# Non-local $BV$ functions and a denoising model with $L^1$ fidelity

Giorgio Stefani

(in collaboration with Konstantinos Bessas)

Shape Optimization, Geometric Inequalities, and Related Topics

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Napoli

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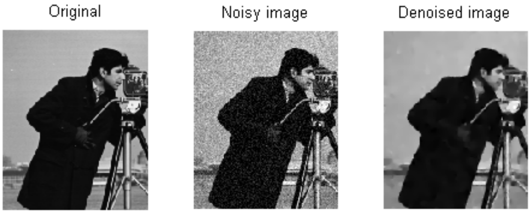


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where $p \in [1, \infty)$ and $\Lambda > 0$ is the fidelity.

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Applications: gravitational-waves (2018) and black hole in Messier 87 galaxy (2019)

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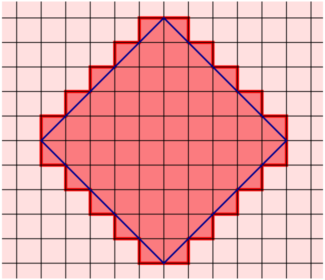

Source: Dipierro-Valdinoci (2018)

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Others: [Buades-Coll-Morel], [Kindermann-Osher-Jones], [Gilboa-Osher], [Antil-Díı́az-Jing-Schikorra] using [Comi-S.] and more...

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STEP 3. We study the associated non-local Cheeger problem.

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## Basic properties

- isometries: $[\,\cdot\,]_K$ is translation invariant, homogeneous and $[c]_K = 0$
- min-max: $[u \wedge v]_K + [u \vee v]_K \leq [u]_K + [v]_K$
- Fatou: $u_k \to u$ in $L^1_{loc}(\mathbb{R}^n) \implies [u]_K \leq \liminf_k [u_k]_K$
- coarea formula: $[u]_K = \displaystyle\int_{\mathbb{R}} P_K(\{u > t\}) \, dt$
- $BV \subset BV^K$: $[u]_K \leq \max\left\{ \|u\|_{L^1}, \frac{1}{2}[u]_{BV} \right\} \displaystyle\int_{\mathbb{R}^n} (1 \wedge |x|) \, K(x) \, dx$

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for $\eta_\delta = K_\delta / \|K_\delta\|_{L^1}$ and $K_\delta = K \, \mathbf{1}_{\mathbb{R}^n \setminus B_\delta}$. Note that $\|K_\delta\|_{L^1} \to \infty$ as $\delta \to 0^+$. □

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Open problem: find isoperimteric sets for $K \notin L^1$ NOT radially symmetric!

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$$K \text{ radially symmetric decreasing} \implies P_K(E) \geq P_K(B^{|E|}), \text{ with } |B^{|E|}| = |E|$$

$$\text{equality} \iff E \text{ is a ball, if } K \text{ radial}^+ \text{ in a ngbh of the origin}$$

Idea of proof: Apply Riesz rearrangement inequality to

$$P_K(E) = \int_0^{\|K\|_{L^\infty}} |E|\,|\{K > t\}| - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathbf{1}_E(x)\,\mathbf{1}_E(y)\,\mathbf{1}_{\{K>t\}}(x-y)\,dx\,dy\,dt$$

noticing that $\{K > t\} = B_{R(t)}$ is a ball for some $R(t) \in [0, \infty]$. □

**Open problem**: find isoperimteric sets for $K \notin L^1$ NOT radially symmetric!

**Corollary [Bessas-S.]**

$$K \text{ radially symmetric decreasing} \implies [u]_K \geq [u^\star]_K$$

$$\text{equality} \iff u \geq 0,\ \{u > t\} \text{ is a ball, if } K \text{ radial}^+ \text{ in a ngbh of the origin}$$

where $u^\star$ is the symmetric decreasing rearrangement of $u$ (apply coarea formula).

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$$P_K(\lambda E) = \int_{\lambda E} \int_{(\lambda E)^c} K(x-y)\,dx\,dy = \lambda^{2n} \int_E \int_{E^c} K(\lambda(x-y))\,dx\,dy$$

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## Gagliardo-Nirenberg-Sobolev for finite support [Bessas-S.]

$u \in BV^K$ with $|\mathrm{supp}(u)| < \infty \implies \|u\|_{L^{\frac{n}{2n-q},1}} \leq C^{iso}_{n,q,|\mathrm{supp}(u)|}\,[u]_K.$

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$K$-Archimedes: $A \subset B$ with $A$ convex and $|B| < +\infty \implies P_K(A) \le P_K(B)$

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We study the functional $K$-variation $L^1$ denoising problem

$$(\text{FP}) \qquad \min_{u \in L^1_{\text{loc}}(\mathbb{R}^n)} [u]_{BV^K} + \Lambda \int_{\mathbb{R}^n} |u - f| \, d\nu$$

where $\nu \in \mathcal{W}(\mathbb{R}^n) = \{\nu = w\mathscr{L}^n : w \in L^\infty, \inf_{\mathbb{R}^n} w > 0\}$ an $L^\infty$-weight measure.

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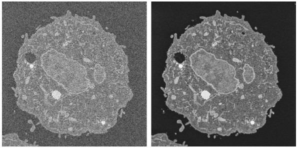
**Why $L^\infty$-weight measures?** $\rightsquigarrow$ deep learning

- do not alter the $L^1$ nature of the approximation term
- more flexibility, adding a degree of freedom in the fidelity
- $\Lambda > 0$ keeps its role of global Lagrangian multiplier
- $\nu$ secondary local fidelity parameter (emphasis on specific regions only)



Source: Sun-Parwani

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Call FSol$(f, \Lambda, \nu)$ the set of solutions of the functional problem

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### Basic properties of F-solutions

- FSol$(f, \Lambda, \nu) \subset L^1_{\text{loc}}$ is convex and closed
- $u_j \in$ FSol$(f_j, \Lambda, \nu)$, $f_j \to f$ in $L^1$, $u_j \to u$ in $L^1_{\text{loc}} \implies u \in$ FSol$(f, \Lambda, \nu)$
- FSol$(f + c, \Lambda, \nu) =$ FSol$(f, \Lambda, \nu) + c$
- FSol$(cf, \Lambda, \nu) = c\,$FSol$(f, \Lambda, \nu)$
- $u \in$ FSol$(f, \Lambda, \nu) \implies u^+ \in$ FSol$(f^+, \Lambda, \nu)$, $u^- \in$ FSol$(f^-, \Lambda, \nu)$
- $u \in$ FSol$(f, \Lambda, \nu) \implies u \wedge c \in$ FSol$(f \wedge c, \Lambda, \nu)$, $u \vee c \in$ FSol$(f \vee c, \Lambda, \nu)$

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We also study the geometric $K$-variation $L^1$ denoising problem ($f = \chi_E$, $u = \chi_U$)

$$(\mathrm{GP}) \qquad \min_{U \subset \mathbb{R}^n} P_K(U) + \Lambda \nu(U \triangle E)$$

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- $U \in \mathrm{GSol}(E, \Lambda, \nu) \implies U + x \in \mathrm{GSol}(E + x, \Lambda, \nu_x)$, $\nu_x(A) = \nu(A - x)$
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## Relation between F-solutions and G-solutions

- $u \in \mathrm{FSol}(f, \Lambda, \nu) \implies \{u > t\} \in \mathrm{GSol}(\{f > t\}, \Lambda, \nu)$ for all $t \in \mathbb{R} \setminus \{0\}$
- $\{u > t\} \in \mathrm{GSol}(\{f > t\}, \Lambda, \nu)$ for a.e. $t \in \mathbb{R} \implies u \in \mathrm{FSol}(f, \Lambda, \nu)$

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Moreover, if $|E| < \infty$, then:

- $U \in \mathrm{GSol}(E, \Lambda, \nu) \implies \chi_U \in \mathrm{FSol}(\chi_E, \Lambda, \nu)$
- $u \in \mathrm{FSol}(\chi_E, \Lambda, \nu) \implies 0 \leq u \leq 1$ a.e., $\{u > t\} \in \mathrm{GSol}(E, \Lambda, \nu)$ for $t \in (0, 1)$

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Moreover, if also $K \notin L^1$, then

(2) $E$ bounded convex $\implies \mathrm{FSol}(\chi_E, \Lambda, \nu) = \{\chi_{U_\Lambda}\}$ for a.e. $\Lambda > 0$ with $U_\Lambda \subset E$

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(1) $E \subset B_R \implies U \subset B_R$ for all $U \in \mathrm{GSol}(E, \Lambda, \nu)$

Moreover, if also $K \notin L^1$, then

(2) $E$ bounded convex $\implies \mathrm{FSol}(\chi_E, \Lambda, \nu) = \{\chi_{U_\Lambda}\}$ for a.e. $\Lambda > 0$ with $U_\Lambda \subset E$

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(1) $\nu((U \cap B_R) \cap E) \leq \nu(U \cap E)$ and $P_K(U \cap B_R) \leq P_K(U)$, since $B_R$ convex.

(2) Consider monotone maps $\Lambda \mapsto \inf / \sup\{\|u - \chi_E\|_{L^1(\nu)} : u \in \mathrm{FSol}(\chi_E, \Lambda, \nu)\}$.

Prove that $\mathrm{FSol}(\chi_E, \Lambda, \nu) = \{u_\Lambda\}$ for $\Lambda > 0$ outside countable jump set.

Observe that $u = \chi_U$ for some $U \subset E$ by basic properties.

Since $\mathrm{FSol}(\chi_E, \Lambda, \nu)$ is convex, $U$ is unique. □

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Assume $K \notin L^1(\mathbb{R}^n)$, $K \in L^1(\mathbb{R}^n \setminus B_r)$ for all $r > 0$. If $|E| < \infty$, then (GP) admits a minimal and a maximal solution $E^-, E^+ \in \mathrm{GSol}(E, \Lambda, \nu)$ w.r.t. inclusion.

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## Comparison principle for (GP) [Bessas-G.]

Assume $K \notin L^1(\mathbb{R}^n)$, $K \in L^1(\mathbb{R}^n \setminus B_r)$ for all $r > 0$, $K$ symmetric and $K > 0$.
If $P_K(E_i) < \infty$ and $\min\{|E_i|, |E_i^c|\} < \infty$ for $i = 1, 2$, then

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Remark $K > 0$ can be weaken to get a comparison principle at small scales.

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**High fidelity for $C^{1,1}$ regular sets [Bessas-S.]**

Let $E$ be $C^{1,1}$ regular open set with $\min\{|E|, |E^c|\} < \infty$. There is $\bar{\Lambda} > 0$ such that

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Arguing via level sets, one can extend the previous result to functions.

**High fidelity for uniformly $C^{1,1}$ regular functions [Bessas-S.]**

Let $f \in L^1$ have uniformly $C^{1,1}$ regular superlevel sets. There is $\bar{\Lambda} > 0$ such that

$$\mathrm{FSol}(f, \Lambda, \nu) = \{f\} \quad \text{for all } \Lambda > \bar{\Lambda}.$$

uniformly $C^{1,1}$ regular superlevels = inner/outer radius of $\{f > t\}$ uniform in $t \in \mathbb{R}$

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The trick is to estimate $[u]_K \gtrsim_h \| u(\cdot + h) - u \|_{L^1} = 2 \| u \|_{L^1} \gtrsim_\nu \| u \|_{L^1(B_R, \nu)}$ for $2R \leq |h| \leq \frac{D}{2}$.

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$$|u(x) - u(y)| \le \omega_{K,D}(|x - y|)\, \big(\mathbf{D}_K u(x) + \mathbf{D}_K u(y)\big),$$

$$\mathbf{D}_K u(x) = \frac{1}{2} \int_{\mathbb{R}^n} |u(x) - u(z)|\, K(x - z)\, dz \quad \text{and} \quad \omega_{K,D} \text{ modulus of continuity}$$

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Let $K$ radial, $K \in L^1(\mathbb{R}^n \setminus B_r)\ \forall r > 0$, $K \notin L^1$ and $q$-decreasing with $q < n+1$. Cheeger sets $E$ exist (hence $h_{K,\nu}(\Omega) > 0$) with

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Moreover, $\partial E \cap \partial \Omega \neq \emptyset$ for $\nu = \mathscr{L}^n$, $\Omega$ open and $K$ $n$-decreasing$^{+}$.

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$$|E|^{\frac{q}{n}-1} \geq C^{iso}_{|\Omega|,n,q,\nu}\, h_{K,\nu}(\Omega).$$

Moreover, $\partial E \cap \partial \Omega \neq \emptyset$ for $\nu = \mathscr{L}^n$, $\Omega$ open and $K$ $n$-decreasing+.

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**Further properties for $\nu = \mathscr{L}^n$ [Bessas-S.]**

- calibrability: balls are self-Cheeger sets
- $K$-Faber-Krahn inequality: $h_K(\Omega) \geq h_K(B^{|\Omega|})$ where $|B^{|\Omega|}| = |\Omega|$

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For $\nu = \mathscr{L}^n$ and $E =$ ball $B$, such result can be improved as

$$\mathrm{GSol}(B, \Lambda, \mathscr{L}^n) = \begin{cases} \{\emptyset\} & \text{for } \Lambda < \Lambda_0 \\ \{\emptyset, B\} & \text{for } \Lambda = \Lambda_0 \\ \{B\} & \text{for } \Lambda > \Lambda_0 \end{cases} \qquad \text{where } \Lambda_0 = \frac{P_K(B)}{|B|}$$

# THANK YOU FOR YOUR ATTENTION!

Slides available via giorgio.stefani.math@gmail.com or giorgiostefani.weebly.com.

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