About maximal distance minimizers

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Naples, January 30–31, 2023
Shape Optimization, Geometric Inequalities, and Related Topics

Problem

For a given compact set $M \subset \mathbb{R}^n$ and a given number $r > 0$ find a closed connected Σ, such that

$$\begin{cases} M \subset \overline{B}_r(\Sigma) \\ \mathcal{H}^1(\Sigma) \text{ is minimal.} \end{cases}$$

The problem was stated at 2003 and was actively reseached in works by Miranda, Paolini, Butazzo and Stepanov (in $\mathbb{R}^n$). They proved that a minimizer Σ exists and that a minimizer can not contain a loop.

Today I am going to talk about:

- The statement of maximal distance minimizer problem;
- Explicit examples;
- Regularity properties of maximal distance minimizers;
- Energetic points: most important points of minimizers;
- A few words about Steiner tree problem;
- Sketch of the one proof for one example;
- Inverse problem and magic (if I will have time).

Problem (Statement 1)

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Problem (Statement 2)

For a given compact set $M \subset \mathbb{R}^n$ and a given number $r > 0$ find a closed connected Σ, such that

$$\begin{cases} F_M(\Sigma) := \max_{y \in M} \mathit{dist}(y, \Sigma) \leq r \\ \mathcal{H}^1(\Sigma) \text{ is minimal.} \end{cases}$$

Problem (Dual statement)

For a given compact set $M \subset \mathbb{R}^n$ and a given number $l > 0$ find a closed connected Σ, such that

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Example for two points at a distance $R > 2r$ apart:

One example for three points:

Each tripod Σ is a minimizer for some three points and $r > 0$. But not vice versa.

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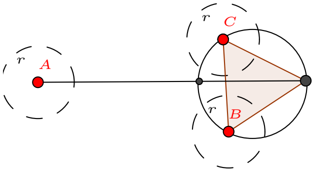

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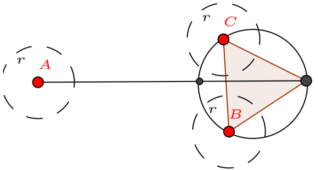

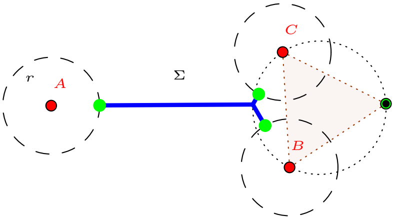

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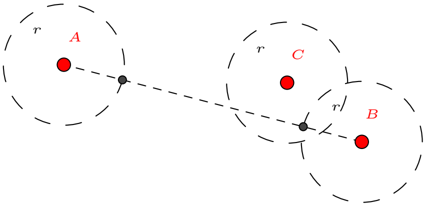

A segment Σ is minimizer for the border (or closure) of its r-neighbourhood.

Every maximal distance minimizer Σ for a set M and number $r > 0$ is also a minimizer for r-neighbourhood of Σ. Uniqueness is an open question here.

The simplest examples

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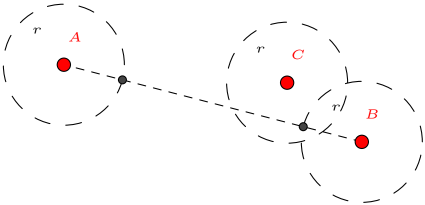

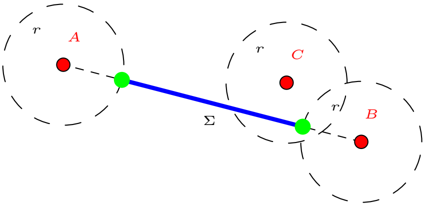

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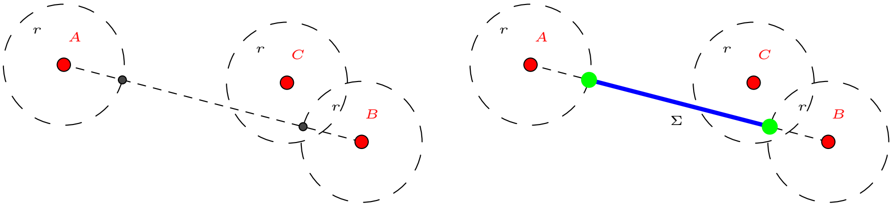

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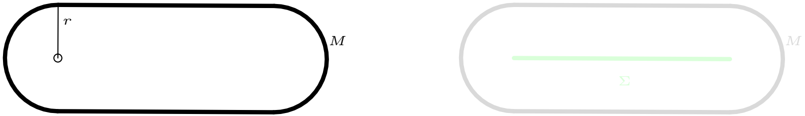

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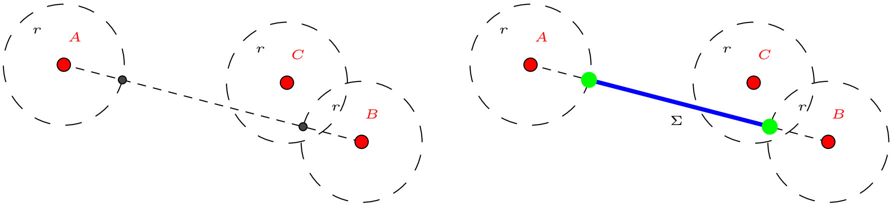

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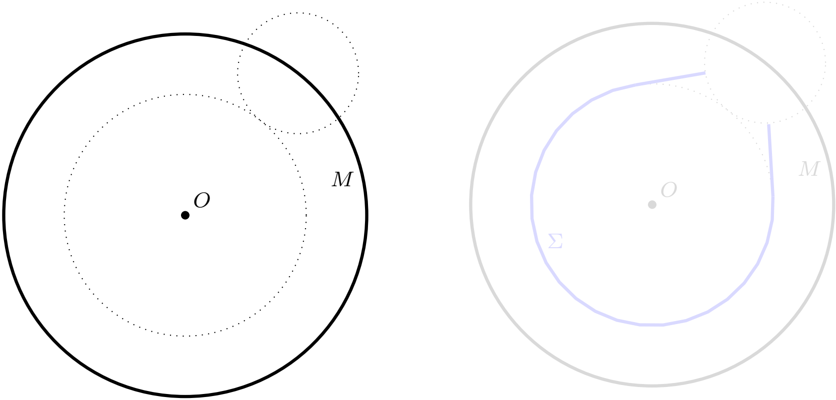

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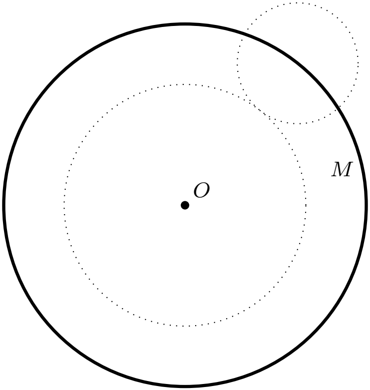

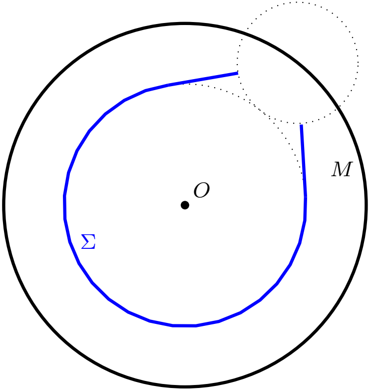

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Solution for a concrete M. A curve with a great curvature radius

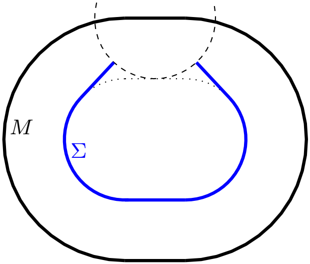

Figure: The solution for the set M with big radius of curvature

Theorem (Cherkashin, T., 2016)

For every closed convex curve M with minimal radius of curvature R and for every $r < R/5$ the set of minimizers contains only horseshoes. For the circumference $M = \partial B_R(O)$ the claim is true for $r < R/4.98$.

Still unknown: what is minimizer for a circle with $R > r > R/4.98$? (it conjectured for a circle by Paolini, Miranda and Stepanov that the answer still is a horseshoe)

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Still unknown:

1. What if $R > r > R/4.98$? (it conjectured for a circle by Paolini, Miranda and Stepanov that the answer still is a horseshoe)
2. What if M is a narrow stadium? (it is not a horseshoe!)

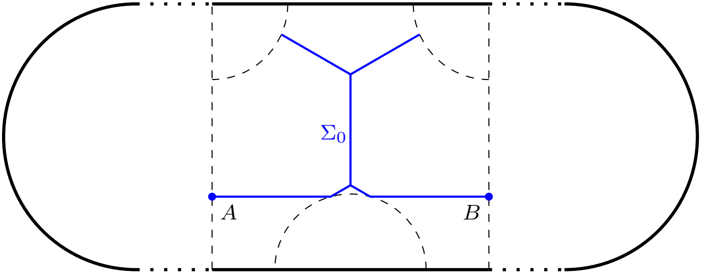
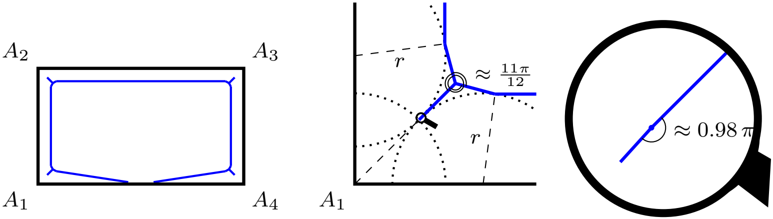


Figure: Horseshoe is not a minimizer for long enough stadium with $R < 1.75r$.



When M is a rectangle, we described the topology of maximal distance minimizers (see our preprint arXiv:2106.00809).

Theorem (Cherkashin–Gordeev–Strukov–T,2021)

Let $M = A_1A_2A_3A_4$ be a rectangle, $r > 0$ be chosen small enough depending on M. Then any maximal distance minimizer has the topology depicted in the left part of Fig. **??**. *The middle part of the picture contains enlarged fragment of the minimizer near A_1; the labeled angles are equal to $\frac{2\pi}{3}$. The rightmost part contains much more enlarged fragment of minimizer near A_1. A minimizer consists of 21 segments; an approximation of the length of a minimizer is $Per - 8.473981r$, where Per is the perimeter of the rectangle.*

Definition

We say that the ray $(ax]$ is a *tangent ray* of the set Σ at the point $x \in \Sigma$ if there exists a non stabilized sequence of points $x_k \in \Sigma$ such that $x_k \to x$ and $\angle x_k x a \to 0$.

Theorem (Gordeev, T., 2022)

Let Σ be a maximal distance minimizer for a compact set $M \subset \mathbb{R}^n$ and an $r > 0$ be fixed. Then

(i) *the angle between each pair of tangent rays at every point of Σ is at least $2\pi/3$. The number of tangent rays at every point of Σ is not greater than 3.*

(ii) *In planar case Σ is a union of a finite number of injective images of the segment $[0, 1]$ with non-intersecting interiors;*

Corollary

In planar case the number of triple points is finite.

Remark. It is not true for a Steiner tree, i. e. there exists an indecomposable Steiner tree with infinite number of triple points.

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Figure: Four cases of one-sided tangent lines in $\mathbb{R}^n$

At the plane also:

- finiteness number of branching points;
- continuity of one-sided tangent rays;
- regular tripod in a neighbourhood of a branching point.

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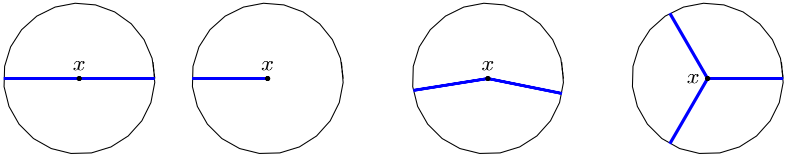

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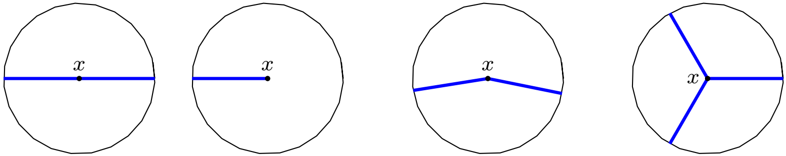

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Main property. For every energetic point $x \in \Sigma$ there exists an $y \in M$ such that $|x - y| = r$ and $B_r(y) \cap \Sigma = \emptyset$.

Figure: The rightest can not be energetic; two middle should be energetic; the leftest can be both

Let us call an *isolated energetic point* of Σ such a point that it has a neighbourhood without any other energetic points. Every isolated point has one of first three depicted behaviours.

Note that in some sense, any minimizer in $\mathbb{R}^n$, $n > 2$ does not have non-energetic points in a larger dimension:

Example

Let Σ be a (local) minimizer for a compact set $M \subset \mathbb{R}^n$ and $r > 0$. Then $\bar{\Sigma} := \Sigma \times \{0\} \subset \mathbb{R}^{n+1}$ is a (local) minimizer for $\bar{M} = (M \times \{0\}) \cup (\Sigma \times \{r\}) \subset \mathbb{R}^{n+1}$ and $E_{\bar{\Sigma}} = \bar{\Sigma}$.

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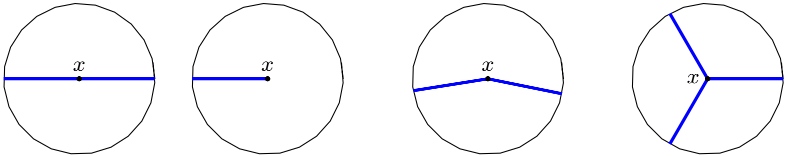

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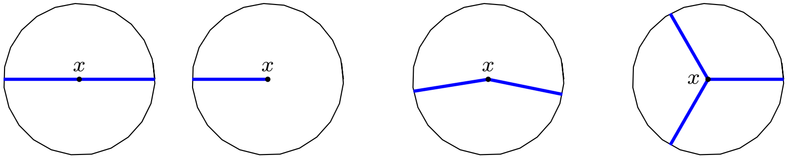

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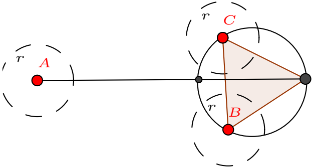
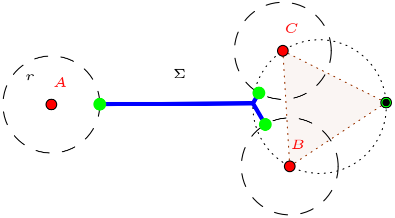
The energetic points. Examples

Given set M is red. The energetic points of Σ are green. Non-energetic points of Σ are blue.





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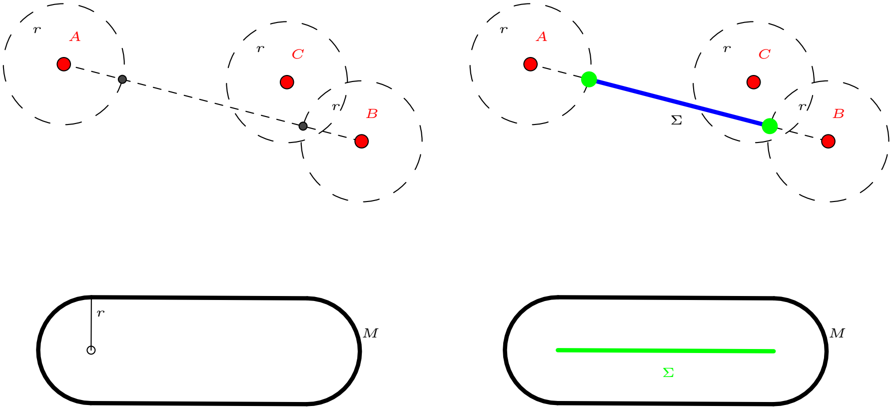

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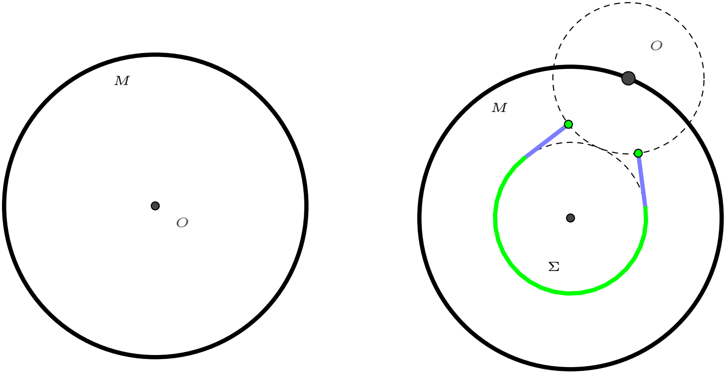

Problem (Steiner tree problem)

$C = \{C_1, C_2 \dots C_m\} \subset \mathbb{R}^n$. To find such a compact set $S : C \subset S$, S is connected, $\mathcal{H}^1(S)$ is the smallest.

Some properties

- $\overline{S}$ contains no loops;
- S is a finite union of segments with pairwise angle at least $2\pi/3$.
- Each point of $S \setminus C$ is a center of a segment or of a regular tripod (see two left cases on the picture)
- A point of C can also be an endpoint or a cornerpoint.

Then S is usually called **Steiner tree**, and it is called **indecomposable**, when $S \setminus C$ is connected.

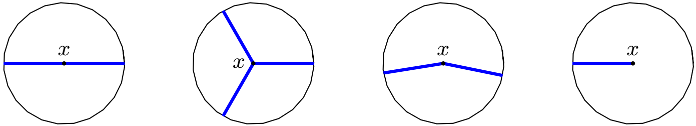

Figure: Four cases of local behaviour of Steiner tree

Parallels between maximal distance and Steiner problems

We have: a set of points at the plane.
We should: construct the connected set arriving at the distance $\leq r$ to every points.

Example for two points with big distance $R > 2r$ between them:

Our problem (to find minimizers of the maximum distance): to connect $r-$neighbourhoods of the points by the shortest connected set.

Steiner problem: to connect set of points by the shortest set:

C— subset of complete metric space. To find S: $\{S \cup C$ — connected$\} =:$ St(C)

$$\mathcal{H}^1(S) \leq \mathcal{H}^1(S'), \forall S' \in \text{St}(C)$$

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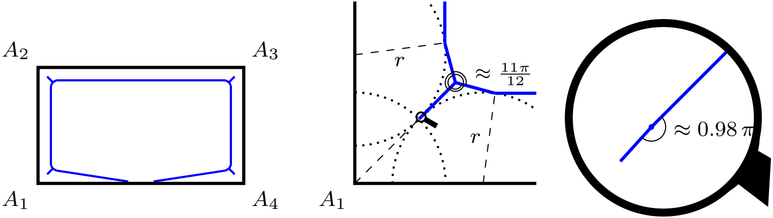

Theorem (Cherkashin–Gordeev–Strukov–T,2021)

If M is a rectangle and $r > 0$ is sufficiently small, then a maximal distance minimizer has topology at depicted at the left figure.

Sketch of the proof:

- Is empty inside (no energetic points and no long segments $\rightarrow$ nothing).
- Angles and stripes. In each angle Σ is connected (we want to win almost $2r$).
- Σ almost contains cycle $\mathcal{C}$ which should be convex polygon.
- $\Sigma \cap \mathcal{C}$ in the angle has 5 vertices and exactly 1 of them is a branching point on $\mathcal{C}$.
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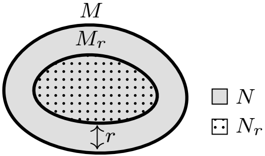

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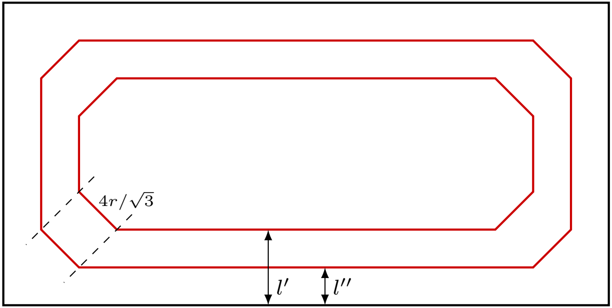

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For a polygonal chain $B_1, \ldots, B_n$ define its *turning* as follows

$$\text{turn}(B_1, \ldots, B_n) := \sum_{i=1}^{n-2} \angle \left([B_i B_{i+1}), [B_{i+1}, B_{i+2}) \right).$$

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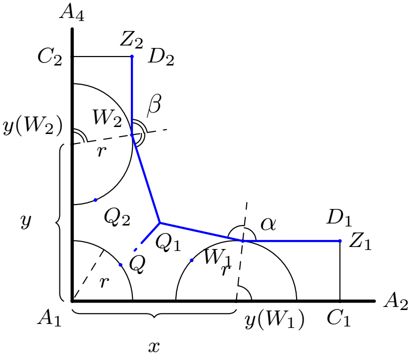

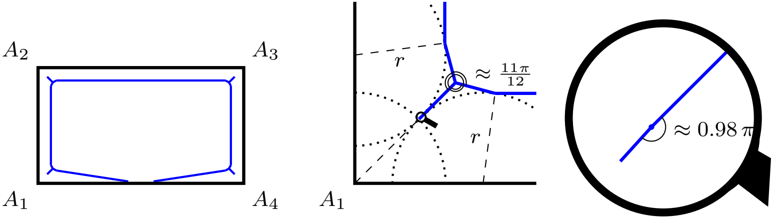

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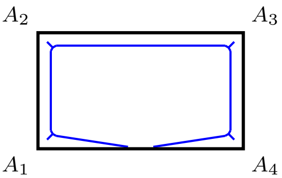

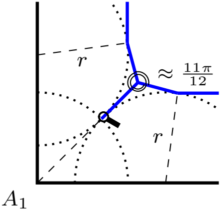

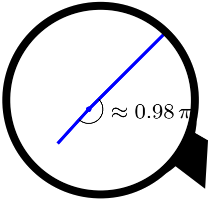

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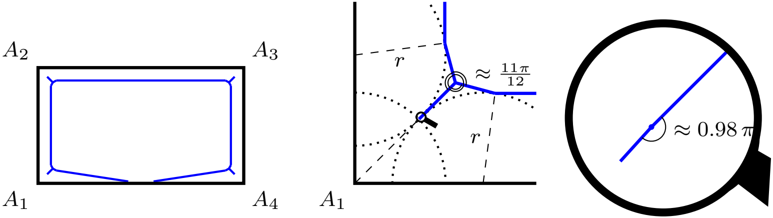

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- If M is a finite set of points. Then Σ is a Steiner tree on at most $\sharp M$ terminals.

Problem (Steiner tree problem)

$C = \{C_1, C_2 \dots C_m\} \subset \mathbb{R}^n$. To find such a compact set $S : C \subset S$, S is connected, $\mathcal{H}^1(S)$ is the smallest.

Some properties

- $\overline{S}$ contains no loops;
- S is a finite union of segments with pairwise angle at least $2\pi/3$.
- Each point of $S \setminus C$ is a center of a segment or of a regular tripod (see two left cases on the picture)
- A point of C can also be an endpoint or a cornerpoint.

What about inverse problem? We want to construct M if Σ is Steiner tree.

Theorem

Cherkashin, T., 2022 Let St be a Steiner tree for terminals $A = (A_1, ..., A_m)$, $A_i \in \mathbb{R}^n$ such that every Steiner tree for an n-tuple in the closed $2r$-neighbourhood of A has the same topology as St for some positive r. Then St is an r-minimizer for an m-tuple M and such M is unique.

Usually the condition holds:

Theorem (Basok, Cherkashin, T., 2022)

For $m \geq 4$ the set of set of terminals with non unique Steiner trees has the Hausdorff dimension $2m - 1$.

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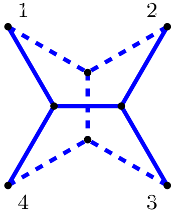
It turns out that a Steiner tree for the vertices of a square is not a maximal distance minimizer for every set of four points:



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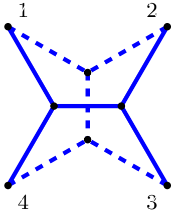

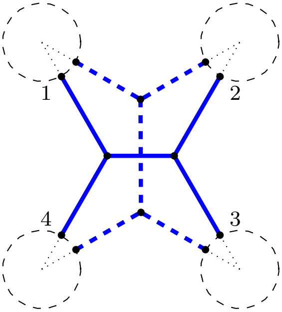

Inverse problem (for simple M)

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- When M is an r-neighbourhood of smooth curve (for sufficiently small $r > 0$)

Theorem

Let γ be a $C^{1,1}$-curve. Then γ is a maximal distance minimizer for a small enough r and $M = \overline{B_r(\gamma)}$.

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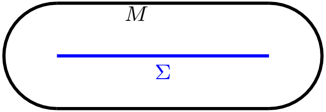

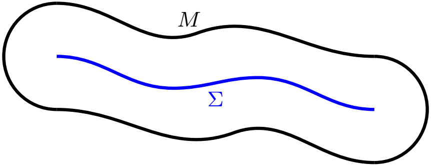

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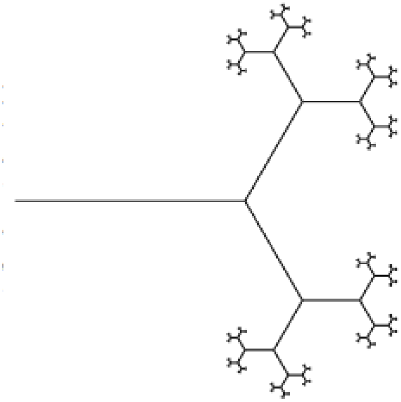


Figure: Indecomposable Steiner tree with infinite number of branching points. Can be a self-similar fractal since 2023

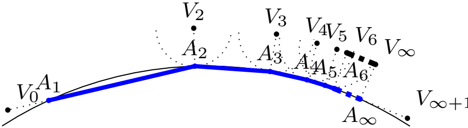
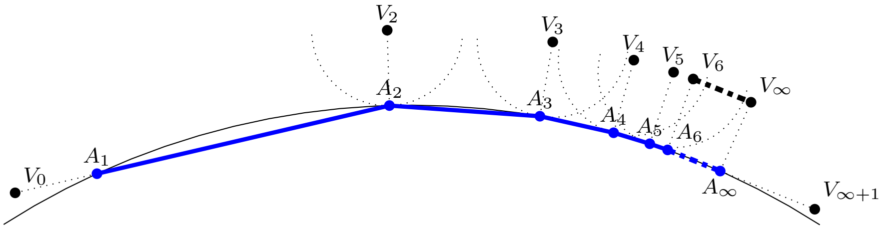


Figure: The example of a minimizer with an infinite number of corner points

- Find the minimizers for a circumference of radius $r < R < 4.98r$. Find the minimizers for a ball.
- Find the explicit estimate for the curvature radius at the horseshoe theorem
- Find the minimizers for a given stadium.
- Can maximal distance minimizer in Euclidean space have infinite many branching points?
- If Σ is a minimizer for some M then Σ is a minimizer for $\overline{B_r(\Sigma)}$. Is Σ the unique minimizer for $\overline{B_r(\Sigma)}$?



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Lemma (Gordeev, Teplitskaya, 2021)

Let Σ be a local minimizer for a compact set $M \subset \mathbb{R}^2$ and $r > 0$ and let $x \in \Sigma$. Let Σ_1 be a connected component of $\Sigma \setminus \{x\}$ with one-sided tangent $(ax]$ and let $\bar{x} \in \Sigma_1$.

1. *For any one-sided tangent $(\bar{a}\bar{x}]$ of Σ at $\bar{x}$ the equality $\angle((\bar{a}\bar{x}), (ax)) = o_{|\bar{x}x|}(1)$ holds.*
2. *Let $(\bar{a}\bar{x}]$ be a one-sided tangent at $\bar{x}$ of any connected component of $\Sigma \setminus \{\bar{x}\}$ not containing x. Then $\angle((\bar{a}\bar{x}], (ax]) = o_{|\bar{x}x|}(1)$.*

For maximal distance minimizers in Euclidean space the following objects coincide due to regularity theorem

Definition

We will say that the ray $(ax]$ is a *one-sided tangent* of the set $\Gamma \subset \mathbb{R}^n$ at the point $x \in \Gamma$ if there exists a connected component Γ_1 of $\Gamma \setminus \{x\}$ with the property that any sequence of points $x_k \in \Gamma_1$ such that $x_k \to x$ satisfies $\angle x_k x a \to 0$. In this case we will also say that $(ax]$ is tangent to the connected component Γ_1.