Controllabilità moltiplicativa per equazioni diffusive nonlineari

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WORKSHOP
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Outline

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1. Introduction
   - Control theory
   - Additive vs multiplicative controllability
   - Motivations: Energy balance models in climatology
2. Multiplicative controllability
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   - State of art: Nonnegative controllability
   - 1-D reaction-diffusion equations with sign change
     - Main ideas for the proof of the main result
   - m-D reaction-diffusion equations with radial symmetry
     - Problem formulation and main results
3. Open problems

Outline

Control theory & Reaction-diffusion equations

$\Omega \subseteq \mathbb{R}^m$ bounded

$$\begin{cases} u_t \; = \; \Delta u \; + \; v(x,t)u \; + \; f(u) & \text{in} \quad Q_T := \Omega \times (0,T) \\ u_{|\partial\Omega} = 0 & t \in (0,T) \\ u_{|t=0} \; = u_0 & \end{cases} \tag{1}$$

$v \in L^\infty(Q_T)$, $f : \mathbb{R} \to \mathbb{R}$ Lipschitz, $\exists f'(0)$ and $f(0) = 0$.

Well-posedness result

$$u_0 \in L^2(\Omega) \Longrightarrow \exists! u \in L^2(0,T; H_0^1(\Omega)) \cap C([0,T]; L^2(\Omega));$$

$$u_0 \in H_0^1(\Omega) \Longrightarrow u \in H^1(0,T; L^2(\Omega)) \cap C([0,T]; H_0^1(\Omega)) \cap L^2(0,T; H^2(\Omega)).$$

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$$\begin{cases} u_t = \Delta u + v(x,t)u + f(u) & \text{in} \quad Q_T := \Omega \times (0,T) \\ u_{|\partial\Omega} = 0 & t \in (0,T) \\ u_{|t=0} = u_0 & \end{cases} \tag{1}$$

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Controllability: linear case ($f(u)=0$)

$$\begin{cases} u_t = \Delta u + vu + h(x,t)1\!\!1_\omega \\ u_{|\partial\Omega} = 0 \qquad (\omega \subset \Omega) \\ u_{|t=0} = u_0 \end{cases}$$

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Additive controls (locally distributed source terms)

Boundary controls

Bilinear controls (global!) (multiplicative controllability)

Definition (Exact controllability)

$\forall u_0 \in H_0, u^* \in H^*, (H_0, H^* \subseteq L^2(\Omega)), \exists$ *"a control function"*, $T > 0$ *such that* $u(\cdot, T) = u^*$.

Definition (Approximate controllability)

$\forall u_0 \in H_0, u^* \in H^*, (H_0, H^* \subseteq L^2(\Omega)), \forall \varepsilon > 0, \exists$ *"a control function"*, $T > 0$ *such that*

$$\|u(\cdot, T) - u^*\|_{L^2(\Omega)} < \varepsilon.$$

Regularizing effect of the heat equation and obstruction to exact controllability: $H^* \subset H_0 = L^2(\Omega)$:

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Additive vs multiplicative controllability

Additive controls

$$u_t - \Delta u = v(x,t)u + h(x,t)1\!\!1_\omega, \qquad \omega \subset \Omega$$

$\forall u_0 \in L^2(\Omega), u^* \in H(\text{“suitable”}H \subset L^2(\Omega)), \exists \omega \subseteq \Omega, h, T > 0$ such that $u(\cdot, T) = u^*$.

Reference

H. Fattorini, D. Russell Exact controllability theorems for linear parabolic equations in one space dimension Arch. Rat. Mech. Anal., **4**, *(1971) 272–292*

Multiplicative controllability and Applied Mathematics

$$\Rightarrow \quad \text{rather than} \qquad \downarrow$$

$$u_t - \Delta u = \quad v(x,t) \quad u + h(x,t)$$

use $\quad\uparrow\quad$ as control variable

Remark

$$\Phi : \text{“control”} \longmapsto \text{“solution”}$$

Additive controls — *vs* — *Bilinear controls*

$\Phi : h \longmapsto u$ *is a linear map;* $\Phi : v \longmapsto u$ *is a nonlinear map.*

Additive vs multiplicative controllability

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Additive controllability by a duality argument (J.L. Lions, 1989): observability inequality and Hilbert Uniqueness Method (HUM).

Reference

P. Baldi, G.F., E. Haus, Exact controllability for quasi-linear perturbations of KdV, To appear on Analysis & PDE.

- nonlinear problem: Nash-Moser theorem (Hörmander version);
- controllability of the linearized problem;
- observability inequality by classical Ingham inequality

Outline

The Budyko-Sellers model

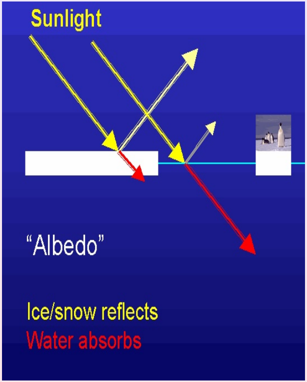
$\mathcal{M}$ compact surface without boundary (typically S^2)

$$u_t - \Delta_{\mathcal{M}} u = R_a(t,x,u) - R_e(t,x,u)$$

where $u(t,x)$ = temperature distribution

- $R_a(x,u) = Q(t,x)\beta(u)$ $\begin{cases} Q = \text{insolation function} \\ \beta = \text{coalbedo} = 1 - \text{albedo} \end{cases}$
-
  - $R_e(x,u) = A(t,x) + B(t,x)u$ Budyko
  - $R_e(x,u) \simeq c\, u^4$ Sellers

Albedo and Coalbedo



- Budyko

$$\beta(u) = \begin{cases} \beta_0 & u < -10 \\ [\beta_0, \beta_1] & u = -10 \\ \beta_1 & u > -10 \end{cases}$$

- Sellers

$$\beta(u) = \begin{cases} \beta_0 & u < u_- \\ \text{line} & u_- \leq u \leq u_+ \\ \beta_1 & u > u_+ \end{cases}$$

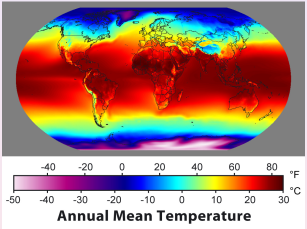
$$u_\pm = -10 \pm \delta$$

One-dimensional BS model

on $\mathcal{M} = \Sigma^2$

$$\Delta_{\mathcal{M}} u = \frac{1}{\sin\phi}\left\{\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) + \frac{1}{\sin\phi}\frac{\partial^2 u}{\partial\lambda^2}\right\}$$

ϕ = colatitude λ = longitude



taking average at $x = \cos\phi$ BS model reduces to

$$\begin{cases} u_t - \left((1-x^2)u_x\right)_x = g(t,x)\,h(u) + f(t,x,u) & x \in]-1,1[\\ (1-x^2)u_x|_{x=\pm 1} = 0 \end{cases}$$

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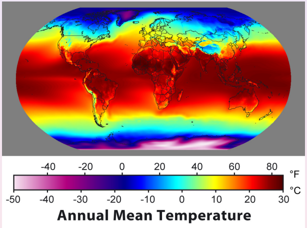

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Progetti di Ricerca GNAMPA coordinati:

- **GNAMPA 2014**: "Controllo moltiplicativo per modelli diffusivi nonlineari".
  Membri: P. Cannarsa (Università di Roma "Tor Vergata"), A. Porretta (Università di Roma "Tor Vergata"), E. Priola (Università di Torino), A. Cutrì (Università di Roma "Tor Vergata"), E.M. Marchini (Politecnico di Milano), C. Pignotti (Università di L'Aquila), R. Guglielmi (University of Bayreuth, Germany).
- **GNAMPA 2015**: "Analisi e controllo di equazioni a derivate parziali nonlineari" .
  Membri: P. Cannarsa (Università di Roma "Tor Vergata"), F. Bucci (Università di Firenze), A. Cutrì (Università di Roma "Tor Vergata"), G. Fragnelli (Università di Bari), C. Pignotti (Università di L'Aquila), R. Guglielmi (Ricam, Università di Linz, Austria), T. Scarinci (Università di Roma "Tor Vergata" & Paris 6).

"Progetto Premiale 2012" del Miur "La Matematica per la società e l'innovazione tecnologica", CNR-INdAM

- Dr. Roberto Natalini - Prof.Tommaso Ruggeri (coordinatori)
- Dr.ssa Daniela Mansutti - Prof. Piermarco Cannarsa (modelli differenziali climatici)

References on multiplicative controllability

Some references on bilinear control of PDEs

- Ball, Marsden and Slemrod (1982)
  [rod and wave equation]
- Coron, Beauchard, Boscain
  [Scrödinger equation]
- Fernández (2001), Lin, Gao and Liu (2006)
  [parabolic equations]
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Outline

Strong maximum principle

$f(0) = 0$, $f(u)$ is differentiable at 0 $\Longrightarrow$ $\frac{f(u)}{u} \in L^\infty(Q_T)$

$$u_t = \Delta u + \left(v + \frac{f(u)}{u}\right) u$$

Thus, the strong maximum principle (SMP) can be extend to semilinear parabolic system (1).

Definition (Approximate controllability)

An evolution system is called globally approximately controllable, if any initial state u_0 in H_0 can be steered into any neighborhood of any target state $u^* \in H^*$ at time T, by a suitable control.

SMP and obstruction to multiplicative controllability: $H^* \neq H_0^1(\Omega)$

$$u_0(x) = 0 \Longrightarrow u(x,t) = 0$$

$$u_0(x) \geq 0 \Longrightarrow u(x,t) \geq 0$$

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Definition

We say that the system (1) *is nonnegatively globally approximately controllable in* $L^2(\Omega)$, *if for every* $\eta > 0$ *and for any nonnegative* $u_0,\ u^* \in L^2(\Omega)$, *with* $u_0 \neq 0$ *there are a* $T = T(\eta, u_0, u^*) \geq 0$ *and a bilinear control* $v \in L^\infty(Q_T)$ *such that for the corresponding solution* u *of* (1) *we obtain*

$$\|u(T,\cdot) - u^*\|_{L^2(\Omega)} \leq \eta\,.$$

Reference

P. Cannarsa, G. F., Approximate multiplicative controllability for degenerate parabolic problems with robin boundary conditions, , CAIM, (2011).

Reference

P. Cannarsa, G. F., Approximate controllability for linear degenerate parabolic problems with bilinear control, Proc. Evolution Equations and Materials with Memory 2010, vol. Sapienza Roma, 2011, pp. 19–36.

Reference

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Nonnegative controllability & Semilinear degenerate problems

$$\begin{cases} u_t - (a(x)u_x)_x = v(t,x)u + f(t,x,u) & \text{in } Q_T := (0,T)\times(-1,1) \\ \begin{cases} \begin{cases} \beta_0 u(t,-1) + \beta_1 a(-1)u_x(t,-1) = 0 & t\in(0,T) \\ \gamma_0\, u(t,1) + \gamma_1\, a(1)\, u_x(t,1) = 0 & t\in(0,T) \end{cases} & \text{(for } WDP) \\ a(x)u_x(t,x)|_{x=\pm 1} = 0 \qquad t\in(0,T) & \text{(for } SDP) \end{cases} \\ u(0,x) = u_0(x) & x\in(-1,1)\,. \end{cases}$$

$$a \in C^0([-1,1]) \,:\, a(x) > 0 \,\forall\, x\in(-1,1),\; a(-1) = a(1) = 0$$

We distinguish two cases:

- ⋆ $\frac{1}{a} \in L^1(-1,1)$ (WDP);
- ⋆ $\frac{1}{a} \notin L^1(-1,1)$ (SDP).

Initial states that change sign:

- 1-D degenerate equations (work in progress with C. Nitsch and C. Trombetti).

Outline

Changes of sign

Reference

P. Cannarsa, G. F., A. Y. Khapalov Multiplicative controllability for semilinear reaction-diffusion equations with finitely many changes of sign , To appear on Journal de Mathématiques Pures et Appliquées, ArXiv: 1510.04203.

$$\begin{cases} u_t = u_{xx} + v(x,t)u + f(u) & \text{in } Q_T = (0,1)\times(0,T), \\ u(0,t) = u(1,t) = 0, & t \in (0,T), \\ u\mid_{t=0} = u_0 \in H_0^1(0,1). \end{cases} \tag{2}$$

We assume that $u_0 \in H_0^1(0,1)$ has *finitely many zeros*, that is, there exist points $0 = x_0^0 < x_1^0 < \cdots < x_n^0 < x_{n+1}^0 = 1$ such that

$$u_0(x) = 0 \iff x = x_l^0, \quad l = 0, \ldots, n+1.$$

$$u_0(x)u_0(y) < 0, \quad \forall x \in \left(x_{l-1}^0, x_l^0\right), \forall y \in \left(x_l^0, x_{l+1}^0\right).$$

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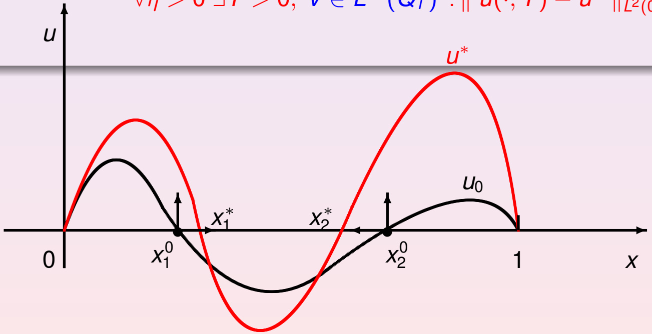
Let $u_0 \in H_0^1(0,1)$ have finitely many points of sign change.

Theorem (P. Cannarsa, G.F., A.Y. Khapalov, JMPA)

Consider any $u^* \in H_0^1(0,1)$ which has exactly as many points of sign change in the same order as u_0. Then,

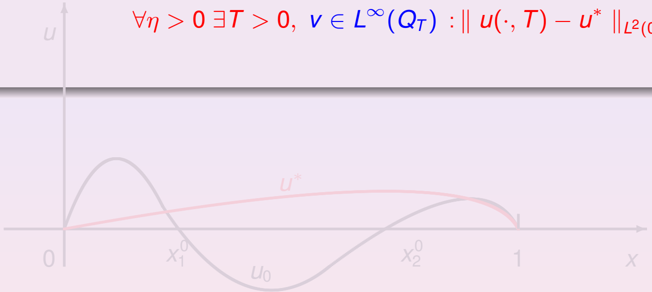
$$\forall \eta > 0 \; \exists T > 0, \; v \in L^\infty(Q_T) \; : \| \, u(\cdot, T) - u^* \, \|_{L^2(0,1)} \leq \; \eta.$$



Corollary (P. Cannarsa, G.F., A.Y. Khapalov, JMPA)

Consider any $u^* \in H_0^1(0,1)$, whose amount of points of sign change is less than or equal to the amount of such points for u_0 and this points are organized in any order of sign change. Then,

$$\forall \eta > 0\ \exists T > 0,\ v \in L^\infty(Q_T)\ : \| u(\cdot, T) - u^* \|_{L^2(0,1)} \leq \eta.$$



Reference

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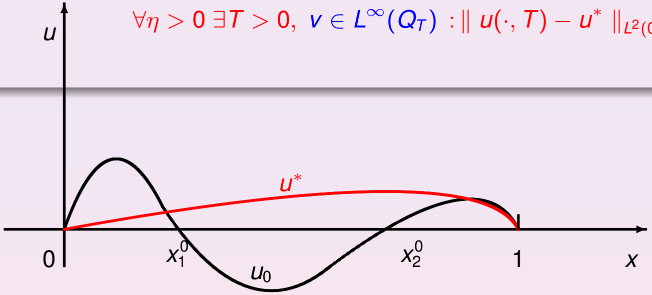

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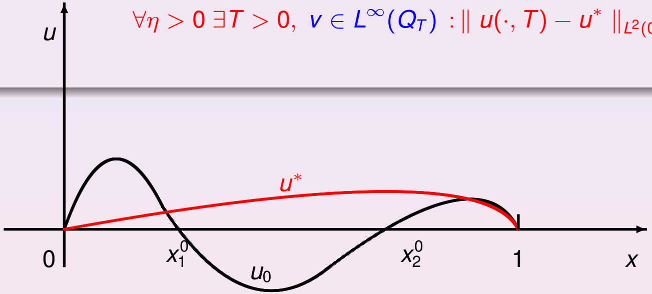

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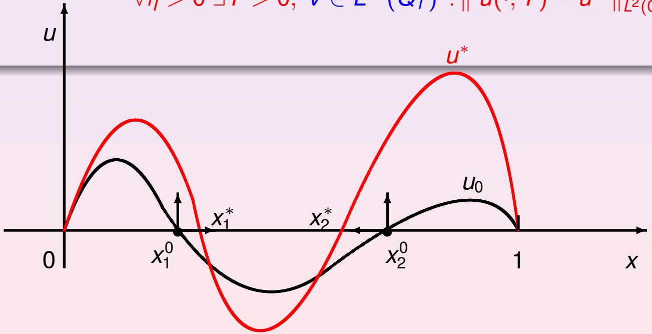
Let $u_0 \in H_0^1(0,1)$ have finitely many points of sign change.

Theorem (P. Cannarsa, G.F., A.Y. Khapalov, JMPA)

Consider any $u^* \in H_0^1(0,1)$ which has exactly as many points of sign change in the same order as u_0. Then,

$$\forall \eta > 0 \ \exists T > 0, \ v \in L^\infty(Q_T) \ : \| \ u(\cdot, T) - u^* \ \|_{L^2(0,1)} \leq \ \eta.$$



Control strategy

Given $N \in \mathbb{N}$, we consider the following partition of $[0, T_N]$ in $2N$ intervals:

$$[0, S_1] \cup [S_1, T_1] \cup \cdots \cup [T_{k-1}, S_k] \cup [S_k, T_k] \cup \cdots \cup [T_{N-1}, S_N] \cup [S_N, T_N].$$

$$v_1 \neq 0 \quad 0 \quad \cdots \quad v_k \neq 0 \quad 0 \quad \cdots \quad v_N \neq 0 \quad 0$$

Construction of the zero curves

- On $[S_k, T_k]$ $(1 \leqslant k \leqslant N)$ we use of the Cauchy datum w_k in

$$\begin{cases} w_t \;=\; w_{xx} \;+\; f(w), & \text{in} \quad (0,1)\times[S_k,T_k], \\ w(0,t) = w(1,t) = 0, & t\in[S_k,T_k], \\ w\mid_{t=S_k} = w_k(x), & w_k''(x)\mid_{x=0,1} = 0, \end{cases}$$

  as a control parameter to be chosen to generate the curves of sign change

- The ℓ-th curve of sign change $(1 \leqslant \ell \leqslant n)$ is given given by solution ξ_ℓ^k

$$\begin{cases} \dot{\xi}_\ell(t) \;=\; -\frac{w_{xx}(\xi_\ell(t),t)}{w_x(\xi_\ell(t),t)}, & t\in[S_k,T_k] \\ \xi_\ell(S_k) = x_\ell^k \end{cases}$$

  where the x_ℓ^k's are the zeros of w_k and so $w(\xi_\ell^k(t),t) = 0$

Construction of the bilinear control

- To fill the gaps between two successive $[S_k, T_k]$'s, on $[T_{k-1}, S_k]$ we construct v_k that steers the solution of

$$\begin{cases} u_t = u_{xx} + v_k(x,t)u + f(u), & \text{in } (0,1)\times[T_{k-1}, S_k], \\ u(0,t) = u(1,t) = 0, & t \in [T_{k-1}, S_k], \\ u\ |_{t=T_{k-1}} = u_{k-1} + r_{k-1}, & \end{cases}$$

from $u_{k-1} + r_{k-1}$ to w_k, where u_{k-1} and w_k have the same points of sign change, and $\|r_{k-1}\|_{L^2(0,1)}$ is small.

Closing the loop

- The distance-from-target function satisfies for some $C_1, C_2 > 0$ and $0 < \theta < 1$

$$J(\{S_k\},\{T_k\}) = \sum_{\ell=1}^{n} |\xi_\ell^N(T_N) - x_\ell^*|$$
$$\leqslant \quad \sum_{\ell=1}^{n} |x_\ell^0 - x_\ell^*| + C_1 \sum_{k=1}^{N} \frac{1}{k^{1+\frac{\vartheta}{2}}} - C_2 \sum_{k=n+1}^{N} \frac{1}{k}$$

- So the distances of each branch of the null set of the solution from its target points of sign change decreases at a linear-in-time rate while the error caused by the possible displacement of points already near their targets is negligible
- This ensures that $J(\{S_k\},\{T_k\}) < \epsilon$ within a finite number of steps

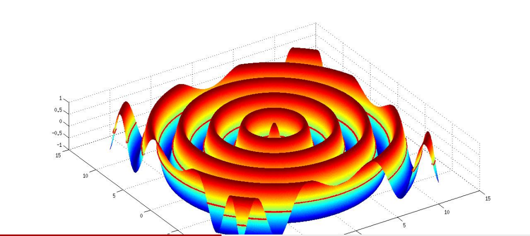
Outline

m-d radial case

$$\Omega = \{x \in \mathbb{R}^m : |x| = \sqrt{x_1^2 + \ldots + x_m^2} \leq 1\}$$

$$\begin{cases} u_t = \Delta u + v(x,t)u + f(u) & \text{in} \quad Q_T := \Omega \times (0,T) \\ u_{|\partial\Omega} = 0 & t \in (0,T) \\ u_{|t=0} = u_0 & \end{cases}$$

u_0 and $v(\cdot, t)$ radial functions. Moreover, all possible lines of change of sign of u_0 are circles with center at the origin.



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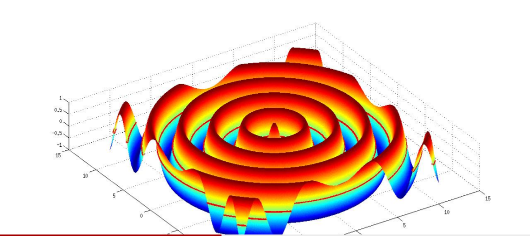

Main results

Radial Assumption holds

INITIAL STATES: All possible lines of change of sign of u_0 are circles with center at the origin.

Theorem (G.F.)

Let $u_0 \in H^2(\Omega) \bigcap H_0^1(\Omega)$. Assume that $u^* \in H^2(\Omega) \bigcap H_0^1(\Omega)$ has as many lines of change of sign in the same order as $u_0(x)$. Then,

$$\forall \varepsilon > 0 \, \exists T > 0, v \in L^\infty(Q_T) \text{ such that } \| u(\cdot, T) - u^* \|_{L^2(\Omega)} < \varepsilon.$$

Corollary (G.F.)

The result of Theorem extends to the case when u^* has a lesser amount of change of lines of signs which can be obtained by merging those of u_0.

u_0 and $v(\cdot, t)$ radial functions:

$$u_0(x) = z_0(r) \quad \text{and} \quad v(x,t) = V(r,t) \qquad \forall x \in \Omega, \ \forall t \in [0,T]$$

where $r = |x|$. Then,

$$\begin{cases} z_t = z_{rr} + \frac{m-1}{r} z_r + V(r,t)z + f(z) & \text{in} \quad (0,1) \times (0,T) \\ \lim\limits_{r \to 0^+} r^{\frac{m-2}{2}} z_r(0,t) = 0 = z(1,t) & t \in (0,T) \\ u_{|_{t=0}} = z_0. \end{cases}$$

z_0 has finitely many points of change of sign in $[0,1]$, that is, there exist points

$$0 = r_0^0 < r_1^0 < \cdots < r_n^0 < r_{n+1}^0 = 1$$

such that $\lim\limits_{r \to 0^+} r^{\frac{m-2}{2}} z'(r) = 0$ and

$$z_0(r) = 0 \quad \Longleftrightarrow \quad r = r_l^0, \quad l = 1, \ldots, n+1$$

$$z_0(r) z_0(s) < 0, \quad \forall r \in \left(r_{l-1}^0, r_l^0\right), \forall s \in \left(r_l^0, r_{l+1}^0\right) \qquad l = 1, \ldots, n.$$

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Open problems

- Initial states that change sign:
  - 1-D degenerate equations (work in progress with C. Nitsch and C. Trombetti).
- To investigate problems in higher space dimensions on domains with specific geometries.
  - m-D non-degenerate case and initial condition that change sign. (with N. Fusco)
  - m-D degenerate case with radial symmetry
- Periodic solutions of the thermostat problem (with C. Nitsch and C. Trombetti)
- To extend this approach to other nonlinear systems of parabolic type
  - Porous media equation (with C. Nitsch and C. Trombetti)
  - the equations of fluid dynamics and swimming models: "Swimming models for incompressible Navier-Stokes equations" (with P. Cannarsa and A. Y. Khapalov).
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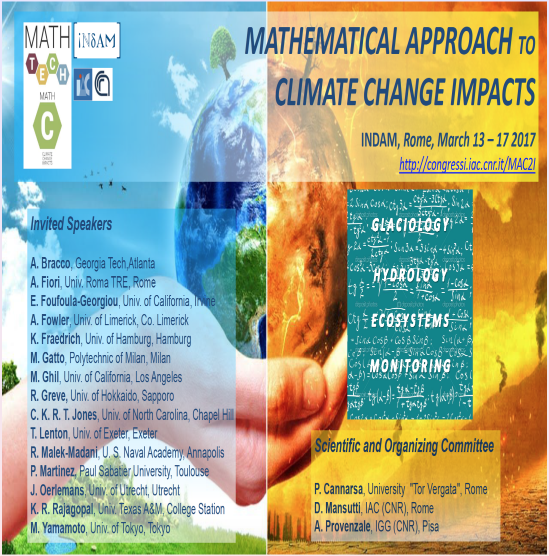

GRAZIE PER LA VOSTRA ATTENZIONE