# TOWARD THE QUANTUM INTERNET: A DIRECTIONAL-DEPENDENT NOISE MODEL FOR QUANTUM SIGNAL PROCESSING

**(INVITED PAPER)**

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**ABSTRACT**

After decades of *pure science phase*, the research on quantum technologies is finally reaching the *engineering phase*, getting out of the labs into business reality. Quantum technologies relies on quantum bits, aka *qubits*, which are the equivalent of classical bits used in classical information processing. Similarly to bits, the information stored in qubits can be corrupted by classical noise. Differently from bits, qubits are also vulnerable to *quantum noise*, a type of noise with no counterpart in the classical world. Hence, it becomes crucial to understand, from an engineering perspective, how the quantum noise corrupts the information stored within a qubit. To this aim, in this invited paper, we overview the effects of the quantum noise on an arbitrary qubit from a signal-processing perspective.

***Index Terms*—** Quantum Signal Processing, Quantum Internet, Quantum Noise.

## 1. INTRODUCTION

Nowadays, there is a growing consensus about considering Quantum Information and Signal Processing and Quantum Internet [1–8] as disruptive paradigms for the next generation of information technologies. Hence, researchers worldwide are focusing on the engineering challenges arising with their design and deployment.

The building block of both Quantum Signal Processing and Quantum Internet is the quantum bit, or *qubit*, describing a discrete two-level quantum state. As widely known, a classical bit encodes one of two mutually exclusive states – 0 or 1 – being in only one state at any time. Conversely, a qubit can be in a *superposition* of the two states, being so simultaneously 0 and 1 at a certain time. Hence, a qubit offer richer, more complex opportunities for information carrying and information processing. And this *quantum advantage* over classical information grows exponentially with the number of qubits. In fact, thanks to the superposition principle, $n$ qubits can simultaneously encode $2^n$ quantum states at once. Differently, $n$ classical bits can encode only one of $2^n$ possible states at any time.

Unfortunately, quantum states are very fragile and they are easily corrupted by *quantum noise*. Specifically, any interaction with the environment irreversibly affects a quantum state, causing a degradation of its quantum properties in a process called *decoherence* - a type of noise with no counterpart in the classical world.

In the following, we revise the effects of the decoherence on an arbitrary qubit from a signal-processing perspective. To this aim, first in Sec. 2 we provide some preliminaries on closed and open quantum systems. Then, in Sec. 3 we overview from a signal-processing perspective the effects of the quantum noise, by describing a directional-dependent model for two different quantum noises. We also simulate such effects by visualizing them on the Bloch Sphere. Finally, we conclude the paper in Section 4.

## 2. BACKGROUND

### 2.1. Closed Quantum Systems

Accordingly to the first postulate of the quantum mechanics, associated to any isolated or closed quantum physical system is a complex Hilbert space. The system is completely described by its state vector, which is a unit vector in the system's state space [9].

The simplest quantum system is the qubit, whose state space is two-dimensional and whose geometrical representation on the Bloch sphere is depicted in Fig. 1. In the following, we adopt the conventional *bra-ket* notation[1] for denoting a qubit.

Closed quantum systems evolve in time according to deter-

[1] The *bra-ket* notation (also known as Dirac's notation) is a standard notion for describing quantum states. In a nutshell, a ket $|\cdot\rangle$ represents a column vector, whereas a bra $\langle\cdot|$ represents the Hermitian conjugate of the corresponding ket. Hence, the standard basis $|0\rangle, |1\rangle$ is equivalent to a couple of 2-dim orthonormal vectors, e.g., $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$ and $|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$. The generic qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, with $\alpha, \beta \in \mathcal{C} : |\alpha|^2 + |\beta|^2 = 1$, is equivalent to

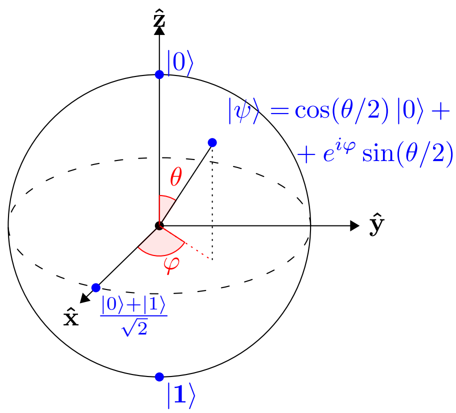


**Fig. 1**. Bloch Sphere: geometrical representation of a qubit. A pure state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ is represented by a point on the sphere surface, with $\alpha = \cos\frac{\theta}{2}$ and $\beta = e^{i\phi}\sin\frac{\theta}{2}$.

**Table 1**. Popular Quantum Gates

| **Gate** | I (identity) | $\sigma_x$ (NOT) | $\sigma_y$ | $\sigma_z$ |
|---|---|---|---|---|
| **Matrix** | $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ | $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ | $\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$ | $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$ |

ministic, reversible unitary operations [10]. That is, the state $|\psi(t)\rangle$ of the system at time $t$ is related to the initial state $|\psi(0)\rangle$ of the system at the initial time 0 through a unitary operator $U(\cdot)$, which depends only on the times $t$ and 0:

$$|\psi(t)\rangle = U(|\psi(0)\rangle) \tag{1}$$

**Remark 1.** *Any linear operator $A : V \rightarrow W$ between vector spaces $V$ and $W$ admits a matrix representation that is completely equivalent to the operator $A(\cdot)$. Hence the matrix representation and the operator are interchangeable. Hence, in the following, we will use the same symbol to denote both.*

The postulates and principles of quantum mechanics can be rewritten in terms of density operator $\rho(t)$ (or, equivalently, density matrix) [9]. For a pure state $|\psi(t)\rangle$, the density matrix is defined as $\rho(t) \triangleq |\psi(t)\rangle\langle\psi(t)|$, with $|\cdot\rangle\langle\cdot|$ denoting the outer product. Accordingly, the time evolution of a closed quantum system (1) is described in terms of $\rho(t)$ as:

$$\rho(t) = U\rho(0)U^{\dagger} \tag{2}$$

Unfortunately, real systems suffer from unwanted interactions with the outside world, called *environment*. Hence, real quantum systems constitute *open* rather than closed physical systems. And the unwanted interactions with the environment show up as noise in quantum signal/information processing. Hence, understanding such noise processes is mandatory to build effective quantum information/signal processing techniques.

### 2.2. Open Quantum Systems

The dynamics of an open quantum system can be regarded as arising from an interaction between the system $\mathcal{S}$ of interest and the environment $\mathcal{E}$, which together form the closed quantum system $\mathcal{SE}$ [9]. Seen as a whole, the system and the environment $\mathcal{SE}$ evolves according to a unitary transformation: $\rho_{\mathcal{SE}}(t) = U\rho_{\mathcal{SE}}(0)U^{\dagger}$.
The status of the system $\mathcal{S}$ of interest can be recovered by tracing out the environment via the partial trace operator $\mathrm{Tr}_{\mathcal{E}}(\cdot)$ over the environment $\mathcal{E}$, i.e.:

$$\rho_{\mathcal{S}}(t) = \mathrm{Tr}_{\mathcal{E}}\left[\rho_{\mathcal{SE}}(t)\right] = \mathrm{Tr}_{\mathcal{E}}\left[U\rho_{\mathcal{SE}}(0)U^{\dagger}\right] \tag{3}$$

where $\rho_{\mathcal{S}}(t)$ is referred to as *reduced density matrix*. From (3), it results that, due to the complex interactions between system and environment, in general $\rho_{\mathcal{S}}(t)$ may not be related through a unitary transformation to the initial state $\rho_{\mathcal{S}}(0)$.
It is very difficult to evaluate (3), since it requires to determine the dynamics $\rho_{\mathcal{SE}}(t)$ of the composite system $\mathcal{SE}$. And the status of the environment is always unknown or not possible to be controlled in reality.
However, by applying some approximations, it is often possible to derive directly the approximate time evolution of $\rho_{\mathcal{S}}(t)$ via the master equation formalism [10]. Accordingly, the time evolution of the system of interest can be written in the Lindblad form[2] as a time-local first-order differential equation system [10, 11], as shown in (4) at the top of the next page.
In (4), we note the presence of two components. The unitary evolution component depends on the Planck's constant $\hbar$, whose value must be experimentally determined, and the Hermitian operator $H_s$, referred to as *Hamiltonian* of the system. The Hamiltonian of the system determines the unitary part of the time evolution. In the following, we assume without loss of generality $\hbar = 1$. The non-unitary evolution component is as a consequence of the non-unitary nature of the trace operation used to obtain the reduced density matrix, and it depends on the Lindblad operators $L_k = \sqrt{\gamma_k}\sigma_k$, with $\{\sigma_k\}_{k=x,y,z}$ denoting the Pauli matrices reported in Table 1.

## 3. A DIRECTIONAL-DEPEND NOISE MODEL

### 3.1. Z-noise

A quantum noise process, with no counterpart in the classical world, is the *phase damping* (or *phase flip*). It describes the loss of quantum information without loss of energy,

the vector $\begin{bmatrix}\alpha\\\beta\end{bmatrix}$, whereas the bra $\langle\psi| = |\psi\rangle^{\dagger}$ is equivalent to the row vector $\begin{bmatrix}\alpha^* & \beta^*\end{bmatrix}$.

[2] In the following, $[A, B]$ denotes the commutator between two operators and it is defined as $[A, B] = AB - BA$. Similarly, $\{A, B\}$ denotes the anti-commutator between two operators and it is defined as $\{A, B\} = AB + BA$.

$$\frac{d}{dt}\rho_{\mathcal{S}}(t) = \overbrace{-\frac{i}{\hbar}[H_s, \rho_{\mathcal{S}}(t)]}^{\text{unitary evolution}} + \overbrace{\sum_k \left( L_k \rho_{\mathcal{S}}(t) L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho_{\mathcal{S}}(t)\} \right)}^{\text{non-unitary evolution}} \tag{4}$$

$$\frac{d\rho_{\mathcal{S}}(t)}{dt} = -\frac{i\Omega}{2}\begin{bmatrix} 0 & 2\rho_{\mathcal{S}}^{01}(t) \\ -2\rho_{\mathcal{S}}^{10}(t) & 0 \end{bmatrix} + \gamma_z \begin{bmatrix} 0 & -2\rho_{\mathcal{S}}^{01}(t) \\ -2\rho_{\mathcal{S}}^{10}(t) & 0 \end{bmatrix} = \begin{bmatrix} 0 & -(i\Omega + 2\gamma_z)\rho_{\mathcal{S}}^{01}(t) \\ (i\Omega - 2\gamma_z)\rho_{\mathcal{S}}^{10}(t) & 0 \end{bmatrix} \tag{5}$$

$$\rho_{\mathcal{S}}(t) = R_{\Omega t}\rho_{\mathcal{S}}(t)R_{\Omega t}^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\Omega t} \end{bmatrix} \rho_{\mathcal{S}}(t) \begin{bmatrix} 1 & 0 \\ 0 & e^{i\Omega t} \end{bmatrix} = \begin{bmatrix} \rho_{\mathcal{S}}^{00}(0) & \rho_{\mathcal{S}}^{01}(0)e^{-2\gamma_z t} \\ \rho_{\mathcal{S}}^{10}(0)e^{-2\gamma_z t} & \rho_{\mathcal{S}}^{11}(0) \end{bmatrix} \tag{8}$$

$$\rho_{\mathcal{S}}(t) = \frac{1}{2}\begin{bmatrix} \rho_{\mathcal{S}}^{00}(0)(1+e^{-2\gamma_x t}) + \rho_{\mathcal{S}}^{11}(0)(1-e^{-2\gamma_x t}) & \rho_{\mathcal{S}}^{01}(0)(1+e^{-2\gamma_x t}) + \rho_{\mathcal{S}}^{10}(0)(1-e^{-2\gamma_x t}) \\ \rho_{\mathcal{S}}^{10}(0)(1+e^{-2\gamma_x t}) + \rho_{\mathcal{S}}^{01}(0)(1-e^{-2\gamma_x t}) & \rho_{\mathcal{S}}^{11}(0)(1+e^{-2\gamma_x t}) + \rho_{\mathcal{S}}^{00}(0)(1-e^{-2\gamma_x t}) \end{bmatrix} \tag{12}$$

and it is one of the most common noises in quantum signal/information processing. The phase damping noise is described by the Lindblad operator $L_z = \sqrt{\gamma_z}\sigma_z$.
By assuming an Hamiltonian $H_s = \frac{\Omega}{2}\sigma_z$ dominated only by the unperturbed qubit energy splitting [10, 12], from (4) it results (5) shown at the top of this page. The diagonal elements $\rho_{\mathcal{S}}^{jj}(t)$ are constant in time, i.e., $\rho_{\mathcal{S}}^{jj}(t) = \rho_{\mathcal{S}}^{jj}(0)\,\forall t$ with $j = 0, 1$. Differently, the off-diagonal elements $\rho_{\mathcal{S}}^{ij}(t)$, with $i \neq j \in \{0, 1\}$ are:

$$\rho_{\mathcal{S}}^{01}(t) = \rho_{\mathcal{S}}^{01}(0)e^{-(i\Omega+2\gamma_z)t} \tag{6}$$

$$\rho_{\mathcal{S}}^{10}(t) = \rho_{\mathcal{S}}^{10}(0)e^{-(-i\Omega+2\gamma_z)t} = \left(\rho_{\mathcal{S}}^{01}(t)\right)^* \tag{7}$$

From (6)-(7), it results that an arbitrary qubit accumulates a phase $\Omega t$ due to the energy difference $\Omega$ between the states $|0\rangle$ and $|1\rangle$ as shown in Fig. 2. By knowing $\Omega$, this phase evolution can be compensated through a Phase shift gate $R_\phi$ with $\phi = \Omega t$, as shown in (8) at the top of this page.
In the following, we will focus our attention on the expression of $\rho_{\mathcal{S}}(t)$ given in (8), since it embeds the effects of the unknown noise. Specifically, from (8) it results that the off-diagonal elements of $\rho_{\mathcal{S}}(t)$ decay exponentially in time due to the noise. Hence the information about the initial quantum state embedded in these elements is lost exponentially in time.
To better understand from a signal-processing perspective the noise effects, let us consider the state of a single qubit in the Bloch representation, mapping any pure/mixed quantum state represented by a point within the Bloch sphere to its Cartesian coordinates via the vector $\mathbf{r} = (r_x, r_y, r_z) \in \mathcal{R}^3$ [13]. According to this representation, $\rho_{\mathcal{S}}(t)$ can be rewritten as:

$$\rho_{\mathcal{S}}(t) = \frac{1}{2}\begin{bmatrix} 1 + r_z(t) & r_x(t) - ir_y(t) \\ r_x(t) + ir_y(t) & 1 - r_z(t) \end{bmatrix} \tag{9}$$

By exploiting (8) and (9), after some algebraic manipulations, the vector components $\mathbf{r}(t) = (r_x(t), r_y(t), r_z(t))$ of the qubit at time $t$ change, with respect to the vector components $\mathbf{r}(0) = (r_x(0), r_y(0), r_z(0))$ at time 0, due to the Z-noise according to the following law:

$$r_x(t) = r_x(0)e^{-2\gamma_z t},\ r_y(t) = r_y(0)e^{-2\gamma_z t},\ r_z(t) = r_z(0) \tag{10}$$

as shown in Figs. 3 and 4.

**Remark 2.** *From* (10)*, it results that the quantum noise due to dephasing is multiplicative on the different space directions. In particular, it is directional-dependent since it does not affect the z-component of the qubit, whereas it affects exponentially and equally both the x- and y- components.*

**Remark 3.** *Furthermore and very interesting, from* (10)*, it results that the initial pure state* $|\psi(0)\rangle$ *is transformed in a mixed state, being* $||\mathbf{r}(t)|| \leq 1$*. Hence, the qubit at time* $t$ *lies in the interior of the Bloch Sphere.*

To clearly show the effects of the phase-damping, in Fig. 3 and 4 we plot the time evolution of the qubit subject to Z-noise, with the red sphere denoting the coordinates $\mathbf{r}(0) = \frac{1}{\sqrt{3}}(1, 1, 1)$ of the initial quantum state $|\psi(0)\rangle$. We observe that, in agreement with (10), the z-component remains unchanged. Differently, the x- and the y-components exponentially decays to zero and, consequently, the qubit asymptotically evolves toward the mixed state represented by the point $\mathbf{r} = (0, 0, r_z(0))$. The characteristic orbital evolution around the z-axis of the x- and y-components is a consequence of the phase accumulation induced by the system Hamiltonian $H_s = \frac{\Omega}{2}\sigma_z$, as shown in Fig. 2.

### 3.2. X-Noise

Let us now consider the case when the noise can be modeled with the Lindblad operator $L_x = \sqrt{\gamma_x}\sigma_x$. By neglecting the self-Hamiltonian $H_s = \frac{\Omega}{2}\sigma_z$ for the previously highlighted

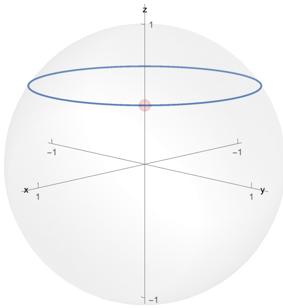

**Fig. 2**. Free time evolution of $|\psi(t)\rangle$ with Hamiltonian $H_s = \frac{\Omega}{2}\sigma_z$.

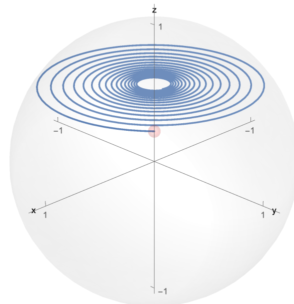

**Fig. 3**. Time evolution of the qubit subject to Z-noise with $H_s = \frac{\Omega}{2}\sigma_z$.

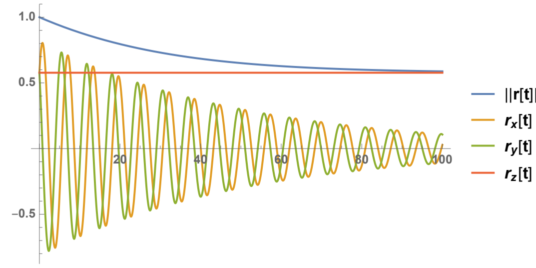


**Fig. 4**. Time evolution of $\mathbf{r}(t)$ subject to Z-noise with $H_s = \frac{\Omega}{2}\sigma_z$.

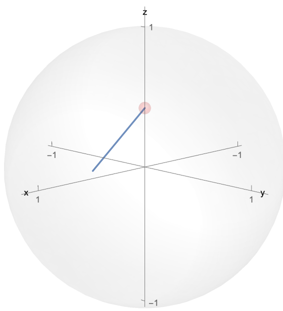

**Fig. 5**. Time evolution of the qubit subject to X-noise.

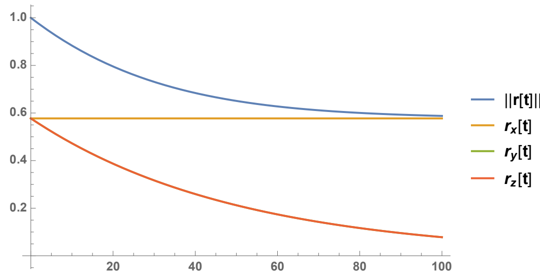


**Fig. 6**. Time evolution of of $\mathbf{r}(t)$ subject to X-noise.

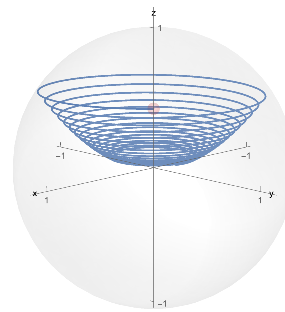

**Fig. 7**. Time evolution of the qubit subject to X-noise with $H_s = \frac{\Omega}{2}\sigma_z$.

reasons, from (4) it results:

$$\frac{d}{dt}\rho_S(t) = \gamma_x \begin{bmatrix} \rho_S^{11}(t) - \rho_S^{00}(t) & \rho_S^{10}(t) - \rho_S^{01}(t) \\ \rho_S^{01}(t) - \rho_S^{10}(t) & \rho_S^{00}(t) - \rho_S^{11}(t) \end{bmatrix} \qquad (11)$$

By solving this system of first-order differential equations, one obtains (12) reported at the beginning of the previous page. From (9), after some algebraic manipulations, it results that the vector $\mathbf{r}(t) =$ changes in time due to the X-noise according to the following law:

$$r_x(t) = r_x(0),\; r_y(t) = r_y(0)e^{-2\gamma_x t},\; r_z(t) = r_z(0)e^{-2\gamma_x t} \qquad (13)$$

**Remark 4.** *From* (13) *it results that the X-noise is again multiplicative on the different space directions. In particular such a noise is directional-dependent, since it does not affect the x-component of the qubit. While it affects exponentially and equally both the y- and z- components. Moreover, as for the Z-noise, it also results that the initial pure state $|\psi(0)\rangle$ is transformed in a mixed state, being $||\mathbf{r}(t)|| \leq 1$. Hence, the qubit at time $t$ lies in the interior of the Bloch Sphere.*

To clearly show the effects of the X-noise, in Fig. 5 and 6 we plot the time evolution of the qubit subject to X-noise neglecting the phase accumulation induced by the system Hamiltonian $H_s$ and represented in Fig. 2. We observe that, in agreement with (13), the x-component remains unchanged whereas the y- and the z-components exponentially decay to zero with a decay-rate driven by the coupling factor $\gamma_x$. The qubit asymptotically evolves toward the mixed state represented by the point $\mathbf{r} = (r_x(0), 0, 0)$. Finally, in Fig. 7 we plot the time evolution of the qubit by accounting for the phase accumulation as well, and it is easy to recognize the characteristic orbital evolution around the z-axis as a consequence of the phase accumulation induced by the system Hamiltonian.

## 4. CONCLUSIONS

In this invited paper, we overviewed the effects of the quantum noise on an arbitrary qubit from a signal-processing perspective. Specifically, we considered two quantum noises, Z-noise or phase damping and X-noise, and we highlighted the directional-dependent effects of such noises on the quantum information embedded in the qubit. In particular, we highlighted that the considered noises transform a pure state in a mixed state, as a consequence of the interaction between the qubit and the environment.

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