)} F(x,d)\,dx} = \frac{7\cdot 37}{5}\frac{1}{\frac{2^2}{5}d^5 + \frac{2^2}{d}} \leq \frac{7\cdot 37}{4}d = \frac{7\cdot 37}{4}10^{-14} < 10^{-12}$$

$$< \frac{1}{\max\left\{T^{p^-}; T^{p^+}\right\}}\frac{c^{p^-}}{\int_{\Omega} F(x,c)dx} = \frac{5}{(\sqrt{2}+\sqrt[5]{5}+\sqrt[6]{6})^6 2^{15}3^4}$$

Some consequences

$$\begin{cases} -\Delta_{\vec{p}} u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (AD_{\lambda}^{\vec{p}})$$

Put

$$\mathcal{K}^* = \frac{\omega_R}{2^N |\Omega|} \mathcal{K}.$$

(AR_1^+) *there exist constants* $\mu > p^+$ *and* $M > 0$ *such that,* $0 < \mu F(t) \leq tf(t)$ *for all* $t \geq M$.

Theorem

Let $f : [0, +\infty[\to [0, +\infty[$ be a continuous function such that the (AR_1^+)−condition holds. Moreover, assume that there are two positive constants c and d, with $d < 1 \leq c$, such that

$$\frac{F(c)}{c^{p^-}} < \mathcal{K}^* \frac{F(d)}{d^{p^-}}. \qquad (12)$$

Then, for each

$$\lambda \in \tilde{\Lambda}_1 := \left] \frac{1}{\max\left\{T^{p^-}; T^{p^+}\right\}} \frac{1}{|\Omega|} \frac{1}{\mathcal{K}^*} \frac{d^{p^-}}{F(d)}, \frac{1}{\max\left\{T^{p^-}; T^{p^+}\right\}} \frac{1}{|\Omega|} \frac{c^{p^-}}{F(c)} \right[,$$

the problem $(AD_{\lambda}^{\vec{p}})$ has at least two positive weak solutions.

Some consequences

$$\begin{cases} -\Delta_{\vec{p}} u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (AD_{\lambda}^{\vec{p}})$$

(AR_1^+) *There exist constants* $\mu > p^+$ *and* $M > 0$ *such that,* $0 < \mu F(t) \leq tf(t)$ *for all* $t \geq M$.

Theorem

Let $f : [0, +\infty[\to [0, +\infty[$ be a continuous function such that the (AR_1^+)−condition holds. Assume that

$$\limsup_{t \to 0^+} \frac{F(t)}{t^{p^-}} = +\infty. \tag{13}$$

Put $\lambda^* = \dfrac{1}{\max\left\{T^{p^-}; T^{p^+}\right\}} \dfrac{1}{|\Omega|} \sup_{c \geq 1} \dfrac{c^{p^-}}{F(c)}$.

Then, for each $\lambda \in]0, \lambda^*[$, the problem $(AD_{\lambda}^{\vec{p}})$ admits at least two positive weak solutions.

Remark

$$\lambda^* = \frac{1}{\max\left\{T^{p^-}; T^{p^+}\right\}} \frac{1}{|\Omega|} \max\left\{\sup_{c \geq 1} \frac{c^{p^-}}{F(c)}; \sup_{0 < c < 1} \frac{c^{p^+}}{F(c)}\right\}.$$

Some consequences

Theorem

Fix s, q such that $0 \leq s < p^- - 1$ and $p^+ - 1 < q$. Put

$$\eta^* = \min\left\{ \frac{1-\frac{p^+}{q+1}}{\frac{p^+}{s+1}-1}, \left[\frac{(s+1)(q+1)}{\max\left\{T^{p^-};T^{p^+}\right\}|\Omega|} \frac{\left(\frac{p^+}{s+1}-1\right)^{\frac{p^+-(s+1)}{q-s}} \left(1-\frac{p^+}{q+1}\right)^{\frac{(q+1)-p^+}{q-s}}}{(q+1)\left(1-\frac{p^+}{q+1}\right)+(s+1)\left(\frac{p^+}{s+1}-1\right)} \right]^{\frac{q-s}{(q+1)-p^+}} \right\}.$$

Then, for each $\eta \in]0, \eta^*[$ the problem

$$\begin{cases} -\Delta_{\vec{p}} u = \eta u^s + u^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (AD_\eta^{\vec{p}})$$

has at least two positive weak solutions.

Example 2: $N = 2, \Omega = B(0,1), p_1 = 3$ and $p_2 = 4$

For each $\eta \in \left]0, \frac{3}{2^8(2^{\frac{1}{2}}+3^{\frac{1}{3}})^8}\right[$, the problem

$$\begin{cases} -\frac{\partial}{\partial x_1}\left(\left|\frac{\partial u}{\partial x_1}\right|\frac{\partial u}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(\left|\frac{\partial u}{\partial x_2}\right|^2\frac{\partial u}{\partial x_2}\right) = \eta u + u^5 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits at least two positive weak solutions.
Indeed

$$m_{p^-} = \left(\frac{2}{\pi}\right)^{\frac{1}{3}}, \quad T_0 = \frac{2}{\pi^{\frac{1}{4}}}, \quad T = (3^{\frac{1}{3}} + 4^{\frac{1}{4}})\frac{2}{\pi^{\frac{1}{4}}},$$

$$\max\left\{T^{p^-}; T^{p^+}\right\}|\Omega| = (3^{\frac{1}{3}} + 4^{\frac{1}{4}})^4 2^4, \quad (s+1)(q+1) = 12,$$

$$\frac{\left(\frac{p^+}{s+1} - 1\right)^{\frac{p^+-(s+1)}{q-s}}\left(1 - \frac{p^+}{q+1}\right)^{\frac{(q+1)-p^+}{q-s}}}{(q+1)\left(1 - \frac{p^+}{q+1}\right) + (s+1)\left(\frac{p^+}{s+1} - 1\right)} = \frac{1}{3^{\frac{1}{2}}4},$$

$$\eta^* = \min\left\{\frac{1}{3}; \left[\frac{3^{\frac{1}{2}}}{(3^{\frac{1}{3}} + 4^{\frac{1}{4}})^4 2^4}\right]^2\right\} = \frac{3}{(3^{\frac{1}{3}} + 4^{\frac{1}{4}})^8 2^8}.$$

Second part

Non-variational elliptic equations

D. Motreanu, A. Sciammetta, E. Tornatore, *A sub-super solutions approach for Neumann boundary value problems with gradient dependence*, Nonlinear Anal. Real World Appl. **54** (2020) 1–12.

$$\begin{cases} -\mathrm{div}(A(x,\nabla u)) + \alpha(x)|u|^{p-2}u = f(x,u,\nabla u) & \text{in } \Omega \\ A(x,\nabla u)\cdot \nu(x) = 0 & \text{su } \partial\Omega. \end{cases} \tag{P}$$

1. $A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous map;
2. $\Omega \subset \mathbb{R}^N$ is a nonempty bounded domain with boundary $C^{1,\gamma}$ for $\gamma \in]0,1[$;
3. $1 < p < +\infty$ with $p < N$;
4. $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function;
5. $\alpha \in L^\infty(\Omega)$, with $\alpha \geq 0$ and $\alpha \not\equiv 0$;
6. ν is the unit outward normal vector to $\partial\Omega$ at each point $x \in \partial\Omega$.
7. $X = W^{1,p}(\Omega)$;
8. $\|u\| = \left(\int_\Omega |\nabla u|^p \, dx + \int_\Omega \alpha(x)|u|^p \, dx\right)^{\frac{1}{p}}$, which is equivalent to the usual one

$$\|u\|_p = \left(\|\nabla u\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}$$

[1] V.G. Mazja, *S*obolev Spaces, Springer Ser. Soviet Math., Springer-Verlag, Berlin, 1985.

G. Barletta, P. Candito, D. Motreanu, *Constant sign and sign-changing solutions for quasilinear elliptic equations with Neumann boundary condition*, J. Convex Anal. **21** (2014), 53–66.

G. Bonanno, P. Candito, *Three solutions to a Neumann problem for elliptic equations involving the p-Laplacian*, Arch. Math. **80** (2003), 424–429.

F. Faraci, D. Motreanu, D. Puglisi, *Positive solutions of quasi-linear elliptic equations with dependence on the gradient*, Calc. Var. Partial Differential Equations **54** (2015), 525–538.

D. Motreanu, V.V. Motreanu, N. Papageorgiou, *Multiple constant sign and nodal solutions for nonlinear Neumann eigenvalue problems*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **(5)** 10 (2011), 729–755.

D. Motreanu, M. Tanaka, *Existence of solutions for quasilinear elliptic equations with jumping nonlinearities under the Neumann boundary condition*, Calc. Var. Partial Differential Equations **43** (2013), 231–264.

D. Motreanu, P. Winkert, *Existence and asymptotic properties for quasilinear elliptic equations with gradient dependence*, Appl. Math. Lett. **95** (2019), 78–84.

M. Tanaka, *Existence of a positive solution for quasilinear elliptic equations with a nonlinearity including the gradient*, Bound. Value Probl., **173** (2013), 11 pp.

P. Winkert, *Multiple solution results for elliptic Neumann problems involving set-valued nonlinearities*, J. Math. Anal. Appl. 377 (1) (2011) 121–134.

Basic notations

Definition

A function $u : \Omega \to \mathbb{R}$ is a weak solution of problem (P) if $u \in W^{1,p}(\Omega)$ satisfies the following condition for all $v \in W^{1,p}(\Omega)$

$$\int_\Omega A(x, \nabla u) \cdot \nabla v dx + \int_\Omega \alpha(x) \left|u\right|^{p-2} uv dx = \int_\Omega f(x, u, \nabla u) v dx.$$

A function $\overline{u} \in W^{1,p}(\Omega)$ is a *supersolution* of problem (P) if $u \in W^{1,p}(\Omega)$ satisfies the following condition

$$\int_\Omega \left(A(x, \nabla \overline{u}) \cdot \nabla v + \alpha(x)|\overline{u}|^{p-2}\overline{u}v\right) dx \geq \int_\Omega f(x, \overline{u}, \nabla \overline{u}) v \, dx$$

for all $v \in W^{1,p}(\Omega)$, with $v \geq 0$ a.e. in Ω.
A function $\underline{u} \in W^{1,p}(\Omega)$ is a *subsolution* of problem (P) if $u \in W^{1,p}(\Omega)$ satisfies the following condition

$$\int_\Omega \left(A(x, \nabla \underline{u}) \cdot \nabla v + \alpha(x)|\underline{u}|^{p-2}\underline{u}v\right) dx \leq \int_\Omega f(x, \underline{u}, \nabla \underline{u}) v \, dx$$

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for all $v \in W^{1,p}(\Omega)$, with $v \geq 0$ a.e. in Ω.

(H) There exists a function $\sigma \in L^{\gamma'}(\Omega)$ with $\gamma \in (1, p^*)$ and $a > 0$ and $\beta \in [0, \frac{p}{(p^*)'})$ such that

$$|f(x, s, \xi)| \leq \sigma(x) + a|\xi|^\beta \text{ for a.e. } x \in \Omega, \text{ all } s \in [\underline{u}(x), \overline{u}(x)], \ \xi \in \mathbb{R}^N.$$

Put $\lambda > 0$ and we consider the following auxiliary Neumann problem:

$$\begin{cases} -\mathrm{div}(A(x,\nabla u)) + \alpha(x)|u|^{p-2}u + \lambda\Pi(u) = N_f(Tu) & \text{in } \Omega, \\ A(x,\nabla u)\cdot\nu(x) = 0 & \text{su } \partial\Omega. \end{cases} \qquad (T_\lambda)$$

1. $N_f : [\underline{u},\overline{u}] \to (W^{1,p}(\Omega))^*$ is the Nemytskij operator corresponding to the function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ in (P), that is
$$\langle N_f(u), v\rangle = \int_\Omega f(x,u(x),\nabla u(x))v(x)dx;$$
2. for all $u \in W^{1,p}(\Omega)$, truncation operator $T : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$
$$Tu(x) = \begin{cases} \overline{u}(x) & \text{if} \quad u(x) > \overline{u}(x), \\ u(x) & \text{if} \quad \underline{u}(x) \le u(x) \le \overline{u}(x), \\ \underline{u}(x) & \text{if} \quad u(x) < \underline{u}(x) \end{cases} \qquad (14)$$
3. cut-off function $\pi : \Omega \times \mathbb{R} \to \mathbb{R}$
$$\pi(x,s) = \begin{cases} (s-\overline{u}(x))^{\frac{\beta}{p-\beta}} & \text{if} \quad s > \overline{u}(x), \\ 0 & \text{if} \quad \underline{u}(x) \le s \le \overline{u}(x), \\ -(\underline{u}(x)-s)^{\frac{\beta}{p-\beta}} & \text{if} \quad s < \underline{u}(x), \end{cases} \qquad (15)$$
4. $\Pi : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is the Nemytskij operator corresponding to the function $\pi : \Omega \times \mathbb{R} \to \mathbb{R}$
$$\Pi(u) = \pi(\cdot, u(\cdot)).$$
5. for each $\lambda > 0$ the operator $A_\lambda : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is defined as
$$A_\lambda(u) = -\mathrm{div}(A(x,\nabla u)) + \alpha(x)|u|^{p-2}u + \lambda\Pi(u) - N_f(Tu)$$

Main tool - Surjectivity theorem

Theorem (see [1, Theorem 2.99])

Let X be a real reflexive Banach space and let $A_\lambda : X \to X^*$ be an operator which satisfies following conditions:

1. A_λ is **bounded**, that is A_λ maps bounded sets to bounded sets;
2. A_λ is **coercive**, that is
$$\lim_{\|u\|\to\infty} \frac{\langle A_\lambda u, u\rangle}{\|u\|} = +\infty;$$
3. A_λ is **pseudomonotone**, that is let $\{u_n\} \in X$ be such that
$$u_n \rightharpoonup u \quad \text{in} \quad X \quad \text{and} \quad \limsup_{n\to\infty} \langle A_\lambda u_n, u_n - u\rangle \leq 0,$$
then $\forall w \in X, \quad \langle A_\lambda u, u - w\rangle \leq \liminf_{n\to\infty} \langle A_\lambda u_n, u_n - w\rangle$.

Then A_λ is **surjective**, i.e. for every $b \in X^*$ the equation $A_\lambda x = b$ has at least one solution $x \in X$.

[1] S. Carl, V.K. Le, D. Motreanu, *Nonsmooth variational problems and their inequalities. Comparison principles and applications*, Springer, New York, 2007.

Hypothesis on A

$A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous and verifies the following condition:

(A) There exist constants $0 < c_1 \leq c_2$ such that

$$A(x,\xi) \cdot \xi \geq c_1 |\xi|^p \text{ and } |A(x,\xi)| \leq c_2(|\xi|^{p-1} + 1)$$

for a.e. $x \in \Omega$, all $\xi \in \mathbb{R}^N$. $A(x,\cdot)$ is monotone on $\mathbb{R}^N$, i.e.

$$(A(x,\xi) - A(x,\eta)) \cdot (\xi - \eta) \geq 0 \text{ for all } \xi, \eta \in \mathbb{R}^N.$$

Remark

We do not require that A has to be a potential operator.

Example: $A(x,\xi) = |\xi|^{p-2}\xi + g(x,\xi)|\xi|^{q-2}\xi$

- $1 < q < p < +\infty$;
- $g : \Omega \times \mathbb{R}^N \to \mathbb{R}$ nonnegative, continuous function such that

$$|g(x,\xi)| \leq c_0(1 + |\xi|^{p-q})$$

  for a constant $c_0 > 0$ for all $x \in \Omega$, for all $\xi \in \mathbb{R}^N$;
- $g(x,\cdot)$ monotone on $\mathbb{R}^N$ for a.e. $x \in \Omega$.

If $g \equiv 0 \Longrightarrow \Delta_p u := \text{div}\left(|\nabla u|^{p-2}\nabla u\right)$, $p-$Laplacian operator.

If $g \equiv 1 \Longrightarrow \Delta_p u + \Delta_q u := \text{div}\left(|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u\right)$, $(p,q)-$Laplacian operator.

Hypothesis on π

$$|\pi(x,s)| \leq c|s|^{\frac{\beta}{p-\beta}} + \varrho(x) \text{ for a.e. } x \in \Omega, \text{ all } s \in \mathbb{R}, \tag{16}$$

with $c > 0$ and $\varrho \in L^{\frac{p}{\beta}}(\Omega)$.
From definition of $\pi : \Omega \times \mathbb{R} \to \mathbb{R}$ we obtain that

$$\int_\Omega \pi(x,u(x))u(x)\,dx \geq r_1 \|u\|^{\frac{p}{p-\beta}}_{L^{\frac{p}{p-\beta}}(\Omega)} - r_2 \text{ for all } u \in W^{1,p}(\Omega) \tag{17}$$

$$\int_\Omega |\pi(x,u(x))||v(x)|\,dx \leq r_3 \|u\|^{\frac{\beta}{p-\beta}}_{L^{\frac{p}{p-\beta}}(\Omega)} \|v\|_{L^{\frac{p}{p-\beta}}(\Omega)} + r_4 \|v\|_{L^{\frac{p}{p-\beta}}(\Omega)} \text{ for all } u,\ v \in W^{1,p}(\Omega), \tag{18}$$

with r_1, r_2, r_3 and r_4 positive constants.

Theorem 1 (Esistence of a solution of auxiliary problem (T_λ))

Assume that $\underline{u}$ and $\overline{u}$ are a subsolution and a supersolution of problem (P) respectively, with $\underline{u} \leq \overline{u}$ a.e. in Ω such that hypotheses (A) and (H) are fulfilled. Then there exists $\lambda_0 > 0$ such that whenever $\lambda \geq \lambda_0$ there is a solution of auxiliary problem (T_λ).

Sketch of Proof: $A_\lambda : W_0^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$

$$\langle A_\lambda u, v\rangle = \int_\Omega A(x, \nabla u) \cdot \nabla v \, dx + \int_\Omega \alpha(x)|u|^{p-2} uv \, dx + \int_\Omega \pi(x, u)v \, dx - \int_\Omega f(x, Tu, \nabla Tu)v \, dx.$$

- A_λ **is bounded**. From (A), (H), estimate (18), and since $\alpha \in L^\infty(\Omega)$.
- A_λ **is pseudomonotone**. Let $\{u_n\} \subset W^{1,p}(\Omega)$ be a sequence satisfies

$$u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty} \langle A_\lambda u_n, u_n - u\rangle \leq 0.$$

From our assumption on f, T, π, α, Hölder inequality and R-K compact embedding theorem we get

$$\lim_{n\to\infty} \int_\Omega f(x, Tu_n, \nabla(Tu_n))(u_n - u)dx = 0, \lim_{n\to\infty} \int_\Omega \pi(x, u_n)(u_n - u)dx = 0, \lim_{n\to\infty} \int_\Omega \alpha(x)|u_n|^{p-1}(u_n - u)dx = 0.$$

Then

$$\limsup_{n\to\infty} \int_\Omega A(x, \nabla u_n) \cdot \nabla(u_n - u)dx \leq 0$$

$(S)_+$ − property

$$\Longrightarrow u_n \to u \text{ in } W^{1,p}(\Omega) \Longrightarrow \quad \begin{array}{l} A_\lambda u_n \rightharpoonup A_\lambda u, \\ \langle A_\lambda u_n, u_n\rangle \to \langle A_\lambda u, u\rangle \end{array}$$

- A_λ **is coercive**.

$$\begin{aligned}\langle A_\lambda u, u\rangle &= \int_\Omega A(x,\nabla u)\cdot\nabla v u dx + \int_\Omega \alpha(x)|u|^p dx + \int_\Omega \pi(x,u)u dx - \int_\Omega f(x,Tu,\nabla Tu) v dx \\ &\geq \int_\Omega A(x,\nabla u)\cdot\nabla u dx + \int_\Omega \pi(x,u)u dx - \int_\Omega f(x,Tu,\nabla Tu) u dx \\ &\geq (c_1-\varepsilon)\|u\|^p + (\lambda r_1 - c(\varepsilon))\|u\|^{\frac{p}{p-\beta}}_{L^{\frac{p}{p-\beta}}(\Omega)} - d\|u\| - \lambda r_2,\end{aligned}$$

with positive constants $c(\varepsilon), c_1, r_1, d$. Choose $\varepsilon \in (0,c)$ and $\lambda > \frac{c(\varepsilon)}{r_1}$, then

$$\lim_{\|u\|\to+\infty} \frac{\langle A_\lambda u, u\rangle}{\|u\|} = +\infty.$$

- Since the operator $A_\lambda : W^{1,p}(\Omega \to (W^{1,p}(\Omega))^*$ is bounded, pseudomonotone and coercive, it is surjective (see [1, p. 40]). Therefore we can find $u \in W^{1,p}(\Omega)$ that solves

$$\begin{cases} -\mathrm{div}(A(x,\nabla u)) + \alpha(x)|u|^{p-2}u + \lambda\Pi(u) = N_f(Tu) & \text{in } \Omega, \\ A(x,\nabla u)\cdot\nu(x) = 0 & \text{on } \partial\Omega. \end{cases} \qquad (T_\lambda)$$

[1] S. Carl, V.K. Le, D. Motreanu, *Nonsmooth variational problems and their inequalities. Comparison principles and applications*, Springer, New York, 2007.

Theorem 2 (the solution of problem (T_λ) is a solution of (P))

Let $\underline{u}$ and $\overline{u}$ be a subsolution and a supersolution of (P), respectively, with $\underline{u} \leq \overline{u}$ a.e. in Ω such that hypotheses (A) and (H) are fulfilled. Then problem (P) possesses a solution $u \in W^{1,p}(\Omega)$ located in the ordered interval $[\underline{u}, \overline{u}]$.

- From Theorem 1, there is a solution $u \in W_0^{1,p}(\Omega)$ of auxiliary problem provided $\lambda > 0$ sufficiently large

$$\begin{cases} -\text{div}(A(x,\nabla u)) + \alpha(x)|u|^{p-2}u + \lambda\Pi(u) = N_f(Tu) & \text{in } \Omega, \\ A(x,\nabla u)\cdot\nu(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{T_λ}$$

- Using comparison arguments we prove that every solution $u \in W_0^{1,p}(\Omega)$ of auxiliary problem satisfies $\underline{u} \leq u \leq \overline{u}$ *a.e. in* Ω;
- The solution u of the auxiliary truncated problem satisfies $Tu = u$ and $\Pi(u) = 0$, so it is a solution of the original problem

$$\begin{cases} -\text{div}(A(x,\nabla u)) + \alpha(x)|u|^{p-2}u = f(x,u,\nabla u) & \text{in } \Omega \\ A(x,\nabla u)\cdot\nu(x) = 0 & \text{su } \partial\Omega. \end{cases} \tag{P}$$

Put

$$\omega(x,s) := \alpha(x)s^{p-1} - f(x,s,0) \text{ whenever } (x,s) \in \Omega \times (0,+\infty),$$

Theorem

Assume that condition (A) holds and there exist two positive constants a_1 and a_2 with $a_1 < a_2$ for which

$$\omega(x,a_1) \leq 0 \text{ and } \omega(x,a_2) \geq 0 \text{ for a.e. } x \in \Omega,$$

$$|f(x,s,\xi)| \leq \sigma(x) + a|\xi|^{\beta} \text{ for a.e. } x \in \Omega, \text{ for all } s \in [a_1,a_2],\ \xi \in \mathbb{R}^N,$$

for $\sigma \in L^{\gamma'}(\Omega)$ with $\gamma' = \frac{\gamma}{\gamma-1}$, $\gamma \in (1,p^*)$, $a > 0$ e $\beta \in [0, \frac{p}{(p^*)'})$.
Then (P) admits at least a (positive) solution $u \in W_0^{1,p}(\Omega)$ satisfying the a priori estimate $a_1 \leq u(x) \leq a_2$ for all $x \in \Omega$.

Sketch of Proof:

- $\underline{u} = a_1 \Longrightarrow \omega(x,a_1) = \alpha(x)a_1^{p-1} - f(x,a_1,0) \leq 0 \Longleftrightarrow \alpha(x)a_1^{p-1} \leq f(x,a_1,0)$
  $$\int_\Omega \big(\underbrace{A(x,0)\cdot\nabla v}_{=0} + \alpha(x)|a_1|^{p-2}a_1 v\big)dx \leq \int_\Omega f(x,a_1,0)v\,dx,\ \forall v \in W^{1,p}(\Omega) \text{ with } v \geq 0 \text{ a.e. on } \Omega.$$
- $\overline{u} = a_2 \Longrightarrow \omega(x,a_2) = \alpha(x)a_2^{p-1} - f(x,a_2,0) \geq 0 \Longleftrightarrow \alpha(x)a_2^{p-1} \geq f(x,a_2,0)$
  $$\int_\Omega \big(\underbrace{A(x,0)\cdot\nabla v}_{=0} + \alpha(x)|a_2|^{p-2}a_2 v\big)\,dx \geq \int_\Omega f(x,a_2,0)v\,dx,\ \forall v \in W^{1,p}(\Omega) \text{ with } v \geq 0 \text{ a.e. on } \Omega.$$
- From Theorem 2, problem (P) possesses a solution $u \in W^{1,p}(\Omega)$ such that $u \in [\underline{u},\overline{u}]$.

Put

$$\omega(x,s) := \alpha(x)s^{p-1} - f(x,s,0) \text{ whenever } (x,s) \in \Omega \times (0,+\infty),$$

and that condition (A) holds.

Theorem

If there exist positive constants a_i ($i = 1, \ldots, 2m$) with $a_1 < a_2 < a_3 < \ldots < a_{2m-1} < a_{2m}$ for which

$$\omega(x, a_{2j-1}) \le 0 \text{ and } \omega(x, a_{2j}) \ge 0 \text{ for a.e. } x \in \Omega, \text{ for all } j = 1, \ldots, m,$$

$$|f(x,s,\xi)| \le \sigma(x) + a|\xi|^{\beta} \text{ for a.e. } x \in \Omega, \text{ for all } s \in \cup_{j=1}^{m}[a_{2j-1}, a_{2j}],\ \xi \in \mathbb{R}^N,$$

for $\sigma \in L^{\gamma'}(\Omega)$, $\gamma' = \frac{\gamma}{\gamma-1}$, $\gamma \in (1, p^*)$, $a > 0$ and $\beta \in [0, \frac{p}{(p^*)'})$.
Then (P) admits at least m (positive) solutions $u_j \in W_0^{1,p}(\Omega)$, satisfying the a priori estimate $a_{2j-1} \le u_j(x) \le a_{2j}$ for all $x \in \Omega$, $j = 1, \ldots, m$.

Put

$$\omega(x,s) := \alpha(x)s^{p-1} - f(x,s,0) \text{ whenever } (x,s) \in \Omega \times (0,+\infty),$$

and that condition (A) holds.

Theorem

If there exists a strictly increasing sequence of positive numbers $\{a_j\}_{j\geq 1}$ such that

$$\omega(x, a_{2j-1}) \leq 0 \text{ and } \omega(x, a_{2j}) \geq 0 \text{ for a.e. } x \in \Omega, \text{ for all } j \geq 1,$$

$$|f(x,s,\xi)| \leq \sigma(x) + a|\xi|^{\beta} \text{ for a.e. } x \in \Omega, \text{ for all } s \in \cup_{j=1}^{\infty}[a_{2j-1}, a_{2j}],\ \xi \in \mathbb{R}^N,$$

for $\sigma \in L^{\gamma'}(\Omega)$, $\gamma' = \frac{\gamma}{\gamma-1}$, $\gamma \in (1, p^*)$, $a > 0$ and $\beta \in [0, \frac{p}{(p^*)'})$.
Then (P) admits infinitely many (positive) solutions $u_j \in W_0^{1,p}(\Omega)$, satisfying the a priori estimate $a_{2j-1} \leq u_j(x) \leq a_{2j}$ for all $x \in \Omega$, $j \geq 1$.

Example (infinitely many solutions)

Given constants $s_0 > 0$, $\beta_1, \beta_2 \in \left[0, \frac{p}{(p^*)'}\right[$, $\eta \in L^\infty(\Omega)$, ed

$$f(x,s,\xi) = (\alpha(x)s^{p-1} + \sin s)(1 + |\xi|^{\beta_1}) + \eta(x)|\xi|^{\beta_2}$$

for a.e. $x \in \Omega$, for all$s \geq s_0$, $\xi \in \mathbb{R}^N$.

$$f(x,s,\xi) = f(x,s_0,\xi),$$

for a.e. $x \in \Omega$, for all $s < s_0$, $\xi \in \mathbb{R}^N$.

$$\omega(x,s) = -\sin s,$$

for a.e. $x \in \Omega$, for all $s \geq s_0$.

Thank you for your kind attention