Time-Dependent Seismic Reliability of Damage-Cumulating Non-Evolutionary Bilinear Systems

I. Iervolino & E. Chioccarelli  
Università degli Studi di Napoli Federico II, Italy.

M. Giorgio  
Seconda Università di Napoli, Italy.

SUMMARY:
A model for time-dependent reliability assessment of structures, subject to cumulating damage due to point seismic overloads, is presented. The deteriorating structural parameter is the capacity of the structure expressed in terms kinematic (monotonic) ductility to conventional collapse. A stochastic process is employed to model the progressive loss of capacity during a time interval of interest. The capacity reduction due to earthquake shocks is analysed for a simple bilinear structural system, for which earthquake damage increments are non-negative independent and identically distributed random variables. The widely used Gamma distribution is adopted to model damage increments. Some approximations are proposed to get handy expressions for the reliability function. Based on these, close-form solutions for life-cycle structural assessment are derived and discussed with respect to different possible knowledge levels. Finally, illustrative examples show applicability within the performance-based earthquake engineering framework.

Keywords: ductility, earthquakes, gamma distribution, deterioration stochastic processes.

1. INTRODUCTION

Variation in time of seismic structural risk may involve all the three components in which it is usually subdivided within the performance-based earthquake engineering (PBEE) framework, that is, hazard, vulnerability, and loss; e.g., Cornell and Krawinkler (2000). Time-dependency of seismic hazard is often considered to be related to non-stationary occurrence of earthquakes on individual faults, clustering of earthquake sequences, and fault interactions. On the other hand, when many sources contribute to hazard, the resulting occurrence process of earthquakes at a given site may show stationary increments, which is the basis of classical probabilistic seismic hazard analysis (PSHA) familiar to engineers (e.g., McGuire, 2004). Earthquake losses may be time-dependent mainly because of investment costs which require financial discounting (e.g., Yeo and Cornell, 2009). Seismic structural vulnerability, finally, is commonly considered to be affected by two categories of phenomena which may determine time-dependency: (1) continuous deterioration of material characteristics or ageing, and (2) cumulating damage because of repeated overloading due to earthquake shocks (e.g., Sanchez-Silva et al., 2011).

Continuous deterioration due to ageing is often related to aggressive environment which may worse mechanical features of structural elements. In fact, to be able to predict evolution of this kind of wear is especially important in design of maintenance policies for infrastructures such as bridges. Continuous deterioration may also show an effect in increasing seismic fragility (e.g., Rao et al., 2010).

Earthquake shocks potentially lead the hit structure to cumulate damage during its lifetime, unless partial or total restoration; i.e., within a cycle. The cumulative effect of shocks is seen as a different category with respect to ageing basically because it may be conveniently represented by a marked point process, as its occurrence is instantaneous with respect to the structural life. On the other hand, not all earthquake shocks are necessarily damaging. Moreover, loss of capacity is also random, following some distribution depending on the structure and its state at the moment of ground shaking. This kind of vulnerability is often evaluated by means of state-dependent fragility curves; e.g., Luco et al. (2004).
If damage accumulation is measured by means of structural seismic capacity reduction, for example the available ductility to collapse or $\mu(t)$, then the process may be susceptible of the graphical representation in Figure 1. In the figure, a certain threshold corresponding to a limit state of interest is also reported.

![Figure 1](image.png)

**Figure 1.** Seismic cycle representation for a structure subjected to repeated earthquake shocks, when degradation affects residual failure capacity expressed in terms of monotonic ductility to collapse.

Estimation of the probability that the structural capacity falls from the initial value (e.g., *as-new*) to the threshold, is the complement to one of the structural reliability within the life-cycle. To model such a risk is the subject of the study presented in the following.

It is investigated how to probabilistically model the cumulative earthquake effect on residual capacity for simple non-evolutionary single degree of freedom (SDoF) systems, for which it may be shown that reductions of residual seismic capacity are random, non-negative (i.e., shocks cannot result in damage relief), independent, and identically distributed random variables (RVs). Under these conditions, if the additional assumption of Gamma-distributed damage increments is tenable, handy solutions for the reliability assessment result.

The paper is structured such that formulation of the problem, in terms of the variables commonly used in the PBEE framework, is set first. Then, advantages of gamma representation of damage increments are discussed, and analytical expressions for marginal and conditional structural reliability are derived. Subsequently, the damage increment distribution is calibrated for a bilinear SDoF system. Finally, illustrative applications, considering a structure located in a comparatively high-seismicity region in central Italy, and earthquake occurrence following a homogeneous Poisson process (HPP), are carried out.
2. FORMULATION

The scalar capacity parameter considered to be affected by damage accumulation is residual kinematic ductility to collapse. Therefore, the degradation process is that in Eqn. (1.1), where $\mu_0$ is the initial capacity in the cycle, and $D(t)$ is the cumulated damage due to all earthquake events, $N(t)$, occurred until time $t$. Note that both damage in each event, $\Delta \mu_i$, and $N(t)$ are RVs.

$$\mu(t) = \mu_0 - D(t) = \mu_0 - \sum_{i=1}^{N(t)} \Delta \mu_i$$  \hspace{1cm} (1.1)

Given Eqn. (1.1), the probability that the structure fails within $t$, $P_f(t)$, or the complement to one of the structural reliability $R(t)$, is the probability that the structure passes a threshold related to a certain limit state, $\mu_{LS}$, Eqn. (1.2). In other words, it is the probability that in $(0,t)$ the capacity reduces travelling the distance between initial capacity and the threshold, $\mu$. Eqn. (1.2) also represents the cumulated probability function of the lifetime.

$$1 - R(t) = P_f(t) = P[\mu(t) \leq \mu_{LS}] = P[D(t) > \mu_0 - \mu_{LS}] = P[D(t) > \bar{\mu}]$$  \hspace{1cm} (1.2)

The failure probability may be written, via the total probability theorem, as in Eqn. (1.3). In the classical case, where the occurrence of seismic events is described by a HPP, $N(t)$ has a Poisson distribution with rate equal to $\lambda$. Considering the cumulated damage as dependent on the vector of the corresponding ground motion intensities, $IM$, of occurring earthquakes, such conditional probability may be computed as in Eqn. (1.3), where the integral is of $k$-th order.

$$P_f(t) = P[D(t) \geq \bar{\mu}] = \sum_{k=0}^{\infty} P[D(t) \geq \bar{\mu} | N(t) = k] \cdot P[N(t) = k] = \sum_{k=0}^{\infty} \int P[k] \cdot \sum_{i=1}^{k} \Delta \mu_i \geq \bar{\mu} | IM = im, N(t) = k \cdot f_{im}(im) \cdot d(im) \cdot \frac{(\lambda \cdot t)^k}{k!} \cdot e^{-\lambda \cdot t}$$  \hspace{1cm} (1.3)

2.1 Reliability under gamma-distributed damage increments

For earthquakes should be $\Delta \mu_i \geq 0 \forall i$, therefore, deterioration process in subsequent shocks should show non-negative increments. To model cumulative damage proceeding in one direction only, an attractive option is the Gamma distribution. In fact, although processes with other distributions (e.g., lognormal) may show non-negative increments, the main advantage of modelling deterioration processes through this distribution is that it often yields handy reliability solutions. The gamma probability density function (PDF) is given in Eqn. (2.1), where $\gamma$ is the scale parameter, $\alpha$ is the shape parameter, and $\Gamma$ is the gamma function.

$$f_{\Delta \mu} (\delta \mu) = \int_{im} f_{\Delta \mu} (\delta \mu | IM = im) \cdot f_{im}(im) \cdot d(im) = \frac{\gamma^\alpha (\gamma \cdot \delta \mu)^{\alpha-1}}{\Gamma(\alpha)} \cdot e^{-\gamma \cdot \delta \mu}$$  \hspace{1cm} (2.1)
Note that, such continuous RV is suitable to model only the damage in earthquakes which factually determine loss of capacity, not those events whose effect is not large enough (to follow). If increments of damage are independent and identically distributed\(^1\), \(f_{\lambda t_0} (\delta \mu) = f_{\lambda_0} (\delta \mu)\), given the reproductive property of the Gamma random variable the failure probability results as in Eqn. (2.2), where the term conditional to occurrence of \(k_D\) shocks is the incomplete gamma (for which tabular solutions exist). Finally, it is worth to remark that \(k_D\) refers to the filtered process of damaging earthquakes of parameter \(\lambda_D\).

\[
P_j (t) = P \left[ D(t) \geq \bar{\mu} \right] = \\
\sum_{k_D=0}^{\infty} \int P \left[ \sum_{i=1}^{k_D} \Delta \mu \geq \bar{\mu} \mid \lambda_D = k_D \right] \cdot f_{\lambda_D} (\bar{\mu}) \cdot d(\bar{\mu}) \cdot \frac{(\lambda_D \cdot t)^{k_D}}{k_D!} \cdot e^{-\lambda_D \cdot t} =
\]

\[
= \sum_{k_D=0}^{\infty} \sum_{\mu_i=0}^{\infty} \frac{\bar{\mu} \cdot (\bar{\mu} \cdot \delta \mu)^{k_D-\alpha-1}}{\Gamma (k_D - \alpha)} \cdot e^{-\bar{\mu} \cdot \delta \mu} \cdot d(\delta \mu) \cdot \frac{(\lambda_D \cdot t)^{k_D}}{k_D!} \cdot e^{-\lambda_D \cdot t}
\]

Therefore, the approximation of the failure probability in Eqn. (2.2) results as in Eqn. (2.3), recalling the expected number of events according to the HPP model.

\[
\begin{align*}
P \left[ D(t) \geq \bar{\mu} \right] & = E \left[ N_D (t) \right] = E \left[ \sum_{i=1}^{\infty} \Delta \mu \right] = \int_0^{\infty} \frac{\bar{\mu} \cdot (\bar{\mu} \cdot \delta \mu)^{t-\alpha-1}}{\Gamma (t - \alpha)} \cdot e^{-\bar{\mu} \cdot \delta \mu} \cdot d(\delta \mu) \\
E \left[ N_D (t) \right] & = \lambda_D \cdot t
\end{align*}
\]

2.2 Conditional probabilities

Formulations at the beginning of the section refer to marginal reliability in the \((0, t)\) interval. In fact, it may be the case the structure is inspected at \(t^*\), for example after an earthquake felt in the region where the structure is located, and the current residual capacity is measured, \(\mu (t^*)\). In this case, the failure probability conditional to inspection has the same expression of above, just, replacing \(\bar{\mu}\) and \(t\) of Eqn. (1.2), with \(\bar{\mu} = \mu (t^*) - \mu_{i_S}\) and \(t - t^*\), respectively, as in Eqn. (2.4). This is because the structure has now to undergo a smaller capacity reduction to fail.

The same relationship may be used if the residual capacity \(\bar{\mu}^*\) is obtained via a repair at \(t^*\), as it is equivalent to modify the observed residual capacity in the case of an inspection.

\[
P_j (t - t^*) \approx \int_0^{\infty} \frac{\bar{\mu} \cdot (\bar{\mu} \cdot \delta \mu)^{(t-t^*)-\alpha-1}}{\Gamma (t - t^*) - \alpha} \cdot e^{-\bar{\mu} \cdot \delta \mu} \cdot d(\delta \mu), \quad t \geq t^*
\]

Also interesting is the case in which one wants to include in the reliability assessment the information that the structure is still surviving at \(t^*\), but with unknown damage condition. It may be computed via, Eqn. (2.5), plugging in previous results.

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\(^1\) Which, in particular, implies increment of damage of a structure in an earthquake is independent of its state.
Finally, the probability of failure given survival at \( t' \) and the number, \( N(t') = k_D \), of damaging earthquakes until \( t' \), yields to Eqn.(2.6), where the last approximation is analogous to those taken above.

\[
P[\text{failure in } t > t' \mid \text{survival in } t'] = 1 - P[\text{survival in } t > t' \mid \text{survival in } t'] = 1 - \frac{P[\text{survival in } t > t' \cap \text{survival in } t']}{P[\text{survival in } t']} = 1 - \frac{R(t')}{R(t)} = 1 - \frac{1 - P_f(t)}{1 - P_f(t')}
\]

\[
(2.5)
\]

3. CUMULATIVE DAMAGE OF NON-EVOLUTIONARY BILINEAR SYSTEMS

The structural system considered is a non-evolutionary bilinear elastic-perfectly-plastic SDoF, with an elastic period equal to 0.5s. Weight is 100 kN and the yielding force \( F_y \) is equal to 4.17 kN, which correspond to a strength reduction factor equal to 6 when the mass acceleration is equal to 0.25g. Chosen engineering demand parameter (EDP) is the kinematic ductility \( \mu \) defined in terms of displacement: i.e., the maximum ductility demand during each shock. In fact, choosing such an EDP is equivalent to assume that the collapse of the structural model is expected for maximum plastic displacement, independent on the number of plastic cycles and the amount of dissipated energy (Cosenza and Manfredi, 2000). It is also assumed that damage accumulations, due to subsequent events, is monotonic; i.e., for each event, maximum ductility demand is associated in the same direction (Figure 2).

![Figure 2](image)

**Figure 2.** Accumulation of damage in shock sequence with respect to kinematic ductility for a bilinear elastic-perfectly-plastic system.
The considered limit state (LS) is collapse prevention (CP) derived from FEMA 356 (2000), which assumes conventional collapse of concrete structures when maximum drift ratio is equal to 0.04. For the considered system it corresponds to an initial ductility capacity \( \mu_0 \) equal to about 15.

Once structural system and limit state conditions are defined, it is possible to address the hypothesis of damage increments, or capacity reductions, which are independent and identically distributed for this particular case. In fact, due to its force-displacement relationship, considered SDoF has a deformation response, during a single earthquake, independent on the conditions at the beginning of the analysis, and, therefore, on the conditions in which the i-th shock finds the structure in. Thus, damage increments are i.i.d. for the considered SDoF, that is, a given earthquake produces the same amount of ductility demand, or equivalently of capacity reduction, independently of the structural conditions. Moreover, it also follows that a single set of incremental dynamic analyses (IDA) can be used for computing all necessary distributions of damages, as shown in the following section.

### 3.1 Distribution of damage increments

In this section estimation of distribution of kinematic ductility demand is addressed. IDA results are studied in terms of structural ductility response normalized by \( \mu_0 \) in a way that the ductility demand is equal to 1 when the CP-LS is attained. Thus, \( \Delta \mu \), the measure of structural damage increments, may be defined as in Eqn. (3.1), where \( \mu_{before} \) and \( \mu_{after} \) refer to ductility residual capacity before and after a generic earthquake shock. It is apparent that \( \Delta \mu \) varies between zero and one.

\[
\Delta \mu = \frac{\mu_{before} - \mu_{after}}{\mu_0} \tag{3.1}
\]

The PDF of \( \Delta \mu \) may be computed via Eqn. (3.2), where \( f_{IM}(im) \) is derived from the HPP hazard curve for the site where the structure is supposed to be located, assuming elastic spectral acceleration at the elastic period of the SDoF \( (Sa(0.5s)) \) as an IM. On the other hand, \( f_{\Delta \mu \mid IM = im} (\delta \mu | IM = im) \), which is the distribution of damage due to a given value of ground motion intensity, can be derived from IDAs.

\[
f_{\Delta \mu} (\delta \mu) = \int f_{\Delta \mu \mid IM = im} (\delta \mu | IM = im) \cdot f_{IM}(im) \cdot d(im) \tag{3.2}
\]

In fact, Eqn. (3.2) is not perfectly appropriate as \( \Delta \mu \) is not a continuous variable. Because not all earthquakes are damaging, its distribution has the segmented expression in Eqn. (3.3).

\[
P[\Delta \mu] = \begin{cases} 
P_0 & \Delta \mu = 0 \\
\frac{P_0}{f_{\Delta \mu} (\delta \mu)} \cdot d(\delta \mu) & \Delta \mu > 0
\end{cases} \tag{3.3}
\]

Ninety records for IDAs were selected via REXEL v 3.3 beta (Iervolino et al., 2012), with moment magnitude between 5 and 7, epicentral distances lower than 30km, and site class A according to Eurocode 8 (CEN, 2003); details may be found in Chioccarelli and Iervolino (2012).

Figure 3a shows IDA output in which recalling the normalization of ductility demand, collapse limit is equal to 1. It is to note that mean of \( \Delta \mu \) is larger than zero only for values of spectral acceleration larger than about 0.4 m/s\(^2\), which is, in fact, the acceleration of yielding for the considered structure.
Figure 3b reports some distributions of damage increments conditional to ground motion intensity, \( f_{\Delta \mu|M} (\Delta \mu | IM = im) \). Marginalization of the distribution of damage increments with \( f_{IM} (im) \), as per Eqn. (3.2), is left to the application in the next section.

![Figure 3. Ductility demand from IDA analyses (a) and some distributions of structural damage conditional to ground motion intensity (b).](image)

### 4. ILLUSTRATIVE APPLICATION

Considered site is Sulmona (13.96 Lon.; 42.05 Lat.), in central Italy; PSHA was carried out by a FORTRAN software, specifically developed and described in Iervolino at al. (2011), to which the reader should refer for details. The distribution for the spectral ordinate corresponding to the SDoF elastic period, is reported in Figure 4a. Note that, it is not exactly the hazard curve, while it is the distribution of ground motion intensity given the occurrence of an earthquake (it was obtained by the hazard curve normalizing it with respect to the rate of occurrence of events at the site equal to 1.95). In fact, this is required to obtain the marginal distribution of capacity reduction as per Eqn. (3.2). In Figure 4b, the result of such a marginalization is reported.

As discussed in the previous section, the distribution of damage increments is characterized by two different applicability ranges: a continuous probability density function for \( \mu > 0 \) and a mass probability for \( \mu = 0 \). However, also \( \Delta \mu > 1 \) has an interesting meaning. In fact, \( P[\Delta \mu = 0] \) accounts for the probability earthquakes are not strong enough to damage the structure, while \( P[\Delta \mu > 1] \) is the marginal probability the structure fails. In this application mass probabilities associated to \( \Delta \mu = 0 \) and \( \Delta \mu > 1 \) are equal to 0.91 and 0.007 respectively, meaning that only 9% of earthquakes are expected to be damaging, while less than 1% is expected to be catastrophic; i.e., directly causing collapse.

The expected value of \( \Delta \mu \), \( E[\Delta \mu] = 0.015 \), is also reported in Figure 4b. This means about 1.5% average capacity reduction is expected in each earthquake for the considered structure at the considered site (considering both damaging and undamaging earthquakes). Referring to the seismic hazard of Sulmona, given that mean number of earthquakes in one year is equal to 1.95, considered structure is subjected to an average capacity reduction per year equal to 0.015 \times 1.95 = 0.03 and structural capacity drops to zero, on average, after about 30 years, according to the considered damage criterion.
Figure 4. Distribution of ground motion intensity given earthquake occurrence at the site of interest (a); marginal distribution of $\Delta \mu$ for the structure and the site of interest (b).

The continuous part of the distribution of capacity reduction is approximated by a gamma distribution calibrating its parameter via the mean and the variance computed from the application for the case $\Delta \mu > 0$, recalling that $E[\Delta \mu | Damage] = \alpha / \gamma$ and $Var[\Delta \mu | Damage] = \alpha / \gamma^2$. The resulting scale and shape parameters are, 0.521 and 0.094 respectively$^2$.

Given this gamma approximation it is possible to compute the failure probabilities in the following illustrative cases: (1) failure probability in 50 yr, Eqn. (2.3); (2) failure probability in the next 25 yr, given that 70% of as-new capacity has been measured in the last inspection, Eqn. (2.4); (3) failure probability in 25 yr given that no collapse was recorded in the first 25 yr, Eqn. (2.5); and (4) failure probability in the next 25 yr given that one earthquake hit the structure without causing collapse, Eqn. (2.6). Results are given in Eqn. (4.1), where it also recalled how the expected number of damaging earthquake is computed following the filtering of the all earthquakes HPP.

$$
\begin{align*}
(1) & \quad P_f(50) = 0.48 \\
(2) & \quad P_f(k_D=0.3, 25) = 0.48 \\
(3) & \quad P_f(k_D=0.3, 25, k_D > 0.3) = 0.31 \\
(4) & \quad P_f(k_D=0.3, 25, k_D < 0.3, n=1) = 0.26
\end{align*}
$$

(4.1)

$$
\begin{align*}
\lambda_D \cdot t = P_0 \cdot \lambda \cdot 0.09 \cdot 1.95 \cdot t
\end{align*}
$$

In Figure 5 the cumulative distribution function of the lifetime of the structure, $F_T(t)$, according to the model in Eqn. (2.2) is also reported.

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$^2$ The Gamma distribution obtained with these parameters, if plotted, appears as an ugly approximation of the distribution in Figure 4b, even if matching mean and variance. This may indicate that such damage data may be not well represented by this probability model and other functions may be required for this particular case. Nevertheless, to address this issue is outside the scope of the illustrative purposes of this work.
5. CONCLUSIONS

An analytical-form model for reliability analysis of structures cumulating seismic damage was introduced. The time-frame of interest is that of the seismic cycle, that is, the time between two subsequent structural replacements or restorations. The structural damage measure considered is the kinematic ductility to collapse. Homogeneous Poisson process is considered to model earthquake occurrence at the site of the structure.

The reliability assessment was formulated for non-evolutionary bilinear system, for which it was found that the capacity reduction (in terms of kinematic ductility) due to earthquakes is state-invariant; i.e., the damage increments are independent and identically distributed.

The stochastic approach considered makes use of the gamma distribution, which is especially suitable to represent cumulating damage in engineering systems because of its non-negative feature and reproductive characteristics. In fact, the first two moments as well as the whole distribution of fractional capacity reduction were retrieved, conditional to the input ground motion intensity, for a non-evolutionary bilinear SDoF system. Suitability of the Gamma model for this specific case, although relevant for the application, was left out of the study, which was mainly related to discussion of stochastic wear processes for seismic damage accumulation.

The illustrative application refers to the life-cycle analysis of a structure with respect to collapse prevention. Because damage is site-dependent, a relatively high seismicity site in central Italy was considered and distribution of ground motion intensity, given the occurrence of an earthquake, was retrieved.

It has, finally, to be discussed that the main assumption of the model is that reliability is not state-dependent, this means that it is not able, for example, to account properly for those cases in which the damage in a shock also depends on the state in which that specific seismic event finds the structure in. This may be the case of structures with evolutionary hysteretic behavior; e.g., stiffness- and/or strength-degrading. However, on one side, state-dependent models can be developed based on the same premises and, on the other hand, structural systems such as the one investigated herein, usually yield results of general earthquake engineering interest.

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REFERENCES


1. Second line in Eq. (4.1) should be \( P_{f|\mu(25)=0.3, \mu(50)=0.31} \)

2. Last line in Eq. (4.1) should be \( k_D = \lambda_D \cdot t = (1-P_0) \cdot \lambda \cdot t = 0.09 \cdot 1.95 \cdot t \)

3. Reference
