# QUASI-PERIODIC SOLUTIONS OF COMPLETELY RESONANT FORCED WAVE EQUATIONS

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Abstract. We prove existence of quasi-periodic solutions with two frequencies of completely resonant, periodically *forced* nonlinear wave equations with periodic spatial boundary conditions. We consider both the cases the forcing frequency is: (Case A) a rational number and (Case B) an irrational number.

Keywords: Nonlinear Wave Equation, Quasi-Periodic Solutions, Variational Methods, Lyapunov-Schmidt reduction, Infinite Dimensional Hamiltonian Systems.[1]

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## 1. Introduction

We prove existence of small amplitude quasi-periodic solutions for completely resonant forced nonlinear wave equations like

$$\begin{cases} v_{tt} - v_{xx} + f(\omega_1 t, v) = 0 \\ v(t,x) = v(t, x+2\pi) \end{cases} \tag{1.1}$$

where the nonlinear forcing term

$$f(\omega_1 t, v) = a(\omega_1 t) v^{2d-1} + O(v^{2d}), \qquad d > 1,\ d \in \mathbb{N}^+$$

is $2\pi/\omega_1$-periodic in time. We shall consider both the cases

- $A)$ the forcing frequency $\omega_1 \in \mathbb{Q}$
- $B)$ the forcing frequency $\omega_1 \in \mathbb{R} \setminus \mathbb{Q}$.

Existence of periodic solutions for completely resonant forced wave equations was first proved in the pioneering papers [R1], [R2] (with Dirichlet boundary conditions) if the forcing frequency is a rational number ($\omega_1 = 1$ in [R1]-[R2]). This requires to solve an infinite dimensional bifurcation equation which lacks compactness property; see [BN], [C], [BBi]-[BBi1] and references therein for other results. If the forcing frequency is an irrational number existence of periodic solutions has been proved in [PY]-[Mc]: here the bifurcation equation is trivial but a "small divisors problem" appears.

To prove existence of small amplitude quasi-periodic solutions for completely resonant PDE's like (1.1) one generally has to deal with a small divisor problem as well; however the main difficulty is to understand from which solutions of the linearized equation at $v = 0$,

$$v_{tt} - v_{xx} = 0\,,$$

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quasi-periodic solutions branch-off: such linearized equation possesses only $2\pi$-periodic solutions $q_+(t+x)+ q_-(t-x)$ where $q_+(\cdot)$, $q_-(\cdot)$ are $2\pi$-periodic (completely resonant PDE).

Here is the main difference w.r.t non-resonant PDE's for which a developed existence theory of periodic and quasi-periodic solutions has been established, see e.g. [K], [Wa], [CW], [P], [B1] and references therein.

For completely resonant autonomous PDE's, existence of periodic solutions has been proved in [LS], [BP], [BB1], [BB2], [BB3], [GMP], [GP], and quasi-periodic solutions with 2-frequencies have been recently obtained in [P1]-[P2] for the specific nonlinearities $f = u^3 + O(u^5)$. Here the bifurcation equation is solved by ODE methods.

In this paper we prove existence of quasi-periodic solutions with two frequencies $\omega_1$, $\omega_2$ for the completely resonant forced equation (1.1) in both the two cases: Case $A$): $\omega_1 \in \mathbb{Q}$; Case $B$): $\omega_1 \in \mathbb{R} \setminus \mathbb{Q}$.

The more interesting case is $\omega_1 \in \mathbb{Q}$ (case A) when the forcing frequency $\omega_1$ enters in resonance with the linear frequency 1. To find out from which solutions of the linearized equation quasi-periodic solutions of (1.1) branch-off, requires to solve an infinite dimensional bifurcation equation which can not be solved in general by ODE techniques (it is a system of integro-differential equations). However, exploiting the variational nature of equation (1.1) like in [BB1]-[BB2], the bifurcation problem can be reduced to finding critical points of a suitable action functional which, in this case, possesses the infinite dimensional linking geometry [BR].

1.1. **Main results.** We look for quasi-periodic solutions $v(t,x)$ of equation (1.1) of the form

$$\begin{cases} v(t,x) = u(\omega_1 t, \omega_2 t + x) \\ u(\varphi_1 + 2k_1\pi, \varphi_2 + 2k_2\pi) = u(\varphi_1, \varphi_2), \qquad \forall k_1, k_2 \in \mathbb{Z} \end{cases} \tag{1.2}$$

with frequencies

$$\omega = (\omega_1, \omega_2) = (\omega_1, 1+\varepsilon)\,,$$

imposing the frequency $\omega_2 = 1 + \varepsilon$ to be close to the linear frequency 1.

Writing $\partial_{tt} - \partial_{xx} = (\partial_t - \partial_x) \circ (\partial_t + \partial_x)$ we get

$$\Big[\omega_1 \partial_{\varphi_1} + (\omega_2 - 1)\partial_{\varphi_2}\Big] \circ \Big[\omega_1 \partial_{\varphi_1} + (\omega_2 + 1)\partial_{\varphi_2}\Big] u + f(\varphi_1, u) = 0 \tag{1.3}$$

and therefore

$$\Big[\omega_1^2 \partial_{\varphi_1}^2 + (\omega_2^2 - 1)\partial_{\varphi_2}^2 + 2\omega_1\omega_2 \partial_{\varphi_1}\partial_{\varphi_2}\Big] u(\varphi) + f(\varphi_1, u) = 0\,. \tag{1.4}$$

We assume that the forcing term $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$

$$f(\varphi_1, u) = a_{2d-1}(\varphi_1) u^{2d-1} + O(u^{2d}), \qquad d \in \mathbb{N}^+,\ d > 1$$

is analytic in $u$ but has only finite regularity in $\varphi_1$. More precisely

- **(H)** $f(\varphi_1, u) := \sum_{k=2d-1}^{\infty} a_k(\varphi_1) u^k$, $d \in \mathbb{N}^+$, $d > 1$ and the coefficients $a_k(\varphi_1) \in H^1(\mathbb{T})$ verify, for some $r > 0$, $\sum_{k=2d-1}^{\infty} |a_k|_{H^1} r^k < \infty$. The function $f(\varphi_1, u)$ is not identically constant in $\varphi_1$.

We look for solutions $u$ of (1.4) in the Banach space[2]

$$\mathcal{H}_{\sigma,s} := \Big\{ u(\varphi) = \sum_{l\in\mathbb{Z}^2} \hat{u}_l e^{\mathrm{i} l\cdot\varphi} \ \ : \ \hat{u}_l^* = \hat{u}_{-l} \ \text{ and } \ |u|_{\sigma,s} := \sum_{l\in\mathbb{Z}^2} |\hat{u}_l| e^{|l_2|\sigma} [l_1]^s < +\infty \Big\}$$

where $[l_1] := \max\{|l_1|, 1\}$ and $\sigma > 0$, $s \geq 0$.

The space $\mathcal{H}_{\sigma,s}$ is a Banach algebra with respect to multiplications of functions (see Lemma 4.1 in the Appendix), namely

$$u_1, u_2 \in \mathcal{H}_{\sigma,s} \quad \Longrightarrow \quad u_1 u_2 \in \mathcal{H}_{\sigma,s} \quad \text{and} \quad |u_1 u_2|_{\sigma,s} \leq C |u_1|_{\sigma,s} |u_2|_{\sigma,s} \, .$$

We shall prove the following Theorems.

**Theorem A.** *Let $\omega_1 = n/m \in \mathbb{Q}$. Assume that $f$ satisfies assumption* (**H**) *and $a_{2d-1}(\varphi_1) \neq 0$, $\forall \varphi_1 \in \mathbb{T}$. Let $\mathcal{B}_\gamma$ be the uncountable[3] zero-measure Cantor set*

$$\mathcal{B}_\gamma := \Big\{ \varepsilon \in (-\varepsilon_0, \varepsilon_0) \ : \quad |l_1 + \varepsilon l_2| > \frac{\gamma}{|l_2|} \, , \ \forall l_1, l_2 \in \mathbb{Z} \setminus \{0\} \Big\}$$

*where $0 < \gamma < 1/6$.*

*There exist constants $\overline{\sigma} > 0$, $\overline{s} > 2$, $\bar{\varepsilon} > 0$, $\overline{C} > 0$, such that $\forall \varepsilon \in \mathcal{B}_\gamma$, $|\varepsilon|\gamma^{-1} \leq \bar{\varepsilon}/m^2$, there exists a classical solution $u(\varepsilon, \varphi) \in \mathcal{H}_{\overline{\sigma},\overline{s}}$ of (1.4) with $(\omega_1, \omega_2) = (n/m, 1+\varepsilon)$ satisfying*

$$\Big| u(\varepsilon,\varphi) - |\varepsilon|^{\frac{1}{2(d-1)}} \bar{q}_\varepsilon(\varphi) \Big|_{\overline{\sigma},\overline{s}} \leq \overline{C} \frac{m^2 |\varepsilon|}{\gamma\, \omega_1^3} |\varepsilon|^{\frac{1}{2(d-1)}} \tag{1.5}$$

*for an appropriate function $\bar{q}_\varepsilon \in \mathcal{H}_{\sigma,s} \setminus \{0\}$ of the form $\bar{q}_\varepsilon(\varphi) = \bar{q}_+(\varphi_2) + \bar{q}_-(2m\varphi_1 - n\varphi_2)$.*

*As a consequence, equation (1.1) admits the quasi-periodic solution $v(\varepsilon, t, x) := u(\varepsilon, \omega_1 t, x + \omega_2 t)$ with two frequencies $(\omega_1, \omega_2) = (n/m, 1+\varepsilon)$ and the map $t \to v(\varepsilon, t, \cdot) \in H^{\overline{\sigma}}(\mathbb{T})$ has the form*[4]

$$\Big| v(\varepsilon,t,x) - |\varepsilon|^{\frac{1}{2(d-1)}} \Big[ \bar{q}_+(x + (1+\varepsilon)t) + \bar{q}_-((1-\varepsilon)nt - nx) \Big] \Big|_{H^{\overline{\sigma}}(\mathbb{T})} = O\Big( \frac{m^2}{\gamma\, \omega_1^3} |\varepsilon|^{\frac{2d-1}{2(d-1)}} \Big) \, .$$

At the first order the quasi-periodic solution $v(\varepsilon, t, x)$ of equation (1.1) is the superposition of two waves traveling in opposite directions (in general, both components $q_+$, $q_-$ are non trivial).

The bifurcation of quasi-periodic solutions looks quite different if $\omega_1$ is irrational.

**Theorem B.** *Let $\omega_1 \in \mathbb{R} \setminus \mathbb{Q}$. Assume that $f$ satisfies assumption* (**H**), *$\int_0^{2\pi} a_{2d-1}(\varphi_1) d\varphi_1 \neq 0$ and $f(\varphi_1, u) \in H^s(\mathbb{T})$, $s \geq 1$, for all $u$.*

[2]Given $z \in \mathbb{C}$, $z^*$ denotes its complex conjugate.
[3]The proof that $\mathcal{B}_\gamma \cap (0, \varepsilon_0)$ and $\mathcal{B}_\gamma \cap (-\varepsilon_0, 0)$ are both uncountable $\forall \varepsilon_0 > 0$ is like in [BP].
[4]We denote $H^\sigma(\mathbb{T}) := \{ u(\varphi) = \sum_{l\in\mathbb{Z}} \hat{u}_l e^{\mathrm{i}l\varphi} \ : \ \hat{u}_l^* = \hat{u}_{-l} \, , \ |u|_{H^\sigma(\mathbb{T})} := \sum_{l\in\mathbb{Z}} |\hat{u}_l| e^{\sigma|l|} < +\infty \}$.

*Let* $\mathcal{C}_\gamma \subset D \equiv (-\varepsilon_0, \varepsilon_0) \times (1,2)$ *be the uncountable zero-measure Cantor set*[5]

$$\mathcal{C}_\gamma := \left\{ \begin{array}{c} (\varepsilon, \omega_1) \in D : \ \omega_1 \notin \mathbb{Q}, \quad \dfrac{\omega_1}{\omega_2} \notin \mathbb{Q}, \quad |\omega_1 l_1 + \varepsilon l_2| > \dfrac{\gamma}{|l_1| + |l_2|}, \\ |\omega_1 l_1 + (2+\varepsilon) l_2| > \dfrac{\gamma}{|l_1|+|l_2|}, \ \forall\, l_1, l_2 \in \mathbb{Z} \setminus \{0\} \end{array} \right\}. \tag{1.6}$$

*Fix any* $0 < \overline{s} < s - 1/2$. *There exist positive constants* $\overline{\varepsilon}$, $\overline{C}$, $\overline{\sigma} > 0$, *such that,* $\forall (\varepsilon, \omega_1) \in \mathcal{C}_\gamma$ *with* $|\varepsilon|\gamma^{-1} < \overline{\varepsilon}$ *and* $\varepsilon \int_0^{2\pi} a_{2d-1}(\varphi_1)\, d\varphi_1 > 0$, *there exists a nontrivial solution* $u(\varepsilon, \varphi) \in \mathcal{H}_{\overline{\sigma}, \overline{s}}$ *of equation (1.4) with* $(\omega_1, \omega_2) = (\omega_1, 1+\varepsilon)$ *satisfying*

$$\Big| u(\varepsilon, \varphi) - |\varepsilon|^{\frac{1}{2(d-1)}} \bar{q}_\varepsilon(\varphi_2) \Big|_{\overline{\sigma}, \overline{s}} \leq \overline{C} \frac{|\varepsilon|}{\gamma} |\varepsilon|^{\frac{1}{2(d-1)}} \tag{1.7}$$

*for some function* $\bar{q}_\varepsilon(\varphi_2) \in H^{\overline{\sigma}}(\mathbb{T}) \setminus \{0\}$.

*As a consequence, equation (1.1) admits the non-trivial quasi-periodic solution* $v(\varepsilon, t, x) := u(\varepsilon, \omega_1 t, x + \omega_2 t)$ *with two frequencies* $(\omega_1, \omega_2) = (\omega_1, 1+\varepsilon)$ *and the map* $t \to v(\varepsilon, t, \cdot) \in H^{\overline{\sigma}}(\mathbb{T})$ *has the form*

$$\Big| v(\varepsilon, t, x) - |\varepsilon|^{\frac{1}{2(d-1)}} \bar{q}_\varepsilon(x + (1+\varepsilon)t) \Big|_{H^{\overline{\sigma}}(\mathbb{T})} = O\Big( \gamma^{-1} |\varepsilon|^{\frac{2d-1}{2(d-1)}} \Big).$$

**Remark 1.** *Imposing in the definition of* $\mathcal{C}_\gamma$ *the condition* $\omega_1/\omega_2 = \omega_1/(1+\varepsilon) \in \mathbb{Q}$ *we obtain, by Theorem B the existence of periodic solutions of equation (1.1). They are reminiscent, in this completely resonant context, of the Birkhoff-Lewis periodic orbits with large minimal period accumulating at the origin, see* [BaB], [BBV].

**Remark 2. (Non existence)** *In Theorem B, existence of quasi-periodic solutions could follow by other hypotheses on* $f$, *see remark 3. However the hypothesis that the leading term in the nonlinearity* $f$ *is an odd power of* $u$ *is not of purely technical nature. If* $f(\varphi_1, u) = a(\varphi_1) u^D$ *with* $D$ *even and* $\int_0^{2\pi} a(\varphi_1)\, d\varphi_1 \neq 0$, *then,* $\forall R > 0$ *there exists* $\varepsilon_0 > 0$ *such that* $\forall \sigma \geq 0$, $\overline{s} > s - 1/2$, $\forall (\varepsilon, \omega_1) \in \mathcal{C}_\gamma$ *with* $|\varepsilon| < \varepsilon_0$, *equation (1.4) does not possess solutions* $u \in \mathcal{H}_{\sigma, \overline{s}}$ *in the ball* $|u|_{\sigma, \overline{s}} \leq R |\varepsilon|^{1/(D-1)}$, *see Proposition 3.*

To prove Theorems A-B, instead of looking for solutions of equation (1.4) in a shrinking neighborhood of 0, it is a convenient devise to perform the rescaling

$$u \to \delta u \qquad \text{with} \qquad \delta := |\varepsilon|^{1/2(d-1)}$$

enhancing the relation between the amplitude $\delta$ and the frequency $\omega_2 = 1 + \varepsilon$. We obtain the equation

$$\mathcal{L}_\varepsilon u + \varepsilon f(\varphi_1, u, \delta) = 0 \tag{1.8}$$

where, see (1.3),

$$\begin{aligned} \mathcal{L}_\varepsilon &:= \Big[ \omega_1 \partial_{\varphi_1} + \varepsilon \partial_{\varphi_2} \Big] \circ \Big[ \omega_1 \partial_{\varphi_1} + (2+\varepsilon) \partial_{\varphi_2} \Big] \\ &= \Big[ \omega_1^2 \partial_{\varphi_1}^2 + 2\omega_1 \partial_{\varphi_1} \partial_{\varphi_2} \Big] + \varepsilon \Big[ (2+\varepsilon) \partial_{\varphi_2}^2 + 2\omega_1 \partial_{\varphi_1} \partial_{\varphi_2} \Big] \end{aligned}$$

[5] See Lemma 3.1.

and

$$f(\varphi_1, u, \delta) := \text{sign}(\varepsilon) \frac{f(\varphi_1, \delta u)}{\delta^{2(d-1)}} = \text{sign}(\varepsilon) \Big( a_{2d-1}(\varphi_1) u^{2d-1} + \delta a_{2d}(\varphi_1) u^{2d} + \dots \Big) \tag{1.9}$$

and $\text{sign}(\varepsilon) := 1$ if $\omega_2 > 1$ and $\text{sign}(\varepsilon) := -1$ if $\omega_2 < 1$.

To find solutions of equation (1.8) we shall apply the Lyapunov-Schmidt decomposition method which leads to solve separately a "range equation" and a "bifurcation equation".

In order to solve the range equation (avoiding small divisor problems) we restrict $\varepsilon$ to the uncountable zero-measure set $\mathcal{B}_\gamma$ for Theorem A, resp. $(\varepsilon, \omega_1) \in \mathcal{C}_\gamma$ for Theorem B, and we apply the Contraction Mapping Theorem; similar non-resonance conditions have been employed e.g. in [LS], [BP], [BB1]-[BB2], [Mc], [P1].

To solve the *infinite* dimensional bifurcation equation we proceed in different ways in case A) and case B).

As already said, in case A) we follow the variational approach of [BB1],[BB2] noting that the bifurcation equation is the Euler-Lagrange equation of a "reduced action functional" which turns out to have the geometry of the infinite dimensional linking theorem of Benci-Rabinowitz [BR]. However we can not directly apply the linking theorem because the reduced action functional is defined only in a ball centered at the origin (where the range equation is solved). Moreover the infinite dimensional linking theorem of [BR] requires the compactness of the gradient of the functional, property which is not preserved by extending the functional in the whole infinite dimensional space.

In order to overcome these difficulties we perform a further finite dimensional reduction of Galerkin type inspired to [BB3] on a subspace of dimension $N$, with $N$ large but independent of $\varepsilon$, see the equations (2.3)-(2.4)-(2.5).

We shall have to solve the (2.4)-(2.5) equations in a sufficiently large domain of $q_1$ (Lemma 2.3), consistent with the $|\cdot|_{H^1}$ bounds on the solution $q_1$ of the bifurcation equation that can be obtained by the variational arguments, see Lemma 2.6.

Another advantage of this method is that allows to prove the analiticity of the solution $u$ in the variable $\varphi_2$.

In case B) the bifurcation equation could be solved through variational methods as in case A). However there is a simpler technique available. The bifurcation equation reduces, in the limit $\varepsilon \to 0$, to a super-quadratic Hamiltonian system with one degree of freedom. We prove existence of a non-degenerate solution by phase-space analysis. Therefore it can be continued by the Implicit function Theorem to a solution of the complete bifurcation equation for $\varepsilon$ small.

The paper is organized as follows. For simplicity of exposition we prove first Theorem A in the case $\omega_1 = 1$. We deal with the general case $\omega_1 = \frac{n}{m} \in \mathbb{Q}$ at the end of section 2. In section 3 we prove Theorem B.

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## 2. Case A: $\omega_1 \in \mathbb{Q}$

Equation (1.8) becomes, for $\omega_1 = 1$

$$\mathcal{L}_\varepsilon u + \varepsilon f(\varphi_1, u, \delta) = 0 \tag{2.1}$$

where

$$\begin{aligned}\mathcal{L}_\varepsilon &:= \Big[\partial_{\varphi_1} + \varepsilon \partial_{\varphi_2}\Big] \circ \Big[\partial_{\varphi_1} + (2+\varepsilon)\partial_{\varphi_2}\Big] \\ &= \Big[\partial^2_{\varphi_1} + 2\partial_{\varphi_1}\partial_{\varphi_2}\Big] + \varepsilon\Big[(2+\varepsilon)\partial^2_{\varphi_2} + 2\partial_{\varphi_1}\partial_{\varphi_2}\Big] \equiv L_0 + \varepsilon L_1 \,.\end{aligned}$$

To fix notations we shall prove Theorem A in the case $a_{2d-1}(\varphi_1) > 0$ and $\varepsilon > 0$, i.e. $\mathrm{sign}(\varepsilon) > 0$.

By the assumption **(H)** on the nonlinearity $f$ and by the Banach algebra property of $\mathcal{H}_{\sigma,s}$ the Nemitskii operator

$$u \to f(\varphi_1, u, \delta) \in \; C^\infty(B_\rho, \mathcal{H}_{\sigma,s}) \,, \qquad\qquad 0 < s < \frac{1}{2}$$

where $B_\rho$ is the ball of radius $\rho\delta^{-1}$ in $\mathcal{H}_{\sigma,s}$ and $\rho$ is connected to the analiticity radius $r$ of $f$ (note that since $a_k(\varphi_1) \in H^1(\mathbb{T})$ then $a_k(\cdot) \in \mathcal{H}_{\sigma,s}$, $\forall \sigma > 0$, $0 < s < 1/2$).

Equation (2.1) is the Euler-Lagrange equation of the *Lagrangian action functional* $\Psi_\varepsilon \in C^1(\mathcal{H}_{\sigma,s}, \mathbb{R})$ defined by

$$\begin{aligned}\Psi_\varepsilon(u) &:= \int_{\mathbb{T}^2} \frac{1}{2}(\partial_{\varphi_1} u)^2 + (\partial_{\varphi_1} u)(\partial_{\varphi_2} u) + \frac{\varepsilon(2+\varepsilon)}{2}(\partial_{\varphi_2} u)^2 + \varepsilon(\partial_{\varphi_1} u)(\partial_{\varphi_2} u) - \varepsilon F(\varphi_1, u, \delta) \\ &\equiv \Psi_0(u) + \varepsilon \Gamma(u, \delta)\end{aligned}$$

where $F(\varphi_1, u, \delta) := \int_0^u f(\varphi_1, \xi, \delta) d\xi$ and

$$\begin{aligned}\Psi_0(u) &:= \int_{\mathbb{T}^2} \frac{1}{2}(\partial_{\varphi_1} u)^2 + (\partial_{\varphi_1} u)(\partial_{\varphi_2} u) \\ \Gamma(u, \delta) &:= \int_{\mathbb{T}^2} \frac{(2+\varepsilon)}{2}(\partial_{\varphi_2} u)^2 + (\partial_{\varphi_1} u)(\partial_{\varphi_2} u) - F(\varphi_1, u, \delta) \,.\end{aligned}$$

To find critical points of $\Psi_\varepsilon$ we perform a variational Lyapunov-Schmidt reduction inspired to [BB1]-[BB2], see also [AB].

### 2.1. The Variational Lyapunov-Schmidt Reduction.

The unperturbed functional $\Psi_0 : \mathcal{H}_{\sigma,s} \to \mathbb{R}$ possesses an infinite dimensional linear space $Q$ of critical points which are the solutions $q$ of the equation

$$L_0 q = \partial_{\varphi_1}\Big(\partial_{\varphi_1} + 2\partial_{\varphi_2}\Big) q = 0 \,.$$

The space $Q$ can be written as

$$Q = \Big\{ q = \sum_{l \in \mathbb{Z}^2} \hat{q}_l e^{\mathrm{i} l \cdot \varphi} \in \mathcal{H}_{\sigma,s} \;|\; \hat{q}_l = 0 \quad \text{for} \quad l_1(l_1 + 2l_2) \neq 0 \Big\} \,.$$

In view of the variational argument that we shall use to solve the bifurcation equation we split $Q$ as

$$Q = Q_+ \oplus Q_0 \oplus Q_-$$

where[6]

$$\begin{aligned} Q_+ &:= \Big\{ q \in Q \ : \ \hat q_l = 0 \text{ for } l \notin \Lambda_+ \Big\} = \Big\{ q_+ := q_+(\varphi_2) \in H^\sigma_0(\mathbb{T}) \Big\} \\ Q_0 &:= \Big\{ q_0 \in \mathbb{R} \Big\} \\ Q_- &:= \Big\{ q \in Q \ : \ \hat q_l = 0 \text{ for } l \notin \Lambda_- \Big\} = \Big\{ q_- := q_-(2\varphi_1 - \varphi_2), \ q_-(\cdot) \in H^{\sigma,s}_0(\mathbb{T}) \Big\} \end{aligned}$$

and

$$\Lambda_+ := \Big\{ l \in \mathbb{Z}^2 \ : \ l_1 = 0, \ l \neq 0 \Big\}, \quad \Lambda_- := \Big\{ l \in \mathbb{Z}^2 \ : \ l_1 + 2l_2 = 0, \ l \neq 0 \Big\}. \tag{2.2}$$

We shall also use in $Q$ the norm

$$|q|^2_{H^1} = |q_+|^2_{H^1(\mathbb{T})} + q_0^2 + |q_-|^2_{H^1(\mathbb{T})} \sim \sum_{l \in \Lambda_- \cup \{0\} \cup \Lambda_+} \hat q_l^2 (|l|^2 + 1) .$$

We decompose the space $\mathcal{H}_{\sigma,s} = Q \oplus P$ where

$$P \ := \ \Big\{ p = \sum_{l \in \mathbb{Z}^2} \hat p_l e^{\mathrm{i} l \cdot \varphi} \in \mathcal{H}_{\sigma,s} \ | \ \hat p_l = 0 \quad \text{for} \quad l_1(2l_2 + l_1) = 0 \Big\} .$$

Projecting equation (2.1) onto the closed subspaces $Q$ and $P$, setting $u = q + p \in \mathcal{H}_{\sigma,s}$ with $q \in Q$ and $p \in P$, we obtain

$$\begin{cases} L_1[q] + \ \Pi_Q f(\varphi_1, q + p, \delta) = 0 & \qquad (Q) \\ \mathcal{L}_\varepsilon[p] + \varepsilon \Pi_P f(\varphi_1, q + p, \delta) = 0 & \qquad (P) \end{cases}$$

where $\Pi_Q : \mathcal{H}_{\sigma,s} \to Q$, $\Pi_P : \mathcal{H}_{\sigma,s} \to P$ are the projectors respectively onto $Q$ and $P$.

In order to prove analiticity of the solutions and to highlight the compactness of the problem we perform a *finite* dimensional Lyapunov-Schmidt reduction, introducing the decomposition

$$Q = Q_1 \oplus Q_2$$

where

$$Q_1 := Q_1(N) := \Big\{ \ q = \sum_{|l| \leq N} \hat q_l e^{\mathrm{i} l \cdot \varphi} \in Q \Big\} , \ Q_2 := Q_2(N) := \Big\{ q = \sum_{|l| > N} \hat q_l e^{\mathrm{i} l \cdot \varphi} \in Q \Big\} .$$

Setting $q = q_1 + q_2$ with $q_1 \in Q_1$ and $q_2 \in Q_2$, we finally get

$$L_1[q_1] + \Pi_{Q_1} \Big[ f(\varphi_1, q_1 + q_2 + p, \delta) \Big] = 0 \quad \Longleftrightarrow \quad d\Psi_\varepsilon(u)[h] = 0 \ \ \forall h \in Q_1 \quad (Q_1) \tag{2.3}$$

$$L_1[q_2] + \Pi_{Q_2} \Big[ f(\varphi_1, q_1 + q_2 + p, \delta) \Big] = 0 \quad \Longleftrightarrow \quad d\Psi_\varepsilon(u)[h] = 0 \ \ \forall h \in Q_2 \quad (Q_2) \tag{2.4}$$

$$\mathcal{L}_\varepsilon[p] + \varepsilon \Pi_P \Big[ f(\varphi_1, q_1 + q_2 + p, \delta) \Big] = 0 \quad \Longleftrightarrow \quad d\Psi_\varepsilon(u)[h] = 0 \ \ \forall h \in P \quad (P) \tag{2.5}$$

where $\Pi_{Q_i} : \mathcal{H}_{\sigma,s} \to Q_i$ are the projectors onto $Q_i$ $(i = 1, 2)$.

We shall solve first the $(Q_2)$-$(P)$-equations for all $|q_1|_{H^1} \leq 2R$, provided $\varepsilon$ belongs to a suitable Cantor-like set, $|\varepsilon| \leq \varepsilon_0(R)$ is sufficiently small and $N \geq N_0(R)$ is large enough (see Lemma 2.3).

[6] $H^\sigma_0(\mathbb{T})$ denotes the functions of $H^\sigma(\mathbb{T})$ with zero average. $H^{\sigma,s}(\mathbb{T}) := \{ u(\varphi) = \sum_{l \in \mathbb{Z}} \hat u_l e^{\mathrm{i} l \varphi} \ : \ \hat u_l^* = \hat u_{-l}, \ |u|_{H^{\sigma,s}(\mathbb{T})} := \sum_{l \in \mathbb{Z}} |\hat u_l| e^{\sigma |l|} [l]^s < +\infty \}$ and $H^{\sigma,s}_0(\mathbb{T})$ its functions with zero average.

Next we shall solve the $(Q_1)$-equation by means of a variational linking argument, see subsection 2.4.

2.2. **The $(Q_2)$-$(P)$-equations.** We first prove that $\mathcal{L}_\varepsilon$ restricted to $P$ has a bounded inverse when $\varepsilon$ belongs to the uncountable zero measure set

$$\mathcal{B}_\gamma := \Big\{ \varepsilon \in (-\varepsilon_0, \varepsilon_0) \ : \quad |l_1 + \varepsilon l_2| > \frac{\gamma}{|l_2|} \ , \ \forall l_1, l_2 \in \mathbb{Z} \setminus \{0\} \Big\}$$

where $0 < \gamma < 1/6$. $\mathcal{B}_\gamma$ accumulates at 0 both from the right and from the left, see [BP].

The operator $\mathcal{L}_\varepsilon$ is diagonal in the Fourier basis $\{e^{\mathrm{i}l \cdot \varphi} \ , \ l \in \mathbb{Z}^2\}$ with eigenvalues $D_l := (l_1 + \varepsilon l_2)(l_1 + (2+\varepsilon) l_2)$.

**Lemma 2.1.** *For $\varepsilon \in \mathcal{B}_\gamma$ the eigenvalues $D_l$ of $\mathcal{L}_\varepsilon$ restricted to $P$, satisfy*

$$|D_l| = \Big| l_1 + \varepsilon l_2 \Big| \Big| (l_1 + 2l_2) + \varepsilon l_2 \Big| > \gamma \qquad \forall l_1 \neq 0, \ l_1 + 2l_2 \neq 0 \ .$$

*As a consequence the operator $\mathcal{L}_\varepsilon : P \to P$ has a bounded inverse $\mathcal{L}_\varepsilon^{-1}$ satisfying*

$$\Big| \mathcal{L}_\varepsilon^{-1}[h] \Big|_{\sigma,s} \leq \frac{|h|_{\sigma,s}}{\gamma}, \qquad \forall h \in P \, . \tag{2.6}$$

*Proof.* Denoting by $[x]$ the nearest integer close to $x$ and $\{x\} = x - [x]$, we have that $D_l > 1$ if both $l_1 \neq -[\varepsilon l_2]$ and $l_1 + 2l_2 \neq -[\varepsilon l_2]$. If $l_1 = -[\varepsilon l_2]$ then

$$|D_l| \geq \frac{\gamma}{|l_2|} (|2 l_2| - \{\varepsilon l_2\}) \geq \gamma.$$

In the same way if $l_1 + 2l_2 = -[\varepsilon l_2]$ we have $|D_l| \geq \frac{\gamma}{|l_2|}(|2l_2| - \{\varepsilon l_2\}) \geq \gamma$. $\square$

**Lemma 2.2.** *The operator $L_1 : Q_2 \to Q_2$ has bounded inverse $L_1^{-1}$ which satisfies*

$$\Big| L_1^{-1}[h] \Big|_{\sigma,s} \leq \frac{|h|_{\sigma,s}}{N^2}. \tag{2.7}$$

*Proof.* $L_1$ is diagonal in the Fourier basis of $Q$: $e^{il\cdot\varphi}$ with $l \in \Lambda_+ \cup \{0\} \cup \Lambda_-$ (recall (2.2)) with eigenvalues

$$d_l = (2+\varepsilon) l_2^2 \ \text{ if } \ l_1 = 0 \quad \text{and} \quad d_l = (-2+\varepsilon) l_2^2 \ \text{ if } \ l_1 + 2l_2 = 0 \, . \tag{2.8}$$

The eigenvalues of $L_1$ restricted to $Q_2(N)$ verify $|d_l| \geq (2-\varepsilon) N^2$ and (2.7) holds. $\square$

Fixed points of the nonlinear operator $\mathcal{G} : Q_2 \oplus P \to Q_2 \oplus P$ defined by

$$\mathcal{G}(q_2, p; q_1) := \Big( - L_1^{-1} \Pi_{Q_2} f(\varphi_1, q_1 + q_2 + p, \delta), \ -\varepsilon \mathcal{L}_\varepsilon^{-1} \Pi_P f(\varphi_1, q_1 + q_2 + p, \delta) \Big)$$

are solutions of the $(Q_2)$-$(P)$-equations.

Using the Contraction Mapping Theorem we can prove:

**Lemma 2.3. (Solution of the $(Q_2)$-$(P)$ equations)** *$\forall R > 0$ there exist an integer $N_0(R) \in \mathbb{N}^+$ and positive constants $\varepsilon_0(R) > 0$, $C_0(R) > 0$ such that:*

$$\forall |q_1|_{H^1} \le 2R\,, \ \ \forall \varepsilon \in B_\gamma,\ |\varepsilon|\gamma^{-1} \le \varepsilon_0(R)\,, \ \ \forall N \ge N_0(R) \ : \ 0 \le \sigma N \le 1\,, \tag{2.9}$$

*there exists a unique solution $(q_2(q_1), p(q_1)) := (q_2(\varepsilon, N, q_1), p(\varepsilon, N, q_1)) \in Q_2 \oplus P$ of the $(Q_2)$-$(P)$ equations satisfying*

$$|q_2(\varepsilon, N, q_1)|_{\sigma,s} \le \frac{C_0(R)}{N^2}, \quad |p(\varepsilon, N, q_1)|_{\sigma,s} \le C_0(R)|\varepsilon|\gamma^{-1}\,. \tag{2.10}$$

*Moreover the map $q_1 \to (q_2(q_1), p(q_1))$ is in $C^1(B_{2R}, Q_2 \oplus P)$ and*

$$\Big|p'(q_1)[h]\Big|_{\sigma,s} \le C_0(R)|\varepsilon|\gamma^{-1}|h|_{H^1}\,, \quad \Big|q_2'(q_1)[h]\Big|_{\sigma,s} \le \frac{C_0(R)}{N^2}|h|_{H^1} \quad \forall h \in Q_1\,. \tag{2.11}$$

*Proof.* In the Appendix. □

2.3. **The $(Q_1)$-equation.** Once the $(Q_2)$-$(P)$-equations have been solved by $(q_2(q_1), p(q_1)) \in Q_2 \oplus P$ there remains the finite dimensional $(Q_1)$-equation

$$L_1[q_1] + \Pi_{Q_1} f(\varphi_1, q_1 + q_2(q_1) + p(q_1), \delta) = 0\ . \tag{2.12}$$

The geometric interpretation of the construction of $(q_2(q_1),\ p(q_1))$ is that on the finite dimensional sub-manifold $Z \equiv \{q_1 + q_2(q_1) + p(q_1)\ :\ |q_1| < 2R\}$, diffeomorphic to the ball

$$B_{2R} := \{q_1 \in Q_1\ :\ |q_1|_{H^1} < 2R\},$$

the partial derivatives of the action functional $\Psi_\varepsilon$ with respect to the variables $(q_2, p)$ vanish. We claim that at a critical point of $\Psi_\varepsilon$ restricted to $Z$, also the partial derivative of $\Psi_\varepsilon$ w.r.t. the variable $q_1$ vanishes and therefore that such point is critical also for the non-restricted functional $\Psi_\varepsilon : \mathcal{H}_{\sigma,s} \to \mathbb{R}$.

Actually the bifurcation equation (2.12) is the Euler-Lagrange equation of the reduced Lagrangian action functional

$$\Phi_{\varepsilon,N} : B_{2R} \subset Q_1 \to \mathbb{R}, \qquad \Phi_{\varepsilon,N}(q_1) := \Psi_\varepsilon(q_1 + q_2(q_1) + p(q_1))\ .$$

**Lemma 2.4.** *$\Phi_{\varepsilon,N} \in \mathcal{C}^1(B_{2R}, \mathbb{R})$ and a critical point $q_1 \in B_{2R}$ of $\Phi_{\varepsilon,N}$ is a solution of the bifurcation equation (2.12). Moreover $\Phi_{\varepsilon,N}$ can be written as*

$$\Phi_{\varepsilon,N}(q_1) = \text{const} + \varepsilon\Big(\Gamma(q_1) + \mathcal{R}_{\varepsilon,N}(q_1)\Big) \tag{2.13}$$

*where*

$$\Gamma(q_1) := \int_{\mathbb{T}^2} \frac{(2+\varepsilon)}{2}(\partial_{\varphi_2} q_1)^2 + (\partial_{\varphi_1} q_1)(\partial_{\varphi_2} q_1) - a_{2d-1}(\varphi_1)\frac{q_1^{2d}}{2d}$$

$$\begin{aligned}\mathcal{R}_{\varepsilon,N}(q_1) := \int_{\mathbb{T}^2} & F(\varphi_1, q_1, \delta = 0) - F(\varphi_1, q_1 + q_2(q_1) + p(q_1), \delta) \\ & + \frac{1}{2} f(\varphi_1, q_1 + q_2(q_1) + p(q_1), \delta)(q_2(q_1) + p(q_1))\end{aligned}$$

*and, for some positive constant $C_2(R) \geq C_1(R)$,*

$$|\mathcal{R}_{\varepsilon,N}(q_1)| \leq C_2(R)\Big(\delta + |\varepsilon|\gamma^{-1} + \frac{1}{N^2}\Big) \tag{2.14}$$

$$\Big|\mathcal{R}'_{\varepsilon,N}(q_1)[h]\Big| \leq C_2(R)\Big(\delta + |\varepsilon|\gamma^{-1} + \frac{1}{N^2}\Big)|h|_{H^1}, \qquad \forall h \in Q_1 . \tag{2.15}$$

*Proof.* In the Appendix. □

The problem of finding non-trivial solutions of the $Q_1$-equation is reduced to finding non-trivial critical points of the reduced action functional $\Phi_{\varepsilon,N}$ in $B_{2R}$.

By (2.13), this is equivalent to find critical points of the rescaled functional (still denoted $\Phi_{\varepsilon,N}$ and called the reduced action functional)

$$\Phi_{\varepsilon,N}(q_1) = \Gamma(q_1) + \mathcal{R}_{\varepsilon,N}(q_1) \equiv \Big(\mathcal{A}(q_1) - \int_{\mathbb{T}^2} a_{2d-1}(\varphi_1)\frac{q_1^{2d}}{2d}\Big) + \mathcal{R}_{\varepsilon,N}(q_1) \tag{2.16}$$

where the quadratic form

$$\mathcal{A}(q) := \int_{\mathbb{T}^2} \frac{(2+\varepsilon)}{2}(\partial_{\varphi_2} q)^2 + (\partial_{\varphi_1} q)(\partial_{\varphi_2} q)$$

is positive definite on $Q_+$, negative definite on $Q_-$ and zero-definite on $Q_0$. For $q_1 = q_+ + q_0 + q_- \in Q_1$,

$$\mathcal{A}(q_1) = \mathcal{A}(q_+ + q_0 + q_-) = \mathcal{A}(q_+) + \mathcal{A}(q_-) = \frac{\alpha_+}{2}|q_+|^2_{H^1} - \frac{\alpha_-}{2}|q_-|^2_{H^1} \tag{2.17}$$

for suitable positive constants $\alpha_+$, $\alpha_-$, bounded away from 0 by constants independent of $\varepsilon$.

We shall prove the existence of critical points of $\Phi_{\varepsilon,N}$ in $B_{2R}$ of "linking type".

### 2.4. Linking critical points of the reduced action functional $\Phi_{\varepsilon,N}$.

We can not directly apply the linking Theorem because $\Phi_{\varepsilon,N}$ is defined only in the ball $B_{2R}$. Therefore our first step is to extend $\Phi_{\varepsilon,N}$ to the whole space $Q_1$.

**Step 1:** *Extension of* $\Phi_{\varepsilon,N}$. We define the extended action functional $\widetilde{\Phi}_{\varepsilon,N} \in C^1(Q_1, \mathbb{R})$ as

$$\widetilde{\Phi}_{\varepsilon,N}(q_1) := \Gamma(q_1) + \widetilde{\mathcal{R}}_{\varepsilon,N}(q_1)$$

where $\widetilde{\mathcal{R}}_{\varepsilon,N} : Q_1 \to \mathbb{R}$ is

$$\widetilde{\mathcal{R}}_{\varepsilon,N}(q_1) := \lambda\Big(\frac{|q_1|^2_{H_1}}{R^2}\Big)\mathcal{R}_{\varepsilon,N}(q_1)$$

and $\lambda : [0, +\infty) \to [0, 1]$ is a smooth, non-increasing, cut-off function such that

$$\begin{cases} \lambda(x) = 1 & |x| \leq 1 \\ \lambda(x) = 0 & |x| \geq 4 \end{cases} \qquad |\lambda'(x)| < 1 .$$

By definition $\widetilde{\Phi}_{\varepsilon,N} \equiv \Phi_{\varepsilon,N}$ on $B_R := \{q_1 \in Q_1 \ : \ |q_1|_{H^1} \leq R\}$ and $\widetilde{\Phi}_{\varepsilon,N} \equiv \Gamma$ outside $B_{2R}$.

Moreover, by (2.14)-(2.15), there is a constant $C_3(R) \geq C_2(R) > 0$ such that $\forall |q_1|_{H^1} \leq 2R$

$$|\widetilde{\mathcal{R}}_{\varepsilon,N}(q_1)| \leq C_3(R)\Big(\delta + |\varepsilon|\gamma^{-1} + \frac{1}{N^2}\Big) \tag{2.18}$$

$$\Big|\widetilde{\mathcal{R}}'_{\varepsilon,N}(q_1)[h]\Big| \leq C_3(R)\Big(\delta + |\varepsilon|\gamma^{-1} + \frac{1}{N^2}\Big)|h|_{H^1}, \qquad \forall h \in Q_1 . \tag{2.19}$$

In the sequel we shall always assume

$$C_3(R)\Big(\delta + |\varepsilon|\gamma^{-1} + \frac{1}{N^2}\Big) \leq 1 .$$

**Step 2:** $\widetilde{\Phi}_{\varepsilon,N}$ *verifies the geometrical hypotheses of the linking Theorem.*

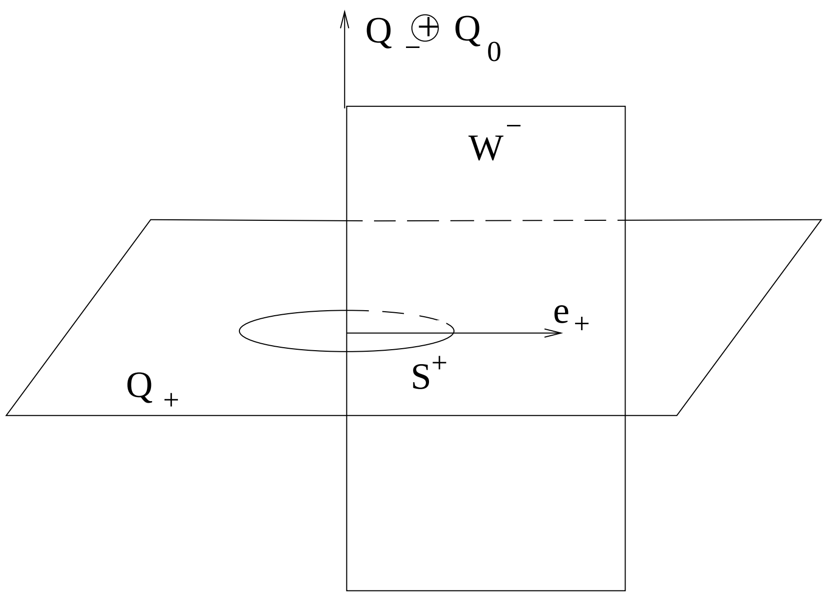


FIGURE 1. The cylinder $W^-$ and the sphere $S^+$ link.

**Lemma 2.5.** *There exist $\varepsilon$-$N$-$\gamma$-independent positive constants $\rho$, $\omega$, $r_1$, $r_2 > \rho$, and $0 < \varepsilon_1(R) \leq \varepsilon_0(R)$, $N_1(R) \geq N_0(R)$ such that, $\forall |\varepsilon|\gamma^{-1} \leq \varepsilon_1(R)$, $\forall N \geq N_1(R)$*

- **(i)** $\widetilde{\Phi}_{\varepsilon,N}(q_1) \geq \omega > 0$, $\forall q_1 \in S^+ := \Big\{ q_1 \in Q_1 \cap Q_+ : |q_1|_{H_1} = \rho \Big\}$,
- **(ii)** $\widetilde{\Phi}_{\varepsilon,N}(q_1) \leq \omega/2$, $\forall q_1 \in \partial W^-$ *where $W^-$ is the cylinder*

$$W^- := \Big\{ q_1 = q_0 + q_- + re^+ \, , \ |q_0 + q_-|_{H_1} \leq r_1 \, , \ q_- \in Q_1 \cap Q_-, \ q_0 \in \mathbb{R} \, , \, r \in [0, r_2] \Big\}$$

*and $e_+ := \cos(\varphi_2) \in Q_1 \cap Q_+$. Note that $\rho, \omega$ are independent of $R$.*

In the following $\kappa_i$, $\kappa_\pm$ will denote positive constants *independent* on $R$, $N$, $\varepsilon$ and $\gamma$.

*Proof.* **(i)** $\forall q_+ \in Q_1 \cap Q_+$ with $|q_+|_{H^1} = \rho < R$ we have

$$\begin{aligned}\widetilde{\Phi}_{\varepsilon,N}(q_+) = \Phi_{\varepsilon,N}(q_+) &= \mathcal{A}(q_+) - \int_{\mathbb{T}^2} a_{2d-1}(\varphi_1)\frac{q_+^{2d}}{2d} + \mathcal{R}_{\varepsilon,N}(q_+) \\ &\geq \frac{\alpha_+}{2}\rho^2 - \kappa_1\rho^{2d} - \Big(\delta + |\varepsilon|\gamma^{-1} + \frac{1}{N^2}\Big)C_3(R). \end{aligned} \tag{2.20}$$

Now we fix $\rho > 0$ small such that $(\alpha_+\rho^2/2) - \kappa_1\rho^{2d} \geq \alpha_+\rho^2/4$. Next, for $(\delta + |\varepsilon|\gamma^{-1} + N^{-2})C_3(R) \leq \alpha_+\rho^2/8$ we get by (2.20)

$$\widetilde{\Phi}_{\varepsilon,N}(q_+) \geq \frac{\alpha_+}{8}\rho^2 =: \omega > 0, \qquad \forall\ q_+ \in Q_1 \cap Q^+ \quad \text{with} \quad |q_+| = \rho\ .$$

**(ii)** Let

$$\begin{aligned} B_1 &:= \Big\{q_1 = q_0 + q_- + r_2 e_+ \text{ with } |q_0 + q_-|_{H^1} \leq r_1, q_- \in Q_1 \cap Q_-\Big\} \subset \partial W^- \\ B_2 &:= \Big\{q_1 = q_0 + q_- + r e_+ \text{ with } |q_0 + q_-|_{H^1} = r_1, q_- \in Q_1 \cap Q_-,\ r \in [0, r_2]\Big\} \subset \partial W^- \end{aligned}$$

and choose $r_1, r_2 > 2R$. For $q_1 = q_0 + q_- + re_+ \in B_1 \cup B_2$

$$\begin{aligned} \widetilde{\Phi}_{\varepsilon,N}(q_1) = \Gamma(q_1) &= \mathcal{A}(q_1) - \int_{\mathbb{T}^2} a_{2d-1}(\varphi_1)(q_0 + q_- + re_+)^{2d} \\ &= -\frac{\alpha_-}{2}|q_-|^2_{H^1} + r^2\mathcal{A}(e_+) - \int_{\mathbb{T}^2} a_{2d-1}(\varphi_1)\frac{(q_0 + q_- + re_+)^{2d}}{2d} \\ &\leq -\frac{\alpha_-}{2}|q_-|^2_{H^1} + r^2\mathcal{A}(e_+) - \alpha\int_{\mathbb{T}^2}(q_0 + q_- + re_+)^{2d} \end{aligned} \tag{2.21}$$

because $a_{2d-1}(\varphi_1)/2d \geq \alpha > 0$. Now, by Hölder inequality and orthogonality

$$\begin{aligned} \int_{\mathbb{T}^2}(q_0 + q_- + re_+)^{2d} &\geq \kappa_2\Big(\int_{\mathbb{T}^2}(q_0 + q_- + re_+)^2\Big)^d \\ &= \kappa_2\Big(\int_{\mathbb{T}^2} q_0^2 + q_-^2 + r^2 e_+^2\Big)^d \\ &\geq \kappa_3(q_0^2 + r^2)^d \geq \kappa_3(q_0^{2d} + r^{2d}) \end{aligned}$$

and by (2.21) we deduce

$$\widetilde{\Phi}_{\varepsilon,N}(q_0 + q_- + re_+) \leq (\kappa_+ r^2 - \kappa_3 r^{2d}) - \Big(\frac{\alpha_-}{2}|q_-|^2_{H^1} + \kappa_3 q_0^{2d}\Big)\ .$$

Now we fix $r_2$ large such that $\kappa_+ r_2^2 - \kappa_3 r_2^{2d} \leq 0$ and therefore

$$\widetilde{\Phi}_{\varepsilon,N}(q_1) \leq \kappa_+ r_2^2 - \kappa_3 r_2^{2d} \leq 0 \qquad \forall q_1 \in B_1\ .$$

Next, setting $M := \max_{r\in[0,r_2]}(\kappa_+ r^2 - \kappa_3 r^{2d})$, we fix $r_1$ large such that

$$\frac{\alpha_-}{2}|q_-|^2_{H^1} + \kappa_3 q_0^{2d} \geq M \qquad \forall\ |q_- + q_0| = r_1$$

and therefore

$$\widetilde{\Phi}_{\varepsilon,N}(q_1) \leq M - \Big(\frac{\alpha_-}{2}|q_-|^2_{H^1} + \kappa_3 q_0^{2d}\Big) \leq 0 \qquad \forall q_1 \in B_2\ .$$

Finally if $q_1 = q_- + q_0$:

$$\begin{aligned}\widetilde{\Phi}_{\varepsilon,N}(q_1) &= \mathcal{A}(q_-) - \int_{\mathbb{T}^2} a_{2d-1}(\varphi_1)\frac{q_1^{2d}}{2d} + \widetilde{\mathcal{R}}_{\varepsilon,N}(q_1) \\ &\le |\widetilde{\mathcal{R}}_{\varepsilon,N}(q_1)| \le C_3(R)(\delta + |\varepsilon|\gamma^{-1} + N^{-2}) \end{aligned} \tag{2.22}$$

and so $\widetilde{\Phi}_{\varepsilon,N}(q_1) \le \omega/2$ if $C_3(R)(\delta + |\varepsilon|\gamma^{-1} + N^{-2}) \le \omega/2$. □

We introduce the minimax class

$$\mathcal{S} := \Big\{ \psi \in C(\overline{W^-}, Q) \ \mid \ \psi = \mathrm{Id} \ \text{ on } \ \partial W^- \Big\}.$$

The maps of $\mathcal{S}$ have an important intersection property, see e.g. Proposition 5.9 of [R3].

**Proposition 1.** **($S^+$ and $W^-$ link with respect to $\mathcal{S}$.)**

$$\psi \in \mathcal{S} \quad \Longrightarrow \quad \psi(W^-) \cap S^+ \neq \emptyset .$$

Define the minimax linking level

$$\mathcal{K}_{\varepsilon,N} := \inf_{\psi \in \mathcal{S}} \max_{q \in W^-} \widetilde{\Phi}_{\varepsilon,N}(\psi(q_1)) .$$

By the intersection property of Proposition 1 and Lemma 2.5-($i$)

$$\max_{q_1 \in W^-} \widetilde{\Phi}_{\varepsilon,N}(\psi(q_1)) \ge \min_{q_1 \in S^+} \widetilde{\Phi}_{\varepsilon,N}(q_1) \ge \omega > 0 \qquad \forall \psi \in \mathcal{S}$$

and therefore

$$\mathcal{K}_{\varepsilon,N} > \omega > 0 .$$

Moreover, since $\mathrm{Id} \in \mathcal{S}$ and (2.18)

$$\begin{aligned}\mathcal{K}_{\varepsilon,N} &\le \max_{q_1 \in W^-} \widetilde{\Phi}_{\varepsilon,N}(q_1) \le \max_{q_1 \in W^-} \Big( \Gamma(q_1) + \widetilde{\mathcal{R}}_{\varepsilon,N}(q_1) \Big) \\ &\le \max_{q_1 \in W^-} \Big( \frac{\alpha_+}{2} |q_+|^2_{H^1} + \frac{\alpha_-}{2} |q_-|^2_{H^1} + \int_{\mathbb{T}^2} \kappa q_1^{2d} \Big) + 1 \le \mathcal{K}_\infty < +\infty \end{aligned} \tag{2.23}$$

where $\mathcal{K}_\infty$ is independent of $N, \varepsilon, \gamma$ since the constants $r_1, r_2$ in the definition of $W^-$ are independent of $N, \varepsilon, \gamma$.

We deduce, by the linking theorem the existence of a (Palais-Smale) sequence $(q_j) \in Q_1$ at the level $\mathcal{K}_{\varepsilon,N}$, namely

$$\widetilde{\Phi}_{\varepsilon,N}(q_j) \to \mathcal{K}_{\varepsilon,N} , \qquad \widetilde{\Phi}'_{\varepsilon,N}(q_j) \to 0 . \tag{2.24}$$

**Step 3:** *Existence of a nontrivial critical point.* Our final aim is to prove that the Palais-Smale sequence $q_j$ converges, up to subsequence, to some non-trivial critical point $\overline{q}_1 \neq 0$ in some open ball of $Q_1$ where $\widetilde{\Phi}_{\varepsilon,N}$ and $\Phi_{\varepsilon,N}$ coincide.

**Lemma 2.6.** *There exists a constant $R_* > 0$, independent on $R$-$\varepsilon$-$N$-$\gamma$, and functions $0 < \varepsilon_2(R) \le \varepsilon_1(R)$, $N_2(R) \ge N_1(R)$ such that for all $|\varepsilon|\gamma^{-1} \le \varepsilon_2(R)$, $N \ge N_2(R)$ the functional $\widetilde{\Phi}_{\varepsilon,N}$ possesses a non-trivial critical point $\overline{q}_1 \in Q_1$ with critical value $\widetilde{\Phi}_{\varepsilon,N}(\overline{q}_1) = \mathcal{K}_{\varepsilon,N}$, satisfying $|\overline{q}_1|_{H_1} \le R_*$.*

*Proof.* Writing $\widetilde{\Phi}_{\varepsilon,N}(q) = \Gamma(q) + \widetilde{\mathcal{R}}_{\varepsilon,N}(q)$ we derive, by (2.18)-(2.19)

$$\begin{aligned}
\widetilde{\Phi}_{\varepsilon,N}(q_j) - \frac{1}{2}\widetilde{\Phi}'_{\varepsilon,N}(q_j)[q_j] &= \Gamma(q_j) - \frac{1}{2}\Gamma'(q_j)[q_j] + \Big(\widetilde{\mathcal{R}}_{\varepsilon,N}(q_j) - \frac{1}{2}\widetilde{\mathcal{R}}'_{\varepsilon,N}(q_j)[q_j]\Big) \\
&= \Big(\frac{1}{2} - \frac{1}{2d}\Big)\int_{\mathbb{T}^2} a_{2d-1}(\varphi_1) q_j^{2d} + \Big(\widetilde{\mathcal{R}}_{\varepsilon,N}(q_j) - \frac{1}{2}\widetilde{\mathcal{R}}'_{\varepsilon,N}(q_j)[q_j]\Big) \\
&\geq \alpha\Big(\frac{1}{2} - \frac{1}{2d}\Big)\int_{\mathbb{T}^2} q_j^{2d} - (\delta + |\varepsilon|\gamma^{-1} + N^{-2})C_3(R) \, .
\end{aligned}$$

Therefore, by (2.23)-(2.24)

$$\mathcal{K}_\infty + 1 + |q_j|_{H_1} \geq \kappa_1 \int_{\mathbb{T}^2} q_j^{2d} := \kappa_1 |q_j|^{2d}_{L^{2d}} \, . \tag{2.25}$$

We also deduce, by (2.25), Hölder inequality and orthogonality

$$\begin{aligned}
\mathcal{K}_\infty + 1 + |q_j|_{H_1} &\geq \kappa_2 \Big(\int_{\mathbb{T}^2} \Big(q_{+,j} + q_{0,j} + q_{-,j}\Big)^2\Big)^d \\
&= \kappa_2 \Big(\int_{\mathbb{T}^2} q_{+,j}^2 + q_{0,j}^2 + q_{-,j}^2\Big)^d \geq \kappa_3 \, (q_{0,j})^{2d}
\end{aligned}$$

and therefore

$$|q_{0,j}| \leq \kappa_4 \Big(1 + |q_j|_{H_1}\Big)^{1/2d} \, . \tag{2.26}$$

By (2.18)-(2.19) and Hölder inequality

$$\begin{aligned}
\widetilde{\Phi}'_{\varepsilon,N}(q_j)[q_{+,j}] &= \alpha_+ |q_{+,j}|^2_{H_1} - \int_{\mathbb{T}^2} a_{2d-1}(\varphi_1) q_j^{2d-1} q_{+,j} + \widetilde{\mathcal{R}}'_{\varepsilon,N}(q_j)[q_{+,j}] \\
&\geq \alpha_+ |q_{+,j}|^2_{H_1} - \kappa_5 |q_{+,j}|_{H_1} \int_{\mathbb{T}^2} |q_j|^{2d-1} - (\delta + \gamma^{-1}|\varepsilon| + N^{-2})C_3(R)|q_{+,j}|_{H_1} \\
&\geq \kappa_6 |q_{+,j}|_{H_1} \Big(|q_{+,j}|_{H_1} - |q_j|^{2d-1}_{L^{2d}} - 1\Big) \, . \qquad (2.27)
\end{aligned}$$

By (2.27) and (2.25), using that $\widetilde{\Phi}'_{\varepsilon,N}(q_j) \to 0$ and simple inequalities, we conclude

$$|q_{+,j}|_{H_1} \leq \kappa_7 \Big(1 + |q_j|^{(2d-1)/2d}_{H_1}\Big) \, .$$

Estimating analogously $\widetilde{\Phi}'_{\varepsilon,N}(q_j)[q_{-,j}]$ we derive

$$|q_{-,j}|_{H_1} \leq \kappa_8 \Big(1 + |q_j|^{(2d-1)/2d}_{H_1}\Big)$$

and by (2.26) we finally deduce

$$|q_j|_{H_1} = |q_{0,j}| + |q_{+,j}|_{H_1} + |q_{-,j}|_{H_1} \leq \kappa_9 \Big(1 + |q_j|^{1/2d}_{H_1} + |q_j|^{(2d-1)/2d}_{H_1}\Big) \, .$$

We conclude that $|q_j|_{H_1} \leq R_*$ for a suitable positive constant $R_*$ independent of $\varepsilon$, $N$, $R$ and $\gamma$.

Since $Q_1$ is finite dimensional $q_j$ converges, up to subsequence, to some critical point $\overline{q}_1$ of $\widetilde{\Phi}_{\varepsilon,N}$ with $|\overline{q}_1|_{H_1} \leq R_*$. Finally, since $\widetilde{\Phi}_{\varepsilon,N}(\overline{q}_1) = \mathcal{K}_{\varepsilon,N} \geq \omega > 0$ we conclude that $\overline{q}_1 \neq 0$. $\square$

We are now ready to prove Theorem A in the case $\omega_1 = 1$.

*Proof of Theorem A for* $\omega_1 = 1$. Let us fix

$$\bar{R} := R_* + 1 \qquad \text{and take} \qquad |\varepsilon|\gamma^{-1} \le \varepsilon_2(\bar{R}) := \overline{\varepsilon}\,.$$

Set $\bar{N} := N_2(\bar{R}) \ge N_0(\bar{R})$.

Applying Lemma 2.3 we obtain, for

$$0 < \sigma \le \frac{1}{N_2(\bar{R})}$$

a solution $(q_2(q_1), p(q_1)) \in (Q_2(\bar{N}) \oplus P) \cap \mathcal{H}_{\sigma,s}$ of the $(Q_2)$-$(P)$ equations $\forall |q_1|_{H_1} \le 2\bar{R}$. By Lemma 2.6 the extended functional $\widetilde{\Phi}_{\varepsilon,N}(q_1)$ possesses a critical point $\bar{q}_1 \ne 0$ with $|\bar{q}_1|_{H_1} \le R_* < \bar{R}$. Since $\widetilde{\Phi}_{\varepsilon,N}(q_1)$ coincides with $\Phi_{\varepsilon,N}(q_1)$ on the ball $B_{\bar{R}}$ we get, by Lemma 2.4, the existence of a nontrivial weak solution $\overline{q}_1 + q_2(\overline{q}_1) + p(\overline{q}_1) \in \mathcal{H}_{\sigma,s}$ of equation (2.1). Finally

$$u = |\varepsilon|^{1/2(d-1)}\Big[\overline{q}_1 + q_2(\overline{q}_1) + p(\overline{q}_1)\Big] \equiv |\varepsilon|^{1/2(d-1)}\Big[\overline{q}_\varepsilon + p(\overline{q}_1)\Big]$$

solves equation (1.4).

The solution $\overline{q}_\varepsilon := \overline{q}_1 + q_2(\overline{q}_1)$ of the $(Q)$-equation belongs to $Q \cap \mathcal{H}_{\sigma,s+2}$ by the regularizing properties of $L_1^{-1}$, see in Lemma 2.2 formula (2.8).

Since $\bar{p} := p(\overline{q}_1)$ solves

$$\Big(\partial^2_{\varphi_1} + 2(1+\varepsilon)\partial_{\varphi_1}\partial_{\varphi_2}\Big)\bar{p} = -\varepsilon\Big[(2+\varepsilon)\partial^2_{\varphi_2}\bar{p} + \Pi_P f(\varphi_1, u, \delta)\Big] \in \mathcal{H}_{\sigma',s} \quad \forall 0 < \sigma' < \sigma \tag{2.28}$$

and the eigenvalues of $\partial^2_{\varphi_1} + 2(1+\varepsilon)\partial_{\varphi_1}\partial_{\varphi_2}$ restricted to $P$ satisfy, for $\varepsilon \in \mathcal{B}_\gamma$,

$$\Big| l_1\Big[(l_1 + 2l_2) + \varepsilon 2 l_2\Big]\Big| \ge \gamma\frac{|l_1|}{2|l_2|} \qquad \forall\, l_1 + 2l_2 \ne 0,\ l_2 \ne 0$$

and we deduce that $\bar{p} \in \mathcal{H}_{\sigma'',s+1}$ for all $0 < \sigma'' < \sigma'$ and $|\partial_{\varphi_1}\bar{p}|_{\sigma'',s} = O(|\varepsilon|\gamma^{-1})$. Now, again by (2.28),

$$\partial^2_{\varphi_1}\bar{p} = -2(1+\varepsilon)\partial_{\varphi_2}\partial_{\varphi_1}\bar{p} - \varepsilon\Big[(2+\varepsilon)\partial^2_{\varphi_2}\bar{p} + \Pi_P f(\varphi_1, u, \delta)\Big] \in \mathcal{H}_{\overline{\sigma},s} \quad \forall 0 < \overline{\sigma} < \sigma''\,.$$

therefore $\bar{p} \in \mathcal{H}_{\overline{\sigma},s+2}$ and $|\bar{p}|_{\overline{\sigma},s+2} = O(|\varepsilon|\gamma^{-1})$. (1.5) follows with $\bar{s} := s + 2 > 2$.

By (1.2), the function $v(\varepsilon, t, x) = u(\varepsilon, t, x + (1+\varepsilon)t)$ is a solution of equation (1.1) with $\omega_1 = 1$. The frequency $\omega_2 = 1 + \varepsilon \notin \mathbb{Q}$ since $\varepsilon \in \mathcal{B}_\gamma$. To show that $v(\varepsilon, t, x)$ is quasi-periodic it remains to prove that $u$ depends on both the variables $(\varphi_1, \varphi_2)$ independently.

We claim that $\bar{q}_1 \notin Q_0 \oplus Q_-$, i.e. $\overline{q}_+(\varphi_2) \in Q_+ \setminus \{0\}$, and therefore $u$ depends on $\varphi_2$. Indeed by Lemma 2.6 we know that $\tilde{\Phi}_{\varepsilon,N}(\bar{q}_1) > \omega > 0$ and $|\overline{q}_1|_{H^1(\mathbb{T})} < \overline{R}$. On the other hand, by (2.22) in Lemma 2.5 $\tilde{\Phi}_{\varepsilon,N}(q_- + q_0) < \omega/2$, for all $|q_- + q_0|_{H^1} \le \bar{R}$, so that necessarily $\bar{q}_1 \notin Q_0 \oplus Q_-$.

We claim that any solution $u$ of (2.1) depending only on $\varphi_2$, namely solving

$$(2+\varepsilon)u''(\varphi_2) + f(\varphi_1, u(\varphi_2), \delta) = 0\,, \tag{2.29}$$

is $u(\varphi_2) \equiv 0$ . Indeed, by definition,

$$\delta^{2(d-1)} f(\varphi_1, u, \delta) = f(\varphi_1, \delta u) = \sum_{k=2d-1}^{\infty} a_k(\varphi_1)(\delta u)^k$$

(recall $\mathrm{sign}(\varepsilon) = 1$). Consider now a smooth zero mean function $g(\varphi_1)$ such that $\int_0^{2\pi} a_k(\varphi_1) g(\varphi_1) \neq 0$ for some $k$ (recall that by assumption (**H**) some of the $a_k(\varphi_2)$ are not constant). By (2.29) we have

$$(2+\varepsilon)u''(\varphi_2)\int_0^{2\pi} g(\varphi_1)d\varphi_1 + \int_0^{2\pi} f(\varphi_1, u(\varphi_2), \delta) g(\varphi_1) d\varphi_1 = 0$$

which implies, by the assumption **(H)** on $f$,

$$\sum_{k=2d-1}^{\infty} [\delta u(\varphi_2)]^k \int_0^{2\pi} a_k(\varphi_1) g(\varphi_1) d\varphi_1 = 0\,. \tag{2.30}$$

The function $G(z) := \sum_{k=2d-1}^{\infty} b_k z^k$ with $b_k := \int_0^{2\pi} a_k(\varphi_1) g(\varphi_1) d\varphi_1$ is a nontrivial analytic function. Therefore equation (2.30), i.e. $G(\delta u(\varphi_2)) = 0$, cannot have a sequence of zeros accumulating to 0. So, for $\delta$ small enough, $u(\varphi_2) \equiv 0$. □

*Proof of Theorem A for any rational frequency* $\omega_1 = \frac{n}{m} \in \mathbb{Q}$. Consider now equation (1.8) with $\omega_1 = n/m$ where $n, m$ are coprime integers.

The space $Q$, formed by the solutions of $\partial_{\varphi_1}(\frac{n}{m}\partial_{\varphi_1} + 2\partial_{\varphi_2})q = 0$ can be written as

$$Q = \Big\{ q = \sum_{l \in \mathbb{Z}^2} \hat{q}_l e^{\mathrm{i} l \cdot \varphi} \in \mathcal{H}_{\sigma,s} \ | \ \hat{q}_l = 0 \quad \text{for} \quad l_1(nl_1 + 2ml_2) \neq 0 \Big\}$$

and is composed by functions of the form

$$q(\varphi) = q_+(\varphi_2) + q_-(2m\varphi_1 - n\varphi_2) + q_0\,.$$

Let $P$ be the supplementary space to $Q$ and perform the Lyapunov-Schmidt decomposition like in (2.3)-(2.4)-(2.5).

For $\varepsilon$ in the Cantor set $\mathcal{B}_\gamma$, the eigenvalues

$$D_l = \Big(\frac{n}{m} l_1 + \varepsilon l_2\Big)\Big(\frac{n}{m} l_1 + 2l_2 + \varepsilon l_2\Big)$$

of the linear operator $\mathcal{L}_\varepsilon$ can be bounded, arguing as in Lemma 2.1, by

$$|D_l| = \frac{|(nl_1 + \varepsilon m l_2)(nl_1 + 2ml_2 + \varepsilon m l_2)|}{m^2} > \frac{\gamma}{m^2} \qquad \forall l_1 \neq 0, \ nl_1 + 2ml_2 \neq 0\,.$$

As a consequence

$$\Big| \mathcal{L}_\varepsilon^{-1}[h] \Big|_{\sigma,s} \leq \frac{m^2 |h|_{\sigma,s}}{\gamma}, \qquad \forall h \in P\,,$$

and, in solving the $(Q_2)$-$(P)$ equations as in Lemma 2.3, we obtain the new restriction

$$\gamma^{-1}|\varepsilon| \leq \frac{\varepsilon_0(R)}{m^2}\,, \qquad N \geq N_0(R)$$

and the bound (compare with (2.10)) $|p(q_1)|_{\sigma,s} \leq C_0(R)|\varepsilon|\gamma^{-1} m^2$.

The corresponding reduced action functional has again the form (2.13)-(2.16) with the different quadratic part

$$\mathcal{A}(q_1) = \mathcal{A}(q_+ + q_0 + q_-) = \mathcal{A}(q_+) + \mathcal{A}(q_-) = \frac{\alpha_+}{2}|q_+|^2_{H^1} - n^2\frac{\alpha_-}{2}|q_-|^2_{H^1}$$

and therefore it still possesses a linking critical point $\bar{q}_1 \in Q_1$.

To prove the bound (1.5) note that the eigenvalues of $\omega_1^2\partial^2_{\varphi_1} + 2\omega_1(1+\varepsilon)\partial_{\varphi_1}\partial_{\varphi_2}$ ($\omega_1 = n/m$) restricted to $P$ satisfy, for $\varepsilon \in \mathcal{B}_\gamma$,

$$\omega_1\Big|l_1\frac{(nl_1 + 2l_2m) + \varepsilon 2l_2m}{m}\Big| \geq \frac{\omega_1|l_1|\gamma}{2|l_2|m^2} \qquad \forall\, nl_1 + 2ml_2 \neq 0,\ l_2 \neq 0$$

and therefore $\bar{p} \in \mathcal{H}_{\overline{\sigma},s+2}$ and $|\bar{p}|_{\overline{\sigma},s+2} = O(|\varepsilon|m^2/\omega_1^3\gamma)$. □

## 3. **Case B:** $\omega_1 \notin \mathbb{Q}$

We now look for solutions of equation (1.8) when the forcing frequency $\omega_1$ is an irrational number.

To fix notations we shall prove Theorem B when $\int_0^{2\pi} a_{2d-1}(\varphi_1)d\varphi_1 > 0$ and therefore $\varepsilon > 0$, i.e. $\text{sign}(\varepsilon) = 1$ .

Fixed $0 < \overline{s} < s - 1/2$, the Nemitskii operator $u \to f(\varphi_1, u, \delta) \in C^\infty(B_\rho, \mathcal{H}_{\sigma,\overline{s}})$ since, if $a_k(\varphi_1) \in H^s(\mathbb{T})$, then $a_k(\cdot) \in \mathcal{H}_{\sigma,\overline{s}}$, $\forall\sigma > 0$, $0 < \overline{s} < s - 1/2$.

For $\varepsilon = 0$ equation (1.8) reduces to

$$\omega_1\partial_{\varphi_1}\Big(\omega_1\partial_{\varphi_1} + 2\partial_{\varphi_2}\Big)q = 0 \tag{3.1}$$

and its solutions $q$ form, by the irrationality of $\omega_1$, the infinite dimensional subspace

$$Q := \Big\{q \in \mathcal{H}_{\sigma,\overline{s}}\ :\ \partial_{\varphi_1}q \equiv 0\Big\} = \Big\{q = q(\varphi_2) \in H^\sigma(\mathbb{T})\Big\}\ . \tag{3.2}$$

To find solutions of (1.8) for $\varepsilon \neq 0$, we perform a Lyapunov-Schmidt reduction and we decompose the space

$$\mathcal{H}_{\sigma,\overline{s}} = Q \oplus P$$

where $Q \equiv H^\sigma(\mathbb{T})$ and

$$P \ := \ \Big\{p = \sum_{l\in\mathbb{Z}^2}\hat{p}_l e^{\mathrm{i}l\cdot\varphi} \in \mathcal{H}_{\sigma,\overline{s}}\ |\ \hat{p}_l = 0 \quad \text{for} \quad l_1 = 0\Big\}\ .$$

Projecting equation (1.8) onto the closed subspaces $Q$ and $P$, setting $u = q + p \in \mathcal{H}_{\sigma,\overline{s}}$ with $q \in Q$, $p \in P$ we obtain

$$(2+\varepsilon)\ddot{q} + \Pi_Q\Big[f(\varphi_1, q+p, \delta)\Big] \ = \ 0 \qquad (Q) \tag{3.3}$$

$$\mathcal{L}_\varepsilon[p] + \varepsilon\,\Pi_P\Big[f(\varphi_1, q+p, \delta)\Big] \ = \ 0 \qquad (P) \tag{3.4}$$

where $\ddot{q} = \partial^2_{\varphi_2}q$, $\Pi_Q : \mathcal{H}_{\sigma,\overline{s}} \to Q$ is the projector onto $Q$,

$$(\Pi_Q u)(\varphi_2) := \frac{1}{2\pi}\int_0^{2\pi} u(\varphi_1,\varphi_2)\ d\varphi_1\,,$$

and $\Pi_P = \text{Id} - Q$ is the projector onto $P$.

We could proceed now as in the previous section performing a finite dimensional reduction and applying variational methods. However, in this case, the infinite dimensional $(Q)$-equation can be directly solved by the Implicit Function Theorem in a space of analytic functions.

For this, it is useful to consider the parameter $\delta$ (and $\varepsilon = \delta^{2(d-1)}$) in the right hand side of (3.4), as an independent parameter $\delta = \eta$, $\varepsilon = \eta^{2(d-1)}$, and to solve the equation

$$\mathcal{L}_\varepsilon[p] + \eta^{2(d-1)}\, \Pi_P\Big[f(\varphi_1, q+p, \eta)\Big] = 0 \qquad (P_\eta) \tag{3.5}$$

for $(\varepsilon, \omega_1)$ in the Cantor set $\mathcal{C}_\gamma$ and *for all* $\eta$ small. In this way we highlight the smoothness of the solution $p(\eta, \varepsilon, \cdot)$ of the $(P_\eta)$-equation (3.5) in the variable $\eta$.

3.1. **Solution of the** $(P_\eta)$**-equation.** We first prove that the operator $\mathcal{L}_\varepsilon : P \to P$ has a bounded inverse when $(\varepsilon, \omega_1)$ belongs to the Cantor set $\mathcal{C}_\gamma$ defined in (1.6).

**Lemma 3.1.** *For any $\varepsilon_0 > 0$ the Cantor set $\mathcal{C}_\gamma$ is uncountable.*

*Proof.* Consider the set $\overline{\mathcal{C}}$ of couples $x_1$, $x_2 \in \mathcal{B}_\gamma$ such that:

$$x_1 \in (-\varepsilon_1, \varepsilon_1)\,, \quad x_2 \in \Big(1+\varepsilon_1, 2-\varepsilon_1\Big)\,, \quad x_1 + x_2 \notin \mathbb{Q}\,, \quad x_1 - x_2 \notin \mathbb{Q}\,.$$

where $\varepsilon_1 = \varepsilon_0/2$. $\overline{\mathcal{C}}$ is an uncountable subset of $\mathbb{R}^2$ since for all $x_1 \in \mathcal{B}_\gamma$ the conditions $x_1 \pm x_2 \notin \mathbb{Q}$ exclude only a countable set of values $x_2$. The Lemma follows since $\mathcal{C}_\gamma$ contains $\psi^{-1}\overline{\mathcal{C}}$ where $\psi : (\varepsilon, \omega_1) \to (\varepsilon/\omega_1,\ (2+\varepsilon)/\omega_1)$ is an invertible map for $(\varepsilon, \omega_1) \in (-\varepsilon_0, \varepsilon_0) \times (1, 2)$. $\square$

The operator $\mathcal{L}_\varepsilon$ has eigenvalues $D_l = (\omega_1 l_1 + \varepsilon l_2)(\omega_1 l_1 + 2l_2 + \varepsilon l_2)$.

**Lemma 3.2.** *For $(\varepsilon, \omega_1) \in \mathcal{C}_\gamma$ the eigenvalues $D_l$ of $\mathcal{L}_\varepsilon$ restricted to $P$ satisfy*

$$|D_l| = \Big|(\omega_1 l_1 + \varepsilon l_2)(\omega_1 l_1 + 2l_2 + \varepsilon l_2)\Big| > \gamma\,, \qquad \forall l_1 \neq 0\,. \tag{3.6}$$

*As a consequence, the operator $\mathcal{L}_\varepsilon : P \to P$ has a bounded inverse $\mathcal{L}_\varepsilon^{-1}$ satisfying*

$$\Big|\mathcal{L}_\varepsilon^{-1}[p]\Big|_{\sigma,\bar{s}} \leq \frac{|p|_{\sigma,\bar{s}}}{\gamma}, \qquad \forall p \in P\,. \tag{3.7}$$

*Proof.* Estimate (3.6) is trivially satisfied if $-l_1 \neq \frac{\varepsilon}{\omega_1} l_2$ and $-l_1 \neq \frac{2+\varepsilon}{\omega_1} l_2$. Now, if $-l_1 = [\frac{\varepsilon}{\omega_1} l_2]$, then $|(2+\varepsilon)l_2 + \omega_1 l_1| > |(2+\varepsilon)l_2 - \varepsilon l_2| - \frac{1}{2} > |l_2|$. Therefore, using $|\omega_1 l_1 + \varepsilon l_2| > \gamma/|l_2|$, we get (3.6). The same estimate (3.6) holds if $-l_1 = [\frac{2+\varepsilon}{\omega_1} l_2]$ since, in this case, $|\omega_1 l_1 + \varepsilon l_2| > |(2+\varepsilon)l_2 - \varepsilon l_2| - \frac{1}{2} > |l_2|$. $\square$

Fixed points of the nonlinear operator $\mathcal{G} : P \to P$ defined by

$$\mathcal{G}(\eta, p) := -\eta^{2(d-1)}\, \mathcal{L}_\varepsilon^{-1} \Pi_P f(\varphi_1, q+p, \eta)$$

are solutions of the $(P_\eta)$-equation.

**Lemma 3.3.** *Assume* $(\varepsilon, \omega_1) \in \mathcal{C}_\gamma$. $\forall R > 0$ *there exists* $\eta_0(R)$, $C_0(R) > 0$ *such that* $\forall |q|_{H^\sigma(\mathbb{T})} \le R$, $0 < \eta\gamma^{-c} \le \eta_0(R)$, *with* $c = 1/2(d-1)$, *there exists a unique* $p(\eta, q) \in P \cap \mathcal{H}_{\sigma,\bar{s}}$ *solving the* $(P_\eta)$*-equation (3.5) and satisfying*

$$|p(\eta, q)|_{\sigma,\bar{s}} \le C_0(R)\eta^{2(d-1)}\gamma^{-1} \tag{3.8}$$

*and the equivariance property*

$$p(\eta, q_\theta)(\varphi_1, \varphi_2) = p(\eta, q)(\varphi_1, \varphi_2 - \theta), \qquad \forall \theta \in \mathbb{T} \tag{3.9}$$

*where* $q_\theta(\varphi_1, \varphi_2) := q(\varphi_1, \varphi_2 - \theta)$. *Moreover* $p(\cdot, \cdot) \in C^1((0, \eta_0(R)) \times Q; P)$.

*Proof.* In the Appendix. □

3.2. **The** $(Q)$**-equation.** Once the $(P_\eta)$-equation has been solved by $p(\eta, q) \in P$ there remains the infinite dimensional bifurcation equation

$$(2+\varepsilon)\ddot{q} + \Pi_Q\Big[f(\varphi_1, q + p(\eta, q), \eta)\Big] = 0\,. \tag{3.10}$$

Recalling (1.9), the $(Q)$-equation (3.10) evaluated at $\eta = 0$ reduces to the ordinary differential equation

$$(2+\varepsilon)\ddot{q} + \langle a_{2d-1}\rangle\, q^{2d-1} = 0 \tag{3.11}$$

where $\langle a_{2d-1}\rangle := (1/2\pi)\int_0^{2\pi} a_{2d-1}(\varphi_1)\, d\varphi_1$.

Equation (3.11) is a superlinear autonomous Hamiltonian system with one degree of freedom and can be studied by a direct phase-space analysis.

**Lemma 3.4.** *There exists* $\overline{\sigma} > 0$ *such that, equation (3.11) possesses a* $2\pi$*-periodic, analytic solution* $\overline{q}(\varphi_2) \in H^{\overline{\sigma}}(\mathbb{T})$. *Morevoer,* $\overline{q}(\varphi_2)$ *is non-degenerate up to time translations, i.e. the linearized equation on* $\overline{q}$

$$(2+\varepsilon)\ddot{h} + (2d-1)\langle a_{2d-1}\rangle \overline{q}^{2(d-1)} h = 0 \tag{3.12}$$

*possesses a one-dimensional space of* $2\pi$*-periodic solutions, spanned by* $\dot{\overline{q}}$.

*Proof.* Up to a rescaling, equation (3.11) can be written as $\ddot{x} = -V'(x)$ with potential energy $V(x) := x^{2d}$. All solutions of such system are analytic and periodic with period

$$T(E) = 4\int_0^{E^{\frac{1}{2d}}} \frac{dx}{\sqrt{2(E - x^{2d})}} = 4E^{\frac{1}{2d}-\frac{1}{2}}\int_0^1 \frac{dx}{\sqrt{2(1 - x^{2d})}}\,.$$

The equation $T(E) = 2\pi$ has a solution $\bar{q}(\varphi_2)$ which is in $H^{\overline{\sigma}}(\mathbb{T})$ for some appropriate $\bar{\sigma} > 0$. The non-degeneracy of the corresponding $2\pi$-periodic solution follows by

$$\frac{dT}{dE} = 2\Big(\frac{1}{d} - 1\Big)E^{\frac{1}{2d}-\frac{3}{2}}\int_0^1 \frac{dx}{\sqrt{2(1 - x^{2d})}} \ne 0$$

and the next Proposition proved in the Appendix.

**Proposition 2.** *Suppose the autonomous second order equation* $-\ddot{x} = V'(x)$, $x \in \mathbb{R}$, *possesses a continuous family of non-constant periodic solutions* $x(E,t)$ *with energy* $E$ *and period* $T(E)$ *satisfying the anysocronicity condition* $\frac{dT(E)}{dE} \neq 0$. *Then* $x(E,t)$ *is non-degenerate up to time translations, i.e. the* $T(E)$*-periodic solutions of the linearized equation*

$$-\ddot{h} = D^2 V(x(E,t))h \tag{3.13}$$

*form a one dimensional subspace spanned by* $(\partial_t x)(E,t)$.

□

From now on we fix $\bar{R} := |\overline{q}|_{H^{\overline{\sigma}}(\mathbb{T})} + 1$ in Lemma 3.3 and take $0 < \eta\gamma^{-c} \le \eta_0(\bar{R})$.

By Lemma 3.4 and (3.9), we can construct solutions of the infinite dimensional bifurcation equation (3.10) by means of the Implicit Function Theorem:

**Lemma 3.5.** *There exist* $0 < \eta_1 \le \eta_0(\bar{R})$, $C_1 > 0$ *such that for all* $0 < \eta\gamma^{-c} \le \eta_1$, *equation (3.10) has a unique (up to translations) solution* $\bar{q}_\eta(\varphi_2) \in H^{\overline{\sigma}}(\mathbb{T})$ *satisfying*

$$|\bar{q}_\eta - \overline{q}|_{H^{\overline{\sigma}}(\mathbb{T})} \le C_1 |\eta| \,.$$

*Proof of Theorem B.* Setting again $\delta \equiv \eta$, $\bar{q}_\varepsilon(\varphi_2) + p(\varepsilon, \bar{q}_\varepsilon)$ solves (1.8) and

$$u(\varepsilon, \varphi) = |\varepsilon|^{1/2(d-1)} \Big[ \bar{q}_\varepsilon(\varphi_2) + p(\varepsilon, \bar{q}_\varepsilon) \Big]$$

is a non trivial solution of (1.4). The bound (1.7) follows by (3.8). As in Theorem A the solution $u$ depends on both the variables $(\varphi_1, \varphi_2)$. Finally, the solution $v(\varepsilon, t, x) := u(\varepsilon, \omega_1 t, x + \omega_2 t)$ of (1.1) is quasi-periodic since, by the definition of $\mathcal{C}_\gamma$, $\omega_1/\omega_2 = \omega_1/(1+\varepsilon) \notin \mathbb{Q}$. □

**Remark 3.** *To prove existence of solutions of (1.8), i.e. (1.1), it is sufficient that the second order equation (3.11) possessess a continuous, nonisocronous family of non-constant periodic orbits one of them having period* $2\pi/j$, *see Proposition 2.*

The hypothesys that the leading term in the nonlinearity $f$ is an odd power of $u$ is not of technical nature. The following non-existence result holds:

**Proposition 3. (Non-existence)** *Let* $f(\varphi_1, u) = a(\varphi_1) u^D$ *with* $D$ *even and* $\int_0^{2\pi} a(\varphi_1)\, d\varphi_1 \neq 0$. $\forall R > 0$, *there exists* $\varepsilon_0 > 0$ *such that* $\forall \sigma \ge 0$, $\overline{s} > s - \frac{1}{2}$, $\forall (\varepsilon, \omega_1) \in \mathcal{C}_\gamma$ *with* $|\varepsilon| < \varepsilon_0$, *equation (1.4) does not possess solutions* $u \in \mathcal{H}_{\sigma,\overline{s}}$ *in the ball* $|u|_{\sigma,\overline{s}} \le R|\varepsilon|^{1/(D-1)}$.

*Proof.* We first rescale equation (1.4) with $u \to |\varepsilon|^{1/(D-1)} u$ obtaining

$$\mathcal{L}_\varepsilon u + |\varepsilon| a(\varphi_1) u^D = 0 \,. \tag{3.14}$$

Write any solution $u_\varepsilon \in B_{\sigma,\overline{s}}(R) := \{u \in \mathcal{H}_{\sigma,\overline{s}} \; : \; |u|_{\sigma,\overline{s}} \le R\}$ of (3.14) as $u_\varepsilon = q_\varepsilon + p_\varepsilon$ with $q_\varepsilon \in Q$, $p_\varepsilon \in P$. $p_\varepsilon$ satisfies the $(P)$-equation $\mathcal{L}_\varepsilon p + |\varepsilon| \Pi_P a(\varphi_1) u^D = 0$ and therefore $|p_\varepsilon|_{\sigma,\overline{s}} \le C(R)|\varepsilon|$. Then, for $\varepsilon$ small enough, $p_\varepsilon \equiv p(\varepsilon, q_\varepsilon)$ where $p(\varepsilon, q_\varepsilon)$ is constructed as in Lemma 3.3 and satisfies the estimate $|p(\varepsilon, q_\varepsilon)|_{\sigma,\overline{s}} \le C|\varepsilon| |q_\varepsilon|^D_{H^\sigma(\mathbb{T})}$.

The projection $q_\varepsilon$ satisfies the $(Q)$-equation

$$(2+\varepsilon)\ddot{q}_\varepsilon + \mathrm{sign}(\varepsilon)\Pi_Q\Big[a(\varphi_1)(q_\varepsilon + p(\varepsilon, q_\varepsilon))^D\Big] = 0 \tag{3.15}$$

and therefore $|q_\varepsilon|_{H^{\sigma,2}(\mathbb{T})} \le C(R)$.

We claim that $q_\varepsilon \to 0$ in $H^\sigma(\mathbb{T})$ (and so in $\mathcal{H}_{\sigma,\overline{s}}$) for $\varepsilon \to 0$. Indeed, from any subsequence $q_\varepsilon$, we can extract by the compact embedding $H^{\sigma,2}(\mathbb{T}) \hookrightarrow H^\sigma(\mathbb{T})$ another convergent subsequence $q_{\varepsilon_n}$ such that $q_{\varepsilon_n} \to \overline{q} \in H^\sigma(\mathbb{T})$. By (3.15), we deduce that

$$2\ddot{\overline{q}} + \mathrm{sign}(\varepsilon)\langle a\rangle \overline{q}^D = 0$$

where $\langle a\rangle := \int_0^{2\pi} a(\varphi_1)\, d\varphi_1 \neq 0$. Such equation does not possess non-trivial periodic solutions for both $\mathrm{sign}(\varepsilon) = \pm 1$, i.e. $\varepsilon > 0$ and $\varepsilon < 0$, and we conclude that $\overline{q} = 0$.

We finally prove that equation (3.15) does not possess non-trivial periodic solutions in a small neighborhood of the origin.

Linearizing equation (3.15) at $q = 0$ we get $(2+\varepsilon)\ddot{h} = 0$ whose solutions in $H^\sigma(\mathbb{T})$ are the constants. We can perform another Lyapunov-Schmidt reduction close to 0 decomposing $H^\sigma(\mathbb{T}) = \{\text{constants}\} \oplus \{\text{zero average functions}\}$, namely $q_\varepsilon = \rho + w$. By the Implicit function Theorem we get that for any constant $|\rho| \le \rho_0$ small enough (independently of $\varepsilon$) there exists a unique zero average function $w_\rho$ with $|w_\rho|_{H^\sigma(\mathbb{T})} = O(\rho^D)$ solving

$$(2+\varepsilon)\ddot{w}_\rho + \Big[a(\varphi_1)(\rho + w_\rho + p(\varepsilon, q_\varepsilon))^D - \Big\langle a(\varphi_1)(\rho + w_\rho + p(\varepsilon, q_\varepsilon))^D\Big\rangle\Big] = 0\,.$$

Hence $\rho$ is such that

$$0 = \Big\langle a(\varphi_1)(\rho + w_\rho + p(\varepsilon, q_\varepsilon))^D\Big\rangle = \langle a\rangle \rho^D + o(\rho^D)\,.$$

This implies $\rho = 0$ since $\langle a\rangle \neq 0$ and so $q_\varepsilon = \rho + w_\rho = 0$. □

## 4. Appendix

**Lemma 4.1.** *$\mathcal{H}_{\sigma,s}$ is a Banach algebra for $\sigma, s \ge 0$.*

*Proof.* By the product Cauchy formula

$$uv = \sum_{j\in\mathbb{Z}^2}\Big(\sum_{k\in\mathbb{Z}^2} u_{j-k}v_k\Big)e^{\mathrm{i}j\cdot\varphi}$$

and therefore

$$\begin{aligned}
|uv|_{\sigma,s} &:= \sum_{j\in\mathbb{Z}^2} e^{\sigma|j_2|}[j_1]^s\Big|\sum_{k\in Z^2} u_{j-k}v_k\Big| \le \sum_{j\in\mathbb{Z}^2} e^{\sigma|j_2|}[j_1]^s \sum_{k\in\mathbb{Z}^2}|u_{j-k}|\,|v_k| \\
&\le \sum_{k\in\mathbb{Z}^2}|v_k|\sum_{j\in\mathbb{Z}^2}|u_{j-k}|e^{\sigma|j_2|}[j_1]^s \\
&\le 2^s\sum_{k\in\mathbb{Z}^2}|v_k|e^{\sigma|k_2|}[k_1]^s\sum_{j\in\mathbb{Z}^2}|u_{j-k}|e^{\sigma|j_2-k_2|}[j_1-k_1]^s := 2^s|u|_{\sigma,s}|v|_{\sigma,s}
\end{aligned}$$

since $e^{\sigma|j_2|} \le e^{\sigma|j_2-k_2|}e^{\sigma|k_2|}$ and $[j_1] \le 2[j_1-k_1][k_1]$ for all $k, j \in \mathbb{Z}^2$. □

*Proof of Lemma 2.3.* Let us consider

$$B := \Big\{ (q_2, p) \in Q_2 \oplus P : \ |q_2|_{\sigma,s} \le \rho_1 \, , \ |p|_{\sigma,s} \le \rho_2 \Big\}$$

with norm $|(q_2,p)|_{\sigma,s} := |q_2|_{\sigma,s} + |p|_{\sigma,s}$. We claim that, under the assumptions (2.9) there exists $0 < \rho_1, \ \rho_2 < 1$, see (4.6), such that the map $(q_2,p) \to \mathcal{G}(q_2,p;q_1)$ is a contraction in $B$, i.e.:

(i) $(q_2,p) \in B \implies \mathcal{G}(q_2,p;q_1) \in B$;

(ii) $|\mathcal{G}(q_2,p;q_1) - \mathcal{G}(\widetilde{q}_2, \widetilde{p}; q_1)|_{\sigma,s} \le (1/2)|(q_2,p) - (\widetilde{q}_2,\widetilde{p})|_{\sigma,s} \, , \ \ \forall (q_2,p), (\widetilde{q}_2,\widetilde{p}) \in B$.

In the following $\kappa_i$ will denote positive constants *independent* on $R$, $N$ and $\varepsilon$ (i.e. on $\delta := |\varepsilon|^{1/2(d-1)}$).

By (2.7) and the Banach algebra property of $\mathcal{H}_{\sigma,s}$

$$\begin{aligned} |\mathcal{G}_1(q_2,p;q_1)|_{\sigma,s} &= |L_1^{-1}\Pi_{Q_2} f(\varphi_1, q_1 + q_2 + p, \delta)|_{\sigma,s} \\ &\le \frac{\kappa_1}{N^2}\Big( |q_1|_{\sigma,s}^{2d-1} + |q_2|_{\sigma,s}^{2d-1} + |p|_{\sigma,s}^{2d-1} \Big) \end{aligned} \tag{4.1}$$

provided that $0 \le \delta \le \delta_0(R)$. Similarly, for $\varepsilon \in \mathcal{B}_\gamma$, by (2.6),

$$\begin{aligned} |\mathcal{G}_2(q_2,p;q_1)|_{\sigma,s} &= |\varepsilon \mathcal{L}_\varepsilon^{-1} \Pi_P f(\varphi_1, q_1 + q_2 + p, \delta)|_{\sigma,s} \\ &\le \kappa_2 |\varepsilon| \gamma^{-1} \Big( |q_1|_{\sigma,s}^{2d-1} + |q_2|_{\sigma,s}^{2d-1} + |p|_{\sigma,s}^{2d-1} \Big) \, . \end{aligned} \tag{4.2}$$

For all $q_1 \in Q_1(N)$ and since $0 \le s < 1/2$

$$\begin{aligned} |q_1|_{\sigma,s} &= \sum_{|l_2| \le N} |\hat{q}_{0,l_2}| e^{\sigma |l_2|} + |\hat{q}_{-2l_2,l_2}| e^{\sigma|l_2|} [-2l_2]^s \\ &\le e^{\sigma N} \sum_{|l_2|\le N} |\hat{q}_{0,l_2}| + |\hat{q}_{-2l_2,l_2}| [-2l_2]^s \le \kappa_3 \Big[ \Big( \sum_{|l_2| \le N} |\hat{q}_{0,l_2}|^2 [l_2]^2 \Big)^{1/2} \Big( \sum_{l_2 \in \mathbb{Z}} \frac{1}{[l_2]^2} \Big)^{1/2} \\ &+ \Big( \sum_{|l_2|\le N} |\hat{q}_{-2l_2,l_2}|^2 [l_2]^2 \Big)^{1/2} \Big( \sum_{l_2 \in \mathbb{Z}} \frac{1}{[l_2]^{2(1-s)}} \Big)^{1/2} \Big] \le \kappa_4 |q_1|_{H^1} \end{aligned} \tag{4.3}$$

whenever $0 \le \sigma N \le 1$.

Substituting in (4.1)-(4.2) we get $\forall |q_1|_{H^1} \le 2R$, $\forall |q_2|_{\sigma,s} \le \rho_1$, $\forall |p|_{\sigma,s} \le \rho_2$

$$|\mathcal{G}_1(q_2,p;q_1)|_{\sigma,s} \le \kappa_5 N^{-2} \Big( R^{2d-1} + \rho_1^{2d-1} + \rho_2^{2d-1} \Big) \tag{4.4}$$

$$|\mathcal{G}_2(q_2,p;q_1)|_{\sigma,s} \le \kappa_5 |\varepsilon| \gamma^{-1} \Big( R^{2d-1} + \rho_1^{2d-1} + \rho_2^{2d-1} \Big) \, . \tag{4.5}$$

Now, setting $C_0(R) := \kappa_5 R^{2d-1}$, we define

$$\rho_1 := \frac{2C_0(R)}{N^2} \qquad \rho_2 := 2|\varepsilon|\gamma^{-1} C_0(R) \, . \tag{4.6}$$

By (4.4), (4.5) there exists $N_0(R) \in \mathbb{N}^+$ and $\varepsilon_0(R) > 0$ such that $\forall N \ge N_0(R)$ and $\forall |\varepsilon| \gamma^{-1} \le \varepsilon_0(R)$

$$|\mathcal{G}_1(q_2,p;q_1)|_{\sigma,s} \le \rho_1 \qquad |\mathcal{G}_2(q_2,p;q_1)|_{\sigma,s} \le \rho_2$$

proving (i). Item (ii) is obtained with similar estimates.

By the Contraction Mapping Theorem there exists a unique fixed point $(q_2(q_1), p(q_1)) := (q_2(\varepsilon, N, q_1), \, p(\varepsilon, N, q_1))$ of $\mathcal{G}$ in $B$. The bounds (2.10) follow by (4.6).

Since $\mathcal{G} \in C^1(Q_2 \oplus P \times Q_1; Q_2 \oplus P \times Q_1)$ the Implicit function Theorem implies that the maps $Q_1 \ni q_1 \to (q_2(\varepsilon, N, q_1), p(\varepsilon, N, q_1))$ are $C^1$.

Differentiating $(q_2(q_1), p(q_1)) = \mathcal{G}(q_2(q_1), p(q_1), q_1)$

$$q_2'(q_1)[h] = -L_1^{-1}\Pi_{Q_2}(\partial_u f)(\varphi_1, q_1 + q_2(q_1) + p(q_1), \delta)\Big(h + q_2'(q_1)[h] + p'(q_1)[h]\Big)$$
$$p'(q_1)[h] = -\varepsilon\mathcal{L}_\varepsilon^{-1}\Pi_{Q_2}(\partial_u f)(\varphi_1, q_1 + q_2(q_1) + p(q_1), \delta)\Big(h + q_2'(q_1)[h] + p'(q_1)[h]\Big)$$

and using (2.7), (2.6) and the Banach algebra property of $\mathcal{H}_{\sigma,s}$

$$|q_2'(q_1)[h]|_{\sigma,s} \le C(R)N^{-2}\Big(|h|_{\sigma,s} + |q_2'(q_1)[h]|_{\sigma,s} + |p'(q_1)[h]|_{\sigma,s}\Big)$$
$$|p'(q_1)[h]|_{\sigma,s} \le C(R)|\varepsilon|\gamma^{-1}\Big(|h|_{\sigma,s} + |q_2'(q_1)[h]|_{\sigma,s} + |p'(q_1)[h]|_{\sigma,s}\Big)$$

which implies the bounds (2.11) since

$$\det\begin{vmatrix} 1 - C(R)N^{-2} & -C(R)N^{-2} \\ -C(R)|\varepsilon|\gamma^{-1} & 1 - C(R)|\varepsilon|\gamma^{-1} \end{vmatrix} \ge \frac{1}{2}$$

for $C(R)(|\varepsilon|\gamma^{-1} + N^{-2})$ small enough and (4.3).

*Proof of Lemma 2.4.* By (2.4), (2.5) we have that, at $u := q_1 + q_2(q_1) + p(q_1)$,

$$d\Psi_\varepsilon(u)[h] = 0 \ \ \forall h \in Q_2 \qquad \text{and} \qquad d\Psi_\varepsilon(u)[h] = 0 \ \ \forall h \in P\,. \tag{4.7}$$

Since $q_2'(q_1)[k] \in Q_2$ and $p'(q_1)[k] \in P$ $\forall k \in Q_1$, we deduce

$$d\Phi_{\varepsilon,N}(q_1)[k] = d\Psi_\varepsilon(u)\Big[h + q_2'(q_1)[k] + p'(q_1)[k]\Big] = d\Psi_\varepsilon(u)[k] \quad \forall k \in Q_1$$

and therefore $u := q_1 + p(q_1) + q_2(q_1)$ solves also the $(Q_1)$-equation (2.3).

Write $\Psi_\varepsilon(u) = \Psi_\varepsilon^{(2)}(u) - \varepsilon \int_{\mathbb{T}^2} F(\varphi_1, u, \delta)$ where

$$\Psi_\varepsilon^{(2)}(u) := \int_{\mathbb{T}^2} \frac{1}{2}(\partial_{\varphi_1}u)^2 + (1+\varepsilon)(\partial_{\varphi_1}u)(\partial_{\varphi_2}u) + \frac{\varepsilon(2+\varepsilon)}{2}(\partial_{\varphi_2}u)^2$$

is an homogeneous functional of degree two. By homogeneity:

$$\Psi_\varepsilon(u) = \frac{1}{2}d\Psi_\varepsilon^{(2)}(u)[u] - \varepsilon\int_{\mathbb{T}^2} F(\varphi_1, u, \delta)\,. \tag{4.8}$$

By (2.4), (2.5) (i.e. (4.7))

$$d\Psi_\varepsilon^{(2)}(q_1 + q_2(q_1) + p(q_1))[q_2(q_1) + p(q_1)] = \varepsilon\int_{\mathbb{T}^2} f(\varphi_1, u, \delta)(q_2(q_1) + p(q_1))\,. \tag{4.9}$$

Substituting in (4.8) we obtain, at $u = q_1 + q_2(q_1) + p(q_1)$

$$\begin{aligned}\Phi_{\varepsilon,N}(q_1) &= \Psi_\varepsilon(q_1 + p(q_1) + q_2(q_1)) = \frac{1}{2} d\Psi_\varepsilon^{(2)}(u)[q_1 + p(q_1) + q_2(q_1)] - \varepsilon \int_{\mathbb{T}^2} F(\varphi_1, u, \delta) \\ &= \frac{1}{2} d\Psi_\varepsilon^{(2)}(q_1)[q_1] - \varepsilon \int_{\mathbb{T}^2} F(\varphi_1, u, \delta) + \frac{1}{2} f(\varphi_1, u, \delta)(q_2(q_1) + p(q_1)) \\ &= \Psi_0(q_1) + \varepsilon \int_{\mathbb{T}^2} \frac{(2+\varepsilon)}{2} (\partial_{\varphi_2} q_1)^2 + (\partial_{\varphi_1} q_1)(\partial_{\varphi_2} q_1) - F(\varphi_1, u, \delta) \\ &+ \frac{1}{2} f(\varphi_1, u, \delta)(q_2(q_1) + p(q_1)) = \text{const} + \varepsilon(\Gamma(q_1) + \mathcal{R}_{\varepsilon,N}(q_1))\end{aligned}$$

because $\Psi_0(q_1) \equiv$ const.

By (2.10) the bounds (2.14)-(2.15) follow.

*Proof of Lemma 3.3.* The existence of $p(\eta, q) \in \mathcal{H}_{\sigma,\overline{s}}$ can be proved as in Lemma 2.3 using the Contraction Mapping Theorem. The smoothness of $p(\eta, q)$ follows by the Implicit Function Theorem since $\mathcal{G}(\eta, p)$ is smooth in $\eta$ and $q$.

By the invariance of equation (3.5) under translations in the $\varphi_2$ variable the function $p(\eta, q)(\varphi_1, \varphi_2 - \theta)$ solves

$$p(\eta, q)(\varphi_1, \varphi_2 - \theta) + \eta^{2(d-1)} \mathcal{L}_\varepsilon^{-1} \Pi_P f\Big(\varphi_1, q_\theta + p(\eta, q)(\varphi_1, \varphi_2 - \theta), \eta\Big) = 0$$

and, therefore, by uniqueness (3.9) holds.

*Proof of Proposition 2.* Write $x(E, t) = y(\omega(E)t, E)$ where $y(\varphi, E)$ is $2\pi$-periodic in $\varphi$ and $\omega(E) := 2\pi/T(E)$. The functions $(\partial_t x)(E, t)$ and

$$(\partial_E x)(E, t) = t \frac{d\omega(E)}{dE} (\partial_\varphi y)(\omega(E)t, E) + (\partial_E y)(\omega(E)t, E) \tag{4.10}$$

are two linearly independent solutions of the linearized equation (3.13). $(\partial_t x)(E, t)$ is $2\pi$-periodic while, since

$$\frac{d\omega(T)}{dT} = 2\pi T(E)^{-2} \frac{dE(T)}{dT} \neq 0 \qquad \text{and} \qquad (\partial_\varphi y)(\varphi, E) \not\equiv 0$$

(if not $x(E, t)$ would be constant in $t$), $(\partial_E x)(E, t)$ is not $2\pi$-periodic. We conclude that the space of $T(E)$-periodic solutions of (3.13) form a 1-dimensional linear space spanned by $(\partial_t x)(E, t)$.

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