Geometric Properties for Parabolic and Elliptic PDE's
4th Italian-Japanese Workshop

Palinuro (SA)

May 25th – 29th, 2015

ABSTRACTS of the TALKS

COUPLED PHYSICS INVERSE PROBLEMS AND JACOBIANS OF σ–HARMONIC MAPPINGS

GIOVANNI ALESSANDRINI
DIPARTIMENTO DI MATEMATICA E GEOSCIENZE
UNIVERSITÀ DEGLI STUDI DI TRIESTE

The coupling of different physical imaging modalities has provoked a sharp change in perspective in inverse boundary problems, which stimulated a remarkable shift in attention from problems with overdetermined boundary data (usually highly ill–posed) to inverse problems with interior data (potentially less ill–posed). Typically, this potential advantage is limited by the difficulty of being able to pick arrays of solutions of elliptic equations (the so–called σ–harmonic mappings) whose Jacobian is nontrivial. This issue poses challenging problems of geometrical character. I shall illustrate some recent results in the two–dimensional setting and shall discuss the difficulties encountered in higher dimensions. This is joint work with Vincenzo Nesi, Dipartimento di Matematica, Sapienza, Università di Roma.

Some symmetric boundary value problems and non-symmetric solutions

G. Arioli

Dipartimento di Matematica
Politecnico di Milano
P.zza L. da Vinci 32, 20133 Milano, ITALY
email: gianni.arioli@polimi.it

A classical result by Gidas, Ni, Nirenberg (1979) concerns the symmetry of positive solutions for the equation

$$-\Delta u(z) = f(z, u(z)), \quad \forall z \in \Omega, \qquad u(z) = 0, \quad \forall z \in \partial\Omega \qquad (*)$$

when the domain Ω is symmetric with respect to a $n-1$ dimensional hyperplane and it is convex, and the nonlinearity f has some monotonicity. Since then, many new results have been obtained concerning the symmetry of solutions of (*). As an example, Pacella and Weth proved in (2007) that f Ω is a ball or an annulus, f is radially symmetric, $\partial_u f$ is convex in u and a solution u of $(*)$ has Morse index $(\leq n)$, then it is symmetric.

We presents new results concerning the existence of non-symmetric solutions and the non-existence of symmetric solutions for (*).

A new notion of generalized solution in PDE's

Vieri Benci

April 8, 2015

In many problems of Geometry and Mathematical Physics, the notion of function is not sufficient and it is necessary to extend it. Among people working in partial differential equations, the theory of distribution of Schwartz and the notion of **weak solution** are the main tools to be used when equations do not have classical solutions. However, there are problems which do not have solutions, not even in the space of distributions. As model problem you may think of the Yamabe equation

$$-\triangle u = u^{p-1},\ u > 0,\ p \geq \frac{2N}{N-2} \tag{YE}$$

with Dirichelt boundary conditions in a bounded star-shaped open set.

Moreover, there are equations which have more than one weak solution even when uniqueness is expected. Usually, these equations do not have classical solutions since they develop singularities and the conservation laws are violated. As an example let us consider the Burgers' equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{BE}$$

A classical solution of the Cauchy problem, $u(t,x)$, is unique and, if it has compact support, it preserves the momentum $P = \int u\ dx$ and the energy $E = \frac{1}{2}\int u^2 dx$ as well as other quantities. However, at some time a singularity appears and the solution cannot be longer described by a smooth function. The notion of weak solution is necessary, but the problem of uniqueness becomes a central issue and the energy is not preserved.

Having these problems in mind, we construct a new class of functions called **ultrafunctions** in which the above problems have (generalized) solutions which behave as expected. In this construction, we apply the general ideas of Non Archimedean Mathematics and some techniques of Non Standard Analysis.

STRUCTURAL INSTABILITY OF NONLINEAR PLATES MODELLING SUSPENSION BRIDGES

ELVISE BERCHIO
DIPARTIMENTO DI SCIENZE MATEMATICHE
POLITECNICO DI TORINO

We model the roadway of a suspension bridge as a thin rectangular plate hinged on its short edges and free on its long edges. Then, we analyze the structural instability arising in the nonlinear fourth order hyperbolic problem which describes the dynamics of the bridge. With a finite dimensional approximation, two different kinds of oscillating modes are found: longitudinal and torsional. We prove that the system remains stable at low energies, whereas if the total energy is larger it may transfer from the longitudinal to the torsional components. An explanation of the precise mechanism which governs the energy transfer is also provided.

AN OVERDETERMINED PROBLEM FOR THE ANISOTROPIC CAPACITY

CHIARA BIANCHINI
DIPARTIMENTO DI MATEMATICA E INFORMATICA "U. DINI",
UNIVERSITÀ DEGLI STUDI DI FIRENZE

Abstract

A symmetry result is astablished for the solution to an overdetermined exterior problem which extends Serrin's exterior result [W. Reichel, Arch. Rational Mech. Anal. 1997] to the anisotropic case. The involved anisotropy arises from replacing the Euclidean norm of the gradient with an arbitrary norm in the associated variational integral, which corresponds to the Finsler Laplacian as governing operator in the differential problem. The resulting symmetry of the solution is that of the so-called Wulff-shape.

This is a joint work with G. Ciraolo (Università di Palermo) and Paolo Salani (Università degli Studi di Firenze).

SHARP BOUNDS FOR NEUMANN EIGENVALUES OF THE HERMITE OPERATOR

BARBARA BRANDOLINI

DIPARTIMENTO DI MATEMATICA E APPLICAZIONI "R. CACCIOPPOLI"
UNIVERSITÀ DEGLI STUDI DI NAPOLI FEDERICO II

Let Ω be a convex, possibly unbounded, domain of $\mathbb{R}^N$ and denote by $\mu_1(\Omega)$ the first nontrivial Neumann eigenvalue of the Hermite operator in Ω; we prove that

$$\mu_1(\Omega) \geq 1. \tag{1}$$

The estimate is sharp since equality sign holds if Ω is a N-dimensional strip. We observe that (1) can be viewed as an optimal Poincaré-Wirtinger inequality for functions belonging to the weighted Sobolev space $H^1(\Omega, d\gamma_N)$, where γ_N is the N-dimensional Gaussian measure.
When $N = 2$, under some additional assumptions on Ω, we study the equality case and we show that $\mu_1(\Omega) = 1$ if and only if Ω is any 2-dimensional strip.

Dirichlet-Neumann shape optimization problems

Giuseppe Buttazzo – University of Pisa

Abstract

We consider spectral optimization problems of the form

$$\min\Big\{\lambda_1(\Omega;D):\ \Omega\subset D,\ |\Omega|=1\Big\},$$

where D is a given subset of the Euclidean space $\mathbb{R}^d$. Here $\lambda_1(\Omega;D)$ is the first eigenvalue of the Laplace operator $-\Delta$ with Dirichlet conditions on $\partial\Omega\cap D$ and Neumann or Robin conditions on $\partial\Omega\cap\partial D$. The equivalent variational formulation

$$\begin{aligned}\lambda_1(\Omega;D)=\min\Big\{\int_\Omega|\nabla u|^2\,dx+k\int_{\partial D}u^2\,d\mathcal{H}^{d-1}\ :\\ u\in H^1(D),\ u=0\text{ on }\partial\Omega\cap D,\ \|u\|_{L^2(\Omega)}=1\Big\}\end{aligned}$$

reminds the classical drop problems, where the first eigenvalue replaces the total variation functional. We prove an existence result for general shape cost functionals and we show some qualitative properties of the optimal domains. The case of Dirichlet condition on a *fixed* part and of Neumann condition on the *free* part of the boundary is also considered.

Sobolev inequalities in arbitrary domains

Andrea Cianchi
Dipartimento di Matematica e Informatica "U.Dini", Università di Firenze
Viale Morgagni 67/A, 50134 Firenze, Italy , e-mail: cianchi@unifi.it

A theory of Sobolev inequalities in arbitrary open sets in $\mathbb{R}^n$ is offered. Boundary regularity of domains is replaced with information on boundary traces of trial functions and of their derivatives up to some explicit minimal order. The relevant Sobolev inequalities involve constants independent of the geometry of the domain, and exhibit the same critical exponents as in the classical inequalities on regular domains. Our approach relies upon new representation formulas for Sobolev functions, and on ensuing pointwise estimates which hold in any open set. This is a joint work with V. Maz'ya.

THE ELASTICA PROBLEM UNDER AREA CONSTRAINT

VINCENZO FERONE
UNIVERSITÀ DI NAPOLI "FEDERICO II" - ITALY

ABSTRACT. We discuss the problem of minimizing the elastic energy $E(\gamma)$ of a closed curve γ among all plane simple regular closed curves of given enclosed area $A(\gamma)$, and we show that the minimum is attained for a circumference. Possible extensions of the result to more general functionals of the curvature will also be presented. This a joint work with B. Kawohl and C. Nitsch.

SYMMETRIZATION WITH RESPECT TO THE ANISOTROPIC PERIMETER AND APPLICATIONS

NUNZIA GAVITONE

UNIVERSITÀ DEGLI STUDI DI NAPOLI FEDERICO II,
DIPARTIMENTO DI MATEMATICA E APPLICAZIONI "R. CACCIOPPOLI"

Abstract. In this talk we introduce a new type of symmetrization, which preserves the anisotropic perimeter of the level sets of a suitable concave smooth function, in order to prove sharp comparison results for solutions of a class of homogeneous Dirichlet fully nonlinear elliptic problems of second order and for suitable anisotropic Hessian integrals.

On the variational problem associated with the Hardy inequalities involving a mean oscillation

Norisuke Ioku
Ehime University

The aim of the talk is to present some results on sharp constants in Hardy inequalities for a mean oscillation:

$$C_{N,p} \int_{\mathbb{R}^N} \frac{\left(\tilde{u^\sharp}(x) - u^\sharp(x)\right)^p}{|x|^p} dx \leq \int_{\mathbb{R}^n} |\nabla u(x)|^p \, dx \tag{1}$$

for every $u \in W^{1,p}(\mathbb{R}^N)$, where $N \geq 2$, $1 \leq p < \infty$, $u^\sharp$ is a radially symmetric non-increasing rearrangement of u, and $\tilde{u^\sharp}(x)$ is a mean integral of $u^\sharp$ on $B_{|x|}(0)$. The inequality without its sharp constant essentially proved by Alvino-Trombetti-Lions [1] and Kolyada [4]. It is known that the inequality (1) implies Sobolev-Lorentz embeddings $W^{1,p}(\mathbb{R}^N) \subset L^{p^*,p}(\mathbb{R}^N)$ if $1 \leq p < N$, and the critical log-Hardy inequality if $p = N$.

In this talk, we consider sharp constants of (1) and related variational problems. This is a joint work with Professor Michinori Ishiwata (Osaka University).

References

[1] Alvino, A., Trombetti, G., Lions, P.-L., *On optimization problems with prescribed rearrangements*, Nonlinear Anal. **13** (1989), 185–220.

[2] Ioku, N., *Sharp Sobolev inequalities in Lorentz spaces for a mean oscillation,* J. Funct. Anal. **266** (2014), 2944–2958.

[3] Ioku, N., Ishiwata, M., *On the variational problem associated with the Hardy inequalities involving a mean oscillation,* preprint.

[4] Kolyada, V. I., *Estimates of rearrangements and embedding theorems*, Mat. Sb. **136** (1988), 3–23, (Russian); English transl., Math. USSR-Sb. **55** (1989), 1–21.

e-mail: ioku@ehime-u.ac.jp

Metastability for quasi-linear parabolic equations with drift

Hitoshi Ishii
Faculty of Education and Integrated Arts and Sciences
Waseda University

I discuss metastability results for quasi-linear parabolic PDE with a drift term. M. Freidlin and L. Koralov (ArXiv:0903.0428v2(2012), Probab. Theory Related Fields (2000)) have investigated the asymptotic behavior of solutions $u^\varepsilon(x,t)$ of the quasi-linear parabolic PDE

$$u_t^\varepsilon(x,t) = \sum_{i,j=1}^n a_{ij}(x, u^\varepsilon(x,t))u^\varepsilon_{x_ix_j}(x,t) + \sum_{i=1}^n b_i(x)u^\varepsilon_{x_i}(x,t) \quad \text{in } \ Q := \Omega \times (0,\, \infty).$$

with the Dirichlet condition $u^\varepsilon(x,t) = g(x)$ on the parabolic boundary $\partial_{\mathrm{p}}Q$, where Ω is a smooth bounded domain in $\mathbb{R}^n$, and have established some results by the probabilistic methods. The results are concerned with the asymptotic behavior of $u^\varepsilon(x,t)$ in the logarithmic time scale, that is, the behavior of $u^\varepsilon(x, \exp(\lambda/\varepsilon))$ as $\varepsilon \to 0$ for each fixed $\lambda > 0$. I present a PDE approach to the same asymptotic problem, which is based on joint work with P. E. Souganidis of Chicago University.

On the solvability of maximizing problems associated with Sobolev type embeddings in $\mathbb{R}^N$

Michinori Ishiwata
Osaka University

In this talk, we consider the attainability of a maximizing problem

$$D := \sup_{\|u\|_{H_\gamma^{1,N}}=1} \left(\|u\|_N^N + \alpha \|u\|_p^p \right),$$

where $N \geq 2$, $N < p < \infty$, $\alpha > 0$, $\gamma > 0$ and $\|u\|_{H_\gamma^{1,N}} = \left(\|u\|_N^\gamma + \|\nabla u\|_N^\gamma \right)^{\frac{1}{\gamma}}$. This problem has a close relation with the variational problem associated with the Trudinger-Moser type inequality in $\mathbb{R}^N$ and it turns out that the existence of a maximizer for D is closely related with the exponent γ and α. We give an existence and a nonexistence result for the variational problem.

This is a joint work with H. Wadade, Kanazawa university.

References

[1] M. Ishiwata, *Existence and nonexistence of maximizers for variational problems associated with Trudinger-Moser type inequalities in $\mathbb{R}^N$*, Math. Ann. **351** (2011), 781–804.

[2] M. Ishiwata, H. Wadade, *On a maximizing problem associated with the Sobolev type embeddings in $\mathbb{R}^N$ with lower order terms under equivalent constraints*, submitted.

[3] Y. Li and B. Ruf, *A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^n$*, Indiana Univ. Math. J. **57** (2008), 451–480.

[4] B. Ruf, *A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^2$*, J. Funct. Anal. **219** (2005), 340–367.

Solutions with time-dependent singularities for semilinear parabolic equations

Toru Kan
Tokyo Institute of Technology, Japan

For semilinear parabolic equations with a power nonlinearity, we consider the existence and the behavior of solutions with time-dependent singularities. Here, a time-dependent singularity means a singularity in space the position of which depends on time. Under some condition for the nonlinearity, we construct a solution having time-dependent singularity and show that the singularity is weaker than or equal to that of the fundamental solution of Laplace's equation for almost all time.

This is a joint work with Jin Takahashi (Tokyo Institute of Technology).

When does the heat equation have a solution with a sequence of similar level sets?

Tatsuki Kawakami
Osaka Prefecture University

Abstract. Consider the unique bounded solution $u = u(x,t)$ of the Cauchy problem for the heat equation:

$$\partial_t u = \Delta u \quad \text{in} \quad \mathbb{R}^N \times (0,+\infty), \qquad u = g \geq 0 \quad \text{on} \quad \mathbb{R}^N \times \{0\}, \tag{1}$$

where $N \geq 1$ and g is a non-trivial bounded non-negative function with compact support. It is well known that if g is radially symmetric, then the solution u of (1) must be radially symmetric.

The overdetermined problems, which determine the shape of solutions by using some additional information of solutions, are interesting ones in the study of qualitative properties of solutions of partial differential equations. In [2, Corollary 3.2], problem (1), where g is replaced by a characteristic function of a bounded open set, is considered, and it is shown that if there exists a non-empty stationary isothermic surface of u, then u must be radially symmetric. In this talk we consider another type of overdetermination. Precisely we consider the Cauchy problem (1) which has a solution with a certain sequence of similar level sets.

This is a joint work with Shigeru Sakaguchi (Tohoku University).

References

[1] T. Kawakami and S. Sakaguchi, When does the heat equation have a solution with a sequence of similar level sets?, to appear in Ann. Mat. Pura Appl.

[2] R. Magnanini, J. Prajapat and S. Sakaguchi, Stationary isometric surfaces and uniformly dense domains, Trans. Amer. Math. Soc. **358** (2006), 4821–4841.

On the criteria for the stability of the axisymmetric CMC surfaces

Yoshihito Kohsaka

(Graduate School of Maritime Sciences, Kobe University)

The stability of the stationary surfaces for the geometric evolution equation

$$V = -\Delta_{\Gamma_t} H \tag{1}$$

will be studied. Here V is the normal velocity of the evolving surfaces Γ_t, H is the mean curvature of Γ_t, and Δ_{Γ_t} is the Laplace-Beltrami operator on Γ_t. The geometric evolution equation (1) is called the surface diffusion equation, which was first derived by Mullins [1] to model the motion of interfaces in the case that the motion of interfaces is governed purely by mass diffusion within the interfaces (for simplicity we set the diffusion constant to 1). The surface diffusion equation (1) is obtained as the H^{-1}-gradient flow of the area functional for the evolving surfaces Γ_t, so that (1) has a variational structure that the area of the surface decreases whereas the volume of the region enclosed by the surface is preserved. This provides the constant mean curvature surfaces (CMC surfaces) as the stationary surfaces for (1).

Our goal is to derive the criteria of the stability of the axisymmetric CMC surfaces (in this talk, catenoids, cylinders, and unduloids will be treated) via the linearized stability analysis to the surface diffusion equation (1).

References

[1] W. W. Mullins, *Theory of thermal grooving*, J. Appl. Phys., 28 (1957), 333-339.

THE EVOLUTION OF P-HARMONIC MAPS
– monotonicity estimate and regularity

Masashi Misawa
Kumamoto University, Japan

Abstract We study the regularity of the evolution of $p-$harmonic maps, called the $p-$harmonic map heat flows. Our main ingredient is a monotonicity like estimate for a scaled energy. Based on the monotnicity estimate, the criterion for uniform regularity estimates is established for regular $p-$harmonic map heat flows. As an application of the regularity criterion, the small data global existence of regular $p-$harmonic map heat flow is shown. The localization of regularity estimates may also discuss.

On a second order elliptic equation arising in astrophysics

Ederson Moreira dos Santos
Universidade de São Paulo - São Carlos - Brazil

In this talk, I will consider some problems involving an elliptic equation arising in astrophysics, the so-called Hénon equation. I will recall some results and present new contributions on the ground state solutions and least energy nodal solutions. Moreover, exploiting the symmetry of the problem, I will discuss about solutions that concentrate and blow up at points and around spheres as the concentration parameter tends to infinity.

FINITE TIME BLOW UP AND NON-UNIFORM BOUND FOR A SOLUTION TO THREE DIMENSIONAL DRIFT-DIFFUSION SYSTEM

Takayoshi Ogawa
Mathematical Institute, Tohoku University
Sendai 980-8578, JAPAN

1. Abstract

We consider the Cauchy problem for a system of drift-diffusion equation in three or higher space dimensions:

$$\begin{cases} \partial_t u - \Delta u + \kappa \nabla \cdot (u \nabla \psi) = 0, & t > 0, x \in \mathbb{R}^n, \\ -\Delta \psi = \mu u, \quad \kappa, \ \mu = \pm 1, & t > 0, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $u = u(t,x)$ denote the particle densities and ψ denotes the potential of the particle field. For simplicity we assume the coupling constants μ and κ are normalized as either 1 or -1. The large time behavior of the solution is well understood if the spacial dimension is two and the number of the compornent is one. If the dimension is higher, then detailed behavior of the solution is under the research. We show that under some condition on the initial data, the solution never stays in bounded. Namely either the solution blows up in a finite time or if the solution exists globally then the solution never remains bounded. We show this result under the less restriction on the initial data at spacially infinity. We use a generalized version of the Shanon-Fisher inequality to controle the negative part of the entropy functional.

References

[1] Biler, P., *Existence and nonexistence of solutions for a model of gravitational interaction of particles, III*, Colloq. Math., **68** (1995), 229–239.
[2] Corrias, L., Parthame, B., Zaag, H. *Global solutions of some chemotaxis and angiogenesis system in hight space dimensions*, Milan J. Math. **72** (2004), 1-28.
[3] Keller, E. F., Segel, L. A., *Initiation of slime mold aggregation viewed as an instability,* J. Theor. Biol., **26** (1970), 399-415.
[4] Kurokiba, M., Ogawa, T., *Finite time blow-up of the solution for a nonlinear parabolic equation of drift-diffusion type*, Differential Integral Equations, **16** (2003), 427-452.
[5] Kurokiba, M., Ogawa, T., *Two dimensional drift-diffusion system in a critical weighted space*, Differential Integral Equations, **28**, no. 7-8 (2015), 753-776.
[6] Nagai, T., *Blowup of non-radial solutions to parabolic-elliptic systems modeling chemotaxis in two-dimensional domains*, J. Inequal. Appl., **6** (2001), 37-55.
[7] Nagai, T., Ogawa, T., *Global existence of solutions to a parabolic-elliptic system of drift-diffusion type in* $\mathbb{R}^2$, Funck Eckvac., to appear.
[8] Senba, T., *Blowup behavior of radial solutions to Jäger-Luckhaus system in high dimensional domain*, Funkcilaj Ekvac. **48** (2005), 247-271.
[9] Senba, T., *Blowup in infinite time of radial solutions to parabolic-elliptic system in high-dimensional Euclidean spaces*, Nonlinear Anal. **70** (2009), 2549-2562.

CONVERGENCE TO EQUILIBRIUM OF GRADIENT FLOW DEFINED ON PLANAR CURVES

SHINYA OKABE

The L^2-gradient flow of curves for the total squared curvature, also called *elastic flow*, coupled with different boundary conditions, has been widely studied in the mathematical literature. Long time existence of the evolutions is generally obtained by the smoothing effect of the energy. Regarding the asymptotic behavior as $t \to \infty$, there are general results implying that the solution subconverges to a (possibly nonunique) stationary solution. However, there are few results proving the full convergence of solutions (that is, without passing to a subsequence), and they are mostly obtained in the case of closed curve. In [2, 3, 4], the full convergence is proved with the aid of an additional constraint, the so-called inextensible condition, while in [1, 5, 6] it follows from the uniqueness of the equilibrium state.

In this talk, first we consider the gradient flow of a general geometric functional defined on planar open curves with a natural boundary condition. The purpose of this talk is to prove the full convergence of solutions of the gradient flow under a weaker condition, namely that there are only finitely many equilibrium states at each prescribed energy level. Moreover, we apply the result to the elastic flow under some natural boundary conditions.

References

[1] G. Dziuk, E. Kuwert and R. Schätzle, *Evolution of elastic curves in $\mathbb{R}^n$: existence and computation*, SIAM J. Math. Anal. **33** (2002), 1228–1245.

[2] N. Koiso, *On the motion of a curve towards elastica*, Acta de la Table Ronde de Géométrie Différentielle, (1996), 403–436, Sémin. Congr. 1, Soc. Math. France, Paris.

[3] S. Okabe, *The motion of elastic planar closed curves under the area-preserving condition*, Indiana Univ. Math. J. **56** (2007), no. 4, 1871–1912.

[4] S. Okabe, *The dynamics of elastic closed curves under uniform high pressure*, Calc. Var. Partial Differential Equations **33** (2008), no. 4, 493–521.

[5] A. Polden, *Curves and Surfaces of Least Total Curvature And Fourth-Order Flows*, PhD Thesis, University of Tübingen (1996).

[6] Y. Wen, *Curve straightening flow deforms closed plane curves with nonzero rotation number to circles*, J. Differential Equations **120** (1995), no. 1, 89–107.

Mathematical Institute, Tohoku University, 980-8578 Sendai, Japan

E-mail address: okabe@math.tohoku.ac.jp

Dynamical approach to overdetermined problems in potential theory

Michiaki Onodera (Kyushu University)

A general scheme for deriving a deformation flow for a parametrized overdetermined problem is introduced. I will show how such a flow can be a strong analytical tool for deriving some properties of the original (stationary) overdetermined problem. In particular, the uniqueness of an admissible domain for the mean value formula for harmonic functions will be shown by clarifying the dynamical structure of the corresponding deformation flow.

IMPROVED HARDY INEQUALITIES IN A LIMITING CASE

FUTOSHI TAKAHASHI

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ which contains the origin and put $R = \sup_{x\in\Omega} |x|$. In this talk, we study the Hardy inequality in a limiting case:

$$\int_\Omega |\nabla u|^N dx \geq \left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u|^N}{|x|^N (\log \frac{Re}{|x|})^N} dx,$$

and its sharper version

$$\int_\Omega |\nabla u|^N dx \geq \left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u|^N}{|x|^N (\log \frac{R}{|x|})^N} dx,$$

both hold for any $u \in W_0^{1,N}(\Omega)$. It is known that the constant $\left(\frac{N-1}{N}\right)^N$ is optimal for both inequalities and never attained on $W_0^{1,N}(\Omega)$. In this talk, we show several improvements of the above inequalities by adding nonnegative terms to the right hand sides. This talk is based on recent joint works with Megumi Sano (Osaka City Univ.)

Department of Mathematics, Osaka City University & OCAMI, Sumiyoshi-ku, Osaka, 558-8585, Japan
E-mail address: futoshi@sci.osaka-cu.ac.jp

Entire solution to the generalized parabolic k-Hessian equation

Kazuhiro Takimoto

Department of Mathematics, Graduate School of Science, Hiroshima University
1-3-1 Kagamiyama, Higashi-Hiroshima city, Hiroshima 739-8526, Japan

E-mail : `takimoto@math.sci.hiroshima-u.ac.jp`

In the early 20th century, Bernstein proved the following theorem:

> *If $f \in C^2(\mathbb{R}^2)$ and the graph of $z = f(x, y)$ is a minimal surface in $\mathbb{R}^3$, then f is necessarily an affine function of x and y.*

This theorem gives the characterization of entire solutions to the minimal surface equation defined in the whole plane $\mathbb{R}^2$.

Later, Pogolerov proved this type of theorem for Monge-Ampère equation:

> *Let $u \in C^4(\mathbb{R}^n)$ be a convex soluton to* $\det D^2u = 1$ *in $\mathbb{R}^n$. Then u is necessarily a quadratic polynomial.*

Moreover, Bao, Chen, Guan and Ji extended this result to the so-called *k-Hessian equation* of the form $F_k(D^2u) = 1$ in $\mathbb{R}^n$ for $1 \le k \le n$. Here $F_k(D^2u)$ is defined by

$$F_k(D^2u) = S_k(\lambda_1, \ldots, \lambda_n), \tag{1}$$

where, for a C^2 function u, $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of D^2u, and S_k denotes the k-th elementary symmetric function. Laplace operator Δu and Monge-Ampère operator $\det D^2u$ correspond respectively to the special cases $k = 1$ and $k = n$ in (1).

In this talk, we shall obtain a Bernstein type theorem for the *parabolic k-Hessian equation* of the form

$$u_t = \rho\left(F_k(D^2u)^{\frac{1}{k}}\right) \quad \text{in } \mathbb{R}^n \times (-\infty, 0],$$

where $\rho : (0, \infty) \to \mathbb{R}$ is a function, under some assumptions.

This talk is based on a joint work with Saori Nakamori (Hiroshima University).

A quantitative version of a Theorem of Alexandrov

Luigi Vezzoni, Università di Torino

Abstract

A celebrated theorem of Alexandrov affirms that the only closed embedded constant mean curvature hypersurfaces in the Euclidean space are the round spheres. The talk mainly focus on the following quantitive version of the Alexandrov theorem:

Theorem [Ciraolo - V.]. *Let S be an n-dimensional, C^2-regular, connected, closed hypersurface embedded in $\mathbb{R}^{n+1}$. There exist constants ϵ, $C > 0$ such that if*

$$\operatorname{osc}(H) \leq \epsilon,$$

then there are two concentric balls B_{r_i} and B_{r_e} such that

$$S \subset \overline{B}_{r_e} \setminus B_{r_i},$$

and

$$r_e - r_i \leq C \operatorname{osc}(H).$$

The constants ϵ and C depend only on n and upper bounds on the C^2-regularity and the area of S.

The proof of the theorem makes use of a quantitive study of the method of the moving planes and the result implies a new pinching theorem for hypersurfaces in the Euclidean space. Furthermore, the theorem is optimal in a sense that it will be specified in the talk.

The last part of the talk will be about an on-going study of the stability of the Alexandrov theorem in the hyperbolic space.