Università Degli Studi Di Napoli
"Federico II"

Scuola Politecnica e delle Scienze di Base
Area Didattica di Scienze Matematiche Fisiche e Naturali

Dipartimento di Matematica e Applicazioni
"Renato Caccioppoli"

Corso Di Laurea in Matematica

Tesi Compilativa In Analisi Matematica

On the Regularity of Solutions of Second Order Elliptic Equations

Candidato:

Gianpaolo Piscitelli

matr.: N98000091

Relatore:

Dott.ssa

Anna Verde

Correlatore:

Dott.ssa

Addolorata Marasco

Sessione Estiva - Anno Accademico 2012/13

Dedication

To my parents:
Thanks for your love, precious support and sacrifice in my education. Thanks for the effort to keep a calm atmosphere in the family and to keep good relationships with everyone.

To my brother:
Thanks for your affection, intense respect and esteem.

To the rest of my family:
Thanks for teaching me to resist in painful moments and for being close to me in joyful moments.

To the community of my town:
Thanks for all the experiences I've had with you and for all the people who have made me know.

Acknowledgments

In terms of this work, I would like to take this opportunity to express my respect and acknowledgment to University of Naples, Department of Mathematic "Renato Caccioppoli" for the procure of all facilities needed to complete this work.

Related to scientific matter, my most sincere thanks and gratitude to my supervisor prof. Nicola Fusco for his helpful and precious advises, suggestions and guidance that he gave me during my thesis period.

My special recognition and deep thanks to my tutor in this achievement Dr. Anna Verde for his advices, help and effort to make hard work more easier for me. I will always appreciate her encouragements to overcome setbacks.

Many friends have contributed to create a more familiar spirit through these wonderful years. Their support and care helped me to stay focused on my study. I would like to thank Luigi Q., Annabella, Salvatore, Francesco, Luigi S., for their help and supporting during my studies and others sweet moments that I have spent with them.

Heartfelt appreciation for all the people, that contributes in the achievement of this modest work.

Contents

Introduction

In this thesis we will concentrate on a famous result due to De Giorgi [**2**] on the Hölder regularity of weak solutions of second order elliptic equations in divergence form with bounded and measurable coefficients.
It is worth to mention that his result solved one of the millenium problem proposed by Hilbert in 1900 during the Second International Congress of Mathematicians in Paris, precisely the nineteenth, about the regularity of minimizers of variational functionals:

The solutions of regular problems of the calculus
of variations are always analytical?

His result was independently obtained by Nash [**15**] and later Moser [**14**] gave a different proof. For this reason nowadays this result is known as De Giorgi-Nash-Moser Theorem.

The starting point of this problem can be addressed in the regularity theory of variational integrals of the type:

$$\mathscr{F}(v) := \int_\Omega F(Dv(x))\, dx \tag{0.1}$$

where Ω is an open subset in $\mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}$ belongs to C^2 class and $\forall p \in \mathbb{R}^n$

$$|D^2F(p)| \leq C, \tag{0.2}$$

$$\exists \lambda > 0 : \langle D^2F(p)\xi, \xi\rangle \geq \lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^n. \tag{0.3}$$

In the Sobolev settings we find that minima of functional (0.1) are weak solutions of the Eulero-Lagrange equation

$$D_i(D_{p_i}F(Dv(x))) = 0 \quad \text{in } \Omega, \tag{0.4}$$

that is

$$\int_\Omega D_{p_i}F(Dv(x))D_i\phi \; dx = 0, \quad \forall \phi \in C_0^\infty(\Omega). \tag{0.5}$$

Furthermore, it can be proved that a solution v of (0.5) is in $W^{2,2}_{loc}(\Omega)$, hence choosing the derivative $D_k\phi$ as test function and integrating by parts we get

$$\int_\Omega D_{p_ip_j}F(Dv(x))D_{jk}vD_i\phi = 0. \tag{0.6}$$

Therefore the derivatives D_ku are weak solutions of the following linear equation in divergence form

$$D_i(D_{p_ip_j}F(Dv(x))D_j(D_kv)) = 0. \tag{0.7}$$

that is

$$D_i(a^{ij}(x)D_j(u^k) \; dx = 0. \tag{0.8}$$

where $u^k = D_kv$ and $a^{ij}(x) = D_{p_ip_j}F(u(x)) \; i, j = 1, ..., n$.
By Schauder approach, if $a^{ij} \in C^{k,\alpha}(\Omega)$ and u^k is a weak solution of (0.8) then $Du \in C^{k+1,\alpha}(\Omega)$.
Thus supposing to know that $F \in C^\infty(\Omega)$ and u is Hölder continuous, then the equation (0.8) has $C^{0,\alpha}(\Omega)$ coefficients, and so $Du \in C^{1,\alpha}_{loc}(\Omega)$. It follows that $u \in C^{2,\alpha}(\Omega)$. Now the coefficients of (0.8) are in $C^{1,\alpha}(\Omega)$ and, again using the Schauder estimates, we get $u \in C^{3,\alpha}(\Omega)$. By iterating, with a bootstrap argument, we conclude that $u \in C^\infty(\Omega)$. We have requested that $u \in C^{0,\alpha}$ to start the iterative procedure, but we only know that $u \in W^{2,2}_{loc}(\Omega)$.
In this context De Giorgi Theorem allows us to fill this gap because it proves that only in the hypotheses of bounded and measurable coefficient the solutions are locally Hölder continuous.

The outline of the thesis follows.
In the first Chapter we recall significant results on Hölder and L^p spaces, as well as on cut-off functions and Sobolev spaces, mainly without proofs because they form a necessary background for what follows.
In the second Chapter we present the proof of De Giorgi's Theorem. We start introducing the De Giorgi class DG and we prove that solutions of elliptic equations belong to such class. Next we show fundamental estimates of the essential supremum to establish the local boundedness of these functions. Finally, to deduce the local Hölder regularity of DG functions we prove an estimate reduction of the essential oscillation on balls of decreasing radius.
In the third Chapter we provide another proof of De Giorgi-Nash-Moser Theorem, due to Moser. We use the John-Niremberg result to prove that weak subsolutions satisfies a weak Harnack inequality. The main tool is the use of suitable test functions. The idea is to "truncate" them in those regions where its derivative becomes degenerate.
Fourth Chapter is devoted to present some counterexamples showing that in the vectorial case there is no everywhere regularity and we mention that only "partial regularity" is available.
In the last Chapter we analyze two physical phenomena that can be described by an elliptic system in divergence form: the motion of an isotropic linear elastic material system and the two-dimensional irrotational flow of an incompressible fluid with vortex potential.

Chapter 1

Preliminaries

1. Basic Notation

In this section we present the notations we will use throughout this thesis. As usual, we denote $\mathbb{R}^n$, $n \geq 2$, the Euclidean n-space, $x = (x_1, ..., x_n) = (x_i)$ its points, with $x_i \in \mathbb{R}$ (real numbers) and $|x| = (\sum_{i=1}^n x_i^2)^{(1/2)}$ the euclidean norm. Let $\mathbf{b} = (b_1, ..., b_n)$ be an ordered $n-$tuple, we denote $|\mathbf{b}| = (\sum_{i=1}^n b_i^2)^{(1/2)}$ its norm.

Let Ω be a subset in $\mathbb{R}^n$. If Ω' is a subset of Ω, then we indicate by $\Omega - \Omega'$ the set $\{x \in \Omega : x \notin \Omega'\}$ and the symbol $\Omega' \subset\subset \Omega$ will mean that Ω' has a compact closure in Ω. We call Ω a domain of $\mathbb{R}^n$ if it is a proper, connected, open subset of $\mathbb{R}^n$.
If Ω is a bounded set, we write $|\Omega|$ its volume and we define

$$u_\Omega := \frac{1}{|\Omega|}\int_\Omega u \, dx = \fint_\Omega u \, dx. \tag{1.1}$$

We denote by $B_r(y)$ the ball of radius r centered in y (we will also write B_r when there aren't ambiguity) and ω_n the volume of unit ball in $\mathbb{R}^n$.

Let $u : \Omega \to \mathbb{R}$, we indicate $D_i u$ the partial derivative of u with respect to x_i, $Du = (D_1 u, ..., D_n u)$ the gradient of u, $D_i^2 u$ the second partial derivative of u with respect to x_i twice, $D_{ij} u$ the second partial derivative of u with respect to x_i and x_j. We write $\beta = (\beta_1, ..., \beta_n)$, with β_i integer and $|\beta| =$

$\sum_{i=1}^{n} \beta_i$, a *multi-index*; we define

$$D^\beta u = D_{x_1}^{\beta_1}(\cdots(D_{x_n}^{\beta_n} u)\cdots)$$

We denote by $C^0(\Omega)$ the set of continuous functions on Ω and by $C^0(\overline{\Omega})$ the set of continuous functions on $\overline{\Omega}$. Moreover $C^k(\Omega)$ will be the set of functions having all derivatives of order $\leq k$ continuous in Ω (k integer or $k = \infty$) and $C^k(\overline{\Omega})$ the set of functions in $C^k(\Omega)$ all of whose derivatives of order $\leq k$ have continuous extensions to $\overline{\Omega}$.

We write $\operatorname{supp} u$ the support of u, that is the closure of the set on which $u \neq 0$, $C_0^k(\Omega)$ the set of functions in $C^k(\Omega)$ with compact support in Ω.

$C = C(*, ..., *)$ denotes a constant depending only on the quantities appearing in parentheses. In a given context, the same letter C will (generally) be used to denote different constants depending on the same set of argument.

2. Hölder Spaces

Let x_0 be a point in $\mathbb{R}^n$ and f a function defined on a bounded set Ω containing x_0. If $0 < \alpha < 1$, we say that f is *Hölder continuous with exponent α at x_0* if the quantity

$$[f]_{\alpha;x_0} = \sup_{\Omega} \frac{|f(x) - f(x_0)|}{|x - x_0|^\alpha} \tag{1.2}$$

is finite. We call $[f]_{\alpha;x_0}$ the *α-Hölder coefficient of f at x_0* with respect to Ω. Clearly if f is Hölder continuous at x_0, then f is continuous at x_0. When (1.2) is finite for $\alpha = 1$, f is said to be *Lipschitz continuous* at x_0.
The notion of Hölder continuity is readily extended to the whole of Ω (not necessarily bounded). We call f *uniformly Hölder continuous with exponent α in* Ω if the quantity

$$[f]_{\alpha;\Omega} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|f(x) - f(x_0)|}{|x - x_0|^\alpha}, \quad 0 < \alpha \leq 1 \tag{1.3}$$

is finite; and *locally Hölder continuous with exponent α in* Ω if f is uniformly Hölder continuous with exponent α on compact subsets of Ω.

Definition 1.1 (Hölder spaces)**.** Let Ω be an open set in $\mathbb{R}^n$ and k a non-negative integer. We define the *Hölder spaces* $C^{k,\alpha}(\overline{\Omega})$ and $C^{k,\alpha}(\Omega)$ respectively as the subspaces of $C^k(\overline{\Omega})$ and $C^k(\Omega)$ consisting of functions whose k-th order partial derivatives are uniformly Hölder continuous and locally Hölder continuous.

We also designate by $C_0^{k,\alpha}(\Omega)$ the space of functions on $C^{k,\alpha}(\Omega)$ having compact support in Ω.
Let us define the norms

$$||u||_{C^{k,\alpha}(\overline{\Omega})} := \sum_{j=0}^{k} \sup_{|\beta|=k} \sup_{\Omega} |D^\beta u| + [f]_{\alpha;\Omega}, \qquad k = 0, 1, 2, ... \tag{1.4}$$

3. $\mathbf{L}^p$ Spaces

Definition 1.2. Let Ω be an open set in $\mathbb{R}^n$, $u : \Omega \to \mathbb{R}^N$ a measurable function, $1 \leq p < \infty$. We define the *L^p-norm of u*

$$||u||_{L^p(\Omega)} := \left(\int_\Omega |u|^p dx \right)^{1/p} \tag{1.5}$$

Definition 1.3. Let Ω be an open set in $\mathbb{R}^n$, $u : \Omega \to \mathbb{R}^N$ a measurable function. Then we define the *essential infimum of u*

$$\operatorname*{essinf}_{\Omega} u := \sup\{a \geq 0 : |\{x \in \Omega : |u| \leq a\}| = 0\} \tag{1.6}$$

and the *essential supremum of u*

$$||u||_{L^\infty(\Omega)} = \operatorname*{esssup}_{\Omega} u := \inf\{a \geq 0 : |\{x \in \Omega : |u| \geq a\}| = 0\} \tag{1.7}$$

We will wright $||u||_p$ and $||u||_\infty$ instead of $||u||_{L^p(\Omega)}$ and $||u||_{L^\infty(\Omega)}$ when there are no ambiguity.

Definition 1.4. Let Ω be an open set in $\mathbb{R}^n$, $1 \leq p < \infty$. We define the *space of $p-$integrable functions on Ω*

$$L^p(\Omega) = \{u : \Omega \to \mathbb{R}^N \text{measurable} : \int_\Omega |u|^p dx \ < +\infty\},$$

the *space of locally $p-$integrable functions on Ω*

$$L^p(\Omega) = \{u : \Omega \to \mathbb{R}^N \text{measurable} : \int_K |u|^p dx \ < +\infty, \ \forall K \subset \Omega, \ K\text{compact}\}$$

and the *space of essentially bounded function on* Ω

(1.8) $$L^\infty(\Omega) = \{u : \Omega \to \mathbb{R}^N \text{measurable} : \ ||f||_{L^\infty(\Omega)} < +\infty\}.$$

Theorem 1.5 (Young's inequality)**.** *Let a, b, p, q be positive real numbers satisfying*

$$\frac{1}{p} + \frac{1}{q} = 1,$$

then

(1.9) $$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

The case $p = q = 2$ of inequality (1.9) is known as Chauchy's inequality. Replacing a by $\varepsilon^{1/p}a$, b by $\varepsilon^{-1/p}b$ for positive ε, we obtain a useful interpolation inequality

(1.10) $$ab \leq \frac{\varepsilon a^p}{p} + \frac{\varepsilon^{-q/p} b^q}{q}.$$

Theorem 1.6 (Hölder inequality)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$, $u \in L^p(\Omega)$, $v \in L^q(\Omega)$, $1/p + 1/q = 1$, then*

(1.11) $$\int_\Omega uv \ dx \leq ||u||_p ||v||_q.$$

When $p = q = 2$, Hölder inequality reduces to the well known Schwarz inequality. Let us note a simple consequences of Hölder inequality

(1.12) $$||u||_q \leq ||u||_p^\lambda ||u||_r^{1-\lambda} \text{ for } u \in L^p(\Omega)$$

where $p \leq q \leq r$ and $1/q = \lambda/p + (1-\lambda)/r$. Combining inequalities (1.10) and (1.12), we obtain an interpolation inequality for L^p-norms:

(1.13) $$||u||_q \leq \varepsilon ||u||_r + \varepsilon^{-\mu} ||u||_p,$$

where

$$\mu = \left(\frac{1}{p} - \frac{1}{q}\right) / \left(\frac{1}{q} - \frac{1}{r}\right).$$

4. Mollifiers and cut-off functions

Definition 1.7 (Mollifier)**.** Let ε be a positive constant, a mollifiers is a function $\rho_\varepsilon : \mathbb{R}^n \to [0, +\infty[$ such that

$$\rho_\varepsilon \in C_0^\infty(\mathbb{R}^n) \tag{1.14}$$

$$\operatorname{supp} \rho_\varepsilon \subseteq B_\varepsilon(0) \tag{1.15}$$

$$\int_{\mathbb{R}^n} \rho_\varepsilon(x)\, dx = 1 \tag{1.16}$$

Remark 1.8. If we consider the function

$$\rho : \mathbb{R}^n \to [0, \infty[, \qquad \rho(x) = \begin{cases} ce^{\frac{1}{|x|^2-1}} & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

with c that satisfies (1.16), then

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^n} \rho\left(\frac{x}{\varepsilon}\right) \tag{1.17}$$

is a mollifier.

Definition 1.9. We consider $f \in L^1_{loc}(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, then we define the convolution between f and g as

$$f * g(x) = \int_{\mathbb{R}}^n f(x-y) g(y)\, dy \tag{1.18}$$

Theorem 1.10. *If $f \in L^1_{loc}(\mathbb{R}^n)$, then $f * \rho_\varepsilon \in C^\infty(\mathbb{R}^n)$.*

Definition 1.11 (Cut-off function)**.** Let Ω be an open set in $\mathbb{R}^n$ and $B_r(x_0) \subset\subset B_R(x_0) \subset\subset \Omega$. A function $\eta : \Omega \to [0, 1]$ is said *cut-off between $B_r(x_0)$ and $B_R(x_0)$* if $\eta \equiv 1$ on $B_r(x_0)$ and $\eta \equiv 0$ on $\Omega - B_R(x_0)$ and $\eta \in C_0^\infty(\Omega)$.

Remark 1.12. We consider the function

$$f : \mathbb{R} \to [0, 1], \qquad f(x) = \begin{cases} 1 & |x| < a \\ \frac{1}{a-b}(|x| - b) & a < |x| < b \\ 0 & |x| > b \end{cases}$$

with $a < b$ and a mollifier ρ_ε. We have

$$\operatorname{supp}(f * \rho_\varepsilon) \subseteq [-b - \varepsilon, b + \varepsilon]$$

and

$$f * \rho_\varepsilon(x) = 1 \quad \text{if} \quad |x| < a - \varepsilon.$$

Hence if $|x| > b+\varepsilon$ and $|y| < \varepsilon$ then $f(x-y) = 0$, and therefore

$$f * \rho_\varepsilon(x) = \int_{-\varepsilon}^{\varepsilon} f(x-y)\rho_\varepsilon(y)\ dy = 0, \tag{1.19}$$

while if $|x| < a-\varepsilon$ and $|y| < \varepsilon$ then $f(x-y) = 1$, and therefore

$$f * \rho_\varepsilon(x) = \int_{-\varepsilon}^{\varepsilon} f(x-y)\rho_\varepsilon(y)\ dy = \int_{-\varepsilon}^{\varepsilon} \rho_\varepsilon(y)\ dy = 1. \tag{1.20}$$

If we set $a = r+\varepsilon$, $b = R-\varepsilon$, $\varepsilon < \frac{R-r}{2}$, $\eta : \mathbb{R}^n \to [0,1]$ $\eta(x) = f * \rho_\varepsilon(|x|)$, then we obtain a cut-off function η between $B_r(0)$ and $B_R(0)$ where fixed r and R. We observe that

$$|D\eta(x)| < \frac{C}{R-r}, \tag{1.21}$$

indeed we have

$$\begin{aligned} |D\eta(x)| = |(f*\rho_\varepsilon)'(x)| = |f*\rho_\varepsilon'(x)| = \int_{-\varepsilon}^{\varepsilon} |f(x-y)|\rho_\varepsilon'(y)\ dy \\ \leq \int_{-\varepsilon}^{\varepsilon} \rho_\varepsilon'(y)\ dy = \int_{-\varepsilon}^{\varepsilon} \frac{1}{\varepsilon^2}\rho'(y)\ dy \leq 2\varepsilon\frac{1}{\varepsilon^2}\max_{\mathbb{R}} \rho'(x). \end{aligned}$$

If we set $\varepsilon = R - r$ we have the desidered result.

5. Sobolev Spaces

Let Ω be a bounded domain in $\mathbb{R}^n$

Definition 1.13 (Weak Derivative)**.** Let u be in $L^1_{loc}(\Omega)$ and α be a multi-index. A locally integrable function v is called the $\alpha^{th}-$weak derivative of u if it satisfies

$$\int_\Omega \phi v\ dx = (-1)^{|\alpha|}\int_\Omega uD^\alpha\phi\ dx \quad \forall\phi \in C_0^{|\alpha|}(\Omega). \tag{1.22}$$

We write $v = D^\alpha u$ and note that $D^\alpha u$ is uniquely determined up to sets of measure zero. We call a function k *times weakly differentiable* if all its weak derivatives exist for orders up to including k. Let us denote the linear space of k times weakly differentiable functions by $W^k(\Omega)$.
The $W^{k,p}(\Omega)$ space are Banach spaces analogous in a certain sense to the $C^{k,\alpha}(\overline{\Omega})$ spaces. In the $W^{k,p}(\Omega)$ spaces, continuous differentiability is replaced by weak differentiability and Hölder continuity by $p-$integrability.

Definition 1.14. Let $p \geq 1$ and k a non negative integer, we define

$$W^{k,p}(\Omega) := \{u \in W^k(\Omega);\ D^\alpha u \in L^p(\Omega)\ \forall |\alpha| \leq k\}$$

and

$$W^{k,p}_{loc}(\Omega) := \{u|_K \in W^k(K);\ D^\alpha u|_K \in L^p(K)\ \forall |\alpha| \leq k,\ \forall K \subset \Omega,\ K \text{compact}\}$$

where $f|_K$ denotes the restriction of f to the set K.

A norm is introduced by defining

$$||u||_{W^{k,p}(\Omega)} := \left(\int_\Omega \sum_{|\alpha| \leq k} |D^\alpha u|^p\ dx \right)^{1/p} \tag{1.23}$$

or equivalently

$$||u||_{W^{k,p}(\Omega)} := \sum_{|\alpha| \leq k} ||D^\alpha u||_p\ dx. \tag{1.24}$$

$W^{k,p}$ is a Banach space under (1.23) or (1.24). Another Banach space is:

$$W^{k,p}_0(\Omega) := \overline{C^k_0(\Omega)} \tag{1.25}$$

The spaces $W^{k,p}(\Omega)$ and $W^{k,p}_0(\Omega)$ do not coincide for bounded Ω.

Theorem 1.15 (Meyers-Serrin)**.** *Let Ω be an open set in $\mathbb{R}^n$ and $p \geq 1$, then $C^\infty(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$.*

Theorem 1.16. *Let Ω be an open set in $\mathbb{R}^n$, α be a multiindex, $u \in L^1_{loc}(\Omega)$. Then $v \in L^1_{loc}(\Omega)$ is the α^{th} weak deriative of u if and only if there exists a sequence $u_n \in C^{|\alpha|}(\Omega)$ such that*

$$u_n \longrightarrow u \text{ in } L^1_{loc}(\Omega)$$
$$D^\alpha u_n \longrightarrow v \text{ in } L^1_{loc}(\Omega).$$

The following results are analogous to classic derivation rules for composition and product of functions.

Theorem 1.17. *Let Ω be an open set in $\mathbb{R}^n$, $u \in W^{1,p}(\Omega)$, $f \in C^1(\mathbb{R})$ such that $f' \in L^\infty(\mathbb{R})$. Then $f \circ u \in W^{1,p}(\Omega)$ and*

$$D(f \circ u) = f'(u)Du. \tag{1.26}$$

Theorem 1.18. *Let Ω be an open set in $\mathbb{R}^n$ and $u \in W^{1,p}(\Omega)$. Then $u^+ \in W^{1,p}(\Omega)$ and*

$$Du^+ = \begin{cases} Du, & \text{in } \{u > 0\}, \\ 0, & \text{in } \{u \le 0\}. \end{cases} \tag{1.27}$$

Theorem 1.19. *Let Ω be an open set in $\mathbb{R}^n$, $u \in W^{1,p}(\Omega)$ and $\eta \in C_0^\infty(\Omega)$. Then $u\eta \in W_0^{1,p}(\Omega)$ and*

$$D(u\eta) = \eta Du + uD\eta. \tag{1.28}$$

The following results will be useful in the sequel.

Definition 1.20. Let us consider $p \ge 1$, we define

$$p^* := \frac{np}{n-p} \tag{1.29}$$

the *conjugate Sobolev exponent of* p.

Let us observe that if $p < n$ then $p^* > p$. Now we state an imbedding Theorem and the related inequalities.

Theorem 1.21 (Sobolev inequalities)**.** *Let Ω be an open set in $\mathbb{R}^n$.*

1. *If $p < n$ then $W_0^{1,p}(\Omega) \subset L^{np/(n-p)}(\Omega)$, and*

$$||u||_{L^{p^*}(\Omega)} \le C||Du||_{L^p(\Omega)}. \tag{1.30}$$

2. *If $p = n$ then $W_0^{1,p}(\Omega) \subset L^q(\Omega)\ \ \forall q \in [p, +\infty)$, and*

$$||u||_{L^q(\Omega)} \le C(n,p)||u||_{W^{1,n}(\Omega)}. \tag{1.31}$$

3. *If $p > n$ then $W_0^{1,p}(\Omega) \subset L^\infty \cup C^{0,\alpha}(\overline{\Omega})$, where $\alpha = 1 - \frac{n}{p}$, and*

$$||u||_{L^\infty(\Omega)} \le C(n,p)||u||_{W^{1,p}(\Omega)}, \tag{1.32}$$

$$||u||_{C^{0,\alpha}(\Omega)} \le C(n,p)||Du||_{L^p(\Omega)}. \tag{1.33}$$

Chapter 2

The De Giorgi Method

In this chapter we will concern with the elliptic equation in divergence form

$$D_i(a^{ij}(x)D_ju) = 0 \quad \text{in } \Omega, \tag{2.1}$$

where Ω is an open subset in $\mathbb{R}^n$ and the matrix $a^{ij}(x)$ satisfies the hypothesis of boundedness

$$a^{ij} \in L^\infty(\Omega) \quad i,j = 1,...,n, \tag{2.2}$$

ellipticity

$$a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^n \tag{2.3}$$

and it is also symmetric

$$a^{ij} = a^{ji}. \tag{2.4}$$

In this thesis, we will adopt the following point of view: given a weak solution of elliptic equation, we will study the additional regularity properties in the interior of Ω. So we will not address existence problems, for these we can refer to Gilbarg and Trudinger [**7**] and Giusti [**8**].
We now introduce the following notions:

Definition 2.1. Let Ω be an open subset in $\mathbb{R}^n$, $u \in W^{1,2}(\Omega)$ is called a weak solution (supersolution, subsolution) of (2.1) if it satisfies

$$\int_\Omega a^{ij}D_juD_iv\ dx = 0\ (\geq 0, \leq 0). \tag{2.5}$$

for all non-negative functions $v \in C_0^1(\Omega)$.

Remark 2.2. The boundedness of the matrix a^{ij} allow us to assert that (2.5) is valid $\forall v \in W_0^{1,2}(\Omega)$.
Indeed, let us fix $v \in W_0^{1,2}(\Omega)$ such that u is a weak solution of (2.1). By definition of $W_0^{1,p}$ (1.25), there exists a sequence $(v_n)_{n\in\mathbb{N}}$ in $C_0^1(\Omega)$ such that v_n tends to v in $W^{1,2}(\Omega)$. Therefore we have

$$\int_\Omega a^{ij} D_j u D_i v_n \, dx = 0 \quad \forall n \in \mathbb{N}.$$

Hence

$$\begin{aligned}\left|\int_\Omega a^{ij} D_j u D_i v \, dx\right| &= \left|\int_\Omega a^{ij} D_j u D_i v \, dx - \int_\Omega a^{ij} D_j u D_i v_n \, dx\right| \\ &\le \int_\Omega |a^{ij} D_j u D_i (v - v_n)| \, dx \\ &\le \Lambda ||Du||_{L^2(\Omega)} ||Dv - Dv_n||_{L^2(\Omega)} \, dx \longrightarrow 0\end{aligned}$$

where $n \to \infty$. So we have that $\int_\Omega a^{ij} D_j u D_i v_n \, dx = 0$, and similarly we can obtain the result for supersolutions and subsolutions.

1. De Giorgi Set

Let us start with the following definition.

Definition 2.3 (De Giorgi set)**.** Let Ω be an open subset in $\mathbb{R}^n$, $x_0 \in \Omega$, $r \in \mathbb{R}^+$, $k \in \mathbb{R}$.
We set $A(k, x_0, r) := B_r(x_0) \cap \{u > k\}$.
A function $u \in W_{loc}^{1,2}(\Omega)$ is in the De Giorgi's set $DG(\Omega)$ if it satisfies

$$\int_{A(k,x_0,r)} |Du(x)|^2 \, dx \le \frac{C}{(R-r)^2} \int_{A(k,x_0,R)} |u(x) - k|^2 \, dx \tag{2.6}$$

$\forall k \in \mathbb{R}$, for a.e. $x_0 \in \Omega$ and $\forall r, R$ such that $0 < r < R < dist(x_0, \partial\Omega)$.

Lemma 2.4. *Let Ω be an open subset in $\mathbb{R}^n$, $x_0 \in \Omega$, $r \in \mathbb{R}^+$, $k \in \mathbb{R}$, $u \in W_{loc}^{1,2}$, then (2.6) is equivalent to*

$$\int_{B_r(x_0)} |D(u(x) - k)^+|^2 \, dx \le \frac{C}{(R-r)^2} \int_{B_r(x_0)} |(u(x) - k)^+|^2 \, dx \tag{2.7}$$

$\forall k \in \mathbb{R}$, *for a.e.* $x_0 \in \Omega$ *and* $\forall r, R$ *such that* $0 < r < R < dist(x_0, \partial\Omega)$ *where*

$$(u(x)-k)^+ := \max\{0, u(x)-k\} = \begin{cases} u(x)-k, & x \in \{u > k\}, \\ 0, & x \in \{u \le k\}. \end{cases} \tag{2.8}$$

Proof. By Theorem 1.18 we observe that

$$D(u(x)-k)^+ = \begin{cases} Du, & x \in \{u > k\}, \\ 0, & x \in \{u \le k\}, \end{cases} \tag{2.9}$$

and that the integrals are calculated on $B_r(x_0) \cap \{u > k\} = A(k, x_0, r)$ and $B_R(x_0) \cap \{u > k\} = A(k, x_0, R)$ in order to obtain the result. □

Throughout this chapter we will write B_R and $A(k, R)$ instead of $B_R(x_0)$ and $A(k, x_0, R)$. The following result concerns the connection between the De Giorgi's class and the solution of (2.1).

Theorem 2.5. *Let* Ω *be an open subset in* $\mathbb{R}^n$*, if* $u \in W^{1,2}_{loc}(\Omega)$ *is a subsolution of (2.1), then* $u \in DG(\Omega)$.

Proof. We consider $x_0 \in \Omega$, r, R such that $0 < r < R < dist(x_0, \partial\Omega)$ and let η be a cut-off function between B_r and B_R with $|D\eta| \le \frac{c}{R-r}$. If we consider the test function $\phi = v\eta^2$ where $v = (u-k)^+$, then we obtain

$$\int_\Omega a^{ij} D_j u D_i(\eta^2 v)\, dx = \int_\Omega \eta^2 a^{ij} D_j u D_i v\, dx + 2\int_\Omega \eta v a^{ij} D_j u D_i v\, dx \le 0$$

Hence

$$\begin{aligned} \int_\Omega \eta^2 a^{ij} D_j u D_i v\, dx &\le -2\int_\Omega \eta v a^{ij} D_j u D_i v\, dx \\ &\le 2\int_\Omega |\eta v||a^{ij} D_j u D_i v|\, dx \le 2\Lambda \int_\Omega \eta v |Du||Dv|\, dx. \end{aligned}$$

The integrals are calculated on $\Omega \cap \{u > k\}$ and, by Theorem 1.18, on this set we have $Dv = Du$. Hence, by (2.3) and (1.11) we can write

$$\begin{aligned} \lambda \int_\Omega \eta^2 |Dv|^2\, dx \le \int_\Omega \eta^2 a^{ij} D_j v D_i v\, dx &\le 2\Lambda \int_\Omega \eta v |Dv||Dv|\, dx \\ &\le 2\Lambda \left(\int_\Omega \eta^2 |Dv|^2\, dx\right)^{1/2} \left(\int_\Omega |v|^2 |D\eta|^2\, dx\right)^{1/2} \end{aligned}$$

Dividing by $\lambda||\eta Dv||_{L^2(\Omega)}$ we gain

(2.10)
$$\int_\Omega \eta^2|Dv|^2\,dx \le \left(\frac{2\Lambda}{\lambda}\right)^2\int_\Omega |v|^2|D\eta|^2\,dx \le \left(\frac{2\Lambda}{\lambda}\right)^2\int_{B_R}|v|^2|D\eta|^2\,dx$$
$$\le \left(\frac{2\Lambda}{\lambda}\right)^2\frac{c^2}{(R-r)^2}\int_{B_R}|v|^2\,dx$$

because η is a cut-off function. Moreover, since $\eta^2|Dv|^2 \ge 0$ and $\eta \equiv 1$ on B_r, we have

$$\int_\Omega \eta^2|Dv|^2\,dx \ge \int_{B_r}\eta^2|Dv|^2\,dx = \int_{B_r}|Dv|^2\,dx. \tag{2.11}$$

We get the result by (2.10) and (2.11) if we set $C = \left(\frac{2\Lambda c}{\lambda}\right)^2$. □

Theorem 2.6. *Let Ω be an open subset in $\mathbb{R}^n$, if $u \in W^{1,2}_{loc}(\Omega)$ is a supersolution of (2.1), then $-u \in DG(\Omega)$.*

Proof. Since u is a supersolution, we have

$$\int_\Omega a^{ij}D_j u D_i v\,dx \ge 0 \quad \forall v \in W^{1,2}_0(\Omega),$$

hence

$$\int_\Omega a^{ij}D_j(-u)D_i v\,dx \le 0 \quad \forall v \in W^{1,2}_0(\Omega),$$

so $-u$ is a subsolution and, by Theorem 2.5, $-u \in DG(\Omega)$. □

2. Local Boundedness

In this section we will prove the local boundedness of the functions in De Giorgi's class. We set, $\forall h \in \mathbb{R}$, $\forall r > 0$ such that $B_r \subset\subset \Omega$

$$u(h,r) := \int_{A(h,r)}|u(x)-h|^2\,dx \tag{2.12}$$

Lemma 2.7. *Let Ω be an open subset in $\mathbb{R}^n$ and $u \in DG(\Omega)$ then $\forall k \in \mathbb{R}$, $\forall x_0 \in \Omega$ and $\forall r$, R such that $0 < r < R < dist(x_0, \partial\Omega)$ we have*

$$\int_{B_r}|(u(x)-k)^+|^2\,dx \le \frac{C_n}{(R-r)^2}|A(k,R)|^{\frac{2}{n}}\int_{B_R}|(u(x)-k)^+|^2\,dx \tag{2.13}$$

where C_n depends only on n.

Proof. Let us consider $k \in \mathbb{R}$. Let us set $v = (u-k)^+$ and $\overline{r} = \frac{R+r}{2}$, hence $r < \overline{r} < R$. Since $u \in DG(\Omega)$, we have

$$\int_{B_{\overline{r}}} |Dv|^2 \, dx \leq \frac{C}{(R-\overline{r})^2} \int_{B_R} |v|^2 \, dx = \frac{C}{(R-r)^2} \int_{B_R} |v|^2 \, dx. \tag{2.14}$$

Let η be a cut-off function between $B_{\overline{r}}$ and B_r with $|D\eta| \leq \frac{c}{R-r}$, by derivative product rules we have $D(v\eta) = vD\eta + \eta Dv$, hence

$$|D(v\eta)|^2 \leq (|vD\eta| + |\eta Dv|)^2 \leq 2(|vD\eta|^2 + |\eta Dv|^2).$$

and, integrating respect to $B_{\overline{r}}$

$$\begin{aligned}\int_{B_{\overline{r}}} |D(v\eta)|^2 \, dx &\leq 2\left[\int_{B_{\overline{r}}} |v|^2|D\eta|^2 \, dx + \int_{B_{\overline{r}}} |\eta|^2|Dv|^2 \, dx\right] \\ &\leq 2\left[\frac{c^2}{(R-r)^2}\int_{B_{\overline{r}}} |v|^2 \, dx + \int_{B_{\overline{r}}} |Dv|^2 \, dx\right].\end{aligned}$$

Hence, by (2.14), we write

$$\begin{aligned}\int_{B_{\overline{r}}} |D(v\eta)|^2 \, dx &\leq 2\left[\frac{c^2}{(R-r)^2}\int_{B_R} |v|^2 \, dx + \frac{C}{(R-r)^2}\int_{B_R} |v|^2 \, dx\right] \\ &\leq \frac{C}{(R-r)^2}\int_{B_R} |v|^2 \, dx,\end{aligned}$$

therefore we have

$$\int_{B_{\overline{r}}} |D(v\eta)|^2 \, dx \leq \frac{C}{(R-r)^2}\int_{B_R} |v|^2 \, dx, \tag{2.15}$$

Now we can apply the Sobolev inequality

$$\begin{aligned}\left(\int_{B_r} |v|^{2^*} \, dx\right)^{\frac{2}{2^*}} &= \left(\int_{B_r} |\eta v|^{2^*} \, dx\right)^{\frac{2}{2^*}} \\ &\leq C_n \int_{B_r} |D(\eta v)|^2 \, dx \leq C_n \int_{B_{\overline{r}}} |D(v\eta)|^2 \, dx.\end{aligned} \tag{2.16}$$

Furthermore, by Hölder inequality with the exponent $\frac{2^*}{2}$

$$\begin{aligned}\int_{B_r} |v|^2 \, dx &\leq \left(\int_{B_r} |v|^{2\frac{2^*}{2}} \, dx\right)^{\frac{2}{2^*}} |A(k,r)|^{1-\frac{2}{2^*}} \\ &= \left(\int_{B_r} |v|^{2^*} \, dx\right)^{\frac{2}{2^*}} |A(k,r)|^{\frac{2}{n}}\end{aligned} \tag{2.17}$$

Hence, by (2.17), (2.16), (2.15) and since $A(k,r) \subseteq A(k,R)$ we have

$$\begin{aligned}\int_{B_r} |v|^2 \, dx &\leq |A(k,r)|^{\frac{2}{n}} \left(\int_{B_r} |v|^{2^*} \, dx \right)^{\frac{2}{2^*}} \\ &\leq C_n |A(k,r)|^{\frac{2}{n}} \int_{B_{\overline{r}}} |D(v\eta)|^2 \, dx \\ &\leq C_n \frac{C}{(R-r)^2} |A(k,r)|^{\frac{2}{n}} \int_{B_R} |v|^2 \, dx \\ &\leq \frac{C_n}{(R-r)^2} |A(k,R)|^{\frac{2}{n}} \int_{B_R} |v|^2 \, dx. \qquad \square\end{aligned}$$

Before enunciating next Lemma, we set $\forall h \in \mathbb{R}$, $\forall r > 0$ such that $B_r \subset\subset \Omega$

$$\phi(h,r) := |A(h,r)|^{\eta} u(h,r)^{\xi}, \tag{2.18}$$

$$\overline{\phi}(h,r) := |\{u \geq h\} \cap \overline{B_r}|^{\eta} u(h,r)^{\xi}. \tag{2.19}$$

Since $\{u > h\} \cap B_r \subseteq \{u \geq h\} \cap \overline{B_r}$, we have

$$\phi(h,r) \leq \overline{\phi}(h,r). \tag{2.20}$$

Lemma 2.8. *Let Ω be an open subset in $\mathbb{R}^n$ and $u \in DG(\Omega)$ then $\forall h > k$, $\forall x \in \Omega$ and $\forall r$, R such that $0 < r < R < dist(x_0, \partial\Omega)$ we have*

$$|A(h,r)| \leq \frac{1}{(h-k)^2} u(k,R) \tag{2.21}$$

$$u(h,r) \leq \frac{C_n}{(R-r)^2} |A(k,R)|^{\frac{2}{n}} u(k,R). \tag{2.22}$$

Proof. Let us observe that in $A(h,r)$ we have $u(x) > h > k$, hence $u(x) - k > h - k$ and $|u(x)-k|^2 > |h-k|^2$. Therefore, integrating on $A(h,r)$ we have

$$\begin{aligned}|A(h,r)||h-k|^2 &= \int_{A(h,r)} |h-k|^2 \, dx \\ &\leq \int_{A(h,r)} |u(x)-k|^2 \, dx \leq \int_{A(k,r)} |u(x)-k|^2 \, dx,\end{aligned}$$

dividing by $(h-k)^2$ and observing that $A(k,r) \subset A(k,R)$

$$|A(h,r)| = \frac{1}{(h-k)^2} \int_{A(h,r)} |h-k|^2 \, dx \leq \frac{1}{(h-k)^2} \int_{A(k,r)} |u(x)-k|^2 \, dx,$$

that is the inequality (2.21). Now, since we have $(u(x)-h)^+ \leq (u(x)-k)^+$, integrating on $A(h,r)$

$$\int_{A(h,r)} |u(x)-h|^2 \, dx = \int_{B_r} |(u(x)-h)^+|^2 \, dx$$
$$\leq \int_{B_r} |u(x)-k|^2 \, dx = \int_{A(k,r)} |u(x)-k|^2 \, dx,$$

hence by (2.13) we have

$$\int_{A(h,r)} |u(x)-h|^2 \, dx \leq \int_{A(k,r)} |u(x)-k|^2 \, dx$$
$$\leq \frac{C_n}{(R-r)^2} |A(k,R)|^{\frac{2}{n}} \int_{A(k,R)} |u(x)-k|^2 \, dx.$$

that is the inequality (2.22). □

Now we consider two positive real numbers η and ξ and we multiply (2.12) and (2.22) respectively by η and ξ. We have

$$|A(h,r)|^{\eta} u(h,r)^{\xi} \leq \frac{C_n^{\xi}}{(h-k)^{2\eta}(R-r)^{2\xi}} |A(k,R)|^{\frac{2\xi}{n}} u(k,R)^{\xi+\eta}. \tag{2.23}$$

We want write the second member of the inequality (2.23) as power of $\phi(k,R)$. We have to find a $\theta > 0$ such that $\frac{2\xi}{n} = \eta\theta$ and $\xi + \eta = \xi\theta$. By these two equality we find: $\theta^2 - \theta - \frac{2}{n} = 0$ and hence $\theta = \frac{1}{2} + \sqrt{\frac{n+8}{4n}} > 1$. We can now write:

$$\phi(h,r) \leq \frac{C_n^{\xi}}{(h-k)^{2\eta}(R-r)^{2\xi}} |\phi(k,R)|^{\theta}, \quad \theta > 1 \tag{2.24}$$

Remark 2.9. Let us observe that, since $\{u > h\} \cap B_r \subseteq \{u \geq h\} \cap \overline{B_r}$, we have

$$\phi(h,r) \leq \overline{\phi}(h,r). \tag{2.25}$$

Lemma 2.10. *Let $(k_n)_{n\in\mathbb{N}}$, $(r_n)_{n\in\mathbb{N}}$ be two sequence such that $k_n \nearrow k_0$ and $r_n \searrow r_0$, $r_n > 0$, then*

$$\phi(k_n, r_n) \to \overline{\phi}(k_0, r_0) \tag{2.26}$$

where $n \to +\infty$.

Proof. Let us set $a_n = |\{u > k_n\} \cap B_{r_n}|$, we have

$$k_n \nearrow k_0 \Longrightarrow (\{u > k_n\})_{n\in\mathbb{N}} \searrow \{u \geq k_0\}$$
$$r_n \searrow r_0 \Longrightarrow (B_{r_n})_{n\in\mathbb{N}} \searrow \overline{B_{r_0}},$$

hence $(\{u > k_n\} \cap B_{r_n})_{n\in\mathbb{N}} \searrow \{u \geq k_0\} \cap \overline{B_{r_0}}$ and therefore

$$a_n \longrightarrow |\bigcap_{i=1}^{\infty}(\{u > k_n\} \cap B_{r_n})_{n\in\mathbb{N}}| = |\{u \geq k_0\} \cap \overline{B_{r_0}}| \tag{2.27}$$

Let us set

$$h_n := \int_{B_{r_n}} |(u(x) - k_n)^+|^2 \, dx = \int_{B_{r_1}} |(u(x) - k_n)^+|^2 \chi_{B_r} \, dx$$

since $B_{r_1} \supseteq B_{r_n}$ $\forall n \in \mathbb{N}$. We have

$$|(u(x) - k_n)^+|^2 \chi_{B_{r_n}} \longrightarrow |(u(x) - k_n)^+|^2 \chi_{B_{r_0}} a.e.$$

where $n \to \infty$. The sequence $f_n = |(u(x) - k_n)^+|^2$ is non-decresing, hence $|f_n \chi_{B_{r_n}}| \leq |f_n| \leq |f_1| = |(u(x) - k_1)^+|^2 \in L^1(B_{r_1})$. Therefore, by the Lebesgue dominated convergence Theorem

$$h_n \longrightarrow \int_{B_{r_1}} |(u(x) - k_n)^+|^2 \chi_{B_{r_0}} \, dx = \int_{B_{r_0}} |(u(x) - k_n)^+|^2 \, dx \tag{2.28}$$

Finally, by (2.27) and (2.28) we have

$$\phi(k_n, r_n) = |\{u > k_n\} \cap B_{r_n})_{n\in\mathbb{N}}|^{\eta} u(k_n, r_n)^{\xi} \longrightarrow$$
$$|\{u \geq k_0\} \cap \overline{B_{r_0}}|^{\eta} u(k_0, r_0)^{\xi} = \overline{\phi}(k_0, r_0)$$

□

Lemma 2.11. *Let Ω be an open subset in $\mathbb{R}^n$, $R_0 > 0$: $B_{R_0} \subset\subset \Omega$. Then $\forall k_0 \in \mathbb{R}$ and $\forall \sigma \in]0, 1[$ we have that $\exists d \in \mathbb{R}$:*

$$\phi(k_0 + d, R_0 - \sigma R_0) = 0 \tag{2.29}$$

In particular d satisfies the relation

$$d^{2\eta} = \frac{2^{2\xi+2\eta\frac{\theta}{\theta-1}} c^{\xi}}{\sigma^{2\xi} R_0^{2\xi}} \phi(k_0, R_0)^{\theta-1}. \tag{2.30}$$

Proof. Let us consider the sequences

$$k_n = k_0 + d - \frac{d}{2^n} \nearrow k_0 + d \tag{2.31}$$

$$r_n = R_0 - \sigma R_0 + \frac{\sigma R_0}{2^n} \searrow R_0 - \sigma R_0. \tag{2.32}$$

We will show that $\forall n \geq 1$

$$\phi(k_n, r_n) \leq \frac{\phi(k_0, R_0)}{2^{\lambda n}}, \quad \text{where } \lambda = \frac{2\xi + 2\eta}{\theta - 1} \tag{2.33}$$

Indeed, for $n = 1$, since $k_1 > k_0$ and $r_1 < R_0$ we have

$$\begin{aligned}
\phi(k_1, r_1) = \phi(k_0 + \frac{d}{2}, R_0 - \frac{\sigma R_0}{2}) &\leq \frac{c^{\xi}}{(\frac{\sigma R_0}{2})^{2\xi} (\frac{d}{2})^{2\eta}} \ \phi(k_0, R_0)^{\theta} \\
&= \left[\frac{c^{\xi}}{(\sigma R_0)^{2\xi} (d)^{2\eta}} \ \phi(k_0, R_0)^{\theta - 1} \right] 2^{2\xi + 2\eta} \ \phi(k_0, R_0) \\
&= \frac{1}{2^{2\xi + 2\eta \frac{\theta}{\theta - 1}}} \ 2^{2\xi + 2\eta} \ \phi(k_0, R_0) = \frac{1}{2^{\frac{2\xi + 2\eta}{\theta - 1}}} \phi(k_0, R_0) = \frac{\phi(k_0, R_0)}{2^{\lambda}}
\end{aligned}$$

Now let us suppose that (2.33) is valid for n, since $k_{n+1} > k_n$ and $r_{n+1} < r_n$, we have

$$\begin{aligned}
\phi(k_{n+1}, r_{n+1}) &\leq \frac{c^{\xi}}{(\frac{\sigma R_0}{2^{n+1}})^{2\xi} (\frac{d}{2^{n+1}})^{2\eta}} \ \phi(k_n, R_n)^{\theta} \\
&= \left[\frac{c^{\xi}}{(\sigma R_0)^{2\xi} (d)^{2\eta}} \ \phi(k_0, R_0)^{\theta - 1} \right] \ \frac{2^{(2\xi + 2\eta)(n+1)}}{\phi(k_0, R_0)^{\theta - 1}} \phi(k_n, R_n)^{\theta} \\
&\leq \frac{1}{2^{2\xi + 2\eta \frac{\theta}{\theta - 1}}} \frac{2^{(2\xi + 2\eta)(n+1)}}{\phi(k_0, R_0)^{\theta - 1}} \left(\frac{\phi(k_0, R_0)}{2^{\lambda n}} \right)^{\theta} \\
&= \frac{1}{2^{\lambda \theta}} \frac{2^{\lambda(\theta - 1)(n+1)}}{\phi(k_0, R_0)^{\theta - 1}} \frac{\phi(k_0, R_0)^{\theta}}{2^{\lambda n \theta}} \\
&= \frac{1}{2^{\lambda[\theta(1+n) - (\theta - 1)(n+1)]}} \ \phi(k_0, R_0) = \frac{\phi(k_0, R_0)}{2^{\lambda(n+1)}}.
\end{aligned}$$

Hence, by induction, the inequality (2.33) is valid $\forall n \geq 1$. Now, by (2.33) and Lemma 2.10 and by letting n tend to $+\infty$, we obtain

$$\overline{\phi}(k_0 + d, R_0 - \sigma R_0) \leq 0$$

and, by (2.25) we have

$$0 \leq \phi(k_0 + d, R_0 - \sigma R_0) \leq \overline{\phi}(k_0 + d, R_0 - \sigma R_0) \leq 0$$

therefore the thesis. □

By the previous result it follows a general estimate for the essential supremum of a $DG-$function.

Theorem 2.12. *Let Ω be an open subset in $\mathbb{R}^n$, $u \in DG(\Omega)$, $r > 0$: $B_r \subset\subset \Omega$. Then $\forall k_0 \in \mathbb{R}$ we have*

$$\operatorname*{esssup}_{B_{r/2}} u \leq k_0 + c\left(\frac{1}{r^n}\int_{A(k_0,r)} |u-k_0|^2 \, dx\right)^{1/2}\left(\frac{|A(k_0,r)|}{r^n}\right)^{\frac{\theta-1}{2}} \tag{2.34}$$

Proof. We set $\sigma = \frac{1}{2}$ so that, by (2.29), $\exists d \in \mathbb{R}$:

$$\begin{aligned} 0 &= \phi(k_0+d, r/2) \\ &= |\{u > k_0+d\} \cap B_{r/2}|^\eta \left(\int_{A(k_0+d,r/2)} |u-(k_0+d)|^2 \, dx\right)^\xi \end{aligned}$$

hence either

$$|\{u > k_0+d\} \cap B_{r/2}| = 0$$

that is

$$\operatorname*{esssup}_{B_{r/2}} u \leq k_0 + d$$

or

$$\int_{A(k_0+d,r/2)} |u-(k_0+d)|^2 \, dx = 0$$

and since $|u-(k_0+d)| > 0$ in $A(k_0+d, r/2)$ we have again

$$|A(k_0+d, r/2)| = 0.$$

Therefore, by (2.30), we have

$$d = \frac{C}{r^{\xi/\eta}}|A(k_0,r)|^{\frac{\theta-1}{2}}\left(\int_{A(k_0,r)} |u(x)-k_0|^2 \, dx\right)^{(\theta-1)\frac{\xi}{2\eta}},$$

since $\frac{2\xi}{\eta} = \eta\theta$, we have $\xi/\eta = n\frac{\theta}{2} = \frac{n}{2} + n\frac{\theta-1}{2}$, whereas, from $\xi + \eta = \xi\theta$, it follows $\frac{\xi}{2\eta}(\theta-1) = \frac{1}{2}$, and then we have the result. □

Now we can observe that the essential supremum on a ball is controlled by a "quadratic integral average".

Theorem 2.13. *Let Ω be an open subset in $\mathbb{R}^n$ and $u \in DG(\Omega)$. Then $\forall r > 0$ s.t. $B_r \subset\subset \Omega$ we have*

$$\operatorname*{esssup}_{B_{r/2}} u \leq c\left(\fint_{B_r} |u|^2 \, dx\right)^{1/2}. \tag{2.35}$$

Proof. By previous Theorem, if we set $k_0 = 0$, we obtain:

$$\begin{aligned}
\operatorname*{esssup}_{B_{r/2}} u &\leq c \left(\frac{1}{r^n} \int_{A(0,r)} |u|^2 \, dx \right)^{1/2} \left(\frac{|A(0,r)|}{r^n} \right)^{\frac{\theta-1}{2}} \\
&\leq c \left(\frac{1}{r^n} \int_{A(0,r)} |u|^2 \, dx \right)^{1/2} \omega_n^{\frac{\theta-1}{2}} \\
&\leq c\omega_n^{\theta/2} \left(\frac{1}{\omega_n r^n} \int_{A(0,r)} |u|^2 \, dx \right)^{1/2}.
\end{aligned}$$

□

Theorem 2.14. *Let Ω be an open subset in $\mathbb{R}^n$, if $u \in W^{1,2}_{loc}(\Omega)$ is a solution of (2.1) then $u \in L^\infty_{loc}$, that is $\forall r > 0 : B_r \subset\subset \Omega$ we have*

$$||u||_{L^\infty(B_{r/2})} \leq c \left(\fint_{B_r} |u|^2 \, dx \right)^{1/2}. \tag{2.36}$$

Proof. Since u is solution of (2.1) then is both subsolution and supersolution and by Theorems 2.5 and 2.6 u and $-u \in DG(\Omega)$. Therefore we can apply Theorem 2.13 to $-u$ in order to gain

$$\operatorname*{esssup}_{B_{r/2}}(-u) \leq c \left(\fint_{B_r} |-u|^2 \, dx \right)^{1/2}$$

hence

$$\operatorname*{essinf}_{B_{r/2}} u \geq -c \left(\fint_{B_r} |u|^2 \, dx \right)^{1/2}. \tag{2.37}$$

By (2.35) and (2.37) we have the desidered result. □

3. De Giorgi Theorem

Definition 2.15. Let Ω be an open set in $\mathbb{R}^n$, $x \in \Omega$, $r > 0$: $B_r(x) \subset\subset \Omega$ and $u : \Omega \to \mathbb{R}$. We define

$$M(r) := \operatorname*{esssup}_{B_r(x)} u, \quad m(r) := \operatorname*{essinf}_{B_r(x)} u, \quad \omega(r) := M(r) - m(r)$$

where ω is called the *essential oscillation* of the function u on the ball $B_r(x)$.

We note that if the function u is bounded, so its oscillation is.

Lemma 2.16. *Let Ω be an open set in $\mathbb{R}^n$, $u \in DG(\Omega)$, $x \in \Omega$, $r > 0$: $2r < dist(x, \partial\Omega)$ and let us set $K_0 = \frac{M(2r)+m(2r)}{2}$ and $k_i = M(2r) - \frac{M(2r)-K_0}{2^i}$. We have that if $|A(K_0, r)| \le \frac{1}{2}|B_r|$, then $\forall m > \overline{m}$*

$$|A(k_m, r)| \le C_n r^n \left(\frac{1}{m}\right)^N, \quad \text{where } N = \frac{n}{2n-2}. \tag{2.38}$$

Proof. Let us consider $h > k > K_0$, hence we set

$$v(x) := \min\{u(x), h\} - \min\{u(x), k\} = \begin{cases} h-k, & x \in \{u \ge h\} \\ u(x)-k, & x \in \{k < u < h\} \\ 0, & x \in \{u \le h\}. \end{cases}$$

Since $\{u \le k\} \supseteq \{u < K_0\}$, by the hypothesis we have

$$|B_r \cap \{v = 0\}| = |B_r \cap \{u \le k\}| \ge |B_r \cap \{u < K_0\}| \ge \frac{1}{2}|B_r|.$$

Applying the Sobolev inequality to v with $p = 1$ and the Hölder one to 1 and $|Du|$ we obtain

(2.39)

$$\begin{aligned} \left(\int_{B_r} |v|^{\frac{n}{n-1}}\, dx\right)^{\frac{n-1}{n}} &\le C_n \int_{B_r} |Dv|\, dx \\ &= C_n \int_{A(k,r)-A(h,r)} |Du|\, dx \\ &\le C_n |A(k,r) - A(h,r)|^{1/2} \left(\int_{A(k,r)-A(h,r)} |Du|\, dx\right)^{1/2}. \end{aligned}$$

By definition of v, we can write

(2.40)

$$\left(\int_{A(h,r)} |v|^{\frac{n}{n-1}}\, dx\right)^{\frac{n-1}{n}} = \left(\int_{A(h,r)} |h-k|^{\frac{n}{n-1}}\, dx\right)^{\frac{n-1}{n}} = (h-k)|A(h,r)|^{\frac{n-1}{n}}.$$

Since $A(h,r) \subseteq B_r$, by (2.40) and (2.39)

$$\begin{aligned} (h-k)|A(h,r)|^{\frac{n-1}{n}} &\le \left(\int_{A(h,r)} |v|^{\frac{n}{n-1}}\, dx\right)^{\frac{n-1}{n}} \le \left(\int_{B_r} |v|^{\frac{n}{n-1}}\, dx\right)^{\frac{n-1}{n}} \\ &\le C_n |A(k,r) - A(h,r)|^{1/2} \left(\int_{A(k,r)-A(h,r)} |Du|\, dx\right)^{1/2} \end{aligned}$$

and hence

$$(h-k)^2 |A(h,r)|^{\frac{2n-2}{n}} \le C_n |A(k,r) - A(h,r)| \int_{A(k,r)-A(h,r)} |Du(x)|^2\, dx.$$

Since $A(k,r) - A(h,r) \subseteq A(k,r)$ and $u \in DG(\Omega)$ we have

$$\begin{aligned}(2.41)\qquad &(h-k)^2|A(h,r)|^{\frac{2n-2}{n}} \\ &\le C_n|A(k,r)-A(h,r)|\int_{A(k,r)-A(h,r)} |Du(x)|^2\, dx \\ &\le C_n|A(k,r)-A(h,r)|\int_{A(k,r)} |Du(x)|^2\, dx \\ &\le C_n|A(k,r)-A(h,r)|\frac{C}{r^2}\int_{A(k,2r)} |u(x)-k|^2\, dx\end{aligned}$$

and since $u(x)-k \le M(2r)-k$, $|A(k,2r)| \le |B_{2r}| = \omega_n r^n$ and $|A(k,r)-A(h,r)|= |A(k,r)|-|A(h,r)|$ we obtain

$$\begin{aligned}(2.42)\qquad &C_n|A(k,r)-A(h,r)|\frac{C}{r^2}\int_{A(k,2r)} |u(x)-k|^2\, dx \\ &\le \frac{C_n}{r^2}|A(k,r)-A(h,r)||M(2r)-k|^2|A(k,2r)| \\ &\le \frac{C_n}{r^2}|A(k,r)-A(h,r)||M(2r)-k|^2\omega_n r^n \\ &\le C_n r^{n-2}(|A(k,r)|-|A(h,r)|)|M(2r)-k|^2.\end{aligned}$$

Hence, by (2.41) and (2.42)

$$(2.43)\qquad (h-k)^2|A(h,r)|^{\frac{2n-2}{n}} \le C_n r^{n-2}(|A(k,r)|-|A(h,r)|)|M(2r)-k|^2.$$

Now let us set $M := M(2r)$ and consider the sequence $k_i = M - \frac{M-K_0}{2^i} \nearrow M$, then

$$(2.44)\qquad k_i - k_{i-1} = \frac{M-K_0}{2^i}$$

$$(2.45)\qquad M - k_{i-1} = \frac{M-K_0}{2^{i-1}}.$$

and hence, since $k_i > k_{i-1}$, by (2.43) and (2.45)

$$\begin{aligned}(k_i-k_{i-1})^2|A(k_i,r)|^{\frac{2n-2}{n}} &\le C_n r^{n-2}(|A(k_{i-1},r)|-|A(k_i,r)|)|M-k_{i-1}|^2 \\ &= C_n r^{n-2}(|A(k_{i-1},r)|-|A(k_i,r)|)\frac{|M-k_0|^2}{2^{2i-2}}.\end{aligned}$$

Now, using (2.44) we can write

$$\frac{|M-k_0|^2}{2^{2i}}|A(k_i,r)|^{\frac{2n-2}{n}} \le C_n r^{n-2}(|A(k_{i-1},r)|-|A(k_i,r)|)\frac{|M-k_0|^2}{2^{2i-2}},$$

and therefore

$$(2.46)\qquad |A(k_i,r)|^{\frac{2n-2}{n}} \le C_n r^{n-2}(|A(k_{i-1},r)|-|A(k_i,r)|).$$

Let us fix $m \in \mathbb{N}$, then $\forall i \leq m$ we obtain $|A(k_i, r)| \geq |A(k_m, r)|$ and therefore

$$\sum_{i=1}^{m} |A(k_i, r)|^{\frac{2n-2}{n}} \geq \sum_{i=1}^{m} |A(k_m, r)|^{\frac{2n-2}{n}} = m|A(k_m, r)|^{\frac{2n-2}{n}},$$

hence, by (2.46)

$$\begin{aligned} m|A(k_m, r)|^{\frac{2n-2}{n}} &\leq \sum_{i=1}^{m} |A(k_i, r)|^{\frac{2n-2}{n}} \leq C_n r^{n-2} \sum_{i=1}^{m} (|A(k_{i-1}, r)| - |A(k_i, r)|) \\ &= C_n r^{n-2}(|A(k_0, r)| - |A(k_m, r)|) \leq C_n r^{n-2} |A(k_0, r)|. \end{aligned}$$

Now we have

$$\begin{aligned} |A(k_m, r)| &\leq \left(C_n r^{n-2} |A(k_0, r)|\right)^{\frac{n}{2n-2}} \left(\frac{1}{m}\right)^{\frac{n}{2n-2}} \\ &\leq \left(C_n r^{n-2} |B_r|\right)^{\frac{n}{2n-2}} \left(\frac{1}{m}\right)^{\frac{n}{2n-2}} \\ &\leq \left(C_n r^{n-2} \omega_n r^n\right)^{\frac{n}{2n-2}} \left(\frac{1}{m}\right)^{\frac{n}{2n-2}} = C_n r^n \left(\frac{1}{m}\right)^{\frac{n}{2n-2}}. \qquad \square \end{aligned}$$

Now we prove a reduction estimate of $DG-$functions from a fixed ball to a concentric one that has smaller radius.

Theorem 2.17. *Let Ω be an open subset in $\mathbb{R}^n$, u and $-u \in DG(\Omega)$, then $\forall r > 0$: $B_{2r} \subset\subset \Omega$ we have*

$$\omega\left(\frac{r}{2}\right) \leq A\omega(2r) \tag{2.47}$$

where $A < 1$ is a constant, not depending on r.

Proof. We observe that

$$\begin{aligned} K_0(u) &= \frac{1}{2}\left(\operatorname*{esssup}_{B_r} u + \operatorname*{essinf}_{B_r} u\right) \\ &= -\frac{1}{2}\left(\operatorname*{essinf}_{B_r}(-u) + \operatorname*{esssup}_{B_r}(-u)\right) = -K_0(-u) \end{aligned}$$

Hence $\{u \leq K_0(u)\} = \{-u \geq K_0(-u)\}$, and if

$$|B_r \cap \{u \geq K_0(u)\}| \leq \frac{1}{2}|B_r|$$

we have

$$|B_r \cap \{u \leq K_0(u)\}| = |B_r \cap \{-u \geq K_0(-u)\}| > \frac{1}{2}|B_r|.$$

Therefore we can assume that $|B_r \cap \{u \geq K_0(u)\}| > \frac{1}{2}$ (otherwise we can consider $-u$) and we wright $K_0 = K_0(u)$. By Theorem 2.12 with $k_i = M(2r) - \frac{(M(2r)-K_0)}{2^i}$ and $K_0 = \frac{M(2r)+m(2r)}{2}$ we have

$$
\begin{aligned}
M\left(\frac{r}{2}\right) &= \operatorname*{esssup}_{B_{r/2}} u \\
&\leq k_i + \frac{C}{r^{n/2}} \left(\int_{A(k_i,r)} |u(x) - k_i|^2 \, dx \right)^{1/2} \left(\frac{|A(k_i,r)|}{r^n} \right)^{\frac{\theta-1}{2}} .
\end{aligned} \tag{2.48}
$$

Hence on $A(k_i, r)$ we have $u(x) - k_i \leq M(2r) - k_i$ and therefore

$$
\int_{A(k_i,r)} |u(x) - k_i|^2 \, dx \leq (M(2r) - k_i)^2 |A(k_i, r)|.
$$

By (2.48) it follows

$$
\begin{aligned}
M\left(\frac{r}{2}\right) &\leq k_i + C \left(\frac{|A(k_i,r)|}{r^n} \right)^{1/2} (M(2r) - k_i) \left(\frac{|A(k_i,r)|}{r^n} \right)^{\frac{\theta-1}{2}} \\
&= k_i + C(M(2r) - k_i) \left(\frac{|A(k_i,r)|}{r^n} \right)^{\frac{\theta}{2}} .
\end{aligned} \tag{2.49}
$$

By Lemma 2.16

$$
C \left(\frac{|A(k_i,r)|}{r^n} \right)^{\theta/2} \leq C_n \left(\frac{1}{i} \right)^{\frac{n\theta}{4n-4}} \to 0 \quad \text{for } i \to \infty,
$$

hence $\exists \nu \in \mathbb{N}$ such that

$$
C \left(\frac{|A(k_\nu,r)|}{r^n} \right)^{\theta/2} < \frac{1}{2}.
$$

Then by (2.49) we have

$$
\begin{aligned}
M\left(\frac{r}{2}\right) &\leq k_\nu + \frac{M(2r) - k_\nu}{2} = \frac{1}{2} k_\nu + \frac{1}{2} M(2r) \\
&= \frac{1}{2} \left(M(2r) - \frac{M(2r) - K_0}{2^\nu} \right) + \frac{1}{2} M(2r) \\
&= \frac{1}{2} \left(M(2r) - \frac{M(2r) - \frac{M(2r)-m(2r)}{2}}{2^\nu} \right) + \frac{1}{2} M(2r) \\
&= \frac{1}{2} M(2r) - \frac{1}{2} \left(\frac{\frac{2M(2r)-M(2r)-m(2r)}{2}}{2^\nu} \right) + \frac{1}{2} M(2r) \\
&= M(2r) - \frac{M(2r) - m(2r)}{2^{\nu+2}}.
\end{aligned}
$$

Subtracting the quantity $m(\frac{r}{2})$ we gain

$$\omega\left(\frac{r}{2}\right) \leq M(2r) - m(2r) - \frac{M(2r) - m(2r)}{2^{\nu+2}},$$

and since $m(\frac{r}{2}) \geq m(2r)$ we have

$$\omega\left(\frac{r}{2}\right) \leq (M(2r) - m(2r))\left(1 - \frac{1}{2^{\nu+2}}\right) = A\omega(2r)$$

where $A = A_\nu$ depends only on ν and $0 < A = 1 - \frac{1}{2^{\nu+2}} < 1$. □

Now we show two Lemmas useful to prove that function in De Giorgi class are Hölder continuous.

Lemma 2.18. *Let* $(a_n)_{n\in\mathbb{N}}$ *be a non-decresing numerical sequence such that* $a_n \to 0$*,* $t \in \mathbb{R}$*,* $0 < t < a_1$*. Then* $\exists k \in \mathbb{N}$*,* $k > 1$ *such that* $a_k < t \leq a_{k-1}$*.*

Proof. Since $a_n \to 0$ we have

$$\exists \overline{n} \in \mathbb{N} : \quad a_n < t \quad \forall n > \overline{n}.$$

We set

$$k := \min\{\ n > \overline{n} :\ a_n < t\} > 1,$$

hence, since the sequence is non-decresing, we have $a_k < t \leq a_{k-1}$. □

Lemma 2.19. *Let* $R > 0$*,* $\phi : [0, +\infty] \to [0, +\infty]$ *be a non-decreasing function and* $0 < C, A < 1$ *constants satisfying the inequality*

$$\phi(Ct) \leq A\phi(t), \quad \forall t \leq R. \tag{2.50}$$

Then we have

$$\exists \alpha > 0 :\ \phi(t) \leq \frac{1}{A}\left(\frac{t}{R}\right)^\alpha \phi(R), \quad \forall t < R. \tag{2.51}$$

Proof. Since $C < 1$, the sequence $c_n = C^{n-1}$ is strictly decreasing and if $k \to \infty$ we have that $c_n \to 0$.
Let us consider $t \leq R$, then $\frac{t}{R} < 1$. By Lemma 2.18

$$\exists k \in \mathbb{N} : C^{k+1} < \frac{t}{R} \leq C^k.$$

By monotony of ϕ and (2.50) we have

$$\phi(t) \leq \phi(C^k R) \leq A\phi(C^{k-1}R) \leq A^2\phi(C^{k-2}R) \leq ... \leq A^k\phi(R). \tag{2.52}$$

We have $C^{k+1} < \frac{t}{R}$ and hence $k > -1 + \log_C \frac{t}{R}$. Since $A \leq 1$ we gain

$$\begin{aligned} A^k \phi(R) &\leq A^{-1} A^{\log_C(\frac{t}{R})} \phi(R) = \frac{1}{A} (C^{\log_C A})^{\log_C(\frac{t}{R})} \phi(R) \\ &= \frac{1}{A} (C^{\log_C(\frac{t}{R})})^{\log_C A} \phi(R) = \frac{1}{A} \left(\frac{t}{R}\right)^{\log_C A} \phi(R). \end{aligned} \tag{2.53}$$

Hence by (2.52) and (2.53) we obtain the desidered result, setting $\alpha := \log_C A$. □

Remark 2.20. If $C < A$ in the previous Lemma we have $\alpha \in]0, 1[$.

Now we are in position to state the main Theorem of this section

Theorem 2.21 (De Giorgi)**.** *Let Ω be an open subset in $\mathbb{R}^n$, u and $-u \in DG(\Omega)$. Then $\exists \alpha \in]0, 1[: u \in C^{0,\alpha}(\Omega)$.*

Proof. Let us consider $\nu \in \mathbb{N}$, $R > 0 : R < dist(x, \partial\Omega)$. By Theorem 2.17 we have

$$\omega(Cr) \leq A\omega(r) \quad \forall r > 0 : \; r \leq R. \tag{2.54}$$

If we set $C = \frac{1}{4}$ and $A = A_\nu < 1$, we can apply Lemma 2.19 to gain

$$\omega(r) \leq \frac{1}{A} \left(\frac{r}{R}\right)^\alpha \omega(R) \quad \forall r < R. \tag{2.55}$$

We take ν big enough such that $A = 1 - \frac{1}{2^{\nu+2}} > \frac{1}{4} = C$. We consider $x, y \in B_R$, $|x - y| = r < R$, then by (2.55)

$$|u(x) - u(y)| \leq \omega(r) \leq \frac{1}{A} \left(\frac{r}{R}\right)^\alpha \omega(R) = \left(\frac{1}{A} \frac{\omega(R)}{R^\alpha}\right) |x - y|^\alpha = L|x - y|^\alpha,$$

with $L := \frac{1}{A} \frac{\omega(R)}{R^\alpha}$. Then $u \in C^{0,\alpha}(B_R)$ and we have the result by the arbitrariness of R. □

Corollary 2.22. *Let Ω be an open subset in $\mathbb{R}^n$, if $u \in W^{1,2}_{loc}(\Omega)$ is a solution of (2.1) then $\exists \alpha \in]0, 1[: u \in C^{0,\alpha}(\Omega)$.*

Proof. Since u is solution, then u is both a subsolution and a supersolution of (2.1), therefore by Theorems 2.5 and 2.6 we have the desidered result. □

Chapter 3

Moser's Iteration Technique

In this chapter we present an alternative proof, due to Moser [**14**], of De Giorgi's result. The main tool in this analysis is the use of the test function technique to prove a weak Harnack inequality. Throughout this Chapter, more generally than the previous one, we will concentrate on a more general operator L of the form

$$Lu = D_i(a^{ij}(x)D_j u + b^i(x)u) + c^i(x)D_i u + d(x)u \tag{3.1}$$

whose coefficients a^{ij}, b^i, c^i, d $(i, j = 1, ..., n)$ are assumed to be measurable functions on a domain $\Omega \subset \mathbb{R}^n$.

If we assume that the function u is weakly differentiable and that the functions $a^{ij}D_j u + b^i u$, $c^i D_i u + du$, $i = 1, ..., n$ are locally integrable, then u is said to satisfy $Lu = 0 (\geq 0, \leq 0)$ in a *weak* or *generalized* sense respectively in Ω according as

$$\mathscr{L}(u, v) = \int_\Omega \{(a^{ij}D_j u + b^i u)D_i v - (c^i D_i u + du)v\}dx = 0(\leq 0, \geq 0) \tag{3.2}$$

for all non-negative functions $v \in W_0^{1,2}(\Omega)$.
Let f^i, g, $i = 1, ..., n$ be locally integrable functions in Ω. Then a weakly differetiable function u will be called a *weak* or *generalized* solution of the

inhomogeneous equation

$$Lu = g + D_i f^i \tag{3.3}$$

in Ω if

$$\mathscr{L}(u,v) = \mathscr{F}(v) = \int_\Omega (f^i D_i v - gv)dx \quad \forall v \in C_0^1(\Omega). \tag{3.4}$$

We shall assume that L is strictly elliptic in Ω; that is, there exists a positive number λ such that

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad \forall x \in \Omega,\ \xi \in \mathbb{R}^n \tag{3.5}$$

We also assume that L has bounded coefficients; that is for some costants Λ and $\nu \geq 0$ we have for all $x \in \Omega$

$$\sum |a^{ij}(x)|^2 \leq \Lambda^2, \quad \lambda^{-2}\sum(|b^i(x)|^2 + |c^i(x)|^2) + \lambda^{-1}|d(x)| \leq \nu^2. \tag{3.6}$$

1. Structural inequalities

Throughout this chapter Ω will denote a bounded domain in $\mathbb{R}^n$. An interesting feature of the test function technique is that they depend not so much on the linearity of the operator L but rather on a *non linear structure* satisfied by L. To be more explicit we have

$$D_i(a^{ij}(x)D_j u + b^i(x)u) + c^i(x)D_i u + d(x)u = g(x) + D_i f^i(x)$$

and therefore

$$D_i(a^{ij}(x)D_j u + b^i(x)u - f^i(x)) + (c^i(x)D_i u + d(x)u - g(x)) = 0.$$

Hence if we define

$$A^i(x,z,p) := a^{ij}(x)p_j + b^i(x)z - f^i(x),$$
$$B(x,z,p) := c^i(x)p_i + d(x)z - g(x),$$

for $(x,z,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, we can now write (3.3) in the form

$$D_i A^i(x,u,Du) + B(x,u,Du) = 0. \tag{3.7}$$

Let us observe that, as usual, a weakly differentiable function u is called a *weak subsolution (supersolution,solution)* of equation (3.7) in Ω if the functions $A^i(x,u,Du)$ and $B(x,u,Du)$ are locally integrable and

$$\int_\Omega (D_i v A^i(x,u,Du) - vB(x,u,Du))dx \ \leq (\ \geq,\ =)\ 0 \tag{3.8}$$

for all $v \geq 0$, $v \in W_0^{1,2}(\Omega)$.
We write $\mathbf{b} = (b^1, ..., b^n)$, $\mathbf{c} = (c^1, ..., c^n)$, $\mathbf{f} = (f^1, ..., f^n)$. Using condition (3.5) and the Schwarz inequality, we have

$$\begin{aligned} p_i A^i(x,z,p) &= a^{ij}(x) p_j p_i + b^i(x) p_i z - f^i(x) p_i \\ &\geq \lambda |p|^2 - |b^i(x) z||p_i| - |f^i(x)||p_i| \\ &\geq \lambda |p|^2 - \varepsilon |p_i|^2 - \frac{1}{2\varepsilon}(|b^i(x) z|^2 + |f^i(x)|^2) \end{aligned}$$

Choosing $\varepsilon = \lambda/2$, we have the so-called *structural inequalities* for the equation (3.3), i.e.

$$(3.9) \qquad \begin{cases} |\mathbf{A}(x,z,p)| \leq \Lambda |p| + |\mathbf{b} z| + |\mathbf{f}| \\ p_i A^i(x,z,p) \geq \frac{\lambda}{2}|p|^2 - \frac{1}{\lambda}(|b^i(x) z|^2 + |f^i(x)|^2) \\ |B(x,z,p)| \leq |\mathbf{c}||p| + |dz| + |g|. \end{cases}$$

For our purposes, we may even simplify the form of these inequalities by writing

$$(3.10) \qquad \overline{z} = |z| + k, \quad \overline{b} = \lambda^{-2}(|\mathbf{b}|^2 + |\mathbf{c}|^2 + k^{-2}|\mathbf{f}|^2) + \lambda^{-1}(|d| + k^{-1}|g|)$$

for some $k > 0$. Hence, we have

$$|\mathbf{A}(x,z,p)| \leq \Lambda |p| + \lambda \overline{b}^{1/2} |z| + \lambda \mathbf{b}^{1/2} k = \Lambda |p| + \lambda \overline{b}^{1/2} \overline{z},$$

$$\begin{aligned} p_i A^i(x,z,p) &\geq \frac{\lambda}{2}|p|^2 - \frac{1}{\lambda}(\lambda^2 \overline{b}|z|^2 + \lambda^2 k^2 \overline{b}) = \frac{\lambda}{2}|p|^2 - \lambda \overline{b}(|z|^2 + k^2) \\ &= \frac{\lambda}{2}|p|^2 - \lambda \overline{b}(\overline{z}^2 - 2|z|k) = \frac{\lambda}{2}|p|^2 - \lambda \overline{b}\overline{z}^2 + 2\lambda |z|) \\ &\geq \frac{\lambda}{2}\left(|p|^2 - 2\overline{b}\overline{z}^2\right) \end{aligned}$$

and, for any $0 < \varepsilon < 1$,

$$\begin{aligned} |\overline{z} B(x,u,p)| &\leq |\overline{z}||\mathbf{c}||p| + |\overline{z}||dz| + |\overline{z}||g| \leq \lambda \overline{b}^{1/2} \overline{z}|p| + \lambda \overline{b}\overline{z}|z| + \lambda \overline{b}\overline{z} k \\ &\leq \lambda \left(\frac{\varepsilon}{2}|p|^2 + \frac{1}{2\varepsilon}\overline{b}\overline{z}^2\right) = \frac{\lambda}{2}\left(\varepsilon |p|^2 + \frac{1}{\varepsilon}\overline{b}\overline{z}^2\right) \end{aligned}$$

By dividing equation (3.3) by the constant $\lambda/2$, we can assume that $\lambda = 2$ in the structural inequalities. Collecting these inequalities together with the assumptions, we obtain

$$\begin{cases} |\mathbf{A}(x,z,p)| \le \Lambda|p| + 2\bar{b}^{1/2}\bar{z} \\ p_i A^i(x,z,p) \ge |p|^2 - 2\bar{b}\bar{z}^2 \\ |\bar{z}B(x,u,p)| \le \varepsilon|p|^2 + \frac{1}{\varepsilon}\bar{b}\bar{z}^2. \end{cases} \tag{3.11}$$

2. John-Niremberg Theorem

The following Theorem will be very useful in the sequel.

Theorem 3.1 (John-Niremberg). *Let $u \in W^{1,1}(\Omega)$ where Ω is convex, and suppose there exists a constant K such that*

$$\int_{\Omega\cap B_r} |Du|dx \le KR^{n-1}, \quad \textit{for all balls } B_R. \tag{3.12}$$

Then there exist positive constants σ_0 and C, depending only on n, such that

$$\int_\Omega \exp\left(\frac{\sigma}{K}|u - u_\Omega|\right) dx \le C(\operatorname{diam}\Omega)^n \tag{3.13}$$

where $\sigma = \sigma_0|\Omega|(\operatorname{diam}\Omega)^{-n}$.

3. Weak Harnack Inequality

For the sequel of this Section we define the quantity k as

$$k = k(R) = \lambda^{-1}(R^\delta||\mathbf{f}||_q + R^{2\delta}||g||_{\frac{q}{2}}) \tag{3.14}$$

where $R > 0$ and $\delta = 1 - \frac{n}{q}$.
Now we prove two fundamentals Theorems.

Theorem 3.2. *Let the operator L satisfy conditions (3.5), (3.6) and suppose that $f^i \in L^q(\Omega)$, $i = 1, ..., n$, $g \in L^{\frac{q}{2}}(\Omega)$ for some $q > n$. Then if u is a $W^{1,2}(\Omega)$ subsolution (supersolution) of equation (3.3) in Ω, we have for any ball $B_{2R}(y) \subset \Omega$ and $p > 1$,*

$$\sup_{B_R(y)} u(-u) \le C(R^{-\frac{n}{p}}||u^+(u^-)||_{L^p(B_{2R}(y))} + k(R)) \tag{3.15}$$

where $C = C(n, \Lambda/\lambda, \nu R, q, p)$.

The crucial result of Moser approach will be the following *weak Harnack inequality* for supersolution.

Theorem 3.3 (Weak Harnack Inequality)**.** *Let the operator L satisfy conditions (3.5), (3.6) and suppose that $f^i \in L^q(\Omega)$, $i = 1, ..., n$, $g \in L^{\frac{q}{2}}(\Omega)$ for some $q > n$. Then if u is a $W^{1,2}(\Omega)$ supersolution of equation (3.3) in Ω, non-negative in a ball $B_{4R}(y) \subset \Omega$ and $1 \leq p < \frac{n}{n-2}$, we have*

$$R^{-\frac{n}{p}} ||u||_{L^p(B_{2R}(y))} \leq C(\inf_{B_R(y)} u + k(R)) \tag{3.16}$$

where $C = C(n, \Lambda/\lambda, \nu R, q, p)$.

Remark 3.4. In this thesis, for semplicity, we give the proof of Theorem 3.2 only when u is a bounded, non negative weak subsolution of (3.3). The result can be extended to an arbitrary weak subsolution to gain the estimate:

$$\sup_{B_R(y)} u \leq C(R^{-\frac{n}{p}} ||u^+||_{L^p(B_{2R}(y))} + k(R)). \tag{3.17}$$

By this inequality we can easily prove the result for an arbitrary supersolution u, since $-u$ is a subsolution and $(-u)^+ = -u^-$. By (3.17) we have

$$\begin{aligned} \sup_{B_R(y)} (-u) &\leq C(R^{-\frac{n}{p}} ||(-u)^+||_{L^p(B_{2R}(y))} + k(R)) \\ &= C(R^{-\frac{n}{p}} || - u^-||_{L^p(B_{2R}(y))} + k(R)) \\ &= C(R^{-\frac{n}{p}} ||u^-||_{L^p(B_{2R}(y))} + k(R)). \end{aligned}$$

Now, we prove Theorems 3.2 and 3.3 jointly.

Proof. We assume initially that $R = 1$ and $k > 0$. The general case is later recovered throught a simple coordinate transformation: $x \to x/R$ and letting k tend to zero. As usual, in the following we shall abbreviate $B_R(y) = B_R$ for any R, the center y to be understood. Let us define, for $\beta \neq 0$ and a non-negative $\eta \in C_0^1(B_4)$, the test function

$$v = \eta^2 \overline{u}^\beta \quad (\overline{u} = u + k). \tag{3.18}$$

By the derivative product rules, v is a valid test function in (3.8) and

$$Dv = 2\eta D\eta \overline{u}^\beta + \beta \eta^2 \overline{u}^{\beta-1} Du, \tag{3.19}$$

hence, by substitution into (3.8), we obtain

$$\begin{aligned} \beta \int_\Omega \eta^2 \overline{u}^{\beta-1} Du \cdot \mathbf{A}(x, u, Du) dx + 2 \int_\Omega \eta D\eta \cdot \mathbf{A}(x, u, Du) \overline{u}^\beta dx \\ - \int_\Omega \eta^2 \overline{u}^\beta B(x, u, Du) dx \end{aligned} \tag{3.20}$$

$$\leq 0 \quad \text{if } u \text{ is a subsolution,}$$
$$\geq 0 \quad \text{if } u \text{ is a supersolution.}$$

Using the structural inequalities (3.11), we can estimate, for any $0 < \varepsilon \leq 1$,

$$
\begin{aligned}
|\eta D\eta \mathbf{A}(x,u,Du)\overline{u}^{\beta}| &\leq \Lambda\eta|D\eta|\overline{u}^{\beta}|Du| + 2\overline{b}^{1/2}\eta|D\eta|\overline{u}\overline{u}^{\beta} \\
&\leq (\frac{\varepsilon}{2}\eta^2|Du|^2 + \frac{1}{2\varepsilon}\Lambda^2|D\eta|^2)\overline{u}^{\beta} \\
&\quad + (\overline{b}\eta^2 + |D\eta|^2)\overline{u}^{\beta+1} \\
&= \frac{\varepsilon}{2}\eta^2\overline{u}^{\beta-1}|Du|^2 + (1+\frac{\Lambda^2}{2\varepsilon})|D\eta|^2\overline{u}^{\beta+1} \\
&\quad + \overline{b}\eta^2\overline{u}^{\beta+1}
\end{aligned} \tag{3.21}
$$

$$\eta^2\overline{u}^{\beta-1}Du\cdot\mathbf{A}(x,u,Du) \geq \eta^2\overline{u}^{\beta-1}|Du|^2 - 2\overline{b}\eta^2\overline{u}^{\beta+1}$$
$$|\eta^2\overline{u}^{\beta}B(x,u,Du)| \leq \varepsilon\eta^2\overline{u}^{\beta-1}|Du|^2 + \frac{1}{\varepsilon}\overline{b}\eta^2\overline{u}^{\beta+1}.$$

We assume henceforth that $\beta > 0$ if u is a subsolution and $\beta < 0$ if u is a supersolution. We then obtain from (3.20) and (3.21)

$$
\begin{aligned}
\beta\int_\Omega &\eta^2\overline{u}^{\beta-1}|Du|^2 - 2\overline{b}\eta^2\overline{u}^{\beta+1}\ dx \\
&\leq 2\int_\Omega \frac{\varepsilon}{2}\eta^2\overline{u}^{\beta-1}|Du|^2 + (1+\frac{\Lambda^2}{2\varepsilon})|D\eta|^2\overline{u}^{\beta+1} + \overline{b}\eta^2\overline{u}^{\beta+1}\ dx \\
&\quad + \int_\Omega \varepsilon\eta^2\overline{u}^{\beta-1}|Du|^2 + \frac{1}{\varepsilon}\overline{b}\eta^2\overline{u}^{\beta+1}\ dx,
\end{aligned}
$$

and hence

$$
\begin{aligned}
(\beta-2\varepsilon)&\int_\Omega \eta^2\overline{u}^{\beta-1}|Du|^2\ dx \leq \\
&\left(2\beta+2+\frac{1}{\varepsilon}\right)\int_\Omega \overline{b}\eta^2\overline{u}^{\beta+1}\ dx + 2\int_\Omega (1+\frac{\Lambda^2}{2\varepsilon})|D\eta|^2\overline{u}^{\beta+1}\ dx.
\end{aligned}
$$

By choosing $\varepsilon = min\{1, |\beta|/4\}$,

$$\int_\Omega \eta^2\overline{u}^{\beta-1}|Du|^2 dx \leq C(|\beta|)\int_\Omega (\overline{b}\eta^2 + (1+\Lambda^2)|D\eta|^2)\overline{u}^{\beta+1}dx, \tag{3.22}$$

where $C(|\beta|)$ is bounded if $|\beta|$ is bounded away from zero. It is now convenient to introduce a function w defined by

$$w = \begin{cases} \overline{u}^{(\beta+1)/2}, & \text{if } \beta \neq -1, \\ \log\overline{u}, & \text{if } \beta = -1. \end{cases}$$

Therefore we have:

$$Dw = \begin{cases} \frac{\beta+1}{2}\overline{u}^{(\beta-1)/2}Du, & \text{if } \beta \neq -1, \\ \frac{1}{\overline{u}}Du = \overline{u}^{(\beta-1)/2}Du, & \text{if } \beta = -1, \end{cases}$$

hence, letting $\gamma = \beta + 1$, we may rewrite (3.22)

(3.23)
$$\int_\Omega |\eta Dw|^2 dx \leq \begin{cases} C(|\beta|)\gamma^2 \int_\Omega (\bar{b}\eta^2 + (1+\Lambda^2)|D\eta|^2)w^2 dx, & \text{if } \beta \neq -1, \\ \\ C\int_\Omega (\bar{b}\eta^2 + (1+\Lambda^2)|D\eta|^2)dx, & \text{if } \beta = -1 \end{cases}$$

Now, we will develop an iteration process from the first part of (3.23). From the Sobolev inequality (1.30) we have

$$||\eta w||^2_{2\hat{n}/(\hat{n}-2)} \leq C\int_\Omega (|\eta Dw|^2 + |wD\eta|^2)dx \tag{3.24}$$

where $\hat{n} = n$ for $n > 2$, $2 < \hat{2} < q$ and $C = C(\hat{n})$. By (1.11) with $q/2$ and its Sobolev conjugate exponent $q/(q-2)$ and (1.13) with $2 < \frac{2\hat{n}}{\hat{n}-2} < \frac{2q}{q-2}$, we obtain, for any $\varepsilon > 0$,

$$\begin{aligned} \int_\Omega \bar{b}(\eta w)^2 \ dx &\leq ||\bar{b}||_{q/2}||\eta w||^2_{2q/(q-2)} \\ &\leq ||\bar{b}||_{q/2}(\varepsilon||\eta w||_{2\hat{n}/(\hat{n}-2)} + \varepsilon^{-\sigma}||\eta w||_2)^2 \end{aligned} \tag{3.25}$$

where

$$\sigma = \frac{\frac{1}{2} - \frac{q-2}{2q}}{\frac{q-2}{2q} - \frac{\hat{n}-2}{2\hat{n}}} = \frac{\frac{q-q+2}{2q}}{\frac{q\hat{n}-2\hat{n}-q\hat{n}+2q}{2\hat{n}q}} = \frac{\frac{1}{q}}{\frac{q-\hat{n}}{\hat{n}q}} = \frac{\hat{n}}{q-\hat{n}}.$$

Hence, by substituting $(3.23)_1$ into (3.24) and using (3.25)

$$\begin{aligned} ||\eta w||^2_{2\hat{n}/(\hat{n}-2)} &\leq C\left(\int_\Omega |\eta Dw|^2 \ dx + \int_\Omega |wD\eta|^2 \ dx\right) \\ &\leq C\left(\int_\Omega (\bar{b}\eta^2 + (1+\Lambda^2)|D\eta|^2)w^2 dx + \int_\Omega |wD\eta|^2 \ dx\right) \\ &\leq C\left(\int_\Omega (\bar{b}\eta^2 w^2 \ dx + \int_\Omega (1+\Lambda^2)|wD\eta|^2 \ dx + \int_\Omega |wD\eta|^2 \ dx\right) \\ &\leq C\left(||\bar{b}||_{q/2}(\varepsilon||\eta w||_{2\hat{n}/(\hat{n}-2)} + \varepsilon^{-\sigma}||\eta w||_2)^2\right. \\ &\qquad \left. + \int_\Omega (1+\Lambda^2)|wD\eta|^2 \ dx + \int_\Omega |wD\eta|^2 \ dx\right) \\ &\leq C\left(\varepsilon^2||\eta w||^2_{2\hat{n}/(\hat{n}-2)} + \varepsilon^{-2\sigma}||\eta w||_2^2 + ||wD\eta||_2^2\right), \end{aligned}$$

therefore we have

$$
\begin{aligned}
(1 - C\varepsilon^2)||\eta w||^2_{2\hat{n}/(\hat{n}-2)} &\leq C\varepsilon^{-2\sigma} \int_\Omega |\eta w|^2 + |wD\eta|^2 \ dx \\
||\eta w||^2_{2\hat{n}/(\hat{n}-2)} &\leq \frac{C}{1 - C\varepsilon^2} \left(\frac{1}{\varepsilon^2}\right)^\sigma \int_\Omega |\eta w|^2 + |wD\eta|^2 \ dx \\
||\eta w||^2_{2\hat{n}/(\hat{n}-2)} &\leq \frac{C\varepsilon^2}{1 - C\varepsilon^2} \left(\frac{1}{\varepsilon^2}\right)^{\sigma+1} \int_\Omega |\eta w|^2 + |wD\eta|^2 \ dx.
\end{aligned}
$$

By an appropriate choice of ε, we obtain

$$
||\eta w||_{2\hat{n}/(\hat{n}-2)} \leq C(1 + |\gamma|)^{\sigma+1}||(\eta + |D\eta|)w||_2 \tag{3.26}
$$

where $C = C(\hat{n}, \Lambda, \nu, q, |\beta|)$ is bounded away from zero. It is now desiderable to specify the cut-off function η more precisely. Let r_1, r_2 be such that $1 \leq r_1 < r_2 \leq 3$ and set $\eta \equiv 1$ in B_{r_1}, $\eta \equiv 0$ in $\Omega - B_{r_2}$ with $|D\eta| \leq 2/(r_2 - r_1)$. Writing $\chi = \hat{n}/(\hat{n} - 2)$, we have from (3.26)

$$
||w||_{L^{2\chi}(B_{r_1})} \leq \frac{C(1 + |\gamma|)^{\sigma+1}}{r_2 - r_1}||w||_{L^2(B_{r_2})}. \tag{3.27}
$$

For $r < 4$ and $p \neq 0$, let us now introduce the quantities

$$
\Phi(p, r) := \left(\int_{B_r} |\overline{u}|^p \ dx\right)^{1/p}. \tag{3.28}
$$

We have

$$
\Phi(\infty, r) = \lim_{p\to\infty} \Phi(p, r) = \sup_{B_r} \overline{u}, \tag{3.29}
$$

and

$$
\Phi(-\infty, r) = \lim_{p\to-\infty} \Phi(p, r) = \inf_{B_r} \overline{u}. \tag{3.30}
$$

We get, for $\gamma \neq 0$, $w^{2\chi} = (\overline{u}^{(\gamma/2)})^{2\chi} = \overline{u}^{\gamma\chi}$ and $w^2 = (\overline{u}^{(\gamma/2)})^2 = \overline{u}^{\gamma}$. If $\gamma > 0$, we then obtain, from inequality (3.27),

$$\begin{aligned}
\Phi(\chi\gamma, r_1) = \left(\int_{B_{r_1}} |\overline{u}|^{\chi\gamma} \, dx \right)^{\frac{1}{\gamma\chi} \cdot \frac{2}{2}} &= ||w||^{2/\gamma}_{L^{2\chi}(B_{r_1})} \\
&\leq \left(\frac{C(1+|\gamma|)^{\sigma+1}}{r_2 - r_1} \right)^{2/\gamma} ||w||^{2/\gamma}_{L^2(B_{r_2})} \\
&= \left(\frac{C(1+|\gamma|)^{\sigma+1}}{r_2 - r_1} \right)^{2/\gamma} \left(\int_{B_{r_2}} |\overline{u}|^{\gamma} \, dx \right)^{\frac{1}{2} \cdot \frac{2}{\gamma}} \\
&= \left(\frac{C(1+|\gamma|)^{\sigma+1}}{r_2 - r_1} \right)^{2/\gamma} \Phi(\gamma, r_2)
\end{aligned}$$

instead, if $\gamma < 0$, we have

$$\begin{aligned}
\Phi(\gamma, r_2) = \left(\int_{B_{r_2}} |\overline{u}|^{\gamma} \, dx \right)^{\frac{2}{\gamma} \cdot \frac{2}{2}} &= ||w||^{2/\gamma}_{L^2(B_{r_2})} \\
&\leq \left(\frac{C(1+|\gamma|)^{\sigma+1}}{r_2 - r_1} \right)^{2/\gamma} ||w||^{2/\gamma}_{L^{2\chi}(B_{r_1})} \\
&= \left(\frac{C(1+|\gamma|)^{\sigma+1}}{r_2 - r_1} \right)^{2/\gamma} \left(\int_{B_{r_1}} |\overline{u}|^{\chi\gamma} \, dx \right)^{\frac{1}{2\chi} \cdot \frac{2}{\gamma}} \\
&= \left(\frac{C(1+|\gamma|)^{\sigma+1}}{r_2 - r_1} \right)^{2/\gamma} \Phi(\chi\gamma, r_2).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\Phi(\chi\gamma, r_1) &\leq \left(\frac{C(1+|\gamma|)^{\sigma+1}}{r_2 - r_1} \right)^{2/|\gamma|} \Phi(\gamma, r_2) \qquad \text{if } \gamma > 0, \\
\Phi(\gamma, r_2) &\leq \left(\frac{C(1+|\gamma|)^{\sigma+1}}{r_2 - r_1} \right)^{2/|\gamma|} \Phi(\chi\gamma, r_1) \quad \text{if } \gamma < 0.
\end{aligned} \tag{3.31}$$

These inequalities can be iterated to yield the desiderate estimates. If u is a subsolution we have $\beta > 0$ and hence $\gamma > 1$. Therefore, taking $p > 1$ and setting $\gamma = \gamma_m = \chi^m p$ and $r_m = 1 + 2^{-m}$, $m = 0, 1, ...$, we obtain, by inequality $(3.31)_1$,

$$\Phi(\chi^m p, 1 + \frac{1}{2^m}) \leq C\Phi(\chi^{m-1} p, 1 + \frac{1}{2^{m-1}}) \leq ... \leq C\Phi(\chi p, 1 + \frac{1}{2}) \leq C\Phi(p, 2)$$

where $C = C(\hat{n}, \Lambda, \nu, q, |\beta|)$. Consequently, we have, by letting m tend to infinity,

(3.32) $$\sup_{B_1} \overline{u} \leq C||\overline{u}||_{L^p(B_2)}.$$

By means of the transformation: $x \to x/R$ and by letting k tend to 0, the estimate (3.15) is established. For the case when u is a supersolution, that is when $\beta < 0$ and $\gamma < 1$, setting $\gamma = \gamma_m = -\chi^m p_0$ and $r_m = 1 + \frac{2}{2^m}$, $m = 0, 1, ...,$, since $\gamma_m < 0$, we can apply $(3.31)_2$ and write

$$\Phi(-p_0, 3) \leq C\Phi(-\chi p_0, 1+\frac{2}{2}) \leq C\Phi(-\chi^2 p_0, 1+\frac{1}{2^2}) \leq ... \leq C\Phi(-\chi^m p_0, 1+\frac{2}{2^m})$$

and letting m tends to infinity

(3.33) $$\Phi(-p_0, 3) \leq C\Phi(-\infty, 1).$$

Now, if we set $\gamma = \gamma_m = \chi^{-m} p$, for $m = 0, 1, ..., l$, $r_m = 3 - \frac{2}{2^{m+1}}$, $m = 0, 1, ..., l-1$, $r_l = 3$ and $p_0 = \chi^{-l} p = \gamma_l$, we can apply $(3.31)_1$ if $\gamma_m = \chi^{-m} p < 1$, that is $p < \chi^m$,

$$\Phi(p, 2) \leq C\Phi(\frac{p}{\chi}, 3 - \frac{1}{2}) \leq C\Phi(\frac{p}{\chi^2}, 3 - \frac{1}{2^2}) \leq ...$$
$$\leq C\Phi(\frac{p}{\chi^{l-1}}, 3 - \frac{1}{2^l}) \leq C\Phi(\frac{p}{chi^l}, 3) = \Phi(p_0, 3)$$

Since $p_0 = \frac{p}{\chi^m} < p$ and $p < \chi$ implies $p < \chi^m$, we have proved that for any p, p_0 s.t. $0 < p_0 < p < \chi$:

(3.34) $$\Phi(p, 2) \leq C\Phi(p_0, 3)$$

where $C = C(\hat{n}, \Lambda, q, p, p_0)$. The conclusion of Theorem 3.3 will thus follow if we can show that, for some $p_0 > 0$,

(3.35) $$\Phi(p_0, 3) \leq C\Phi(-p_0, 3).$$

In order to establish last inequality we turn to $(3.23)_2$. Let B_{2r} be any ball of radius $2r$, lying in $B_4(y)$ and choose the cut-off function η so that $\eta \equiv 1$ in B_r, $\eta \equiv 0$ in $\Omega - B_4$ and $|D\eta| \leq 2/r$. From $(3.23)_2$, with the aid of Hölder inequality (1.11), we obtain

$$
\begin{aligned}
\int_{B_r} |Dw| \; dx &\le Cr^{n/2} \left(\int_{B_r} |Dw|^2 \; dx \right)^{1/2} \\
&\le Cr^{n/2} \left(\int_{\Omega} (\bar{b}\eta^2 + (1+\Lambda^2)|D\eta|^2) dx \right)^{1/2} \\
&\le Cr^{n/2} \left(r^n + \frac{1}{r^2} r^n \right)^{1/2} \le Cr^{n/2} \left(r^{n-2}(r^2+1) \right)^{1/2} \\
&\le Cr^{n/2} r^{\frac{n-2}{2}} (4^2+1)^{1/2} \le Cr^{n-1}
\end{aligned} \tag{3.36}
$$

where $C = C(n, \Lambda, \nu)$. Hence, by Theorem 3.1, there exists a constant $p_0 > 0$ depending on n, Λ and ν such that, for

$$
w_0 = \frac{1}{|B_3|} \int_{B_3} w \; dx,
$$

we have

$$
\int_{B_3} e^{p_0 |w - w_0|} \; dx \le C(n, \Lambda, \nu).
$$

Since $w - w_0, -w + w_0 < |w - w_0|$ we gain

$$
\int_{B_3} e^{p_0 w - p_0 w_0} \; dx \le C(n, \Lambda, \nu),
$$

$$
\int_{B_3} e^{-p_0 w + p_0 w_0} \; dx \le C(n, \Lambda, \nu),
$$

and thus

$$
\int_{B_3} e^{p_0 w} \; dx \le C e^{p_0 w_0},
$$

$$
\int_{B_3} e^{-p_0 w} \; dx \le C e^{-p_0 w_0}.
$$

Now we can write

$$
\int_{B_3} e^{p_0 w} \; dx \int_{B_3} e^{-p_0 w} \; dx \le C e^{p_0 w_0} e^{-p_0 w_0} = C,
$$

and

$$
\left(\int_{B_3} e^{p_0 w} \; dx \right)^{1/p_0} \le C \left(\int_{B_3} e^{-p_0 w} \; dx \right)^{-1/p_0}.
$$

Recalling the definition of w we have the desidered result

$$
\left(\int_{B_3} |\overline{u}|^{p_0} \; dx \right)^{1/p_0} \le C \left(\int_{B_3} |\overline{u}|^{-p_0} \; dx \right)^{-1/p_0}.
$$

By (3.34), (3.35) and (3.33) we have

$$\Phi(p,2) \le C\Phi(-\infty,1)$$

and consequently

$$||\overline{u}||_{L^p(B_2)} \le C \inf_{B_1} \overline{u}, \tag{3.37}$$

we obtain the estimates (3.35) and consequently Theorem 3.3 with $R = 1$ and $k > 0$. Again, the full result then follows by means of the tranformation: $x \to x/R$ and letting k tends to 0. □

Remark 3.5. Let us observe that by combining Theorems 3.2 and 3.3 we obtain the full Harnack inequality.

Theorem 3.6 (Harnack Inequality)**.** *Let the operator satisfy conditions (3.5) and (3.6), and let $u \in W^{1,2}(\Omega)$ satisfy $u \ge 0$ in Ω and $Lu = 0$ in Ω. Then for any $\Omega' \subset\subset \Omega$, that is Ω' has compact closure in Ω, we have*

$$\sup_{\Omega'} u \le C \inf_{\Omega'} u \tag{3.38}$$

where $C = C(n, \Lambda/\lambda, \nu, \Omega', \Omega)$.

Proof. By combining (3.32) and (3.37) we obtain

$$\sup_{B_1} \overline{u} \le C \inf_{B_1} \overline{u} \tag{3.39}$$

and, as well as we have done before, the desidered result follows by the tranformation $x \to x/R$ and by letting k tend to 0. □

4. De Giorgi-Nash-Moser Theorem

Let us start with the following technical Lemma

Lemma 3.7. *Let ω be a non-decreasing function on an interval $(0, R_0]$ satisfying, for all $R \le R_0$, the inequality*

$$\omega(\tau R) \le \gamma\omega(R) + \sigma(R) \tag{3.40}$$

where σ is also non-decreasing and $0 < \gamma$, $\tau < 1$. Then, for any $\mu \in (0,1)$ and $R \le R_0$, we have

$$\omega(R) \le C\left(\left(\frac{R}{R_0}\right)^{\alpha}\omega(R_0) + \sigma(R^{\mu}R_0^{1-\mu})\right). \tag{3.41}$$

where $C = C(\gamma,\tau)$ and $\alpha = (1-\mu)(\log\gamma/\log\tau)$ are postive constants.

Proof. Let us fix $R_1 \leq R_0$, since σ is non-decreasing, we have

$$\omega(\tau R) \leq \gamma\omega(R) + \sigma(R_1), \quad \forall R \leq R_1$$

. We also have

$$\omega(\tau^2 R) = \omega(\tau(\tau R)) \leq \gamma\omega(\tau R) + \sigma(R_1) \leq \gamma\left[\gamma\omega(R) + \sigma(R_1)\right] + \sigma(R_1)$$

that is

$$\omega(\tau^2 R) \leq \gamma^2\omega(R) + (\gamma+1)\sigma(R_1).$$

We now iterate this inequality to get, for any positive integer m,

$$\begin{aligned}\omega(\tau^m R_1) &\leq \gamma^m\omega(R_1) + \sigma(R_1)\sum_{i=0}^{m-1}\gamma^i \\ &\leq \gamma^m\omega(R_0) + \sigma(R_1)\frac{1-\gamma^m}{1-\gamma} \leq \gamma^m\omega(R_0) + \frac{\sigma(R_1)}{1-\gamma}\end{aligned}$$

For any $R \leq R_1$, we can choose m such that

$$\tau^m R_1 < R < \tau^{m-1} R_1.$$

Hence from the first inequality we have

$$\log(\tau^m) < \log\left(\frac{R}{R_1}\right),$$

and, multiplying for $\log\gamma/\log\tau$:

$$\log(\gamma^m) < \frac{\log\gamma}{\log\tau}\log\left(\frac{R}{R_1}\right),$$

therefore

$$\gamma^m < \log\left(\frac{R}{R_1}\right)^{\frac{\log\gamma}{\log\tau}},$$

Hence

$$\omega(R) \leq \omega(\tau^{m-1}R_1) \leq \gamma^{m-1}\omega(R_0) + \frac{\sigma(R_1)}{1-\gamma} \leq \frac{1}{\gamma}\left(\frac{R}{R_1}\right)^{\frac{\log\gamma}{\log\tau}}\omega(R_0) + \frac{\sigma(R_1)}{1-\gamma}$$

Now letting $R_1 = R_0^{1-\mu}R^\mu$, we have

$$\begin{aligned}\omega(R) \leq& \frac{1}{\gamma}\left(\frac{R}{R_0^{1-\mu}R^\mu}\right)^{\frac{\log\gamma}{\log\tau}}\omega(R_0) + \frac{\sigma(R_0^{1-\mu}r^\mu)}{1-\gamma} \\ =& \frac{1}{\gamma}\left(\frac{R}{R_0}\right)^{(1-\mu)\frac{\log\gamma}{\log\tau}}\omega(R_0) + \frac{\sigma(R_0^{1-\mu}r^\mu)}{1-\gamma}\end{aligned}$$

□

For the purpose in the Theorem below we choice μ such that $\alpha < \mu\delta$. Now we are in position to state the main Theorem of this Chapter.

Theorem 3.8 (De Giorgi-Nash-Moser)**.** *Let the operator L satisfy conditions (3.5),(3.6) and suppose that $f^i \in L^q(\Omega)$, $i = 1, ..., n$, $g \in L^{\frac{q}{2}}(\Omega)$ for some $q > n$. Then if u is a $W^{1,2}(\Omega)$ solution of equation (3.3) in Ω, it follows that u is locally Hölder continuous in Ω, and for any ball $B_{R_0}(y) \subset \Omega$ and $R \le R_0$ we have*

$$\operatorname*{osc}_{B_R(y)} u \le CR^{\alpha}(R_0^{-\alpha} \sup_{B_0} |u| + k) \tag{3.42}$$

where $C = C(n, \Lambda/\lambda, \nu, q, R_0)$ and $\alpha = \alpha(n, \Lambda/\lambda, \nu R_0, q)$ are positive constants, and $k = \lambda^{-1}(||\boldsymbol{f}||_q + ||g||_{\frac{q}{2}})$.

Proof. We may assume without loss of generality that $R \le R_0/4$. Let us write

$$\begin{aligned} M_0 &= \sup_{B_0} |u|, \\ M_4 &= \sup_{B_{4R}} u, \\ m_4 &= \inf_{B_{4R}} u, \\ M_1 &= \sup_{B_R} u, \\ m_1 &= \inf_{B_R} u, \\ \omega(R) &= \operatorname*{osc}_{B_R} u = M_1 - m_1. \end{aligned}$$

Then we have

$$\begin{aligned} L(M_4 - u) &= D_i[a^{ij} D_j(M_4 - u) + b^i(M_4 - u)] + c^i D_i(M_4 - u) + d(M_4 - u) \\ &= D_i[-a^{ij} D_j u - b^i u] + M_4 D_i b^i - c^i D_i u - du + M_4 d \\ &= -\{D_i[a^{ij} D_j u + b^i u] + c^i D_i u + du\} + M_4 D_i b^i + M_4 d \\ &= -Lu + M_4(D_i b^i + d) \\ &= -g - D_i f^i + M_4(D_i b^i + d) \\ &= (-g + M_4 d) + D_i(-f_i + M_4 b^i) \end{aligned}$$

and

$$\begin{aligned}
L(u-m_4) &= D_i[a^{ij}D_j(u-m_4)+b^i(u-m_4)]+c^iD_i(u-m_4)+d(u-m_4)\\
&= D_i[a^{ij}D_ju+b^iu]-m_4D_ib^i+c^iD_iu+du-m_4d\\
&= \{D_i[a^{ij}D_ju+b^iu]+c^iD_iu+du\}-m_4D_ib^i+m_4d\\
&= Lu-m_4(D_ib^i+d)\\
&= g+D_if^i-m_4(D_ib^i+d)\\
&= (g-m_4d)+D_i(f_i-m_4b^i)
\end{aligned}$$

Let us set

$$\begin{aligned}
\overline{k}(R) &= \lambda^{-1}R^{\delta}(||\mathbf{f}||_q+M_0||\mathbf{b}||_q)+\lambda^{-1}R^{2\delta}(||g||_{\frac{q}{2}}+M_0||d||_{\frac{q}{2}}),\\
\delta &= 1-\frac{n}{q}.
\end{aligned}$$

If we apply weak Harnack inequality (3.16) with $p=1$ to the functions M_4-u and $u-m_4$, we obtain:

$$\begin{aligned}
R^{-n}||M_4-u||_{L^1(B_{2R}(y))} &\leq C\left(\inf_{B_R(y)}(M_4-u)+k'(R)\right)\\
R^{-n}||u-m_4||_{L^1(B_{2R}(y))} &\leq C\left(\inf_{B_R(y)}(u-m_4)+k''(R)\right)
\end{aligned}$$

where

$$\begin{aligned}
k' = k'(R) &= \lambda^{-1}\left(R^{\delta}||-\mathbf{f}+M_4\mathbf{b}||_q+R^{2\delta}||-g+M_4d||_{\frac{q}{2}}\right)\\
k'' = k''(R) &= \lambda^{-1}\left(R^{\delta}||\mathbf{f}-m_4\mathbf{b}||_q+R^{2\delta}||g-m_4d||_{\frac{q}{2}}\right).
\end{aligned}$$

Hence we have

$$\begin{aligned}
R^{-n}\int_{B_{2R}(y)}(M_4-u)dx &\leq C\Bigg(M_4+\inf_{B_R(y)}(-u)\\
&\quad +\lambda^{-1}\left(R^{\delta}(||\mathbf{f}||_q+M_4||\mathbf{b}||_q)+R^{2\delta}(||g||_{\frac{q}{2}}+M_4||d||_{\frac{q}{2}})\right)\Bigg)\\
&\leq C\left(M_4-\sup_{B_R(y)}u+\overline{k}(R)\right)\\
&= C\left(M_4-M_1+\overline{k}(R)\right)
\end{aligned}$$

and

$$R^{-n}\int_{B_{2R}(y)}(u-m_4)dx \le C\Bigg(\inf_{B_R(y)} u - m_4$$
$$+\lambda^{-1}\left(R^{\delta}(||\mathbf{f}||_q + m_4||\mathbf{b}||_q) + R^{2\delta}(||g||_{\frac{q}{2}} + m_4||d||_{\frac{q}{2}})\right)\Bigg)$$
$$\le C\left(\inf_{B_R(y)} u - m_4 + \overline{k}(R)\right)$$
$$= C\left(m_1 - m_4 + \overline{k}(R)\right)$$

Thus

$$R^{-n}\int_{B_{2R}(y)}(M_4 - u + u - m_4)dx \le C\left(M_4 - M_1 + m_1 - m_4 + 2\overline{k}(R)\right)$$
$$R^{-n}(2R)^n\omega_n(M_4 - m_4) \le C\left(M_4 - m_4 - (M_1 - m_1) + 2\overline{k}(R)\right)$$
$$2^n\omega_n(M_4 - m_4) \le C\left((M_4 - m_4) - (M_1 - m_1) + \overline{k}(R)\right)$$
$$\frac{1}{C}(M_4 - m_4) \le (M_4 - m_4) - (M_1 - m_1) + \overline{k}(R)$$
$$(M_1 - m_1) \le \left(1 - \frac{1}{C}\right)(M_4 - m_4) + \overline{k}(R)$$
$$\omega(R) \le \gamma\omega(4R) + \overline{k}(R)$$

where $\gamma = 1 - C^{-1}$, $C = C(n, \Lambda/\lambda, \nu R, q)$.
If we set $\rho = 4R$, we have

$$\omega\left(\frac{\rho}{4}\right) \le \gamma\omega(\rho) + \overline{k}\left(\frac{\rho}{4}\right).$$

Hence, by Lemma 3.7 , choosing μ such that $\alpha \le \mu\delta$ we obtain

$$\omega(\rho) \le C\left(\left(\frac{\rho}{R_0}\right)^{\alpha}\omega(R_0) + \overline{k}(\rho^{\mu}R_0^{1-\mu})\right),$$

that is

$$\omega(4R) \le C\left(\left(\frac{4R}{R_0}\right)^{\alpha}\omega(R_0) + \overline{k}((4R)^{\mu}R_0^{1-\mu})\right).$$

Now, as $\omega(R) \le \omega(4R)$, we have

$$\omega(R) \le C\Big(4^{\alpha}R^{\alpha}R_0^{-\alpha}\omega(R_0)$$
$$+\lambda^{-1}(4^{\mu}R^{\mu}R_0^{1-\mu})^{\delta}(||\mathbf{f}||_q + M_0||\mathbf{b}||_q)$$
$$+\lambda^{-1}(4^{\mu}R^{\mu}R_0^{1-\mu})^{2\delta}(||g||_{\frac{q}{2}} + M_0||d||_{\frac{q}{2}})\Big)$$

and consequently

$$\omega(R) \leq C\Bigg(R^\alpha R_0^{-\alpha}\omega(R_0) + \lambda^{-1}\left(\frac{R}{R_0}\right)^{\mu\delta} R_0^{\delta}(||\mathbf{f}||_q + M_0||\mathbf{b}||_q) + \lambda^{-1}\left(\frac{R}{R_0}\right)^{2\mu\delta} R_0^{2\delta}(||g||_{\frac{q}{2}} + M_0||d||_{\frac{q}{2}})\Bigg).$$

Therefore, multiplying and dividing the last two terms for R_0^α, we obtain

$$\omega(R) \leq CR^\alpha\Bigg(R_0^{-\alpha}\omega(R_0) + + \lambda^{-1}\left(\frac{R}{R_0}\right)^{\mu\delta} R^{-\alpha}R_0^{\alpha}R_0^{-\alpha}R_0^{\delta}(||\mathbf{f}||_q + M_0||\mathbf{b}||_q) + \lambda^{-1}\left(\frac{R}{R_0}\right)^{2\mu\delta} R^{-\alpha}R_0^{\alpha}R_0^{-\alpha}R_0^{2\delta}(||g||_{\frac{q}{2}} + M_0||d||_{\frac{q}{2}})\Bigg)$$

that is

$$\omega(R) \leq CR^\alpha\Bigg(R_0^{-\alpha}\omega(R_0) + \lambda^{-1}\left(\frac{R}{R_0}\right)^{\mu\delta-\alpha} R_0^{\delta-\alpha}(||\mathbf{f}||_q + M_0||\mathbf{b}||_q) + \lambda^{-1}\left(\frac{R}{R_0}\right)^{2\mu\delta-\alpha} R_0^{2\delta-\alpha}(||g||_{\frac{q}{2}} + M_0||d||_{\frac{q}{2}})\Bigg).$$

Since $\left(\frac{R}{R_0}\right)^{\mu\delta-\alpha} \leq 1$ we have

$$\omega(R) \leq CR^\alpha\left(R_0^{-\alpha}\omega(R_0) + \lambda^{-1}(||\mathbf{f}||_q + M_0||\mathbf{b}||_q) + \lambda^{-1}(||g||_{\frac{q}{2}} + M_0||d||_{\frac{q}{2}})\right)$$

and

$$\begin{aligned}\omega(R) &\leq CR^\alpha\left(R_0^{-\alpha}M_0(1 + R_0^{\alpha}\lambda^{-1}(||\mathbf{b}||_q + ||d||_{\frac{q}{2}})) + \lambda^{-1}(||\mathbf{f}||_q + ||g||_{\frac{q}{2}})\right)\\ &\leq CR^\alpha\left(R_0^{-\alpha}M_0 + k\right).\end{aligned}$$

Now the result follows arguing as in the final part of the proof of Theorem 2.21. □

By combining Theorems 3.2 and 3.8 we have the following interior Hölder estimate for weak solutions of equation (3.3).

Theorem 3.9. *Let the operator L satisfy conditions (3.5),(3.6) and suppose that $f^i \in L^q(\Omega)$, $i = 1, ..., n$, $g \in L^{\frac{q}{2}}(\Omega)$ for some $q > n$. Then, if $u \in W^{1,2}(\Omega)$ satisfies equation (3.3) in Ω, we have for any $\Omega' \subset\subset \Omega$, the estimate*

$$[u]_{\alpha,\Omega'} \leq C(||u||_{L^2(\Omega)} + k) \tag{3.43}$$

where $C = C(n, \Lambda/\lambda, \nu, q, d')$, $d' = dist(\Omega', \partial\Omega)$, $\alpha = \alpha(n, \Lambda/\lambda, \nu d') > 0$ and $k = \lambda^{-1}(||\boldsymbol{f}||_q + ||g||_{\frac{q}{2}})$.

Proof. Let us take $R_0 = d'$ in Theorem 3.8, hence we have, for all $R \leq d'$

$$[u]_{\alpha,B_R} \leq C(d' \sup_{B_{d'}} |u| + k). \tag{3.44}$$

We estimate $sup_{B_{d'}}|u|$ using Theorem 3.2. If $sup_{B_{d'}}|u| = sup_{B_{d'}}u$, then we have

$$\sup_{B_{\frac{d'}{2}}} u \leq C\left(d'^{-\frac{n}{2}}||u^+||_{L^2(B_{d'})} + k\left(\frac{d'}{2}\right)\right) \leq C\left(d'^{-\frac{n}{2}}||u||_{L^2(B_{d'})} + k\left(\frac{d'}{2}\right)\right),$$

if $sup_{B_{d'}}|u| = sup_{B_{d'}}(-u)$, then

$$\sup_{B_{\frac{d'}{2}}}(-u) \leq C\left(d'^{-\frac{n}{2}}||u^-||_{L^2(B_{d'})} + k\left(\frac{d'}{2}\right)\right) \leq C\left(d'^{-\frac{n}{2}}||u||_{L^2(B_{d'})} + k\left(\frac{d'}{2}\right)\right).$$

Therefore we have

$$[u]_{\alpha,B_R} \leq C\left(d'^{-\frac{n}{2}}||u||_{L^2(B_{d'})} + k\left(\frac{d'}{2}\right) + k\right) \leq C\left(||u||_{L^2(B_{d'})} + k\right)$$

□

Chapter 4

Remarks on Elliptic Systems

Let us start observing that, in the scalar case, the linearity of equation (2.1) played actually no role in the proof of De Giorgi's Theorem, the ideas involved being genuinely non-linear ones. Indeed the result was rapidly extended to a vast class of general nonlinear second order elliptic equations in divergence form of the type

$$D_i a^i(x, u, Du) = 0. \tag{4.1}$$

Moreover no pointwise regularity property of the vector field $a = (a^1, ..., a^n)$ is required with respect to the variables (x, u).

There were still hopes for getting a vectorial version of De Giorgi's Theorem 2.21 around 1967, when De Giorgi himself [**3**] showed that no such extension could take place. The regularity results, we have showed in previous Chapters are not valid for elliptic systems of second order of the type:

$$D_i \left(a_{ij}^{\alpha\beta}(x, u, Du) D_j u^\beta \right) = 0 \quad \alpha = 1, ..., N \tag{4.2}$$

where $u = (u^1, ..., u^N)$. We suppose that the system (4.2) is strictly elliptic

$$a_{ij}^{\alpha\beta} \xi_{\alpha_i} \xi_{\beta_j} \geq |\xi|^2 = \sum_{i=1}^{n} \sum_{\alpha=1}^{N} |\xi_\alpha^i|^2. \tag{4.3}$$

Let us recall that the function u is said a weak solution of (4.2) if

$$A(u,\varphi)=\int_\Omega a_{ij}^{\alpha\beta}(x)D_ju^\beta D_i\varphi^\alpha\ dx=0\quad \forall\varphi\in W_0^{1,2}(\Omega;R^N). \tag{4.4}$$

1. Counterexamples

De Giorgi's example [**3**]

We consider $n=N>2$ and

$$a_{ij}^{\alpha\beta}(x)=\delta_{ij}\delta_{\alpha\beta}+\left[(n-2)\delta_{i\alpha}+n\frac{x_\alpha x_i}{|x|^2}\right]\left[(n-2)\delta_{j\beta}+\frac{x_jx_\beta}{|x|^2}\right] \tag{4.5}$$

the discontinuous coefficients. Then the system is elliptic and the map

$$u(x)=\frac{x}{|x|^p},\quad p=\frac{n}{2}\left[1-\frac{1}{\sqrt{4(n-1)^2+1}}\right] \tag{4.6}$$

is solution of (4.2) in $\mathbb{R}^n-\{0\}$. This solution is not bounded and even continuous in the origin, hence Theorem 3.8 fails for elliptic system with discontinuous coefficients.

Giusti and Miranda's example[**10**]

The analogue of Theorem 3.8 doesn't exist even if we suppose that the coefficients are analytic functions of their arguments. In fact, if we set $n=N>2$ and

$$a_{ij}^{\alpha\beta}(u)=\delta_{ij}\delta_{\alpha\beta}+\left(\delta_{i\alpha}+\frac{4}{n-2}\frac{u_\alpha u_i}{1+|u|^2}\right)\left(\delta_{j\beta}+\frac{4}{n-2}\frac{u_ju_\beta}{1+|u|^2}\right). \tag{4.7}$$

The system is elliptic and the map

$$u(x)=\frac{x}{|x|} \tag{4.8}$$

is solution of (4.2) in $\mathbb{R}^n-\{0\}$. Also this solution is not bounded and even continuous in the origin, hence Theorem 3.8 fails also for elliptic systems with continuous coefficients.

Necas's example[**11**]

In the previous examples the singularity of the solution occurs due to the peculiar way the coefficients mix up with the components of the gradient. When there are no coefficients, we can consider Necas's example, a simple system of the type

$$D_i(D_zF(Du))=0. \tag{4.9}$$

The function $F : \mathbb{R}^{Nn} \to \mathbb{R}^+$ is analytic, with quadratic growth, and satisfies the uniform ellipticity and growth conditions

$$\nu \sum_{i=1}^{n} \sum_{\alpha=1}^{N} |\lambda_\alpha^i|^2 \leq \langle D_{\alpha\beta} D_{ij} F \cdot \lambda_{\alpha_i}, \lambda_{\beta_j} \rangle \leq \Lambda \sum_{i=1}^{n} \sum_{\alpha=1}^{N} |\lambda_\alpha^i|^2. \tag{4.10}$$

The function $F(z)$ is rather complicated, and it can be found in [**11**], formula (3.1). With $\Omega \equiv B_1 \subset \mathbb{R}^n$, we can consider $u : B_1 \to \mathbb{R}^{n^2}$ defined by

$$u^{ij}(x) := \frac{x_i x_j}{|x|} - \frac{1}{n} \delta_{ij} |x| \tag{4.11}$$

The importance of this example lies in the fact that it shows that the irregularity is a peculiar feature of the vectorial case, and is not due to the presence of coefficients. Necas's example only works for $n \geq 5$, while a more recent example for $n \geq 3$ can be found in [**16**].

2. Conclusions

Concerning the pointwise regularity of solutions of second order elliptic systems in the interior of Ω, the so called partial regularity comes into the play. The general principle of partial regularity asserts that there is *pointwise regularity of solutions in an open subset whose complement is negligible.* In other words, one tries to prove that the solution is regular in an open subset $\Omega_u \subset \Omega$ such that $|\Omega - \Omega_u| = 0$; the set

$$\Sigma_u := \Omega - \Omega_u \tag{4.12}$$

is called the singular set of u. For this reason partial regularity is sometimes called *almost everywhere regularity.*
Under certain assumptions of elliptic growth, boundedness and Hölder continuity for $a_{ij}^{\alpha\beta}$ (we can see [**6**]), the singular set for (4.2) is characterized as the *non-Lebesgue point set*, indeed is identified by the equality

$$\Sigma_u = \Sigma_0 \cup \Sigma_1 \tag{4.13}$$

where

$$\Sigma_0 := \{\, x_0 \in \Omega : \liminf_{\rho \searrow 0} \fint_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^p \, dy > 0 \text{ or } \limsup_{\rho \searrow 0} |(Du)_{x_0,\rho}| = +\infty \,\}$$

and

$$\Sigma_1 := \{ \, x_0 \in \Omega : \; \liminf_{\rho \searrow 0} \fint_{B_\rho(x_0)} |u - (u)_{x_0,\rho}|^p \; dy > 0 \text{ or } \limsup_{\rho \searrow 0} |(u)_{x_0,\rho}| = +\infty \, \} \, .$$

Finally let us conclude observing that Σ_u is not only negligible, but in some sense “smaller”, if not empty at all.

Chapter 5

Applications to Continuum Mechanics

In this Chapter we analyze two physical phenomena that can be described by an elliptic system in divergence form. First we introduce some useful notions. Let S be a three-dimensional continuous body and let $R = (O, e_i)$ be a rectangular coordinate system in the Euclidean space E_3 (for more precisely definitions, we can refer to [**17**], Chapter 1). Moreover, let C_* and C denote two configurations of S at different times. Any point of S in the *reference configuration* C_*, identified from now on by the label $\mathbf{X}$, is called a material point; the same point in the *actual or current configuration* C is identified by the position x and is called a spatial point. Furthermore, (x_i) and (X_L), with $i, L = 1, 2, 3$, are respectively the *spatial* and *material coordinates* of the particle $\mathbf{X}$ in (O, e_i). We say $\mathbf{T}$ tensor a linear mapping of the Euclidean vector space (a space in which scalar product is defined) and $I_{\mathbf{T}}$, $II_{\mathbf{T}}$ and $III_{\mathbf{T}}$ the first, second and third invariant, respectively

$$(5.1) \qquad I_{\mathbf{T}} = T^i_i, \quad II_{\mathbf{T}} = \frac{1}{2}(T^i_i T^j_j - T^i_j T^j_i), \quad III_{\mathbf{T}} = det(T^i_j)$$

Now we give the following definitions, useful to describe the deformation process of a continuous body S, as it passes from the initial configuration to the current one.

Definition 5.1. We define a *finite deformation* from C_* to C as the vector function

$$\mathbf{x} = \mathbf{x}(\mathbf{X}) \tag{5.2}$$

which maps any $\mathbf{x} \in C_*$ onto the corrisponding $\mathbf{x} \in C$. We assume that the three scalar functions $x_i = x_i(X_L) \in C^1(C)$. Therefore we also define the *deformation gradient* by the tensor

$$\mathbf{F} = (F^i_L) = (D_{X_L} x_i). \tag{5.3}$$

and his Jacobian as

$$J = det(D_{X_L} x_i). \tag{5.4}$$

Definition 5.2. We define the *displacement field* as

$$\mathbf{u}(\mathbf{X}) = \mathbf{x}(\mathbf{X}) - \mathbf{X} \tag{5.5}$$

and also the *displacement gradient tensor*

$$\mathbf{H} = D\mathbf{u}(\mathbf{X}) = (D_{X_L} u^i). \tag{5.6}$$

Any quantity ψ of the continuous system S can be represented in *Lagrangian* or *Eulerian form*, depending on whether it is expressed as a function of $(\mathbf{X}, t)$ or $(\mathbf{x}, t)$, i.e., if it is a field assigned on the initial configuration C_* or on the current configuration C:

$$\psi = \psi(\mathbf{x}, t) = \psi(\mathbf{x}(\mathbf{X}, t), t) = \tilde{\psi}(\mathbf{X}, t) \tag{5.7}$$

The *velocity* and the *acceleration* of the particle $\mathbf{X} \in C_*$ at time t are given by the partial derivative of the material representation

$$\mathbf{v} = \tilde{\mathbf{v}}(\mathbf{X}, t) = D_t\mathbf{x} = D_t\mathbf{u}, \quad \mathbf{a} = \tilde{\mathbf{a}}(\mathbf{X}, t) = D_{tt}\mathbf{x} = D_{tt}\mathbf{u} \tag{5.8}$$

In a simplified approach to continuum mechanics the external actions on an arbitrary material volume c of a continuous system S, are divided into *mass forces*, continuously distributed over c, and *contact forces*, acting on the boundary ∂c of c. Therefore, we define $\mathbf{b}$ as the *specific force*, defined on c and $\mathbf{t}$ the *traction* or the *stress*, defined on ∂c. We also assume the *Euler-Cauchy postulate* that the vector $\mathbf{t}$ depends on the choice of the surface only through its orientation, i.e.,

$$\mathbf{t} = \mathbf{t}(\mathbf{N}) \tag{5.9}$$

where $\mathbf{N}$ is the outward unit vector normal to the surface. Now we introduce a fundamental result

Theorem 5.3. *If* $\mathbf{t}$*,* $\mathbf{b}$ *and* $\mathbf{a}$ *are regular functions, then the action-reaction principle also holds for stresses, i.e.,*

$$\mathbf{t}(\mathbf{N}) = -\mathbf{t}(-\mathbf{N}). \tag{5.10}$$

There exists a second-order tensor $\mathbf{T}$*, called* Cauchy's stress tensor*, such that* $\mathbf{t}$ *is a linear function of* $\mathbf{N}$*, that is*

$$\mathbf{t}(\mathbf{N}) = \mathbf{T}\mathbf{N}. \tag{5.11}$$

The tensor $\mathbf{T}$ *depends on* (x, t)*, but is independent of* $\mathbf{N}$*.*

The fundamental laws of continuum mechanics are relations expressing conservation or balance of physical quantities: mass conservation, momentum balance, angular momentum balance, energy balance, and so on. In order to study these laws for a continuous system S, we introduce the basic assumption that its mass is continuously distributed over the region $C(t)$ occupied by S at the instant t. In mathematical terms, there exists a function $\rho(x, t)$ called the *mass density*, which is supposed to be of class C^1 on $C(t)$. Similarly we introduce the *specific internal energy* ε, that expresses the density of the internal energy, a scalar function $E(c)$ we can associate to any material volume c of S. The thermal power on c divided is as well as the external actions, is continuously distributed over c and acts on the boundary ∂c of c. Therefore, we say r the *specific heat surce*, defined on c, and s the *thermal power flux*, defined on ∂c. We also write

$$s(\mathbf{x}, t, \mathbf{n}) = -\mathbf{h}(\mathbf{x}, t) \cdot \mathbf{N} \tag{5.12}$$

where $\mathbf{h}$ is the *heat flux vector*. The fundamental laws of continuum mechanics are relations expressing conservation or balance of physical quantities. The local formulation of the mass balance principle is

$$\dot{\rho} + \rho D_h v^h = 0, \tag{5.13}$$

the local expression of the momentum balance is

$$\rho \dot{v^\alpha} = D_h T^\alpha_h + \rho b^\alpha, \quad \alpha = 1, 2, 3, \tag{5.14}$$

and we may give the equations of angular momentum balance, energy balance, and so on. The validity of balance equations does not depend on body properties, but the evolution of a continuous deformable body cannot be completely predicted by knowing the equations of motion, the massa distribution, and the external forces acting on the body. In the Eulerian description, $\mathscr{A} = \{\rho, \mathbf{v}, \mathbf{T}, h, \theta\}$, where θ is the *absolute temperature*, represent the unknown fields, so we need to add relations that connect them with the basic fields: the *constitutive equations*.

We say the system S is *homogeneous* in the reference configuration C_* if $\mathscr{A}$ is independent of $\mathbf{X}$. We say an homogeneous material system *elastic* if $\mathscr{A} = \mathscr{A}(\mathbf{F})$, *isotropic* if it always provides the same response in terms of deformation when subjected to the same effort regardless of the direction of application of such effort.

1. Linear Elasticity

We introduce the linear elasticity to semplify the work of studying elastic behaviour of continuous systems. Let us first remark that the linear approximation is valid only when the continuum is subject to small deformation. If we do the following hypothesis:

- the transformation $C_* \to C$ is infinitesimal,
- in the reference configuration C_* the system is elastic, homogeneous at a constant and uniform temperature, and
- C_* is a stress-free state.

then we are studying an isotropic linear elastic solid. We can refer to [**17**] to prove that in these hypotesis

$$J \approx 1 + I_{\mathbf{H}}, \tag{5.15}$$

$$\rho = \frac{\rho_*}{J} \approx \frac{\rho_*}{1 + I_{\mathbf{H}}} \approx \rho_*(1 - I_{\mathbf{H}}), \tag{5.16}$$

$$\dot{v^\alpha} = D_t v^\alpha + D_\alpha v^h \approx D_t v^\alpha = D_{tt} u^\alpha, \quad \alpha = 1, 2, 3, \tag{5.17}$$

$$\mathbf{T} \approx \lambda I_{\mathbf{E}} \mathbf{I} + 2\mu \mathbf{E}, \tag{5.18}$$

where ρ_* is the density in the reference configuration, $\mathbf{I}$ is the unitary tensor and $\mathbf{E}$ is the linearized deformation tensor $\frac{1}{2}(\mathbf{H} + \mathbf{H}^t)$ and λ and μ are called the *Lamé coefficients*. Now we can write, for $\alpha, h = 1,2,3$,

$$\begin{aligned}
D_h T_h^\alpha &= D_{X_L} T_h^\alpha D_h X_L \approx D_{X_L}(\lambda I_{\mathbf{E}} \delta_h^\alpha + 2\mu E_h^\alpha)(F^{-1})_h^L \\
&\approx D_{X_L}(\lambda I_{\mathbf{E}} \delta_h^\alpha + 2\mu E_h^\alpha)(I - H)_h^L \\
&= D_{X_L}(\lambda I_{\mathbf{E}} \delta_h^\alpha + 2\mu E_h^\alpha)(\delta_h^L - H_h^L) \\
&\approx D_{X_L}(\lambda I_{\mathbf{E}} \delta_h^\alpha + 2\mu E_h^\alpha)\delta_h^L = D_h(\lambda I_{\mathbf{E}} \delta_h^\alpha + 2\mu E_h^\alpha) \\
&= \lambda \delta_h^\alpha D_h E_k^k + 2\mu D_h E_h^\alpha \\
&= \lambda \delta_h^\alpha D_h(\frac{1}{2} D_k u^k + D_k u^k) + 2\mu D_h(\frac{1}{2} D_h u^\alpha + D_\alpha u^h) \\
&= \lambda \delta_h^\alpha D_{hk} u^k + \mu D_{hh} u^\alpha + \mu D_{h\alpha} u^h = \lambda D_{\alpha k} u^k + \mu D_{hh} u^\alpha + \mu D_{h\alpha} u^h \\
&= (\lambda + \mu) D_{\alpha h} u^h + \mu D_{hh} u^\alpha,
\end{aligned}$$

hence we have

$$D_h T_h^\alpha \approx (\lambda + \mu) D_\alpha D_h u^h + \mu D_{hh} u^\alpha \tag{5.19}$$

By substituting (5.16), (5.17) and (5.19) into (5.14) we deduce "Navier-Chauchy" equation for isotropic linear elastic material

$$\rho_* D_{tt} u^\alpha = \mu \left(D_{hh} u^\alpha\right) + (\lambda + \mu) D_\alpha \left(D_h u^h\right) + \rho_* b_\alpha, \quad \alpha = 1,2,3. \tag{5.20}$$

If we analyze the elastostatic case, $\mathbf{u} = \mathbf{u}(\mathbf{X})$, and so

(5.21)

$$-\mu \left(D_{11} u^\alpha + D_{22} u^\alpha + D_{33} u^\alpha\right) - (\lambda + \mu) D_\alpha \left(D_h u^h\right) = \rho_* b_\alpha, \quad \alpha = 1,2,3.$$

This is a vectorial equation, so we can write the three scalar equations

$$\begin{cases}
D_1 \left(-\mu D_1 u^1 - (\lambda + \mu) \left(D_h u^h\right)\right) + D_2(-\mu D_2 u^1) + D_3(-\mu D_3 u^1) = \rho_* b_1 \\
D_1(-\mu D_1 u^2) + D_2 \left(-\mu D_2 u^2 - (\lambda + \mu) \left(D_h u^h\right)\right) + D_3(-\mu D_3 u^2) = \rho_* b_2 \\
D_1(-\mu D_1 u^3) + D_2(-\mu D_2 u^3) + D_3 \left(-\mu D_3 u^3 - (\lambda + \mu) \left(D_h u^h\right)\right) = \rho_* b_3.
\end{cases}$$

and, more extensively,

$$\begin{cases}
D_1 \left(-(\lambda + 2\mu) D_1 u^1 - (\lambda + \mu) D_2 u^2 - (\lambda + \mu) D_3 u^3\right) + D_2(-\mu D_2 u^1) \\
\qquad\qquad\qquad\qquad\qquad + D_3(-\mu D_3 u^1) = \rho_* b_1 \\
D_1(-\mu D_1 u^2) + D_2 \left(-(\lambda + \mu) D_1 u^1 - (\lambda + 2\mu) D_2 u^2 - (\lambda + \mu) D_3 u^3\right) \\
\qquad\qquad\qquad\qquad\qquad + D_3(-\mu D_3 u^2) = \rho_* b_2 \\
D_1(-\mu D_1 u^3) + D_2(-\mu D_2 u^3) \\
\qquad + D_3 \left(-(\lambda + \mu) D_1 u^1 - (\lambda + \mu) D_2 u^2 - (\lambda + 2\mu) D_3 u^3\right) = \rho_* b_3.
\end{cases}$$

Therefore we can write the equation (5.21) in divergence form

$$D_i(a_{ij}^{\alpha\beta} D_j u^\beta) = \rho_* b^\alpha \tag{5.22}$$

where

$$a_{ij}^{\alpha\beta} = \begin{cases} -(\lambda + 2\mu), & \text{if } i = j = \alpha = \beta, \\ -(\lambda + \mu), & \text{if } i = \alpha,\ j = \beta,\ i \neq j, \\ -\mu, & \text{if } i = j,\ \alpha = \beta,\ i \neq \alpha, \\ 0, & \text{otherwise.} \end{cases} \tag{5.23}$$

2. Two-dimensional Steady Flow of a Perfect Fluid with Vortex Potential

We say a fluid S *perfect* if the stress tensor is expressed by the constitutive equation

$$\mathbf{T} = -\mathbf{pI}, \quad p > 0 \tag{5.24}$$

where the positive scalar p is called *pressure*. If the pressure

$$p = p(\rho) \tag{5.25}$$

is a given function of the mass density ρ, then the fluid is called *compressible*. On the other hand, if the fluid is density preserving, so that the pressure does not depend on ρ, then the fluid is said to be *incompressible*. Now let us consider the flow of an incompressible perfect fluid in the hypothesis of steady and irrotational motion. The condition that define an irrotational motion is

$$D_1 u^2 - D_2 u^1 = 0 \tag{5.26}$$

where $\mathbf{v} = \mathbf{v}(\mathbf{x})$. If we do the further hypothesis that the definition set of $\mathbf{v}$ is connected, then the irrotationality condition allows us to deduce the existence of a scalar function, said *kinetic potential* $\varphi = \varphi(\mathbf{x})$ such that

$$\mathbf{v} = D\varphi. \tag{5.27}$$

Since the fluid is *incompressible* we deduce from (5.13) that

$$D_h v^h = 0, \tag{5.28}$$

hence substituting (5.27) in (5.28) we gain

$$D_{hh}\varphi = 0. \tag{5.29}$$

Finally, it is worthwhile to note that, in dealing with a two-dimensional flow, the velocity vector $\mathbf{v}$ at any point is parallel to a plane π and it is independent of the coordinate normal to this plane. In this case, if a system $Oxyz$ is introduced, where the axes x and y are parallel to pi and the z axis is normal to this plane, then we have

$$\mathbf{v} = u(x, y)\mathbf{i} + v(x, y)\mathbf{j} \tag{5.30}$$

where u, v are the components of $\mathbf{v}$ on x and y, and i, j are the unit vectors of these axes. If now C is a simply connected region of the plane Oxy, conditions (5.28) and (5.26) become

$$D_x u + D_y v = 0, \quad D_y u - D_x v = 0 \tag{5.31}$$

These conditions allow us to state that the two differential forms $\omega_1 = udx + vdy$ and $\omega_2 = -vdx + udy$ are integrable, i.e. there is a function φ, called the *velocity potential* or the *kinetic potential*, and a function ψ, called the *stream potential* or the *Stokes potential*, such that $D\varphi = udx + vdy$ and $d\psi = -vdx + udy$. From (5.28) it follows that the curves $\varphi = const$ are at any point normal to the velocity field. Furthermore, since $D\phi \cdot D\psi = 0$, the curves $\psi = const$ are flow lines. Let us observe that (5.31) that the functions φ and ψ satisfy the Cauchy-Riemann conditions

$$D_x\varphi = D_y\psi, \quad D_y\varphi = -D_x\psi, \tag{5.32}$$

so that the complex function

$$F(z) = \varphi(x, y) + i\psi(x, y) \tag{5.33}$$

is holomorphic and is called *complex potential*. Then the complex potential can be defined as the holomorphic function whose real and imaginary parts are the velocity potential φ and the stream potential ψ, respectively. The two functions φ and ψ are harmonic and the derivative of $F(z)$

$$V = F'(z) = D_x\varphi + iD_x\psi = u - iv \tag{5.34}$$

is said *complex velocity*. Within the context of considerations developed in the following discussion, it is relevant to remember that the line integral of a holomorphic function vanishes around any arbitrary closed path in a simply connected region, since the Cauchy-Riemann equations are necessary and sufficient conditions for the integral to be independent of the path (and therefore it vanishes for a closed path). The above remarks lead to the conclusion that a 2D irrotational flow of an incompressible fluid is completely defined if a harmonic function $\phi(x,y)$ or a complex potential $F(z)$ is prescribed, as it is shown in the example below. For example we consider a two dimensional flow defined by the complex potential

(5.35)
$$F(z) = -i\frac{\Gamma}{2\pi}\ln z = -i\frac{\Gamma}{2\pi}\ln(re^{i\theta}) = -i\frac{\Gamma}{2\pi}(\ln r + \ln e^{i\theta}) = -i\frac{\Gamma}{2\pi}\ln r + \frac{\Gamma}{2\pi}\theta$$

where r, θ are polar coordinates. It follows that

$$\varphi(x,y) = \frac{\Gamma}{2\pi}\arctan\frac{x}{y} \tag{5.36}$$

$$\psi = -\frac{\Gamma}{2\pi}\ln(x^2+y^2) \tag{5.37}$$

and the curves $\varphi = const$ are straight lines through the origin, while the curves $\psi = const$ are circles whose center is the origin (see figure below). Accordingly, the velocity components become

$$u = D_x\phi = -\frac{\Gamma}{2\pi}\frac{y}{x^2+y^2} = -\frac{\Gamma}{2\pi}\frac{\sin\theta}{r},$$
$$v = D_y\phi = \frac{\Gamma}{2\pi}\frac{x}{x^2+y^2} = \frac{\Gamma}{2\pi}\frac{\cos\theta}{r},$$

Let us observe that the circulation around a path γ bordering the origin is given by

$$\int_\gamma \mathbf{v}\cdot d\mathbf{s} = \Gamma$$

so that it does not vanish if $\Gamma \neq 0$. This is not in contradiction with the condition (5.26) since the plane without the origin is no longer a simply connected region. The complex potential (5.35) can then be used with advantage to describe the uniform two dimensional flow of particles rotating around the axis through the origin and normal to the plane Oxy.

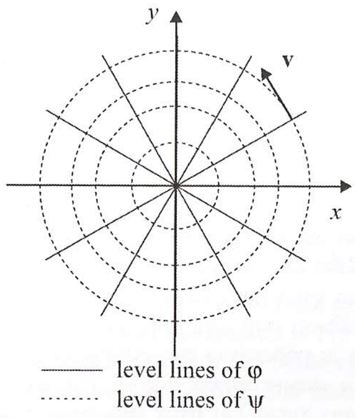

Figure 9.5: Vortex potential

Bibliography

[1] Bernstein S.:*Sur la nature analytique des solutions del equations aux dérivées partielles du second ordre*, Mathematische Annalen 59, 20-76 (1904).

[2] De Giorgi E.: *Sulla differenziabilitá e l'analiticitá delle estremali degli integrali multipli regolari*. Memorie dell'Accademia delle Scienze di Torino classe di scienze fisiche, matematiche e naturali, P. I., III. Ser. 3, 25-43 (1957).

[3] De Giorgi E.: *Un esempio di estremali discontinue per un problema variazionale di tipo ellittico*. Bollettino dell'Unione Matematica Italiana, IV. Ser. 1, 135-137 (1968).

[4] Di Benedetto E., Trudinger N.S.: *Harnack inequalities for quasiminima of variational integrals*. Annales de l'Institute Henri Poincaré, Section C, tome 1, n. 4, 295-308 (1984).

[5] Giaquinta M., Giusti E.: *On the regularity of the minima of variational integrals*. Acta Mathematica 148, 31-46 (1982).

[6] Giaquinta M., Modica G.: *Almost-everywhere regularity for solutions of nonlinear elliptic systems*. Manuscripta Mathematica 28, 109-158 (1979).

[7] Gilbarg D., Trudinger N.S.: *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag (1998).

[8] Giusti E.: *Direct methods in the calculus of variations*. World Scientific (2003).

[9] Giusti E.: *Equazioni Ellittiche del Secondo Ordine*. Pitagora Editrice (1978).

[10] Giusti E., Miranda M.: *Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale regolare del calcolo delle variazioni*. Bollettino dell'Unione Matematica Italiana, IV. Ser. 1, 219-226 (1968).

[11] Hao W., Leonardi S., Necas J.: *An example of irregular solution to a nonlinear Euler-Lagrange elliptic system with real analytic coefficients*. Annali della Scuola Normale Superiore di Pisa Classe delle Scienze, IV. Ser. 23, 57-67 (1996).

[12] Mingione G.: *Regularity of minima: an invitation to the Dark Side of the Calculus of Variations*. Applications of Mathematics, Vol. 51, No. 4, 355-426 (2006).

[13] Malý J., Ziemer W. P.: *Fine Regularity of Solutions of Elliptic Partial Differential Equations.* American Mathematical Society (1997).

[14] Moser J.K.: *A new proof of De Giorgi's Theorem concerning the regularity problem for elliptic differential equations.* Communications on Pure and Applied Mathematics 14, 577-591 (1961).

[15] Nash J.: *Continuity of Solutions of Parabolic and Elliptic Equations.* American Journal of Mathematics 80, 931-954 (1958).

[16] Necas J., John O., Stará J.: *Counterexample to the regularity of weak solution of elliptic systems.* Commentationes Mathematicae Universitatis Carolinae 21, 145-154 (1980).

[17] Romano A., Lancellotta R., Marasco A.: *Continuum Mechanics using Mathematica® - Fundamentals, Applications and Scientific Computing.* Birkhäuser Boston (2006).

[18] Rudin W.: *Real and Complex Analysis.* McGraw-Hill (1987).

[19] Stampacchia G., Équations elliptiques du second ordre á coefficients discontinus. Séminarie Jean Leray, n. 3, 1-77 (1963-1964).