A TEST OF ASYMPTOTIC INTEGRABILITY OF $1+1$ WAVE EQUATIONS

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Abstract

The definition of asymptotic integrability is formulated within the perturbation theory based on multiple-scale expansion. Here we show that a $1+1$ weakly nonlinear and strongly dispersive wave equation can be asymptotically integrable only up to a finite order, and we provide an algorithmic method to test it.

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1. Introduction

Nonlinear wave equations are rarely solvable, and, even if we neglect damping and forcing, our ability to compute nonlinear effects relies mainly on approximation schemes or numerical experiments. One of the approaches is the quasi-monochromatic or slowly varying amplitude approximation that was introduced in nonlinear optics and fluid dynamics more than thirty years ago (references to relevant papers can be found in [1] and [2]). This method applies whenever the solution, at the first order in a small parameter ϵ, is a monochromatic (carrier) wave whose amplitude slowly varies in space, namely is a function of the slow coordinate $\xi = \epsilon x$. Here the smallness parameter ϵ is both the peak amplitude and the relative band-width $\Delta k/k$ of the initial wave-packet in the wave-number (Fourier) variable k (cfr.[1]). The well-known, and broadly applied, result of this analysis is that the amplitude evolves in the slow time variable $t_2 = \epsilon^2 t$ according to the celebrated Nonlinear Schroedinger (NLS) equation, that, since it follows from a generic nonlinear wave equation, has been recently recognized [3] as a universal model of nonlinear wave propagation in the strongly dispersive regime.

Higher order terms in the ϵ- expansion have been recently [2] considered with the purpose of computing inelastic effects in two-solitary wave collisions, and of capturing the evidence that the original nonlinear wave equation under investigation, being quite generic, is indeed nonintegrable. In fact, this last question stems naturally from the observation that the first order reduced equation, namely the NLS equation, is integrable and therefore the nonintegrability of the original nonlinear wave equation should manifest itself at some higher order of the perturbative expansion.

In the following we briefly report on the main results we obtain within the formalism introduced in [1] (for a different approach, see [2]), in particular on the definition of asymptotic integrability up to order $n (A_n$- integrability) which naturally follows from our analysis. As simple examples, we display the conditions for A_1 and A_2 -

integrability, while details, proofs and further examples are provided elsewhere [4].

A distinctive ingredient of our method is the introduction of (finitely or infinetely) many slow time variables, $t_n \equiv \epsilon^n t, n = 1, 2, \cdots$. For pedagogical purpose, we point out first that the occurrence of many slow times is naturally implied by the well-known Poincaré - Lindstedt perturbation scheme. Indeed, consider the single anharmonic oscillator equation

$$d^2q/dt^2 + \omega_o^2 q = c_2 q^2 + c_3 q^3 + \cdots \quad , q = q(t), \tag{1.1}$$

with the initial conditions $q(0) = \epsilon, \quad \dot{q}(0) = 0$, where ϵ is the (small) perturbation parameter. Since, for sufficiently small ϵ, $q(t)$ is periodic, one can set

$$q(t) = f(\theta), \quad \theta = \omega t, \quad f(\theta) = f(\theta + 2\pi), \tag{1.2}$$

together with the two expansions

$$\omega = \omega_o + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots \tag{1.3}$$

$$f = \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + \cdots \tag{1.4}$$

where the coefficients ω_n, for $n \geq 1$, are chosen so as to eliminate the secular terms that appear in the differential equations for the functions $f_n(\theta)$. In this way, the solution $q(t)$ proves to be a function not only of the time t but also of the slow times t_n, since combining (1.2) with (1.3) yields

$$\theta = \omega_o t + \omega_1 t_2 + \omega_2 t_2 + \omega_3 t_3 + \cdots \quad . \tag{1.5}$$

However, this approach does not seem to be convenient when one deals with PDEs; in fact, the following different, yet equivalent, expansion of the solution $q(t)$,

$$q(t) = \sum_{|\alpha|>0} e^{i\alpha\omega_o t} \sum_{n\geq|\alpha|} \epsilon^n q_n^{(\alpha)}(t_1, t_2, \cdots) + \sum_{n\geq 2} \epsilon^n q_n^{(o)}(t_1, t_2, \cdots) \tag{1.6}$$

turns out to be suitable to extension to nonlinear wave equations (see below). Here we note that $q(t)$ is doubly expanded both in harmonics and in powers of ϵ, each amplitude $q_n^{(\alpha)}$ being a function of the slow times only. Moreover, as explicitely shown, only the fundamental harmonics $\exp(\pm i\omega_o t)$ contribute to the first order approximation.

2. Formalism and basic equations

Let

$$Lu = G(u) \tag{2.1}$$

be a $1+1$ wave equation where $u = u(x,t)$ is the dependent variable, L is a linear, first order in time, differential operator with constant coefficients, and $G(u)$ is an analytic function (with no linear terms) of u and its x- derivatives. For instance, we have considered equations with $L = \partial_t - \partial_x^3,\ G(u) = cu_x^3 + (c_2u^2 + c_3u^3 + \cdots)_x$, and $L = \partial_t + \partial_x + a\partial_x^3 + b\partial_x^2\partial_t,\ G(u) = (a_1u^2 + a_2u_x^2 + a_3u_{xx}^2)_x + (a_4u^2 + a_5u_x^2)_{xx} + a_6(u^2)_{xxx}$ this second case being relevant to waves in shallow water. In order to semplify our analysis we assume that the variable u be real, $u = u^*$. For an arbitrarily given (real) wave number k, it is convenient to introduce the plane-wave

$$E \equiv \exp[i(kx - \omega t)] \tag{2.2}$$

where $\omega = \omega(k)$ is the dispersion law that is determined by the condition $LE = 0$. Let us now set the harmonic expansion

$$u = \sum_{\alpha=-\infty}^{+\infty} u^{(\alpha)} E^\alpha \tag{2.3}$$

where the coefficients $u^{(\alpha)}$ depend only on the slow variables ξ and $t_n, n = 1, 2, \cdots$, and let us note that the action of differential operators on u can be easily translated into operations on the functions $u^{(\alpha)}$ via the identities

$$\partial_x(E^\alpha u^{(\alpha)}) = E^\alpha(i\alpha k + \epsilon\partial_\xi)u^{(\alpha)} \quad , \tag{2.4a}$$

$$\partial_t(E^\alpha u^{(\alpha)}) = E^\alpha(-i\alpha\omega + \epsilon\partial_1 + \epsilon^2\partial_2 + \epsilon^3\partial_3 + \cdots)u^{(\alpha)} \quad , \tag{2.4b}$$

where $\partial_n \equiv \partial/\partial t_n$. It is then clear that the equation

$$L(E^\alpha u^{(\alpha)}) = E^\alpha L^{(\alpha)} u^{(\alpha)} \tag{2.5}$$

defines the linear operator $L^{(\alpha)}$, for the α-th harmonics, that is differential in the slow variables ξ and t_n, and possesses a well-defined formal expansion in powers of ϵ, namely

$$L^{(\alpha)} = L_o^{(\alpha)} + \epsilon L_1^{(\alpha)} + \epsilon^2 L_2^{(\alpha)} + \cdots \tag{2.6}$$

Once the expansion (2.3) of u is inserted in the function $G(u)$, then also $G(u)$ turns out to be expressed in terms of harmonics,

$$G(u) = \sum_{\alpha=-\infty}^{+\infty} G^{(\alpha)} E^\alpha \quad , \tag{2.7}$$

where the coefficients $G^{(\alpha)}$ are polynomial expressions of the functions $u^{(\beta)}$ and their ξ - derivatives. As a result of this setting, the original equation (2.1) is equivalent to the (infinite) set of PDEs

$$L^{(\alpha)} u^{(\alpha)} = G^{(\alpha)}, \quad \alpha = 0, \quad \pm 1, \pm 2, \cdots \quad . \tag{2.8}$$

Since the inverse operator $(L^{(\alpha)})^{-1}$ has a formal expansion in powers of ϵ iff $L_o^{(\alpha)} \neq 0$, and $L_o^{(\alpha)}$ is just a number, it is convenient to introduce the following definition: the α-th harmonics is resonant (or, shortly, (α) is a resonance) iff $L_o^{(\alpha)} = 0$. As a consequence, if (α) is not a resonance, then the equation (2.8) can be formally solved by algebraic means, $u^{(\alpha)} = (L^\alpha)^{-1} G^{(\alpha)}$, and only the amplitudes $u^{(\alpha)}$ corresponding to resonances satisfy truly differential equations. Since the reality of u implies the condition $u^{(-\alpha)} = u^{(\alpha)*}$, if we assume, just for the sake of simplicity, that only $\alpha = \pm 1$

are resonances, say $L_o^{(\pm 1)} = 0$, we need to focus our attention only on the equation $L^{(1)}u^{(1)} = G^{(1)}$, while all other equations, after expanding in powers of ϵ,

$$u^{(\alpha)} = \sum_{n=1} \epsilon^n u^{(\alpha)}(n) \qquad (2.9)$$

yield, by recursion, the expression of $u^{(\alpha)}(n)$, for $|\alpha| \neq 1$, in terms of $u^{(1)}(m), u^{(1)*}(l)$ and their ξ- derivatives. As a by-product of the ϵ- expansion, one can show that only the resonant harmonics $u^{(\pm 1)}$ give contributions of order ϵ since $u^{(o)}(1) = 0$ and $u^{(\alpha)}(n) = 0$ for $n< \mid \alpha \mid$. Moreover, because of the special rôle played by the functions $u^{(1)}(n)$, we introduce the following simpler notation $u^{(1)}(n) \equiv u(n), L_n^{(1)} \equiv L_n$ and $G^{(1)}(n) \equiv G(n)$, where, of course,

$$G^{(\alpha)} = \sum_{n=2} \varepsilon^n G^{(\alpha)}(n) \ , \qquad (2.10)$$

with the additional assumption that the functions $u(n)$, $(n = 1, 2, 3 \ldots)$ be infinitely differentiable with respect to the variable ξ, and with the notation

$$u_\ell(n) \equiv \partial_\xi^\ell \ u(n) \ , \qquad u_o(n) = u(n) \ . \qquad (2.11)$$

In order to keep track of the ε-order at which a given expression enters in our equations, it is convenient to denote the ε-order with the symbol O_ε, with self-evident notation such as $O_\varepsilon(u_\ell(n)) = \ell + n$, $O_\varepsilon(L_n) = n$, $O_\varepsilon(G(n)) = n$, etc.. In this way, it is easily seen that the basic equation is the triangular set of PDEs,

$$L_1 \ u(n-1) + L_2 \ u(n-2) + \ldots + L_{n-1} \ u(1) = G(n) \ , \quad n \geq 2 \ , \qquad (2.12)$$

where $G(n)$ is a differential polynomial of the functions $\{u(m), \ u^*(m)\}$ with unit gauge index, this meaning that $G(n) \longrightarrow \exp(i\theta)G(n)$ if $u(n) \longrightarrow \exp(i\theta)u(n)$. More conveniently, we introduce the finite-dimensional vector space $\mathcal{P}(n)$ of differential polynomials that are i) nonlinear in $\{u(m), \ u^*(m)\}$, ii) with unit gauge index, and iii) with $O_\varepsilon = n$. The vector spaces $\mathcal{P}(n)$ can be easily identified by specifying the basis of monomials; thus, for instance, $\mathcal{P}(2)$ is empty, $\mathcal{P}(3)$ is 1-dimensional, $\mathcal{P}(3) = \{u^2(1)\ u^*(1)\}$, $\mathcal{P}(4)$ is 4-dimensional, $\mathcal{P}(4) = \{u(2)\ u(1)\ u^*(1),\ u^2(1)\ u^*(2),\ u_1(1)\ u(1)u^*(1),\ u^2(1)\ u_1^*(1)\}$, and so forth.

3. Secularities and reduced equations

For arbitrarily given initial data, it turns out that the solutions $u(n)$ of the triangular system (2.12) are not bounded as $t_2 \to \infty$ as a consequence of secularities. However, as shown in [1], bounded solutions exist if appropriate conditions are satisfied. In particular, necessary conditions are

$$L_1 u(n) = 0\ , \qquad \partial_n u(1) = K_n[u(1)]\ , \qquad n = 1, 2, \ldots, \tag{3.1}$$

where the vector fields K_n are the commuting flows of the NLS hierarchy, say

$$K_2 = i\omega_2(u_2(1) - 2cu^2(1)u^*(1)) \tag{3.2}$$

is the NLS flow, K_3 is the complex modified Korteweg-de Vries flow and so on (here c is assumed to be real so as to deal with the integrable NLS). This finding clearly implies that $L_n u(1)$ belongs to the vector space $\mathcal{P}(n+1)$, so that the triangular system (2.12) can be recast in the simpler form (note that $L_2 u(1) = G(3)$ is the NLS equation)

$$L_2 u(n-2) + L_3 u(n-3) + \ldots + L_{n-2} u(2) = \tilde{G}(n)\ , \quad n \geq 4\ , \tag{3.3}$$

where now the differential polynomial $\tilde{G}(n) \epsilon\ \mathcal{P}(n)$ depends on $u(\ell)$ only for ℓ up to $n-2$.

The multiple-scale equations (3.3) can be rewritten in a more convenient form by introducing the linear operators

$$M_n \equiv \partial_n - K_n'[u(1)]\ , \qquad n \geq 2\ , \tag{3.4}$$

where $K'_n[u(1)]$ is the Frechét derivative (with respect to $u(1)$) of $K_n[u(1)]$, and by observing that

$$L_n - M_n \epsilon \mathcal{P}(n) \ . \tag{3.5}$$

Indeed, the equations (3.2) now read

$$M_2 u(n-2) + M_3 u(n-3) + \ldots + M_{n-2} u(2) = F(n) \ , \qquad n \geq 4 \ , \tag{3.6}$$

with the twofold virtue that, as for the linear operators L_n, also the operators (3.4) commute with each other,

$$[M_n, M_m] = 0 \ , \tag{3.7}$$

and that $F(n) \epsilon \mathcal{P}(n)$ depends on $u(\ell)$ for ℓ up to $n-3$.

4. Integrability

At this point we note that the reduced multiple-scale equations (3.6) hold for any equation in the general class (2.1), and do not bring, therefore, any information on the integrability properties of the original equation (2.1). However, a relation to integrability stems from the following facts. First, in the linear (and trivially integrable) case, $M_n = L_n$ and $F(n) = 0$, and the triangular system (3.6) splits into the set of evolution (with respect to each slow time) equations $M_\ell u(n) = 0$, $\ell \geq 2$, $n \geq 2$, that are obviously compatible with each other. Second, we show in [4] that, if the original equation (2.1) is C-integrable or S-integrable, then, also in this case, the system (3.6) has the splitting property

$$M_\ell u(n) = f_\ell(n) \epsilon \mathcal{P}(\ell + n) \ , \tag{4.1}$$

where the differential polynomials $f_\ell(n)$ satisfy the compatibility conditions $M_m f_\ell(n)$ $= M_\ell f_m(n)$. In this respect, it is convenient to introduce the subspace $\mathcal{P}_n(m)$ of $\mathcal{P}(m)$ of the differential polynomials that depend on $u(\ell)$ and $u^*(\ell)$ for ℓ up to n. In fact, it turns out that, if the equations (4.1) are satisfied for some ℓ and n, then

$f_\ell(n)\epsilon\mathcal{P}_{n-1}(n+\ell)$. Because of the splitting property (4.1) of integrable equations, and of the fact that the multiple-scale equations obtain in a neighborough of $\varepsilon = 0$, it is natural to define the equation $Lu = G(u)$ asymptotically integrable up to order n, or A_n-integrable, if all the reduced multiple-scale equations $M_2 u(m) + \ldots + M_m\ u(2) = F(m+2)$ for $m = 2, 3, \ldots, n$, but not for $m = n+1$ split into (compatible) equations of the form $M_\ell u(m-\ell+2) = f_\ell(m-\ell+2)$ for $\ell = 2, \ldots, n$, together with the condition $M_3\ f_2(n) = M_2\ f_3(n)$. The case $n = 1$ can be included in this definition by considering that the equation $M_2 u(1) = F(3)$ is the NLS equation for which the integrability condition is Im$c = 0$, where c is the parameter that appears in (3.2).

As for the computational task, one has to solve the equation $M_2\ f_3(n) = M_3\ f_2(n)$ where $M_3\ f_2(n)$ is a given vector in $\mathcal{P}_{n-1}(n+5)$. To this aim, we note that the kernel of the operator M_ℓ in the vector space $\mathcal{P}(n)$ is empty, and that the differential operator is rapresented by a rectangular matrix taking a vector of $\mathcal{P}_{n-1}(n+3)$ into a vector of $\mathcal{P}_{n-1}(n+5)$. As a consequence, the solution $f_3(n)$ of this equation exists iff the vector $f_2(n)$ satisfies appropriate conditions, that are, therefore, those which entail asymptotic integrability. The actual computations have been performed by computer since the algebraic complexity rapidly increases with n. In order to show this, we give below the dimensionality of the first few vector spaces $\mathcal{P}_\ell(n)$ in the notation $\mathcal{P}_\ell(n) \to \dim(\mathcal{P}_\ell(n))$: $\mathcal{P}_1(3) \to 1$, $\mathcal{P}_1(4) \to 2$, $\mathcal{P}_1(5) \to 5$, $\mathcal{P}_1(6) \to 8$, $\mathcal{P}_2(4) \to 4$, $\mathcal{P}_2(5) \to 12$, $\mathcal{P}_2(6) \to 26$, $\mathcal{P}_3(5) \to 14$, $\mathcal{P}_3(6) \to 34$. Thus, the A_2-integrability condition on

$$f_2(2) = (\alpha_1 + i\beta_1)u(1)u_1(1)u^*(1) + (\alpha_2 + i\beta_2)u^2(1)u_1^*(1) \tag{4.2}$$

reads $\beta_1 = \beta_2 = 0$ if $c \neq 0$, while no condition on α_j and β_j is required if $c = 0$. The A_3-integrability condition on the 12-dimensional vector $f_2(3)$ is given by 15 real equations so that the general vector $f_2(3)$ depends on 9 real arbitrary parameters. The explicit expression of these, and higher order, conditions are given in [4]. Once these conditions have been found, they can be used to test the order of the asymptotic integrability of a given wave equation of the form (2.1). Examples of particular wave equations that have been tested by this method are also reported in [4]. Finally, since integrable equations are A_∞-integrable, we conjecture that, conversely, A_∞-integrable equations are indeed (either C or S-) integrable.

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