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Periodic Solutions for a Class of Nonlinear Partial Differential Equations in Higher Dimension

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Abstract: We prove the existence of periodic solutions in a class of nonlinear partial differential equations, including the nonlinear Schrödinger equation, the nonlinear wave equation, and the nonlinear beam equation, in higher dimension. Our result covers cases of completely resonant equations, where the bifurcation equation is infinite-dimensional, such as the nonlinear Schrödinger equation with zero mass, for which solutions which at leading order are wave packets are shown to exist.

1. Introduction

1.1. A brief survey of the literature. The problem of the existence of finite-dimensional tori, i.e. of quasi-periodic solutions, for infinite-dimensional systems, such as nonlinear PDEs, has been extensively studied in the literature. A particularly significant example is the nonlinear Schrödinger equation (NLS)

$$\mathrm{i}v_t - \Delta v + \mu v = f(x, |v|^2)\, v, \tag{1.1}$$

with periodic boundary conditions; here Δ is the Laplacian on $\mathbb{T}^D$, $\mu > 0$ is the "mass", and the function f is real-analytic in a neighbourhood of the origin, where it vanishes. For instance $f(x, |v|^2) = |v|^2$ gives the cubic NLS, which is a widely studied model appearing in many branches of physics, such as the theory of Bose-Einstein condensation, plasma physics, nonlinear optics, wave propagation, theory of water waves [1]. Another physically interesting case is the NLS "with potential", where the mass is substituted by a multiplicative potential $V(x)$. However many results on quasi-periodic solutions are on simplified models, such as

$$\mathrm{i}v_t - \Delta v + M_\sigma v = f(x, |v|^2)\, v, \tag{1.2}$$

where M_σ is a "Fourier multiplier" i.e. a linear operator, depending on a finite number of free parameters σ, which commutes with the Laplacian. The free parameters (as many as the fundamental frequencies of the solution) are chosen in such a way as to impose suitable Diophantine conditions on the "frequencies" of the linearised equation, i.e. the eigenvalues of $-\Delta + M_\sigma$.

Up to recent times, the only available results on quasi-periodic solutions for PDEs were confined to the case of one space dimension ($D = 1$). In this context the first results were obtained by Wayne, Kuksin and Pöschel [27,28,30,35], for the nonlinear Schrödinger equation (1.2) and the nonlinear wave equation (NLW) with Dirichlet boundary conditions, by using KAM techniques. Later on, Craig and Wayne in [16] proved similar result, for both Dirichlet and periodic boundary conditions, with a rather different method based on the Lyapunov-Schmidt decomposition and a Newton scheme. The case of periodic boundary conditions within the framework of KAM theory was then obtained by Chierchia and You [15].

When looking for periodic solutions, one can work directly on Eq. (1.1) and use the mass as a free parameter. However, if the mass vanishes, then the system becomes completely resonant, i.e. all the frequencies of the linearised equation are rationally dependent and there are infinitely many linear solutions with the same period. This makes the problem much harder – already in the case of periodic solutions. The completely resonant case was discussed by several authors, and theorems on the existence of periodic solutions for a large measure set of frequencies were obtained by Bourgain [10] for the NLW with periodic boundary conditions, by Gentile, Mastropietro and Procesi [22] and by Berti and Bolle [4] for the NLW with Dirichlet boundary conditions. In [23], we constructed, for the NLS with Dirichlet boundary conditions, periodic solutions which at leading order are wave packets. The existence of quasi-periodic solutions for the completely resonant NLW with periodic boundary conditions has been proved by Procesi [32] for a zero-measure set of two-dimensional rotation vectors, by Baldi and Berti [3] for a large measure set of two-dimensional rotation vectors, and by Yuan [36] for a large measure set of – at least three-dimensional – rotation vectors.

Finding periodic and quasi-periodic solutions for PDEs in higher space dimensions ($D > 1$) is much harder than in the one-dimensional case, mainly due to the high degeneracy of the frequencies of the linearised equation. The first achievements in this direction were due to Bourgain, and concerned the existence of periodic solutions for NLW [8] and of periodic solutions (also quasi-periodic in $D = 2$) for the NLS [9]. The case of quasi-periodic solutions in arbitrary dimension was solved by Bourgain [11] for the NLW and the NLS with a Fourier multiplier as in (1.2). Bourgain's method is based on a Nash-Moser algorithm.

A proof of existence and stability of quasi-periodic solutions in high dimension was given by Geng and You, using KAM theory. Their result holds for a class of PDEs, with periodic boundary conditions and with nonlinearities which do not depend on the space variable. Both conditions are required in order to ensure a symmetry for the Hamiltonian which simplifies the problem in a remarkable way. Their class of PDEs includes the nonlinear beam equation (NLB) [19] and the NLS with a smoothing nonlinearity [20]

$$\mathrm{i}v_t - \Delta v + M_\sigma v = \Psi(f(|\Psi(v)|^2)\Psi(v)),$$

where Ψ is a convolution operator. This equation has been studied in the mathematical literature (see for instance [31]), because it allows some simplifications, but it does not appear to be a physically interesting model. Their approach does not extend to the NLS with local nonlinearities like (1.2), mainly because it would require a "second Melnikov condition" at each iterative KAM step, and such a condition does not appear to be satisfied by the local NLS.

Successively, Eliasson and Kuksin [17], by using KAM techniques, proved the existence and linear stability of quasi-periodic solutions for the NLS, with local nonlinearities. In their paper the main point is indeed to prove that one may impose a second Melnikov condition at each iterative KAM step. However, their result does not extend to other PDEs, because, in general (see for instance the case of the NLW in $D > 1$), it can be too hard to impose a second Melnikov condition – even on the unperturbed eigenvalues. In Eliasson and Kuksin's paper only finite regularity (in the space variables) is found for the solutions. This is a drawback which does not arise in Bourgain's approach [11], where an exponential decay of the Fourier coefficients is obtained.

Again very recently, Berti and Bolle [7] proved the existence of periodic solutions for PDE systems with C^k nonlinearities – all the other papers are in the analytic setting. They use a Nash-Moser algorithm suited for finitely differentiable nonlinearities, already employed in the one-dimensional case [5], and they find solutions belonging to suitable Sobolev classes.

In [24] we studied the NLS with nonlocal smoothing nonlinearities, and we proved the existence of periodic solutions. In particular we discussed the completely resonant case, for which we obtained for $D = 2$ "wave packet" solutions similar to those found in [23] in the one-dimensional case. The main purpose of [24] was to extend the Lindstedt series method – based on renormalisation group ideas and originally introduced in [21] – to high dimensional PDEs in a simple nontrivial case, i.e. the nonlocal NLS. The proofs however strongly rely on the fact that the nonlinearity is nonlocal and does not cover the local NLS.

In the present paper we prove an "abstract theorem" on the existence of Gevrey-smooth periodic solutions for a wide class of PDEs with analytic nonlinearities satisfying some "abstract conditions". This class of PDEs contains the local NLS as well as the NLB and the NLW; formal statements of both assumptions and results will be given in Sect. 2. Our approach is based on a standard Lyapunov-Schmidt decomposition – which separates the original PDEs into two equations, traditionally called the P and Q equations – combined with renormalised expansions *à la* Lindstedt to handle the small divisor problem. Although the general strategy, at least as far as the small divisor problem is concerned, is similar to [24], we wish to stress that working with local nonlinearities as in (1.1) makes the small divisor problem much more delicate and requires some substantial work to overcome the consequent difficulties. Moreover, not only the result in the present paper is more general, but the proofs are simpler and more compact. More details will be given along the proofs.

As a first application of our abstract theorem we recover various known results discussed in the aforementioned literature. Then – this is the main original result in our paper – we apply the abstract theorem to the completely resonant ($\mu = 0$) local NLS and NLB equations and prove the existence of "wave packet" solutions in the spirit of [23]. Again, proving existence of "wave packet" solutions for the local NLS is much more challenging than for the nonlocal case discussed in [24]. Indeed the proof is completely different – see Subsect. 1.3 for a comparison.

In the following Subsect. 1.2 we provide an informal overview of the main hypotheses and of the Lindstedt series method. Then in Subsect. 1.3 we describe the main applications, with particular attention to the completely resonant cases. We conclude this section by mentioning that periodic solutions are also found in the literature for NLS on $\mathbb{R}^D$ with an external confining potential; see for instance [13,25,34]. In principle, one could expect that confining potentials have an effect similar to that of imposing Dirichlet boundary conditions on a finite domain, but the setting is rather different: the potentials are taken to depend on a small length scale h, and in the limit $h \to 0$ the considered NLS reduce to (1.1). Thus, the unperturbed equation is the same, but on a completely different domain (the full space). For this class of equations solitonic solutions, periodic in time and exponentially decaying in space, are constructed in the quoted references.

1.2. Informal presentation of the main result and techniques. To describe the main hypotheses, consider equations of the form

$$\mathbb{D}(\varepsilon)u(\mathbf{x}) = \varepsilon f(u(\mathbf{x}), \bar{u}(\mathbf{x})), \qquad \mathbf{x} = (t, x), \tag{1.3}$$

where (for instance) $\mathbf{x} \in \mathbb{T}^{D+1}$, ε is a small real parameter, $\mathbb{D}(\varepsilon)$ is a linear operator depending on ε, and $f(u, \bar{u})$ is an analytic function, possibly depending also on x and ε, which is superlinear at $u = 0$. We shall see that our class of Eqs. (2.7) and (2.8) reduce to the form (1.3) after some rescaling.

We require three properties on Eqs. (1.3), which we call Hypotheses 1 to 3 (see Sect. 2 for a precise formulation). Informally, the properties are the following:

1. $\mathbb{D}(\varepsilon)$ is diagonal in the Fourier basis with real eigenvalues $\delta_{\boldsymbol{\nu}}(\varepsilon)$ which are smooth in both $\boldsymbol{\nu}$ and ε and satisfy appropriate bounds on the derivatives.
2. The Q equation at $\varepsilon = 0$ (bifurcation equation) has a non-degenerate solution which is analytic in space and time.
3. For each ε the set of "singular" frequencies $\mathfrak{S}(\varepsilon) := \{\boldsymbol{\nu} \in \mathbb{Z}^{D+1} : |\delta_{\boldsymbol{\nu}}(\varepsilon)| \le 1/2\}$ is of the form $\mathfrak{S}(\varepsilon) = \cup_{j\in\mathbb{N}}\Delta_j(\varepsilon)$ where the $\Delta_j(\varepsilon)$ are disjoint finite sets which are "well separated" and "not too big".

Properties 1 and 3 are assumptions on the linear part, while Property 2 is an assumption on the nonlinearity. In particular the solution of the bifurcation equation identifies the solution of the linearised PDE from which the solution of the full PDE branches off.

Property 1 can probably be weakened to cover cases in which $\mathbb{D}(\varepsilon)$ is not diagonal in the Fourier basis but its eigenfunctions are still "well localised" with respect to the Fourier basis (namely the Fourier coefficients of the eigenfunctions have a uniform exponential decay). Property 2 is required to solve the bifurcation equation by the implicit function theorem. Also this hypothesis could be weakened; see for instance [6] for a discussion of weaker hypotheses in the case of the NLW in dimension $D = 1$. Property 3 was introduced by Bourgain to prove the existence of periodic solutions for the NLS in high dimension [9]. This hypothesis is essential for our proof; a similar, weaker hypothesis appears in [7].

Assuming Properties 1 to 3 we prove our main result, which is the Main Theorem in Sect. 2, by a "renormalised series expansion". The proof of the theorem is performed through two steps, formally described by Propositions 1 and 2 in Subsect. 4.3.

To illustrate our method, write Eqs. (1.3) in Fourier space,

$$\delta_{\boldsymbol{\nu}}(\varepsilon)\, u_{\boldsymbol{\nu}} = \varepsilon f_{\boldsymbol{\nu}}, \qquad f_{\boldsymbol{\nu}} = f_{\boldsymbol{\nu}}(\{u_{\boldsymbol{\nu}'}, \bar{u}_{\boldsymbol{\nu}'}\}_{\boldsymbol{\nu}'\in\mathbb{Z}^{D+1}}), \tag{1.4}$$

where $u_{\boldsymbol{\nu}}$ and $f_{\boldsymbol{\nu}}$ are the $\boldsymbol{\nu}^{\text{th}}$ Fourier coefficients of $u(\mathbf{x})$ and $f(u(\mathbf{x}), \bar{u}(\mathbf{x}))$, respectively. Then one studies (1.4): the presence of the small parameter ε suggests to look for a recursive solution, and hence to write the solution-to-be in the form of a series expansion in powers of ε. This is a very natural approach for any problem in perturbation theory, and leads to a graphical representation of the solution order by order in terms of trees; see the diagrammatic expansion described in Sect. 5.3 and, in particular, Definition 5.7. Thus, the solution can be given a meaning as a formal power series, provided an irrationality condition is assumed on the eigenvalues $\delta_{\boldsymbol{\nu}}(\varepsilon)$; cf. Lemma 5.10 with $M = L = 0$ in Subsect. 5.3.

On the other hand the convergence of the series fails to be proved, and is likely not to hold. This fact suggests to modify Eqs. (1.4) into new equations which have a solution in the form of an absolutely convergent series (which is not a power series in ε). First of all we introduce some notation by rewriting (1.4) as

$$\mathscr{D}(\varepsilon)\, U = \eta\, F, \tag{1.5}$$

where $\eta = \varepsilon$, $U := \{u_{\boldsymbol{\nu}}\}_{\boldsymbol{\nu}\in\mathbb{Z}^{D+1}}$, $F := \{f_{\boldsymbol{\nu}}\}_{\boldsymbol{\nu}\in\mathbb{Z}^{D+1}}$, and $\mathscr{D}(\varepsilon) = \text{diag}\{\delta_{\boldsymbol{\nu}}(\varepsilon)\}_{\boldsymbol{\nu}\in\mathbb{Z}^{D+1}}$. Then, we change (1.5) by considering ε and η as two independent parameters, and adding "corrections" which are linear in U, that is

$$(\mathscr{D}(\varepsilon) + \mathscr{M})\, U = \eta\, F + L\, U, \tag{1.6}$$

where $\mathscr{M} = \mathscr{M}(\varepsilon)$ and $L = L(\eta, \varepsilon)$ are self-adjoint matrices (called "counterterms") with the only restriction that $\mathscr{M}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon) = L_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\eta, \varepsilon) = 0$ if $\boldsymbol{\nu}$, $\boldsymbol{\nu}'$ do not both belong to the same $\Delta_j(\varepsilon)$ for some j. For technical reasons, we shall write $\mathscr{M}(\varepsilon) = \widehat{\chi}_1(\varepsilon) M \widehat{\chi}_1(\varepsilon)$, where the matrices $\widehat{\chi}_1(\varepsilon)$ impose the restriction described above, and M is a matrix of free parameters.

As a matter of fact, the description above does not mention some technical intricacies. For instance we shall need η to be a suitable fractional power of ε (depending on the leading order of the nonlinearity). Moreover for convenience we shall double Eqs. (1.6). Hence both $\mathscr{M}$ and L will carry two further indices; see (2.10) and (4.3).

The reason why we introduce counterterms which are linear in U is that the terms which give problems in the naive power series expansion for (1.5) can be eliminated by adding a linear term to F (this is in the same spirit as Moser's modifying term theorem [29] in KAM theory for finite-dimensional systems). In fact, the terms which may be an obstacle in proving the convergence of the formal power series are classified in a relatively simple way through the language of the diagrammatic expansion. Assume for the time being that $M = 0$. Then it is possible to choose $L = L(\eta, \varepsilon)$ as a convergent power series in η in such a way that the formal power series for the solution of the modified Eq. (1.6) converges, provided the eigenvalues $\delta_{\boldsymbol{\nu}}(\varepsilon)$ satisfy some Diophantine conditions.

Of course, there remains the major problem that the modified Eq. (1.6) with $M = 0$ and $L \neq 0$ is not the original (1.5). Then we introduce $M \neq 0$, and show that for any M it is possible to determine $L = L(\eta, \varepsilon, M)$ as a function of M in such a way that Eqs. (1.6) turn out to be solvable. The proof proceeds essentially in the same way as for $M = 0$, provided ε and M are in an appropriate Cantor set so that the eigenvalues of $\mathscr{D}(\varepsilon) + \mathscr{M}$ satisfy some Diophantine conditions (cf. Definition 5.21 and Lemma 5.24 in Subsect. 5.4). In particular, in order to be able to impose such conditions we shall use in a decisive way the block structure of the matrix $\mathscr{M}$.

This first step is essentially the content of Proposition 1 in Subsect. 4.3. Of course only if $\eta = \varepsilon$ and $\mathscr{M} = L(\eta, \varepsilon, M)$ the modified equation reduces to the original one. Thus, once the first step is accomplished, we are left with the problem of solving the compatibility equation $L(\eta, \varepsilon, M) = \mathscr{M}$. This will be done by showing that, at the cost of further shrinking the set of allowed values for ε, one can choose $M = M(\varepsilon)$ in such a way to solve the compatibility equation. This second step in the proof corresponds to Proposition 2 in Subsect. 4.3.

1.3. Applications of the abstract theorem. In Subsect. 3.1 we consider the NLS, NLW and NLB equations in the non-resonant case (under a Diophantine condition on the mass) and recover the known results on the existence of periodic solutions.

In Subsect. 3.2 – this represents the main novelty of this paper – we discuss cases in which the bifurcation equation is infinite-dimensional, such as the zero-mass NLS and NLB. In the resonant case the linearised equation has an infinite-dimensional space of periodic solutions with the same period, so that in principle we have at our disposal infinitely many linear solutions with the same period which may be extended to solutions of the nonlinear equation. Indeed we find a denumerable infinity of solutions with the same minimal period even in the presence of the nonlinearity. More precisely, we prove the existence of periodic solutions which at leading order involve an arbitrary finite number of harmonics not too far from each other, and which therefore can be described as distorted wave packets. Solutions of this kind are very natural in the case of completely resonant PDEs, where all harmonics are commensurate in the absence of the nonlinearity. An essential ingredient for the existence of such solutions is the particular form of the bifurcation equation: the proof strongly relies on the fact that the leading order of the nonlinearity is cubic and gauge-invariant.

In this latter case the most challenging problem is proving Property 2 in Subsect. 1.2, i.e. the non-degeneracy of the solutions of the bifurcation equation. The problem of the existence of periodic and quasi-periodic solutions in completely resonant systems in higher dimension was already considered by Bourgain in [9], where he constructed quasi-periodic solutions with two frequencies, in $D = 2$, for the resonant NLS with periodic boundary conditions (in contrast with the non-resonant case, where the Fourier multiplier allows to find quasi-periodic solutions with any number of frequencies). However, the non-degeneracy problem is especially complicated in the case of the Dirichlet boundary condition, which we explicitly consider in this paper. To prove non-degeneracy, we require that the nonlinearity does not depend explicitly on the space variables – this is a sufficient condition. Moreover we use a combinatorial Lemma, proved in [24], and some results in algebraic number theory.

In [24], we proved the existence and non-degeneracy of the "wave packet" solutions of the bifurcation equation for the nonlocal NLS, but only in the case $D = 2$. For higher dimension we required some additional condition, which in practice should be checked case-by-case. In fact, the problem was reduced to that of inverting a finite number of matrices of finite – but very big – dimensions, so that a computer-assisted check should be relied upon.

Moreover even in $D = 2$, the proof of the non-degeneracy of the bifurcation equation given in [24] does not extend to the local NLS. The proof in this paper is completely different and much simpler, at the cost of requiring that the nonlinearity does not depend on the space variables. Moreover the result holds in any dimension D.

2. Formal Statement of the Main Result

In this section, we give a rigorous description of the PDE systems we shall consider, and a formal statement of the results that we shall prove in the paper. Throughout the paper we shall call a function $F(t, x)$, with $x = (x_1, \dots, x_D) \in \mathbb{R}^D$ and $t \in \mathbb{R}$, even [resp. odd] in x – or even [resp. odd] *tout court* – if it is even [resp. odd] in each of its arguments x_i.

Let $\mathbb{S}$ be the D dimensional square $[0, \pi]^D$, and let $\partial\mathbb{S}$ be its boundary. We consider for instance the following class of equations:

$$\begin{cases} (\mathrm{i}\partial_t + P(-\Delta) + \mu)\, v = f(x, v, \bar{v}), & (t, x) \in \mathbb{R} \times \mathbb{S}, \\ v(t, x) = 0 & (t, x) \in \mathbb{R} \times \partial\mathbb{S}, \end{cases} \tag{2.1}$$

where Δ is the Laplacian operator, $P : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing convex C^∞ function with $P(0) = 0$, $P(-\Delta)\, \mathrm{e}^{\mathrm{i}m\cdot x} = P(|m|^2)\, \mathrm{e}^{\mathrm{i}m\cdot x}$($\cdot$ denotes the scalar product in $\mathbb{R}^D$), μ is a real parameter which – we can assume – belongs to some finite interval $(0, \mu_0)$, with $\mu_0 > 0$, and $x \to f(x, v(t, x), \bar{v}(t, x))$ is an analytic function which is super-linear in v, $\bar{v}$ and odd (in x) for odd $v(t, x)$, i.e.

$$f(x, v, \bar{v}) = \sum_{r,s\in\mathbb{N}: r+s\geq N+1} a_{r,s}(x)\, v^r \bar{v}^s, \qquad N \geq 1, \tag{2.2}$$

with $a_{r,s}(x)$ even for odd $r + s$ and odd otherwise; notice that the leading order of the nonlinearity is $N+1$. We shall look for odd 2π-periodic solutions with periodic boundary conditions in $[-\pi, \pi]^D$.

We require for f in (2.2) to be of the form

$$f(x, v, \bar{v}) = \frac{\partial}{\partial \bar{v}} H(x, v, \bar{v}) + g(x, \bar{v}), \qquad \overline{H(x, v, \bar{v})} = H(x, v, \bar{v}). \tag{2.3}$$

We also consider the class of equations

$$\begin{cases} \left(\partial_{tt} + (P(-\Delta) + \mu)^2\right) v = f(x, v), & (t, x) \in \mathbb{R} \times \mathbb{S}, \\ v(t, x) = 0, & (t, x) \in \mathbb{R} \times \partial\mathbb{S}, \end{cases} \tag{2.4}$$

and finally the wave equation

$$\begin{cases} (\partial_{tt} - \Delta + \mu)\, v = f(x, v), & (t, x) \in \mathbb{R} \times \mathbb{S}, \\ v(t, x) = 0, & (t, x) \in \mathbb{R} \times \partial\mathbb{S}, \end{cases} \tag{2.5}$$

where $f(x, v)$ is of the form (2.2) with s identically zero and $a_r(x) := a_{r,0}(x)$ real (by parity $a_r(x)$ is even for odd r and odd for even r).

We shall consider also (2.1), (2.4) and (2.5) with periodic boundary conditions: in that case, we shall drop the condition for f to be odd.

Solutions of the linearised equations are superpositions of oscillations, e.g. in case (2.1) they are of the form

$$\sum_{m\in\mathbb{Z}^D} v_m \mathrm{e}^{\mathrm{i}\Omega_m t} \mathrm{e}^{\mathrm{i}m\cdot x}, \qquad \Omega_m = P(|m|^2) + \mu;$$

and similar expressions hold for the other equations. For all these classes of equations we prove a.e. in μ the existence of small periodic solutions with frequency ω close to a given linear frequency $\omega_0 = \Omega_m$ and in an appropriate Cantor set of positive measure. For concreteness we shall focus on the linear oscillation with $m = (1, 1, \ldots, 1)$, which yields the frequency $\omega_0 = P(D)+\mu$ for (2.1) and (2.4) and $\omega_0 = \sqrt{P(D) + \mu}$ for (2.5); of course, the analysis could be easily extended to any other harmonics. For $P(x) = x$ and $\mu = 0$ the system becomes completely resonant: in this case all the harmonics are commensurate with each other. We shall concentrate on the NLS and NLB, and shall prove that there exist periodic solutions which look like perturbations of wave packets, i.e. of superpositions of linear oscillations peaked around given harmonics.

We introduce a smallness parameter by rescaling

$$v(t,x)=\varepsilon^{1/N}u(\omega t,x),\quad \varepsilon>0, \tag{2.6}$$

with $\omega = P(D)+\mu-\varepsilon$ for (2.1) and (2.4) and $\omega^2 = P(D)+\mu-\varepsilon$ for (2.5).

We shall formulate our results in a more abstract context, by considering the following classes of equations with Dirichlet boundary conditions:

$$\text{(I)}\qquad \begin{cases} \mathbb{D}(\varepsilon)\,u=\varepsilon f(x,u,\bar u,\varepsilon^{1/N}), & (t,x)\in\mathbb{T}\times\mathbb{S},\\ u(t,x)=0, & (t,x)\in\mathbb{T}\times\partial\mathbb{S},\end{cases} \tag{2.7a}$$

$$\text{(II)}\qquad \begin{cases} \mathbb{D}(\varepsilon)\,u=\varepsilon f(x,u,\varepsilon^{1/N}), & (t,x)\in\mathbb{T}\times\mathbb{S},\\ u(t,x)=0, & (t,x)\in\mathbb{T}\times\partial\mathbb{S},\end{cases} \tag{2.7b}$$

where $\mathbb{T}:=\mathbb{R}/2\pi\mathbb{Z}$ and $\mathbb{D}(\varepsilon)$ is a linear (possibly integro-)differential wave-like operator with constant coefficients depending on a (fixed once and for all) real parameter ω_0 and on the parameter ε.

We can treat the case of periodic boundary conditions in the same way:

$$\text{(I)}\qquad \mathbb{D}(\varepsilon)\,u=\varepsilon f(x,u,\bar u,\varepsilon^{1/N}),\qquad (t,x)\in\mathbb{T}\times\mathbb{T}^D, \tag{2.8a}$$

$$\text{(II)}\qquad \mathbb{D}(\varepsilon)\,u=\varepsilon f(x,u,\varepsilon^{1/N}),\qquad (t,x)\in\mathbb{T}\times\mathbb{T}^D, \tag{2.8b}$$

with the same meaning of the symbols as in (2.7).

In Case (I) we assume that $f(x,u,\bar u,\varepsilon^{1/N})$ is a rescaling of a function $f(x,u,\bar u)$ defined as in (2.2) and satisfying (2.3). In Case (II) we suppose $\mathbb{D}(\varepsilon)$ real and f real for real u, so that it is natural to look for real solutions $u=\bar u$.

For $\boldsymbol{\nu}\in\mathbb{Z}^{D+1}$ set $\boldsymbol{\nu}=(\nu_0,m)$, with $\nu_0\in\mathbb{Z}$ and $m=(\nu_1,\dots,\nu_D)\in\mathbb{Z}^D$ and $|\boldsymbol{\nu}|=|\nu_0|+|m|=|\nu_0|+|\nu_1|+\ldots+|\nu_D|$. For $\mathbf{x}=(t,x)=(t,x_1,\dots,x_D)\in\mathbb{R}^{D+1}$, set $\boldsymbol{\nu}\cdot\mathbf{x}=\nu_0 t+m\cdot x=\nu_0 t+\nu_1x_1+\ldots+\nu_Dx_D$. Set also $\mathbb{Z}_+=\{0\}\cup\mathbb{N}$ and $\mathbb{Z}_*^{D+1}=\mathbb{Z}^{D+1}\setminus\{\mathbf{0}\}$. Finally denote by $\delta(i,j)$ the Kronecker delta, i.e. $\delta(i,j)=1$ if $i=j$ and $\delta(i,j)=0$ otherwise. Given a finite set $\mathfrak{A}$ we denote by $|\mathfrak{A}|$ the cardinality of the set. Throughout the paper, for $z\in\mathbb{C}$ we denote by $\bar z$ the complex conjugate of z.

Since all the results of the paper are local (that is, they concern small amplitude solutions), we shall always assume that the hypotheses below are satisfied for all ε sufficiently small.

Hypothesis 1 ***(Conditions on the linear part).***

1. *$\mathbb{D}(\varepsilon)$ is diagonal in the Fourier basis $\{e^{i\boldsymbol{\nu}\cdot\mathbf{x}}\}_{\boldsymbol{\nu}\in\mathbb{Z}^{D+1}}$ with real eigenvalues $\delta_{\boldsymbol{\nu}}(\varepsilon)$ which are C^∞ in both $\boldsymbol{\nu}$ and ε.*
2. *For all $\boldsymbol{\nu}\in\mathbb{Z}_*^{D+1}$, one has either $\delta_{\boldsymbol{\nu}}(0)=0$ or $|\delta_{\boldsymbol{\nu}}(0)|\ge\gamma_0|\boldsymbol{\nu}|^{-\tau_0}$, for suitable constants $\gamma_0,\tau_0>0$.*
3. *For all $\boldsymbol{\nu}\in\mathbb{Z}_*^{D+1}$ one has $|\partial_\varepsilon\delta_{\boldsymbol{\nu}}(\varepsilon)|<c_2|\boldsymbol{\nu}|^{c_0}$ and, if $|\delta_{\boldsymbol{\nu}}(\varepsilon)|<1/2$, one has $|\partial_\varepsilon\delta_{\boldsymbol{\nu}}(\varepsilon)|>c_1|\boldsymbol{\nu}|^{c_0}$ as well, for suitable ε-independent constants $c_0,c_1,c_2>0$.*
4. *For all $\boldsymbol{\nu}\in\mathbb{Z}_*^{D+1}$ such that $|\delta_{\boldsymbol{\nu}}(\varepsilon)|<1/2$ one has $|\partial_\varepsilon\partial_{\boldsymbol{\nu}}\delta_{\boldsymbol{\nu}}(\varepsilon)|\le c_3|\boldsymbol{\nu}|^{c_0-1}$, for a suitable ε-independent constant $c_3>0$.*
5. *In Case (I) we require that if for some ε and for some $\boldsymbol{\nu}_1,\boldsymbol{\nu}_2\in\mathbb{Z}^{D+1}$ one has $|\delta_{\boldsymbol{\nu}_1}(\varepsilon)|,|\delta_{\boldsymbol{\nu}_2}(\varepsilon)|<1/2$, then $|\boldsymbol{\nu}_1-\boldsymbol{\nu}_2|\le|\boldsymbol{\nu}_1+\boldsymbol{\nu}_2|$.*

We now pass to the equation for the Fourier coefficients. We write

$$u(t,x) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^{D+1}} u_{\boldsymbol{\nu}}\, \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\mathbf{x}}, \tag{2.9}$$

and introduce the coefficients $u^{\pm}_{\boldsymbol{\nu}}$ by setting $u^{+}_{\boldsymbol{\nu}} := u_{\boldsymbol{\nu}}$ and $u^{-}_{\boldsymbol{\nu}} := \overline{u_{\boldsymbol{\nu}}}$. Analogously we define

$$\begin{aligned} f_{\boldsymbol{\nu}} = f_{\boldsymbol{\nu}}(\{u\},\eta) &:= [f(x,u,\bar{u},\eta)]_{\boldsymbol{\nu}} = \sum_{r,s\in\mathbb{N}:r+s=N+1} [a_{r,s}(x)u^r\bar{u}^s]_{\boldsymbol{\nu}} \\ &+ \sum_{r,s\in\mathbb{N}:r+s>N+1} \eta^{r+s-N-1}[a_{r,s}(x)u^r\bar{u}^s]_{\boldsymbol{\nu}}, \end{aligned}$$

where $\{u\} = \{u^{\sigma}_{\boldsymbol{\nu}}\}^{\sigma=\pm}_{\boldsymbol{\nu}\in\mathbb{Z}^{D+1}}$, $[\cdot]_{\boldsymbol{\nu}}$ denotes the Fourier coefficient with label $\boldsymbol{\nu}$, and we set $f^{+}_{\boldsymbol{\nu}} := f_{\boldsymbol{\nu}}$ and $f^{-}_{\boldsymbol{\nu}} := \overline{f_{\boldsymbol{\nu}}}$. Naturally $f^{\sigma}_{\boldsymbol{\nu}}$ depends also on the Fourier coefficients of the functions $a_{r,s}(x)$, which we denote by $a_{r,s,m}$, with $m \in \mathbb{Z}^D$; we set $a^{+}_{r,s,m} := a_{r,s,m}$ and $a^{-}_{r,s,m} := \overline{a_{r,s,m}}$.

Then in Fourier space Eqs. (2.7) and (2.8) give

$$\delta_{\boldsymbol{\nu}}(\varepsilon)\, u^{\sigma}_{\boldsymbol{\nu}} = \varepsilon f^{\sigma}_{\boldsymbol{\nu}}(\{u\},\varepsilon^{1/N}), \quad \boldsymbol{\nu}\in\mathbb{Z}^{D+1}, \tag{2.10}$$

and in the case of Dirichlet boundary conditions we shall require $u^{\sigma}_{\boldsymbol{\nu}} = -u^{\sigma}_{S_i(\boldsymbol{\nu})}$ for all $i = 1,\ldots,D$, where $S_i(\boldsymbol{\nu})$ is the linear operator that changes the sign of the i^{th} component of $\boldsymbol{\nu}$.

Remark 2.1. The reality condition on H in (2.3) reads

$$(s+1)\, a^{-}_{s+1,r-1,m} = r\, a^{+}_{r,s,-m}. \tag{2.11}$$

Moreover, by the analyticity assumption on the nonlinearity, one has $|a_{r,s,m}| \le A_1^{r+s}\mathrm{e}^{-A_2|m|}$ for suitable positive constants A_1 and A_2 independent of r and s.

Remark 2.2. We have doubled our equations by considering separately the equations for $u^{+}_{\boldsymbol{\nu}}$ and $u^{-}_{\boldsymbol{\nu}}$ – which clearly must satisfy a compatibility condition. In Case (II) one can work only on $u^{+}_{\boldsymbol{\nu}}$, since $u^{-}_{\boldsymbol{\nu}} = u^{+}_{-\boldsymbol{\nu}}$. In other examples it may be possible to reduce to solutions with $u_{\boldsymbol{\nu}}$ real for all $\boldsymbol{\nu}\in\mathbb{Z}^{D+1}$, but we found it more convenient to introduce the doubled equations in order to deal with the general case.

Following the standard Lyapunov-Schmidt decomposition scheme we split $\mathbb{Z}^{D+1}$ into two subsets called $\mathfrak{P}$ and $\mathfrak{Q}$ and treat the equations separately. By definition we call $\mathfrak{Q}$ the set of those $\boldsymbol{\nu}\in\mathbb{Z}^{D+1}$ such that $\delta_{\boldsymbol{\nu}}(0) = 0$; then we define $\mathfrak{P} = \mathbb{Z}^{D+1}\setminus\mathfrak{Q}$. Equations (2.10) restricted to the $\mathfrak{P}$ and $\mathfrak{Q}$ subset are called respectively the P and Q equations.

Hypothesis 2 ***(Conditions on the Q equation).***

1. *For all $\boldsymbol{\nu}\in\mathfrak{Q}$ one has $\lambda_{\boldsymbol{\nu}}(\varepsilon) := \varepsilon^{-1}\delta_{\boldsymbol{\nu}}(\varepsilon) \ge c > 0$, where c is ε-independent.*

2. *The Q equation at $\varepsilon = 0$,*

$$\lambda_{\boldsymbol{\nu}}(0)\, u_{\boldsymbol{\nu}}^{\sigma} = f_{\boldsymbol{\nu}}^{\sigma}(\{u^{\sigma}\}, 0), \quad \boldsymbol{\nu} \in \mathfrak{Q},$$

has a non-trivial non-degenerate solution

$$q^{(0)}(t, x) = \sum_{\boldsymbol{\nu} \in \mathfrak{Q}} u_{\boldsymbol{\nu}}^{(0)} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\mathbf{x}},$$

where non-degenerate means that the matrix

$$J_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'} = \lambda_{\boldsymbol{\nu}}(0)\, \delta(\boldsymbol{\nu}, \boldsymbol{\nu}')\, \delta(\sigma, \sigma') - \frac{\partial f_{\boldsymbol{\nu}}^{\sigma}}{\partial u_{\boldsymbol{\nu}'}^{\sigma'}}(\{q^{(0)}\}, 0)$$

is invertible. Moreover one has $|u_{\boldsymbol{\nu}}^{(0)}| \le \Lambda_0 \mathrm{e}^{-\lambda_0|\boldsymbol{\nu}|}$ *and* $\left|(J^{-1})_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'}\right| \le \Lambda_0 \mathrm{e}^{-\lambda_0|\boldsymbol{\nu}-\boldsymbol{\nu}'|}$, *for suitable constants Λ_0 and λ_0.*

Remark 2.3. The solution of the **bifurcation equation**, i.e. of the Q equation at $\varepsilon = 0$, could be assumed to be only Gevrey-smooth. Note also that, even when $\mathfrak{Q}$ is infinite-dimensional, the number of non-zero Fourier components of $q^{(0)}(t, x)$ can be finite.

The non-degeneracy condition in Hypothesis 2 is required in order to apply implicit function arguments. In principle the assumption can be weakened; see for instance [6]. However, to find optimal conditions is a very difficult task, already in the finite-dimensional case; see for instance [14], where the case of hyperbolic lower-dimensional tori with one normal frequency is investigated for finite-dimensional quasi-integrable systems.

Definition 2.4 (The sets $\mathfrak{S}(\boldsymbol{\varepsilon})$, $\mathfrak{S}$, and $\mathfrak{R}$). *Let ε_0 be a fixed positive constant. For $\varepsilon \in [0, \varepsilon_0]$ we set $\mathfrak{S}(\varepsilon) := \{\boldsymbol{\nu} \in \mathfrak{P} : |\delta_{\boldsymbol{\nu}}(\varepsilon)| < 1/2\}$ and $\mathfrak{S} = \cup_{\varepsilon \in [0,\varepsilon_0]} \mathfrak{S}(\varepsilon)$. Finally we call $\mathfrak{R}$ the subset $\mathfrak{P} \backslash \mathfrak{S}$.*

Remark 2.5. Note that $\boldsymbol{\nu} \in \mathfrak{R}$ means that $|\delta_{\boldsymbol{\nu}}(\varepsilon)| \ge 1/2$ for all $\varepsilon \in [0, \varepsilon_0]$.

The following definitions appear (in a slightly different form) in the papers by Bourgain. We shall use the formulation proposed by Berti and Bolle in [7], in terms of equivalence classes, because it will turn out to be very convenient.

Definition 2.6 (The equivalence relation $\sim$). *Let β and C_2 be two fixed positive constants. We say that two vectors $\boldsymbol{\nu}, \boldsymbol{\nu}' \in \mathfrak{S}(\varepsilon)$ are equivalent, and we write $\boldsymbol{\nu} \sim \boldsymbol{\nu}'$, if the following happens: one has $|\delta_{\boldsymbol{\nu}}(\varepsilon)|, |\delta_{\boldsymbol{\nu}'}(\varepsilon)| < 1/2$ and there exists a sequence $\{\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_K\}$ in $\mathfrak{S}(\varepsilon)$, with $\boldsymbol{\nu}_1 = \boldsymbol{\nu}$ and $\boldsymbol{\nu}_K = \boldsymbol{\nu}'$, such that*

$$\left|\delta_{\boldsymbol{\nu}_k}(\varepsilon)\right| < \frac{1}{2}, \qquad |\boldsymbol{\nu}_k - \boldsymbol{\nu}_{k+1}| \le \frac{C_2}{2}\left(|\boldsymbol{\nu}_k| + |\boldsymbol{\nu}_{k+1}|\right)^{\beta}, \qquad k = 1, \ldots, K-1.$$

Denote by $\Delta_j(\varepsilon)$, $j \in \mathbb{N}$, the equivalence classes with respect to $\sim$.

Remark 2.7. The equivalence relation $\sim$ induces a partition of $\mathfrak{S}(\varepsilon)$ into disjoint sets $\{\Delta_j(\varepsilon)\}_{j\in\mathbb{N}}$. Note also that, if $\boldsymbol{\nu}, \boldsymbol{\nu}' \in \Delta_j(\varepsilon)$, then it is not possible that for some ε' one has $\boldsymbol{\nu} \in \Delta_{j_1}(\varepsilon')$ and $\boldsymbol{\nu}' \in \Delta_{j_2}(\varepsilon')$ with $j_1 \ne j_2$.

Hypothesis 3 ***(Conditions on the set $\mathfrak{S}(\boldsymbol{\varepsilon})$: separation properties).***

There exist four ε-independent positive constants α, β, C_1, C_2, with α small enough and $\beta \le \alpha$, such that the sets $\Delta_j(\varepsilon)$ constructed according to Definition 2.6 satisfy $|\Delta_j(\varepsilon)| \le C_1 p_j^{\alpha}(\varepsilon)$, where $p_j(\varepsilon) = \min_{\boldsymbol{\nu}\in\Delta_j(\varepsilon)} |\boldsymbol{\nu}|$, for all $j \in \mathbb{N}$.

Remark 2.8. The condition that α be small will be essential in the following. On the contrary we could also allow $\beta > \alpha$ and this would also simplify the forthcoming analysis. However we prefer to consider directly the more relevant case $\beta \le \alpha$ because this is the case which arises in all applications. The relation between α and β is dictated by the explicit application one has in mind. On the contrary, it would be interesting to look for optimal bounds on the constant α in Hypothesis 3.

Lemma 2.9. *Hypothesis 3 implies the following properties:*

1. $\mathrm{dist}(\Delta_j(\varepsilon), \Delta_{j'}(\varepsilon)) \ge \dfrac{C_2}{2}\left(p_j(\varepsilon) + p_{j'}(\varepsilon)\right)^{\beta} \qquad \forall j, j' \in \mathbb{N}$ *such that* $j \ne j'$,
2. $\mathrm{diam}(\Delta_j(\varepsilon)) \le C_1 C_2 p_j^{\alpha+\beta}(\varepsilon) \qquad \forall j \in \mathbb{N}$,
3. $\max_{\boldsymbol{\nu}\in\Delta_j(\varepsilon)} |\boldsymbol{\nu}| \le 2p_j(\varepsilon) \qquad \forall j \in \mathbb{N}$,

and, furthermore, we can always assume that $2^{c_0-1} C_1 C_2 p_j^{\alpha+\beta} \le \zeta p_j$, with $\zeta c_3 < c_1/4$, where the constants c_1 and c_3 are defined in Hypothesis 1.

Proof. Properties 1–3 follow immediately from Definition 2.6. Indeed, the bound $|\Delta_j(\varepsilon)| \le C_1 p_j^{\alpha}(\varepsilon)$, used in Definition 2.6, yields $|\boldsymbol{\nu}| \le 2p_j(\varepsilon)$ for all $\boldsymbol{\nu} \in \Delta_j(\varepsilon)$. Then $\mathrm{diam}(\Delta_j(\varepsilon)) \le C_1 p_j^{\alpha} C_2 (4 p_j^{\beta}(\varepsilon))/2$ and, for $\boldsymbol{\nu} \in \Delta_j(\varepsilon)$ and $\boldsymbol{\nu}' \in \Delta_{j'}(\varepsilon)$ with $j \ne j'$, one has $|\boldsymbol{\nu} - \boldsymbol{\nu}'| \ge C_2(|\boldsymbol{\nu}| + |\boldsymbol{\nu}'|)^{\beta}/2 \ge C_2(p_j(\varepsilon) + p_{j'}(\varepsilon))^{\beta}/2$. □

Remark 2.10. The sets $\Delta_j(\varepsilon)$ are locally constant, in the sense that for almost all $\bar{\varepsilon} \in [0, \varepsilon_0]$ there exists an interval $\mathfrak{I}$ containing $\bar{\varepsilon}$ such that $\Delta_j(\varepsilon) = \Delta_j(\bar{\varepsilon})$ for all $\varepsilon \in \mathfrak{I}$.

Definition 2.11. *Given $f : \mathbb{T}^{D+1} \to \mathbb{C}$ define the norm*

$$|f|_{\kappa} := \sup_{\boldsymbol{\nu}\in\mathbb{Z}^{D+1}} |f_{\boldsymbol{\nu}}| \, \mathrm{e}^{\kappa|\boldsymbol{\nu}|^{1/2}} \qquad \text{with} \qquad f(\mathbf{x}) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^{D+1}} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\mathbf{x}} f_{\boldsymbol{\nu}}. \tag{2.12}$$

We can now state our main result.

Main Theorem. *Consider a PDE in the class described by (2.7) and (2.8), such that the Hypotheses 1, 2 and 3 hold. Then there exist two positive constants ε_0 and κ, a Cantor set $\mathfrak{E} \subset [0, \varepsilon_0]$, and a function $u(t, x) = u(t, x; \varepsilon)$ with the following properties:*

1. *$u(t, x; \varepsilon)$ is 2π-periodic in time, Gevrey-smooth both in time and in space, and C^1 in $\varepsilon \in [0, \varepsilon_0]$;*
2. *for all $\varepsilon \in [0, \varepsilon_0]$ one has $\left|u(t, x; \varepsilon) - q^{(0)}(t, x)\right|_{\kappa} \le C\varepsilon$, where $q^{(0)}$ is defined in Hypothesis 2;*
3. *$u(t, x; \varepsilon)$ solves the PDE for $\varepsilon \in \mathfrak{E}$;*
4. *the set $\mathfrak{E}$ has density 1 at $\varepsilon = 0$.*

The result above provides only Gevrey regularity. In [24] we could prove analyticity of the periodic solutions of the non-local NLS only in the non-resonant case. We leave as an open problem whether the solution in the Main Theorem is analytic in time and/or space.

3. Applications

3.1. Non-resonant equations. Let us prove that Eqs. (2.1), (2.4), and (2.5) – in particular the NLS, the NLB and the NLW – comply with all Hypotheses 1 to 3 in Sect. 2 and therefore admit a periodic solution by the Main Theorem.

3.1.1. The NLS equation.

Theorem 3.1. *Consider the nonlinear Schrödinger equation in dimension D*

$$\mathrm{i}\partial_t v - \Delta v + \mu\, v = f(x, v, \bar{v}),$$

with Dirichlet boundary conditions on the square $[0,\pi]^D$*, where* $\mu \in (0, \mu_0) \subset \mathbb{R}$ *and* $f(x, v, \bar{v}) = |v|^2 v + O(|v|^4)$*, that is f is given according to* (2.2) *and* (2.3)*, with* $N = 2$*,* $a_{2,1} = 1$ *and* $a_{r,s} = 0$ *for* r, s *such that* $r + s = 3$ *and* $(r, s) \neq (2, 1)$*. Then there exist a full measure set* $\mathfrak{M} \subset (0, \mu_0)$ *and two positive constants* ε_0 *and* κ *such that the following holds. For all* $\mu \in \mathfrak{M}$ *there exists a Cantor set* $\mathfrak{E}(\mu) \subset [0, \varepsilon_0]$*, such that for all* $\varepsilon \in \mathfrak{E}(\mu)$ *the equation admits a solution* $v(t, x)$*, which is* $2\pi/\omega$*-periodic in time and Gevrey-smooth both in time and in space, and such that*

$$\left| v(t,x) - \sqrt{\varepsilon} q_0 \mathrm{e}^{\mathrm{i}\omega t} \sin x_1 \ldots \sin x_D \right|_{\kappa} \le C\varepsilon, \qquad \omega = D + \mu - \varepsilon, \qquad |q_0| = \left(\frac{4}{3}\right)^{D/2}.$$

The set $\mathfrak{E} = \mathfrak{E}(\mu)$ *has density 1 at* $\varepsilon = 0$.

With the notations of Sect. 2 one has $\delta_{\mathbf{v}}(\varepsilon) = -\omega n + |m|^2 + \mu$, with $\omega = \omega_0 - \varepsilon$ and $\omega_0 = D + \mu$. Then it is easy to check that all items of Hypothesis 1 are satisfied provided μ is chosen in such a way that $|-\omega_0 n + |m|^2| \ge \gamma_0 |n|^{-\tau_0}$. This is possible for μ in a full measure set; cf. Eq. (2.1) in [24]. Then Hypothesis 1 holds with $c_0 = c_2 = c_3 = 1$ and $c_1 = 1/\sqrt{1 + 4\omega_0}$.

The subset $\mathfrak{Q}$ is defined as $\mathfrak{Q} := \{(n, m) \in \mathbb{Z}^{1+D} : n = 1,\ |m_i| = 1\ \forall i = 1, \ldots D\}$, and one can assume q_0 to be real, so that, by the Dirichlet boundary conditions, $\mathfrak{Q}$ is in fact one-dimensional, and $u_{n,m} = \pm q_0$ for all $(n, m) \in \mathfrak{Q}$. The leading order of the Q equation is explicitly studied in [24], where it is proved that Hypothesis 2 is satisfied.

Finally, Hypothesis 3 has been proven by Bourgain [9] (see also Appendix A6 in [24]).

Of course, Theorem 3.1 refers to solutions with $m = (1, 1, \ldots, 1)$, but it easily extends to solutions which continue other harmonics of the linear equation; see comments in [24].

Also, the condition on the nonlinearity can be weakened. In general N can be any integer $N > 1$, and no other conditions must be assumed on the functions $a_{r,s}(x)$ beyond those mentioned after (2.2). In that case (for simplicity we consider the same solution of the linear equation as in Theorem 3.1), the leading order of the Q equation becomes $q_0 = \mathrm{sign}(\varepsilon) A_0 q_0^N$ (again by taking for simplicity's sake q_0 to be real), where A_0 is a constant depending on the nonlinearity. If A_0 is non-zero, this surely has a non-trivial non-degenerate solution q_0 either for positive or negative values of ε. In general the non-degeneracy condition in item 2 of Hypothesis 2 has to be verified case by case by computing A_0.

3.1.2. The NLW equation.

Theorem 3.2. *Consider the nonlinear wave equation in dimension D,*

$$\partial_{tt} v - \Delta v + \mu v = f(x, v),$$

with Dirichlet boundary conditions on the square $[0, \pi]^D$*, where* $\mu \in (0, \mu_0) \subset \mathbb{R}$ *and* $f(x, v) = v^3 + O(v^4)$*, that is* f *is given according to (2.2), with* $s = 0$*,* $N = 2$*,* $a_{3,0} = 1$*. Then there exist a full measure set* $\mathfrak{M} \subset (0, \mu_0)$ *and two positive constants* ε_0 *and* κ *such that the following holds. For all* $\mu \in \mathfrak{M}$ *there exists a Cantor set* $\mathfrak{E}(\mu) \subset [0, \varepsilon_0]$*, such that for all* $\varepsilon \in \mathfrak{E}(\mu)$ *the equation admits a solution* $v(t, x)$*, which is* $2\pi/\omega$*-periodic in time and Gevrey-smooth both in time and in space, and such that*

$$\left|v(t, x) - q_0 \sqrt{\varepsilon} \cos \omega t \sin x_1 \ldots \sin x_D\right|_\kappa \leq C\varepsilon, \quad \omega = \sqrt{D + \mu - \varepsilon}, \quad q_0 = \left(\frac{4}{3}\right)^{(D+1)/2}.$$

The set $\mathfrak{E} = \mathfrak{E}(\mu)$ *has density 1 at* $\varepsilon = 0$*.*

In that case one has $\delta_{\mathbf{v}}(\varepsilon) = -\omega^2 n^2 + |m|^2 + \mu$, with $\omega^2 = \omega_0^2 - \varepsilon$ and $\omega_0^2 = D^2 + \mu$. Once more, it is easy to check that Hypothesis 1 is satisfied provided μ is chosen in a full measure set, with $c_0 = c_2 = c_3 = 1$ and $c_1 = 1/(1 + 4\omega_0^2)$.

The subset $\mathfrak{Q}$ is given by $\mathfrak{Q} := \{(n, m) \in \mathbb{Z}^{1+D} : n = \pm 1,\ |m_i| = 1\ \forall i = 1, \ldots D\}$, and, if one chooses to look for solutions that are even in time, then $\mathfrak{Q}$ is one-dimensional. The Q equation at $\varepsilon = 0$ can be discussed as in the case of the nonlinear Schrödinger equation. For instance for f as in the statement of Theorem 3.2 the non-degeneracy in item 2 of Hypothesis 2 can be explicitly verified. Again, the analysis easily extends to more general situations, under the assumption that the Q equation at $\varepsilon = 0$ admits a non-degenerate solution. For a fixed nonlinearity, this can be easily checked with a simple computation.

Hypothesis 3 has been verified by Bourgain [8], under some strong conditions on ω. Recently the same separation estimates have been proved by Berti and Bolle [7], by only requiring that ω^2 be Diophantine.

3.1.3. Other equations. The separation properties for the NLS equation imply similar separation also for the nonlinear beam (NLB) equation

$$\partial_{tt} v + (\Delta + \mu)^2\ v = f(x, v),$$

and in that case we can also consider nonlinearities with one or two space derivatives.

As in the previous cases one restricts μ to some full measure set, and Hypothesis 1 holds with $c_0 = c_3 = 2$, $c_2 = 1$ and $c_1 = 1/\sqrt{1 + 2\omega_0}$. This implies that the subset $\mathfrak{Q}$ is one-dimensional, provided we look for real solutions which are even in time.

The same kind of arguments holds for all equations of the form (2.1) and (2.4). The separation of the points $(m, |m|^2)$ in $\mathbb{Z}^{D+1}$ implies, by convexity, also the separation of $(m, P(|m|^2))$, with $P(x)$ defined after (2.1).

3.2. Completely resonant equations. Here we describe an application to completely resonant NLS and NLB equations, namely Eqs. (2.1) and (2.4) with $P(x) = x$ and $\mu = 0$, and with Dirichlet boundary conditions (the case of periodic boundary conditions is easier for fully resonant equations). Since the equation is completely resonant we need some assumption on the nonlinearity in order to comply with Hypothesis 2. We set $f(x, v, \bar{v}) = |v|^2 v$ for the NLS and $f(x, v) = v^3$ for the NLB (the NLB falls in Case (II) and we look for real solutions), but our proofs extend easily to deal with higher order corrections which are odd and do not depend explicitly on the space variables. In the case of the NLS we say that the leading term of the nonlinearity is cubic and gauge-invariant.[1]

The validity of Hypothesis 1 can be discussed as in the non-resonant equations of Subsects. 3.1. The separation properties (Hypothesis 3) do not change in the presence of a mass term, and they have been already discussed in the non-resonant examples of Subsect. 3.1. Thus, we only need to prove the non-degeneracy of the solution of the Q equation. Since the nonlinearity does not depend explicitly on x we look for solutions such that $u_{\boldsymbol{\nu}} \in \mathbb{R}$. We follow closely [24], but we set $\omega_0 = 1$. This is done for purely notational reasons, and is due to the fact that a trivial rescaling of time allows us to put $\omega_0 = 1$.

3.2.1. The NLS equation. The subset $\mathfrak{Q}$ is infinite-dimensional, i.e. $\mathfrak{Q} := \{(n, m) \in \mathbb{N} \times \mathbb{Z}^D : n = |m|^2\}$. We set $u_{(n,m)} = q_m = a_m + O(\varepsilon^{1/2})$ for $(n, m) \in \mathfrak{Q}$ and restrict our attention to the case $q_m \in \mathbb{R}$. At leading order, the Q equation is (cf. [24])

$$|m|^2 a_m = \sum_{\substack{m_1,m_2,m_3 \\ m_1+m_2-m_3=m \\ \langle m_1-m_3, m_2-m_3\rangle=0}} a_{m_1} a_{m_2} a_{m_3}. \tag{3.1}$$

Note that in the case of [24], the left-hand side of (3.1) was $|m|^{2+2s} D^{-1} a_m$, with s a free parameter; then (3.1) is recovered by setting $s = 0$ and rescaling by $1/\sqrt{D}$ the coefficients q_m.

By Lemma 17 of [24] – which holds for all values of s – for each $N_0 \geq 1$ there exist infinitely many finite sets $\mathcal{M}_+ \subset \mathbb{Z}^D_+$ with N_0 elements such that Eq. (3.1) admits the solution (due to the Dirichlet boundary conditions we describe the solution in $\mathbb{Z}^D_+$)

$$a_m = \begin{cases} 0, & m \in \mathbb{Z}^D_+ \backslash \mathcal{M}_+ \\ \sqrt{\dfrac{1}{2^{D+1} - 3^D} \left(|m|^2 - c_1 \displaystyle\sum_{m' \in \mathcal{M}_+} |m'|^2 \right)}, & m \in \mathcal{M}_+, \end{cases}$$

with $c_1 = 2^{D+1}/(2^{D+1}(N_0 - 1) + 3^D)$. The set $\mathcal{M}_+$ defines a matrix J on $\mathbb{Z}^D$ such that

$$(JQ)_m = |m|^2 - 2 \sum_{\substack{m_1,m_2,m_3 \\ m_1+m_2-m_3=m \\ \langle m_1-m_3, m_2-m_3\rangle=0}} Q_{m_1} a_{m_2} a_{m_3} - 2 \sum_{\substack{m_1>m_2,m_3 \\ m_1+m_2-m_3=m \\ \langle m_1-m_3, m_2-m_3\rangle=0}} a_{m_1} a_{m_2} Q_{m_3}, \tag{3.2}$$

where $m_1 > m_2$ refers, say, to lexicographic ordering of $\mathbb{Z}^D$; see in particular Eqs. (8.5) and (8.7) of [24].

[1] I.e. the equation up to the third order is invariant under the transformation $v \to v e^{i\alpha}$ for any $\alpha \in \mathbb{R}$.

Moreover we know (Lemma 18 of [24]) that the matrix J is block-diagonal with blocks of size depending only on N_0, D: we denote by $K(N_0, D)$ the bound on such a size. Whatever the block structure, the matrix J has the form $\mathrm{diag}(|m|^2) + 2T$, where all the entries of T are linear combinations of terms $q_{m_i} q_{m_j}$ with integer coefficients. If we multiply J by $z := (2^{D+1} - 3^D)(2^{D+1}(N_0 - 1) + 3^D)$ – which is odd – we obtain a matrix $\widetilde{J} := \mathrm{diag}(z|m|^2) + 2\widetilde{T}$, where all the entries of $\widetilde{T}$ are integral linear combinations of the square roots of a finite number of integers. Let us call the prime factors of such integers $p_0 = 1, p_1, p_2, \ldots$.

Definition 3.3 (The lattice $\mathbb{Z}^D_1$). *Let* $\mathbb{Z}^D_1 := (1, 0, \ldots, 0) + 2\mathbb{Z}^D$ *be the affine lattice of integer vectors such that the first component is odd and the others even. Let* $\mathbb{Z}^D_{1,+}$ *be its intersection with* $\mathbb{Z}^D_+$*. Of course, for all* $m \in \mathbb{Z}^D_1$ *one has* $|m|^2$ *odd.*

Since we are working with odd nonlinearities which do not depend explicitly on the space variables we look for solutions such that $u_{n,m} = 0$ if $m \notin \mathbb{Z}^D_1$.

Let $1, p_1, \ldots, p_k$ be prime numbers (as above), and let $a_1, \ldots, a_K$ be the set of all products of square roots of different numbers p_i, i.e. $a_1 = 1$, $a_2 = \sqrt{p_1}$, $a_3 = \sqrt{p_1 p_2}$, etc. It is clear that the set of integral linear combinations of a_i is a ring (of algebraic integers). We denote it by $\mathfrak{a}$. The following Lemma is a simple consequence of Galois theory [2]. For completeness, the proof is given in Appendix A.

Lemma 3.4. *The numbers* a_i *are linearly independent over the rationals.*

Immediately we have the following corollary (I denotes the identity).

Corollary 3.5. *In* $\mathfrak{a}$ *consider* $2\mathfrak{a}$*, i.e. the set of linear combinations with even coefficients.*

- $2\mathfrak{a}$ *is a proper ideal, and the quotient ring* $\mathfrak{a}/2\mathfrak{a}$ *is thus a non-zero ring.*
- *if a matrix* M *with entries in* $\mathfrak{a}$ *is such that* $M - I$ *has all entries in* $2\mathfrak{a}$*, then* M *is invertible.*

The point of Corollary 3.5 is that the determinant of $M = I + 2\widetilde{M}$, with the entries of $\widetilde{M}$ in $\mathfrak{a}$, is $1 + 2\alpha$, with $\alpha \in \mathfrak{a}$. Hence, by Lemma 3.4, $2\alpha \neq \pm 1$.

Lemma 3.6. *For all* N_0 *and for all* $\mathcal{M}_+ \subset \mathbb{Z}^D_{1,+}$ *the matrix* J *defined by* $\mathcal{M}_+$ *is invertible. Its inverse is a block matrix with blocks of dimension depending only on* N_0, D *so that for some appropriate* C *one has* $(J^{-1})_{m,m'} \leq C$ *if* $|m - m'| \leq K(N_0, D)$*, while* $(J^{-1})_{m,m'} = 0$ *otherwise.*

Proof. We use Corollary 3.4, the fact that the matrix $\widetilde{J}$ has entries in $\mathfrak{a}$ and the fact that $z|m|^2$ is odd for all $m \in \mathbb{Z}^D_{1,+}$. □

Now, we can state our result on the completely resonant NLS.

Theorem 3.7. *Consider the nonlinear Schrödinger equation in dimension* D,

$$\mathrm{i}\partial_t v - \Delta v = f(v, \bar{v}),$$

with Dirichlet boundary conditions on the square $[0, \pi]^D$*, where* f *is given according to* (2.2) *and* (2.3)*, with* $N = 2$*,* $a_{2,1} = 1$*,* $a_{r,s} = 0$ *for* r, s *such that* $r + s = 3$ *and* $(r, s) \neq (2, 1)$*, and* $a_{r,s}(x)$ *independent of* x *for* $r + s > 3$ *(so that in particular* $a_{r,s} = 0$ *for even* $r + s$*). Then for any* $N_0 \geq 1$ *there exist sets* $\mathcal{M}_+$ *of* N_0 *vectors in* $\mathbb{Z}^D_+$ *and real* *amplitudes* $\{a_m\}_{m\in\mathcal{M}_+}$ *such that the following holds. There exist two positive constants* ε_0 *and* κ *and a Cantor set* $\mathfrak{E} \subset [0, \varepsilon_0]$*, such that for all* $\varepsilon \in \mathfrak{E}$ *the equation admits a solution* $v(t, x)$*, which is* $2\pi/\omega$*-periodic in time and Gevrey-smooth both in time and in space, and such that, setting*

$$q_0(t, x) = (2\mathrm{i})^D \sum_{m\in\mathcal{M}_+} a_m \mathrm{e}^{\mathrm{i}|m|^2 t} \sin m_1 x_1 \ldots \sin m_D x_D, \quad \omega = 1 - \varepsilon, \tag{3.3}$$

one has

$$\left|v(t, x) - \sqrt{\varepsilon} q_0(x, \omega t)\right|_\kappa \le C\varepsilon.$$

The set $\mathfrak{E}$ *has density 1 at* $\varepsilon = 0$.

3.2.2. The beam equation. We set $\omega^2 = \omega_0^2 - \varepsilon = 1 - \varepsilon$ (recall that we are assuming $\omega_0 = 1$ by a suitable time rescaling). The subset $\mathfrak{Q}$ is given by $\mathfrak{Q} := \{(n, m) \in \mathbb{N} \times \mathbb{Z}^D : |n| = |m|^2\}$. We set $u_{n,m} = q_m^+$ for $n = |m|^2$ and $u_{n,m} = q_m^-$ for $n = -|m|^2$. We can require that $q_m^+ = q_m^- \equiv q_m$ for all m (we obtain a solution which is even in time). Since we look for real solutions, this implies that $q_m \in \mathbb{R}$ if D is even and $q_m \in \mathrm{i}\mathbb{R}$ if D is odd. Since the nonlinearity does not depend explicitly on x, we can look for solutions $u_{n,m}$ such that $m \in \mathbb{Z}_1^D$ (see Definition 3.3).

Finally the separation properties of the small divisors do not depend on the presence of the mass term, so that we only need to prove the existence and non-degeneracy of the solutions of the bifurcation equation.

The Q equation at leading order is

$$|m|^4 a_m = (-1)^D \sum_{\substack{m_1+m_2+m_3=m \\ \pm|m_1|^2\pm|m_2|^2\pm|m_3|^2=\pm|m|^2}} a_{m_1} a_{m_2} a_{m_3},$$

where we have set $|q_m| = a_m + O(\varepsilon^{1/2})$.

Lemma 3.8. *The condition* $\pm|m_1|^2 \pm |m_2|^2 \pm |m_3|^2 = \pm|m|^2$*, for* $m_i, m \in \mathbb{Z}_1^D$*, is equivalent to* $\langle m_1 + m_3, m_2 + m_3\rangle = 0$.

Proof. The condition $|m_1|^2+|m_2|^2+|m_3|^2 = (m_1+m_2+m_3)^2$ is equivalent to $\langle m_1, m_2+m_3\rangle + \langle m_2, m_3\rangle = 0$, which is impossible since the left hand side is an odd integer. The same happens with the condition $|m_1|^2-|m_2|^2-|m_3|^2 = (m_1+m_2+m_3)^2$. Thus, we are left with $|m_1|^2+|m_2|^2-|m_3|^2 = (m_1+m_2+m_3)^2$, which implies $\langle m_1+m_3, m_2+m_3\rangle = 0$. □

Lemma 3.8 implies that the bifurcation equation, restricted to $\mathbb{Z}_1^D$, is identical to that of a smoothing NLS with $s = 2$; cf. [24]. Indeed by recalling that $q_m = (-1)^D q_{-m}$ one has

$$|m|^4 a_m = \sum_{\substack{m_1+m_2-m_3=m \\ \langle m_1-m_3, m_2-m_3\rangle=0}} a_{m_1} a_{m_2} a_{m_3}. \tag{3.4}$$

Then we can repeat the arguments of the previous subsection. By Lemma 17 of [24] – which holds for all values of s – for each $N_0 \geq 1$ there exist infinitely many finite sets $\mathcal{M}_+ \subset \mathbb{Z}^D_{1,+}$ with N_0 elements such that Eq. (3.4) has the solution

$$a_m = \begin{cases} 0, & m \in \mathbb{Z}^D_+ \setminus \mathcal{M}_+ \\ \sqrt{\dfrac{1}{2^{D+1} - 3^D}\left(|m|^4 - c_1 \displaystyle\sum_{m' \in \mathcal{M}_+} |m'|^4 \right)}, & m \in \mathcal{M}_+, \end{cases},$$

with $c_1 = 2^{D+1}/(2^{D+1}(N_0 - 1) + 3^D)$.

The matrix J is defined as in (3.2), only with $|m|^4$ on the diagonal. We know (Lemma 18 of [24] does not depend on the values of s) that the matrix J is block-diagonal with blocks of size bounded by $K(N_0, D)$ (defined as in Subsect. 3.2.1). Whatever the block structure, the matrix J has the form $\mathrm{diag}(|m|^4) + 2T$, where all the entries of T are linear combinations of terms $a_{m_i} a_{m_j}$ with integer coefficients. If we multiply J by $z := (2^{D+1} - 3^D)(2^{D+1}(N_0 - 1) + 3^D)$ – which is odd – we obtain a matrix $\widetilde{J} := \mathrm{diag}(z|m|^4) + 2\widetilde{T}$, where all the entries of $\widetilde{T}$ are linear combinations of the square roots of a finite number of integers; finally $z|m|^4$ is clearly odd and we can apply Lemma 3.4 to obtain the analogue of Lemma 3.6. Thus, a theorem analogous to Theorem 3.7 is obtained, with $q_0(t, x)$ in (3.3) replaced with

$$q_0(t, x) = 2^{D+1} \sum_{m \in \mathcal{M}_+} a_m \cos |m|^2 t \, \sin m_1 x_1 \dots \sin m_D x_D, \qquad \omega^2 = 1 - \varepsilon.$$

We leave the formulation to the reader.

4. Technical Set-up and Propositions

4.1. Renormalised $\boldsymbol{P}$*-*$\boldsymbol{Q}$ *equations.* Group Eqs. (2.10) for $\boldsymbol{\nu} \in \mathfrak{S}$ as a matrix equation. Setting

$$\begin{aligned} &U = \{u^\sigma_{\boldsymbol{\nu}}\}^{\sigma=\pm}_{\boldsymbol{\nu}\in\mathfrak{S}}, \quad V = \{u^\sigma_{\boldsymbol{\nu}}\}^{\sigma=\pm}_{\boldsymbol{\nu}\in\mathfrak{R}}, \quad Q = \{u^\sigma_{\boldsymbol{\nu}}\}^{\sigma=\pm}_{\boldsymbol{\nu}\in\mathfrak{Q}}, \quad F = \{f^\sigma_{\boldsymbol{\nu}}\}^{\sigma=\pm}_{\boldsymbol{\nu}\in\mathfrak{S}}, \\ &\mathscr{D}(\varepsilon) = \mathrm{diag}\,\{\delta_{\boldsymbol{\nu}}(\varepsilon)\}^{\sigma=\pm}_{\boldsymbol{\nu}\in\mathfrak{S}}, \end{aligned} \tag{4.1}$$

the P equations spell

$$\begin{cases} \mathscr{D}(\varepsilon)\, U = \varepsilon F(U, V, Q, \varepsilon^{1/N}), \\ u^\sigma_{\boldsymbol{\nu}} = \varepsilon \delta^{-1}_{\boldsymbol{\nu}}(\varepsilon)\, f^\sigma_{\boldsymbol{\nu}}(U, V, Q, \varepsilon^{1/N}), \qquad \boldsymbol{\nu} \in \mathfrak{R}, \quad \sigma = \pm, \end{cases} \tag{4.2}$$

with a reordering of the arguments of the coefficients $f^\sigma_{\boldsymbol{\nu}}$. Note that also the first line in (4.2) could be written for components as the second line (and it would look exactly like the second line); however we find more convenient the shortened writing for $\boldsymbol{\nu} \in \mathfrak{S}$.

We shall proceed as follows. We introduce an appropriate "correction" to the left hand side of (4.2). We shall consider self-adjoint matrices $\mathscr{M}(\varepsilon) := \{\mathscr{M}^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon)\}^{\sigma,\sigma'=\pm}_{\boldsymbol{\nu},\boldsymbol{\nu}'\in\mathfrak{S}}$, such that, for each fixed ε, $\mathscr{M}(\varepsilon)$ is block-diagonal on the sets $\Delta_j(\varepsilon)$ (cf. Definition 2.6), namely $\mathscr{M}^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon) \neq 0$ can hold only if $\boldsymbol{\nu}, \boldsymbol{\nu}' \in \Delta_j(\varepsilon)$ for some j. Note that in order to have $u^+_{\boldsymbol{\nu}} = \overline{u^-_{\boldsymbol{\nu}}}$ we must require that $\mathscr{M}^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'} = \mathscr{M}^{-\sigma',-\sigma}_{\boldsymbol{\nu}',\boldsymbol{\nu}}$.

Definition 4.1 (The set $\mathfrak{G}$ and the matrix $\widehat{\chi}_1$). *Call* $\mathfrak{G} = \{1/4 > \bar{\gamma} > 0 : ||\delta_{\boldsymbol{\nu}}(0)| - \bar{\gamma}| \geq \bar{\gamma}_0/|\boldsymbol{\nu}|^{\bar{\tau}_0}$ *for all* $\boldsymbol{\nu} \in \mathbb{Z}_*^{D+1}\}$*, for suitable constants* $\bar{\gamma}_0, \bar{\tau}_0 > 0$*. For* $\bar{\gamma} \in \mathfrak{G}$*, we introduce the step function* $\bar{\chi}_1(x)$ *such that* $\bar{\chi}_1(x) = 0$ *if* $|x| \geq \bar{\gamma}$ *and* $\bar{\chi}_1(x) = 1$ *if* $|x| < \bar{\gamma}$*, and set* $\bar{\chi}_0(x) = 1 - \bar{\chi}_1(x)$*. We then introduce the (ε-dependent) diagonal matrices* $\widehat{\chi}_1 = \text{diag}\{\bar{\chi}_1(\delta_{\boldsymbol{\nu}}(\varepsilon))\}_{\boldsymbol{\nu}\in\mathfrak{S}}^{\sigma=\pm}$ *and* $\widehat{\chi}_0 = \text{diag}\{\bar{\chi}_0(\delta_{\boldsymbol{\nu}}(\varepsilon))\}_{\boldsymbol{\nu}\in\mathfrak{S}}^{\sigma=\pm}$*.*

Remark 4.2. One has $\mathfrak{G} \neq \emptyset$. Moreover, for any interval $\mathfrak{U} \subset (0, 1/4)$, the relative measure of the set $\mathfrak{U} \cap \mathfrak{G}$ tends to 1 as $\bar{\gamma}_0$ tends to 0, provided $\bar{\tau}_0$ is large enough.

Remark 4.3. Notice the difference with respect to [24], where the functions $\bar{\chi}_i(x)$ were smooth. This new definition will allow us to strongly simplify notations and proofs in the forthcoming analysis. In particular one has $\widehat{\chi}_1^2 = \widehat{\chi}_1$ and $\widehat{\chi}_1\widehat{\chi}_0 = 0$, with 0 the null matrix. Roughly speaking, this will allow us to invert matrices without mixing singular and regular frequencies.

Definition 4.4 (Resonant sets). *A set* $\mathcal{N} = \{\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_m\} \subset \mathfrak{S}$ *is* **resonant** *if there exists* $\varepsilon \in [0, \varepsilon_0]$ *and* $j \in \mathbb{N}$ *such that* $\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_m \in \Delta_j(\varepsilon)$*. A resonant set* $\{\boldsymbol{\nu}_1, \boldsymbol{\nu}_2\}$ *with* $m = 2$ *will be called a* **resonant pair**. *Given a resonant set* $\mathcal{N} = \{\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_m\}$ *we call* $\mathcal{C}_{\mathcal{N}}$ *the set of all* $\boldsymbol{\nu} \in \mathfrak{S}$ *such that* $\mathcal{N} \cup \{\boldsymbol{\nu}\}$ *is still a resonant set. Finally set* $\overline{\mathcal{C}}_{\mathcal{N}}(\varepsilon) := \{\boldsymbol{\nu}' \in \mathcal{C}_{\mathcal{N}} : |\delta_{\boldsymbol{\nu}'}(\varepsilon)| < \bar{\gamma}\}$*, with* $\bar{\gamma}$ *introduced in Definition 4.1.*

Define the *renormalised P equation* as

$$\begin{cases} (\mathscr{D}(\varepsilon) + \mathscr{M})\, U = \eta^N F(U, V, Q, \eta) + L\, U, \\ u_{\boldsymbol{\nu}}^{\sigma} = \eta^N\, \delta_{\boldsymbol{\nu}}^{-1}(\varepsilon)\, f_{\boldsymbol{\nu}}^{\sigma}(U, V, Q, \eta), \qquad\qquad \nu \in \mathfrak{R}, \end{cases} \tag{4.3}$$

where η is a real parameter, while $\mathscr{M} = \{\mathscr{M}_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'}\}_{\boldsymbol{\nu},\boldsymbol{\nu}'\in\mathfrak{S}}^{\sigma,\sigma'=\pm}$ and $L = \{L_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'}\}_{\boldsymbol{\nu},\boldsymbol{\nu}'\in\mathfrak{S}}^{\sigma,\sigma'=\pm}$ are self-adjoint matrices with the properties:

1. $\mathscr{M} = \widehat{\chi}_1 M \widehat{\chi}_1$, where $M = \{M_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'}\}_{\boldsymbol{\nu},\boldsymbol{\nu}'\in\mathfrak{S}}^{\sigma,\sigma'=\pm}$;
2. $M_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'} = L_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'} = 0$ if $\{\boldsymbol{\nu}, \boldsymbol{\nu}'\}$ is not a resonant pair;
3. $M_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'} = M_{\boldsymbol{\nu}',\boldsymbol{\nu}}^{-\sigma',-\sigma}$ and $L_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'} = L_{\boldsymbol{\nu}',\boldsymbol{\nu}}^{-\sigma',-\sigma}$.

Remark 4.5. Property 1 above implies that $\mathscr{M}$ has an ε-dependent block structure, which will be crucial in the convergence estimates. On the other hand we need to introduce free (i.e. ε-independent) parameters. Thus, we introduce the elements of the matrix M as ε-independent parameters, with the only restriction that they satisfy the ε-independent Properties 2 and 3. Eventually we shall manage to fix M as a function of the parameter ε, that is $M = M(\varepsilon)$. Moreover an important property for the measure estimates will be that $M(\varepsilon)$ depends smoothly on ε, at least in a large measure set.

The *renormalised Q equation* is defined as

$$u_{\boldsymbol{\nu}}^{\sigma} = \sum_{\boldsymbol{\nu}'\in\mathfrak{Q}} \sum_{\sigma'=\pm} (J^{-1})_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'} f_{\boldsymbol{\nu}'}^{\sigma'}(U, V, Q, \eta), \qquad \boldsymbol{\nu} \in \mathfrak{Q}, \quad \sigma = \pm. \tag{4.4}$$

Remark 4.6. By looking at (4.3) and (4.4) it would be tempting to introduce 2×2 matrices $M_{\boldsymbol{\nu},\boldsymbol{\nu}'}$ and $(J^{-1})_{\boldsymbol{\nu},\boldsymbol{\nu}'}$ instead of carrying along the subscripts σ, σ' through all the equations. However, in the following, to introduce the diagrammatic expansion and check some symmetry properties, we shall have to write everything by components: hence it will be more convenient to keep also the σ labels, in order not to introduce too many notations.

The parameter η and the *counterterms* L will have to satisfy eventually the identities (*compatibility equation*)

$$\eta = \varepsilon^{1/N}, \qquad \mathscr{M} = L. \tag{4.5}$$

We proceed in the following way: first we solve the renormalised P and Q equations (4.3) and (4.4), then we impose the compatibility equation (4.5).

4.2. Matrix spaces. Here we introduce some notations and properties that we shall need in the following.

Definition 4.7 (The Banach space $\boldsymbol{\mathcal{B}_{\kappa,\rho}}$). *We consider the space of infinite-dimensional self-adjoint matrices* $\{M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}\}^{\sigma,\sigma'=\pm}_{\boldsymbol{\nu},\boldsymbol{\nu}'\in\mathfrak{S}}$ *such that* $M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'} = 0$ *if* $\{\boldsymbol{\nu}, \boldsymbol{\nu}'\}$ *is not resonant. For* $\rho, \kappa > 0$ *we equip such a space with the norm*

$$|M|_{\kappa,\rho} := \sup_{\boldsymbol{\nu},\boldsymbol{\nu}'\in\mathfrak{S}} \sup_{\sigma,\sigma'=\pm} \left| M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'} \right| \mathrm{e}^{\kappa|\boldsymbol{\nu}-\boldsymbol{\nu}'|^{\rho}},$$

so obtaining a Banach space that we call $\mathcal{B}_{\kappa,\rho}$*. For L a linear operator on* $\mathcal{B}_{\kappa,\rho}$ *define the operator norm*

$$|L|_{\mathrm{op}} = \sup_{M\in\mathcal{B}_{\kappa,\rho}} \frac{|LM|_{\kappa,\rho}}{|M|_{\kappa,\rho}}.$$

Definition 4.8 (Matrix norms). *Let* A *be a* $d \times d$ *self-adjoint matrix, and denote with* $A(i, j)$ *and* $\lambda^{(i)}(A)$ *its entries and its eigenvalues, respectively. We define the norms*

$$|A|_{\infty} := \max_{1\le i,j\le d} |A(i, j)|, \qquad \|A\| := \frac{1}{\sqrt{d}}\sqrt{\mathrm{tr}(A^2)}, \qquad \|A\|_2 := \max_{|x|_2\le 1} |Ax|_2\,,$$

where, given a vector $x \in \mathbb{R}^d$*, we denote by* $|x|_2$ *its Euclidean norm.*

Lemma 4.9 *Given a* $d \times d$ *self-adjoint matrix* A*, the following properties hold.*

1. *The norm* $\|A\|$ *depends smoothly on the coefficients* $A(i, j)$.
2. *One has* $\|A\|/\sqrt{d} \le |A|_{\infty} \le \sqrt{d}\|A\|$.
3. *One has* $\max_{1\le i\le d} |\lambda^{(i)}(A)|/\sqrt{d} \le \|A\| \le \max_{1\le i\le d} |\lambda^{(i)}(A)|$.
4. *For invertible A one has* $\partial_{A(i,j)} A^{-1}(i', j') = -A^{-1}(i', i)\, A^{-1}(j, j')$ *and* $\partial_{A(i,j)}\|A\| = A(i, j)/d\|A\|$.

Here and henceforth we shall write $A = \mathscr{D}(\varepsilon) + \mathscr{M}$ in (4.3).

Definition 4.10 (Small divisors). *For* $\boldsymbol{\nu} \in \mathfrak{S}$ *define* $A^{\boldsymbol{\nu}}(\varepsilon)$ *as the matrix with entries* $\bar{\chi}_1(\delta_{\boldsymbol{\nu}}(\varepsilon))\, A^{\sigma_1,\sigma_2}_{\boldsymbol{\nu}_1,\boldsymbol{\nu}_2}$ *such that* $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2 \in \overline{\mathcal{C}}_{\boldsymbol{\nu}}(\varepsilon)$ *and* $\sigma_1, \sigma_2 = \pm$*. If* $|\delta_{\boldsymbol{\nu}}(\varepsilon)| < \bar{\gamma}$ *(cf. Definition 4.1), define also* $d^{\boldsymbol{\nu}}(\varepsilon) := 2|\overline{\mathcal{C}}_{\boldsymbol{\nu}}(\varepsilon)|$ *and* $p_{\boldsymbol{\nu}}(\varepsilon) = \min\{|\boldsymbol{\nu}'| : \boldsymbol{\nu}' \in \overline{\mathcal{C}}_{\boldsymbol{\nu}}(\varepsilon)\}$*. For real positive* ξ*, define the* **small divisor**

$$x_{\boldsymbol{\nu}}(\varepsilon) := \frac{1}{p^{\xi}_{\boldsymbol{\nu}}(\varepsilon)} \left\| (A^{\boldsymbol{\nu}}(\varepsilon))^{-1} \right\|^{-1},$$

if A is invertible, and set $x_{\boldsymbol{\nu}}(\varepsilon) = 0$ *if A is not invertible.*

Remark 4.11. Note that for $\boldsymbol{\nu} \in \Delta_j(\varepsilon)$ one has $p_{\boldsymbol{\nu}}(\varepsilon) = p_j(\varepsilon)$, $d_{\boldsymbol{\nu}}(\varepsilon) \le 2|\Delta_j(\varepsilon)|$, and $A^{\boldsymbol{\nu}}(\varepsilon) = A^{\boldsymbol{\nu}'}(\varepsilon)$ for all $\boldsymbol{\nu}' \in \overline{\mathcal{C}}_{\boldsymbol{\nu}}(\varepsilon)$. This shows that $d_{\boldsymbol{\nu}}(\varepsilon)$, $x_{\boldsymbol{\nu}}(\varepsilon)$ and $p_{\boldsymbol{\nu}}(\varepsilon)$ are the same for all $\boldsymbol{\nu}' \in \overline{\mathcal{C}}_{\boldsymbol{\nu}}(\varepsilon)$. Note also that, if $\boldsymbol{\nu} \in \Delta_j(\varepsilon)$ for some $j \in \mathbb{N}$, then one has $\overline{\mathcal{C}}_{\boldsymbol{\nu}}(\varepsilon) = \{\boldsymbol{\nu}' \in \Delta_j(\varepsilon) : |\delta_{\boldsymbol{\nu}'}(\varepsilon)| < \bar{\gamma}\}$. Hypothesis 3 implies $d_{\boldsymbol{\nu}}(\varepsilon) \le 2C_1 p^{\alpha}_{\boldsymbol{\nu}}(\varepsilon)$.

Definition 4.12 *(The sets $\mathfrak{D}_0$, $\mathfrak{D}_1(\gamma)$, $\mathfrak{D}_2(\gamma)$, and $\mathfrak{D}(\gamma)$). We define $\mathfrak{D}_0 = \{(\varepsilon, M) : \varepsilon \in [0, \varepsilon_0],\ |M|_\kappa \le C_0\varepsilon_0\}$, for a suitable positive constant C_0, and, for fixed $\tau, \tau_1 > 0$ and $\gamma < \bar{\gamma}$, we set $\mathfrak{D}_1(\gamma) = \{(\varepsilon, M) \in \mathfrak{D}_0 : x_{\boldsymbol{\nu}} \ge \gamma/p^{\tau}_{\boldsymbol{\nu}}(\varepsilon)$ for all $\boldsymbol{\nu} \in \mathfrak{S}\}$, $\mathfrak{D}_2(\gamma) = \{(\varepsilon, M) \in \mathfrak{D}_0 : ||\delta_{\boldsymbol{\nu}}(\varepsilon)| - \bar{\gamma}| \ge \gamma/|\boldsymbol{\nu}|^{\tau_1}$ for all $\boldsymbol{\nu} \in \mathfrak{S}\}$, and $\mathfrak{D}(\gamma) = \mathfrak{D}_1(\gamma) \cap \mathfrak{D}_2(\gamma)$.*

Definition 4.13 *(The sets $\mathcal{I}_{\mathcal{N}}(\gamma)$ and $\overline{\mathcal{I}}_{\mathcal{N}}(\gamma)$). Given a resonant set $\mathcal{N}$ we define $\overline{\mathcal{I}}_{\mathcal{N}}(\gamma) := \{\varepsilon \in [0, \varepsilon_0] : \exists \boldsymbol{\nu} \in \mathcal{C}_{\mathcal{N}}$ such that $||\delta_{\boldsymbol{\nu}}(\varepsilon)| - \bar{\gamma}| < \gamma|\boldsymbol{\nu}|^{-\tau_1}\}$, and set $\mathcal{I}_{\mathcal{N}}(\gamma) := \{(\varepsilon, M) \in \mathfrak{D}_0 : \varepsilon \in \overline{\mathcal{I}}_{\mathcal{N}}(\gamma)\}$.*

4.3. Main propositions. We state the propositions which represent our main technical results. The Main Theorem in Sect. 2 is an immediate consequence of Propositions 1 and 2 below.

Proposition 1. *There exist positive constants $K_0, K_1, \kappa, \rho, \eta_0$ such that the following holds true. For $(\varepsilon, M) \in \mathfrak{D}(\gamma)$, there exists a matrix $L(\eta, \varepsilon, M) \in \mathcal{B}_{\kappa,\rho}$, such that the following holds:*

1. *For each ε the matrix $L(\eta, \varepsilon, M)$ is block-diagonal so as to satisfy $L(\eta, \varepsilon, M) = \widehat{\chi}_1 L(\eta, \varepsilon, M)\widehat{\chi}_1$. Moreover the $L(\eta, \varepsilon, M)$ is analytic in η for $|\eta| \le \eta_0$, and uniformly bounded for $(\varepsilon, M) \in \mathfrak{D}(\gamma)$ as*
$$|L(\eta, \varepsilon, M)|_{\kappa,\rho} \le |\eta|^N K_0.$$
2. *There exists a uniquely determined solution $u^{\sigma}_{\boldsymbol{\nu}}(\eta, M, \varepsilon)$ of Eqs. (4.3) and (4.4), which is analytic in η for $|\eta| \le \eta_0$, and such that for all $\boldsymbol{\nu} \in \mathbb{Z}^{D+1}$ and $\sigma = \pm$,*
$$\left|u^{\sigma}_{\boldsymbol{\nu}}(\eta, M, \varepsilon)\right| \le |\eta|\, K_0 \mathrm{e}^{-\kappa|\boldsymbol{\nu}|^{1/2}}.$$
3. *The matrix elements $L^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\eta, \varepsilon, M)$ can be extended on the set $\mathfrak{D}_0 \backslash \mathcal{I}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}(\gamma)$ to C^1 functions $L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\eta, \varepsilon, M)$, such that $L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\eta, \varepsilon, M) = L^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\eta, \varepsilon, M)$ for all $(\varepsilon, M) \in \mathfrak{D}(2\gamma)$. Moreover, for all $(\varepsilon, M) \in \mathfrak{D}_0 \backslash \mathcal{I}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}(\gamma)$, the matrix elements $L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\eta, \varepsilon, M)$ satisfy the bounds*
$$\begin{aligned}
&\left|L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\eta, \varepsilon, M)\right| \le \mathrm{e}^{-\kappa|\boldsymbol{\nu}-\boldsymbol{\nu}'|^{\rho}} |\eta|^N K_1,\\
&|\partial_\varepsilon L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\eta, \varepsilon, M)| \le \mathrm{e}^{-\kappa|\boldsymbol{\nu}-\boldsymbol{\nu}'|^{\rho}} |\eta|^N K_1 |p_{\boldsymbol{\nu}}|^{c_0},\\
&|\partial_\eta L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\eta, \varepsilon, M)| \le \mathrm{e}^{-\kappa|\boldsymbol{\nu}-\boldsymbol{\nu}'|^{\rho}} N\, |\eta|^{N-1} K_1.
\end{aligned}$$
4. *For all $(\varepsilon, M) \in \mathfrak{D}_0 \backslash \cup \mathcal{I}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}(\gamma)$, where the union is taken over all the resonant pairs $\{\boldsymbol{\nu}, \boldsymbol{\nu}'\}$, one has*
$$\left|\partial_M L^E(\eta, \varepsilon, M)\right|_{\mathrm{op}} \le \sum_{\boldsymbol{\nu}\in\mathfrak{S}} \sum_{\boldsymbol{\nu}'\in\mathcal{C}_{\boldsymbol{\nu}}} \sum_{\sigma,\sigma'=\pm} \left|\partial_{M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}} L^E(\eta, \varepsilon, M)\right|_{\kappa,\rho} \le |\eta|^N K_1.$$

5. *The functions* $u_{\boldsymbol{\nu}}^{\sigma}(\eta, \varepsilon, M)$ *can be extended on the set* $\mathfrak{D}_0$ *to* C^1 *functions* $u_{\boldsymbol{\nu}}^{E\,\sigma}(\eta, \varepsilon, M)$*, such that* $u_{\boldsymbol{\nu}}^{E\,\sigma}(\eta, \varepsilon, M) = u_{\boldsymbol{\nu}}^{\sigma}(\eta, \varepsilon, M)$ *for all* $(\varepsilon, M) \in \mathfrak{D}(2\gamma)$ *and*
$$\left|u_{\boldsymbol{\nu}}^{E\,\sigma}(\eta, \varepsilon, M)\right| \le |\eta|^N K_1 \mathrm{e}^{-\kappa|\boldsymbol{\nu}|^{1/2}},$$
uniformly for $(\varepsilon, M) \in \mathfrak{D}_0$.

Remark 4.14. In our analysis we choose $M \in \mathcal{B}_{\kappa,\rho}$ because eventually we obtain $L \in \mathcal{B}_{\kappa,\rho}$, but – as the bound on the M-derivative in Item 4 of Proposition 1 suggests – we could also take M in a larger space, say $\mathcal{B}_{\infty}$ with norm $|M|_{\infty} = \sup_{\boldsymbol{\nu},\boldsymbol{\nu}'\in\mathfrak{S}} \sup_{\sigma,\sigma'=\pm} |M_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'}|$. In Item 3 of Proposition 1 we need to work on the matrix elements $L_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'}(\eta, \varepsilon, M)$ since the extensions hold for (ε, M) in the $(\boldsymbol{\nu}, \boldsymbol{\nu}')$-dependent sets $\mathfrak{D}_0\backslash\mathcal{I}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}(\gamma)$.

Once we have proved Proposition 1, we solve the compatibility equation (4.5) for the extended counterterms $L^E(\varepsilon^{1/N}, \varepsilon, M)$, which are well defined provided we choose $\varepsilon \le \varepsilon_0$, with $\varepsilon_0 = \eta_0^N$.

Proposition 2. *There exist functions* $\varepsilon \to (\varepsilon, M_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'}(\varepsilon))$ *from* $[0, \varepsilon_0] \to \mathfrak{D}_0$*, with an appropriate choice of* C_0 *in Definition 4.12, such that the following holds.*

1. *For* $\varepsilon \in \overline{\mathcal{I}}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}(\gamma)$ *one has* $M_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'}(\varepsilon) = 0$*, while for* $\varepsilon \in [0, \varepsilon_0]\backslash\overline{\mathcal{I}}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}(\gamma)$ *the elements* $M_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'}(\varepsilon)$ *are* C^1*, verify the equation*
$$M_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'}(\varepsilon) = L_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{E\,\sigma,\sigma'}(\varepsilon^{1/N}, \varepsilon, M(\varepsilon)), \tag{4.6}$$
and satisfy the bounds
$$\left|M_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'}(\varepsilon)\right| \le K_2\varepsilon\, \mathrm{e}^{-\kappa|\boldsymbol{\nu}-\boldsymbol{\nu}'|^{\rho}}, \qquad \left|\partial_{\varepsilon} M_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'}(\varepsilon)\right| \le K_2 \left(1 + \varepsilon p_{\boldsymbol{\nu}}^{c_0}(\varepsilon)\right) \mathrm{e}^{-\kappa|\boldsymbol{\nu}-\boldsymbol{\nu}'|^{\rho}},$$
for a suitable constant K_2.
2. *The functions* $u_{\boldsymbol{\nu}}^{E}(\varepsilon) := u_{\boldsymbol{\nu}}^{E\,+}(\varepsilon^{1/N}, \varepsilon, M(\varepsilon))$ *are* C^1 *in* $[0, \varepsilon_0]$.
3. *The set* $\mathfrak{E}(2\gamma) := \{\varepsilon \in [0, \varepsilon_0] : (\varepsilon, M(\varepsilon)) \in \mathfrak{D}(2\gamma)\}$ *has density 1 at* $\varepsilon = 0$*, namely*
$$\lim_{\varepsilon\to 0^+} \frac{\mathrm{meas}(\mathfrak{E}(2\gamma) \cap (0, \varepsilon))}{\varepsilon} = 1.$$

4.4. Proof of the Main Theorem. By Items 1 and 2 in Proposition 1 for all $(\varepsilon, M) \in \mathfrak{D}(\gamma)$ we can find a matrix $L(\eta, \varepsilon, M)$ so that there exists a unique solution $u_{\boldsymbol{\nu}}^{\sigma}(\eta, \varepsilon, M)$ of (4.3) and (4.4) for all $|\eta| \le \eta_0$, for a suitable η_0, and for ε_0 small enough. By Items 3 and 5 in Proposition 1 the matrix blocks $L_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'}(\eta, \varepsilon, M)$ and the solution $u_{\boldsymbol{\nu}}^{\sigma}(\eta, \varepsilon, M)$ can be extended to C^1 functions – denoted by $L_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{E\,\sigma,\sigma'}(\eta, \varepsilon, M)$ and $u_{\boldsymbol{\nu}}^{E\,\sigma}(\eta, \varepsilon, M)$ – for all $(\varepsilon, M) \in \mathfrak{D}_0\backslash\mathcal{I}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}(\gamma)$ and for all $(\varepsilon, M) \in \mathfrak{D}_0$, respectively. Moreover $L_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{E\,\sigma,\sigma'}(\eta, \varepsilon, M) = L_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'}(\eta, \varepsilon, M)$ and $u_{\boldsymbol{\nu}}^{E\,\sigma}(\eta, \varepsilon, M) = u_{\boldsymbol{\nu}}^{\sigma}(\eta, \varepsilon, M)$ for all $(\varepsilon, M) \in \mathfrak{D}(2\gamma)$.

Equation (4.3) coincides with our original (4.2) provided the compatibility equation (4.5) is satisfied. Now we fix $\varepsilon_0 < \eta_0^N$ so that $L^E(\varepsilon^{1/N}, \varepsilon, M)$ and $u_{\boldsymbol{\nu}}^{E\,\sigma}(\varepsilon^{1/N}, \varepsilon, M)$ are well defined for $|\varepsilon| < \varepsilon_0$. By Item 1 in Proposition 2, there exists a matrix $M(\varepsilon)$ which satisfies the extended compatibility equation (4.6). Finally by Item 3 in Proposition 2 (we need Item 4 in Proposition 2 to prove it) the Cantor set $\mathfrak{E}(2\gamma)$ is well defined and of large relative measure.

Set $\mathfrak{E} = \mathfrak{E}(2\gamma)$ and $u(\mathbf{x}; \varepsilon) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^{D+1}} u^E_{\boldsymbol{\nu}}(\varepsilon)\mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\mathbf{x}}$. The function $u(t, x; \varepsilon)$ is C^1 in ε for $\varepsilon \in [0, \varepsilon_0]$ by Item 2 in Proposition 2. Moreover it is 2π-periodic and Gevrey-smooth, and satisfies the bound in Item 2, by Item 5 in Proposition 1.

For all $\varepsilon \in \mathfrak{E}$ the pair $(\varepsilon, M(\varepsilon))$ is by definition in $\mathfrak{D}(2\gamma)$, so that by Item 3 in Proposition 1 one has $L_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon^{1/N}, \varepsilon, M(\varepsilon)) = L^E_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon^{1/N}, \varepsilon, M(\varepsilon))$ and by Item 5 in Proposition 1 one has $u^\sigma_{\boldsymbol{\nu}}(\varepsilon^{1/N}, \varepsilon, M(\varepsilon)) = u^{E\,\sigma}_{\boldsymbol{\nu}}(\varepsilon^{1/N}, \varepsilon, M(\varepsilon))$, and hence $u^\sigma_{\boldsymbol{\nu}}(\varepsilon^{1/N}, \varepsilon, M(\varepsilon))$ solves (4.3) for $\eta = \varepsilon^{1/N}$. So, by Item 1 in Proposition 2, $M(\varepsilon)$ solves the true compatibility equation (4.5) for all $\varepsilon \in \mathfrak{E}$. Then $u(t, x; \varepsilon)$ is a true nontrivial solution of (4.3) and (4.4) in $\mathfrak{E}$.

5. Tree Expansion

5.1. Recursive equations. In this section we find a formal solution $u^\sigma_{\boldsymbol{\nu}}, L$ of (4.3) and (4.4) as a power series on η; the solution $u^\sigma_{\boldsymbol{\nu}}, L$ depends on the matrix M and it will be written in the form of a tree expansion.

We shall introduce the trees *in abstracto*, by giving the rules how to construct them, that is, essentially, how to associate labels to unlabelled trees. Then, we shall show that both the solution $u^\sigma_{\boldsymbol{\nu}}$ and the matrix L can be expressed in terms of labelled trees. Of course, the easiest way to see that the construction makes sense is to try to express $u^\sigma_{\boldsymbol{\nu}}$ and L in terms of trees and then check that some constraints and relations must be imposed on the tree labels.

We assume for $u^\sigma_{\boldsymbol{\nu}}(\eta, \varepsilon, M)$ for all $\boldsymbol{\nu} \in \mathfrak{P}$ and for the matrix $L(\eta, \varepsilon, M)$ a formal series expansion in η:

$$u^\sigma_{\boldsymbol{\nu}}(\eta, \varepsilon, M) = \sum_{k=N}^{\infty} \eta^k u^{(k)\sigma}_{\boldsymbol{\nu}}, \quad L(\eta, \varepsilon, M) = \sum_{k=N}^{\infty} \eta^k L^{(k)}, \tag{5.1}$$

with the Ansatz that $L^{(k)\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'} = 0$ if either $\bar{\chi}_1(\delta_{\boldsymbol{\nu}}(\varepsilon))\bar{\chi}_1(\delta_{\boldsymbol{\nu}'}(\varepsilon)) = 0$ or the pair $\{\boldsymbol{\nu}, \boldsymbol{\nu}'\}$ is not resonant, so that $L = \widehat{\chi}_1 L \widehat{\chi}_1$. We set also $u^{(k)\sigma}_{\boldsymbol{\nu}} = 0$ for all $k \le N$ and $\boldsymbol{\nu}, \boldsymbol{\nu}' \in \mathfrak{P}$, and the same for $L^{(k)\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}$ for $\boldsymbol{\nu}, \boldsymbol{\nu}' \in \mathfrak{S}$.

For $\boldsymbol{\nu} \in \mathfrak{Q}$ we set

$$u^\sigma_{\boldsymbol{\nu}}(\eta, \varepsilon, M) = u^{(0)}_{\boldsymbol{\nu}} + \sum_{k=N}^{\infty} \eta^k u^{(k)}_{\boldsymbol{\nu}} \tag{5.2}$$

with $u^{(0)+}_{\boldsymbol{\nu}} = u^{(0)}_{\boldsymbol{\nu}}$ and $u^{(0)-}_{\boldsymbol{\nu}} = \overline{u^{(0)}_{\boldsymbol{\nu}}}$ (cf. Item 2 in Hypothesis 2 for notations). Again we set $u^{(k)\sigma}_{\boldsymbol{\nu}} = 0$ for $0 < k < N$ and $\boldsymbol{\nu} \in \mathfrak{Q}$.

Inserting the series expansions (5.1) and (5.2) into (4.3) we obtain

$$\begin{cases} u^{(k)\sigma}_{\boldsymbol{\nu}} = \dfrac{f^{(k-N)\sigma}_{\boldsymbol{\nu}}}{\delta_{\boldsymbol{\nu}}(\varepsilon)}, & \boldsymbol{\nu} \in \mathfrak{R}, \quad \sigma = \pm, \\ u^{(k)\sigma}_{\boldsymbol{\nu}} = \displaystyle\sum_{\boldsymbol{\nu}'\in\mathfrak{Q},\sigma'=\pm} (J^{-1})^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'} f^{(k)\sigma'}_{\boldsymbol{\nu}'}, & \boldsymbol{\nu} \in \mathfrak{Q}, \quad \sigma = \pm, \\ (\mathscr{D}(\varepsilon) + \mathscr{M})\, U^{(k)} = F^{(k-N)} + \displaystyle\sum_{r=N}^{k-N} L^{(r)} U^{(k-r)}. & \end{cases} \tag{5.3}$$

5.2. Multiscale analysis.

Definition 5.1 (The scale functions). *Let χ be a non-increasing function $C^{\infty}(\mathbb{R}_{+}, [0,1])$, such that $\chi(x) = 0$ if $x \geq 2\gamma$ and $\chi(x) = 1$ if $x \leq \gamma$, with γ given in Definition 4.12; moreover one has $|\partial_x \chi(x)| \leq \Gamma\gamma^{-1}$ for some positive constant Γ. Let $\chi_h(x) = \chi(2^h x) - \chi(2^{h+1}x)$ for $h \geq 0$, and $\chi_{-1}(x) = 1 - \chi(x)$.*

Remark 5.2. In contrast to the functions $\bar{\chi}_i$ in Definition 4.1, the scale functions χ_h are smooth. Indeed, in this case smoothness is important, because we shall need to derivate such functions. On the other hand, the fact that the functions $\bar{\chi}_i$ are sharp implies that the matrix A^{-1} has the same block structure as A.

Recall that for each ε the matrix $A = \mathscr{D}(\varepsilon) + \mathscr{M}$ is block diagonal with a diagonal part whose eigenvalues are larger than $\bar{\gamma} > \gamma$ and a list of $C_1 p_{\boldsymbol{\nu}}^{\alpha}(\varepsilon) \times C_1 p_{\boldsymbol{\nu}}^{\alpha}(\varepsilon)$ blocks $A^{\boldsymbol{\nu}}$ containing small entries. In the following if $A^{\boldsymbol{\nu}}$ is invertible — i.e. if $x_{\boldsymbol{\nu}} \neq 0$ — we will denote the entries of $(A^{\boldsymbol{\nu}})^{-1}$ by $(A^{-1})_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'}$ even though it may be possible that the whole matrix A is not invertible.

Definition 5.3 (Propagators). *For $\boldsymbol{\nu}, \boldsymbol{\nu}' \in \mathfrak{S}$, we define the* **propagators**

$$(G_{i,h})_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'} = \begin{cases} \chi_h(x_{\boldsymbol{\nu}}(\varepsilon))\,\bar{\chi}_1(\delta_{\boldsymbol{\nu}}(\varepsilon))\,\bar{\chi}_1(\delta_{\boldsymbol{\nu}'}(\varepsilon))(A^{-1})_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'}, & \text{if } i=1 \text{ and } \chi_h(x_{\boldsymbol{\nu}}(\varepsilon)) \neq 0, \\ \bar{\chi}_0(\delta_{\boldsymbol{\nu}}(\varepsilon))\,\delta_{\boldsymbol{\nu}}^{-1}(\varepsilon), & \text{if } i=0,\ \boldsymbol{\nu}=\boldsymbol{\nu}',\ \sigma=\sigma' \text{ and } h=-1, \\ 0, & \text{otherwise.} \end{cases}$$

In terms of the propagators we obtain

$$A^{-1} = \sum_{i=0,1} \sum_{h=-1}^{\infty} G_{i,h}, \tag{5.4}$$

which provides the multiscale decomposition. Notice that if $(A^{-1})_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'} \neq 0$ then $x_{\boldsymbol{\nu}}(\varepsilon) = x_{\boldsymbol{\nu}'}(\varepsilon)$ (see Remark 4.11), so that the matrices $G_{i,h}$ are indeed self-adjoint.

Remark 5.4. Only the propagator $G_{1,h}$ can produce small divisors while the propagator $G_{0,-1}$ is diagonal and of order one. Hence, there exists a positive constant C such that we can bound the propagators as

$$\left|G_{0,-1}\right|_{\infty} \leq C\gamma^{-1}, \qquad \left|(G_{1,h})_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'}\right| \leq 2^h C\gamma^{-1} p_{\boldsymbol{\nu}}^{-\xi}(\varepsilon)\sqrt{p_{\boldsymbol{\nu}}^{\alpha}(\varepsilon)}, \tag{5.5}$$

where the condition $d_{\boldsymbol{\nu}}(\varepsilon) \leq 2C_1 p_{\boldsymbol{\nu}}^{\alpha}(\varepsilon)$ – cf. Remark 4.11 – and Item 2 of Lemma 4.9 have been used.

We write $L^{(k)}$ in (5.1) as

$$L_{\boldsymbol{\nu}_1,\boldsymbol{\nu}_2}^{(k)\sigma_1,\sigma_2} = \sum_{h=-1}^{\infty} \chi_h(x_{\boldsymbol{\nu}_1}(\varepsilon)) L_{h,\boldsymbol{\nu}_1,\boldsymbol{\nu}_2}^{(k)\sigma_1,\sigma_2}, \tag{5.6}$$

for all resonant pairs $\{\boldsymbol{\nu}_1, \boldsymbol{\nu}_2\}$; we denote by $L_h^{(k)}$ the matrix with entries $L_{h,\boldsymbol{\nu}_1,\boldsymbol{\nu}_2}^{(k)\sigma_1,\sigma_2}$. Finally we set

$$U^{(k)} = \sum_{i=0,1} \sum_{h=-1}^{\infty} U_{i,h}^{(k)}, \tag{5.7}$$

so that (5.3) gives

$$\begin{cases} u_{\boldsymbol{\nu}}^{(k)\sigma} = \displaystyle\sum_{\boldsymbol{\nu}'\in\mathfrak{Q}} \sum_{\sigma'=\pm} (J^{-1})_{\boldsymbol{\nu},\boldsymbol{\nu}'}^{\sigma,\sigma'} f_{\boldsymbol{\nu}'}^{(k)\sigma'}, & \boldsymbol{\nu} \in \mathfrak{Q}, \quad \sigma = \pm, \\ u_{\boldsymbol{\nu}}^{(k)\sigma} = \dfrac{f_{\boldsymbol{\nu}}^{(k-N)\sigma}}{\delta_{\boldsymbol{\nu}}(\varepsilon)}, & \boldsymbol{\nu} \in \mathfrak{R}, \quad \sigma = \pm, \\ U_{i,h}^{(k)} = G_{i,h} F^{(k-N)} + \delta(i,1)\, G_{1,h} \displaystyle\sum_{h_1=-1}^{\infty} \sum_{r=N}^{k-N} L_h^{(r)} U_{1,h_1}^{(k-r)}, & i = 0, 1, \ h \geq -1, \end{cases} \tag{5.8}$$

which are the recursive equations we want to study.

5.3. Diagrammatic rules. A connected graph $\mathcal{G}$ is a collection of points (vertices) and lines connecting all of them. We denote with $V(\mathcal{G})$ and $L(\mathcal{G})$ the set of nodes and the set of lines, respectively. A path between two nodes is the minimal subset of $L(\mathcal{G})$ connecting the two nodes. A graph is planar if it can be drawn in a plane without graph lines crossing.

Definition 5.5 (Trees). *A* tree *is a planar graph $\mathcal{G}$ containing no closed loops. One can consider a tree $\mathcal{G}$ with a single special node v_0: this introduces a natural partial ordering on the set of lines and nodes, and one can imagine that each line carries an arrow pointing toward the node v_0. We can add an extra (oriented) line ℓ_0 exiting the special node v_0; the added line ℓ_0 will be called the root line and the point it enters (which is not a node) will be called the root of the tree. In this way we obtain a rooted tree θ defined by $V(\theta) = V(\mathcal{G})$ and $L(\theta) = L(\mathcal{G}) \cup \ell_0$. A labelled tree is a rooted tree θ together with a label function defined on the sets $L(\theta)$ and $V(\theta)$.*

We shall call *equivalent* two rooted trees which can be transformed into each other by continuously deforming the lines in the plane in such a way that the latter do not cross each other (i.e. without destroying the graph structure). We can extend the notion of equivalence also to labelled trees, simply by considering equivalent two labelled trees if they can be transformed into each other in such a way that also the labels match.

Given two nodes $v, w \in V(\theta)$, we say that $v \prec w$ if w is on the path connecting v to the root line. We can identify a line with the nodes it connects; given a line $\ell = (w, v)$ we say that ℓ enters w and exits (or comes out of) v, and we write $\ell = \ell_v$. To help himself follow the diagrammatic construction, one can visualise the trees with the root to the left and the end-nodes to the right; in particular given a line (w, v) one has $v \prec w$, with the endpoint w to the left of the endpoint v.

Given two comparable lines ℓ and ℓ_1, with $\ell_1 \prec \ell$, we denote with $\mathcal{P}(\ell_1, \ell)$ the path of lines connecting ℓ_1 to ℓ; by definition the two lines ℓ and ℓ_1 do not belong to $\mathcal{P}(\ell_1, \ell)$. We say that a node v is along the path $\mathcal{P}(\ell_1, \ell)$ if at least one line entering or exiting v belongs to the path. If $\mathcal{P}(\ell_1, \ell) = \emptyset$ there is only one node v along the path (such that ℓ_1 enters v and ℓ exits v).

Definition 5.6 (Lines and nodes). *We call* **internal nodes** *the nodes such that there is at least one line entering them; we call* **internal lines** *the lines exiting the internal nodes. We call* **end-nodes** *the nodes which have no entering line. We denote with $L(\theta)$, $V_0(\theta)$ and $E(\theta)$ the set of lines, internal nodes and end-nodes, respectively. Of course $V(\theta) = V_0(\theta) \cup E(\theta)$.*

As anticipated at the beginning of Subsect. 5.1, we first introduce the trees as abstract structures and prove thereafter that the quantities of interest can be expressed in terms of trees. For further details and to better understand the general strategy of the construction, we refer to the more pedagogical discussion in [24]. In fact, we first develop a naive but very natural tree expansion for the solution. Such an expansion would work for any choice of L if the solution were analytic in ε. However in our case we do not expect that analyticity holds, so we modify the expansion into a new one (renormalised expansion), where we fix L appropriately (see Definition 5.21) in such a way to eliminate the contributions which would cause the divergence of the series.

We associate with the nodes (internal nodes and end-nodes) and lines of any tree θ some labels, according to the following rules.

Definition 5.7 (Diagrammatic rules). *Let θ be a tree. We associate with θ the following labels:*

1. *With each internal line $\ell \in L(\theta)$ one associates a label q, p or r. We say that ℓ is a p-line, a q-line or an r-line, respectively, and we call $L_q(\theta)$, $L_p(0)$ and $L_r(0)$ the set of internal lines $\ell \in L(\theta)$ which are q-lines, p-lines and r-lines, respectively.*
2. *With each line $\ell \in L(\theta)$ one associates the* **type** *label $i_\ell = 0, 1$ and the* **scale** *label $h_\ell \in \mathbb{N} \cup \{-1, 0\}$.*
3. *With each line $\ell \in L(\theta)$ except the root line ℓ_0 one associates a* **sign** *label $\sigma_\ell = \pm$.*
4. *With each internal line $\ell \in L(\theta)$ one associates the* **momenta** *$(\boldsymbol{\nu}_\ell, \boldsymbol{\nu}'_\ell) \in \mathbb{Z}^{D+1} \times \mathbb{Z}^{D+1}$.*
5. *With each line $\ell \in L(\theta)$ exiting an end-node one associates the* **momentum** *$\boldsymbol{\nu}_\ell$.*
6. *For each node v there are $p_v \geq 0$ entering lines. If $p_v = 0$ then $v \in E(\theta)$, if $p_v > 0$ then either $p_v = 1$ or $p_v \geq N+1$ and $v \in V_0(\theta)$, where N is introduced in (2.1). If $L(v)$ is the set of lines entering v one has $p_v = |L(v)|$.*
7. *With each end-node $v \in E(\theta)$ one associates the* **mode** *label $\boldsymbol{\nu}_v \in \mathfrak{Q}$, the* **order** *label $k_v = 0$, and the* **sign** *label $\sigma_v = \pm$.*
8. *With each internal node $v \in V_0(\theta)$ one associates the* **mode** *label $m_v \in \mathbb{Z}^D$, the* **order** *label $k_v \in \mathbb{N}$, and the* **sign** *label $\sigma_v = \pm$, and one defines r_v as the number of lines $\ell \in L(v)$ with $\sigma_\ell = \sigma_v$, and one sets $s_v = p_v - r_v$.*

The following constraints and relations will be imposed on the labels.

9. *Given an internal node $v \in V_0(\theta)$, if $p_v = 1$ let ℓ_1 be the line entering v and ℓ be the line exiting v. Then ℓ and ℓ_1 are both p-lines. Moreover one has $i_{\ell_1} = i_\ell = 1$ and $\{\boldsymbol{\nu}'_\ell, \boldsymbol{\nu}_{\ell_1}\}$ is a resonant pair.*
10. *If a line $\ell \in L(\theta)$ is not a p-line one sets $i_\ell = 0$.*
11. *If a line $\ell \in L(\theta)$ has $i_\ell = 0$, then $h_\ell = -1$.*
12. *Let $\ell \in L(\theta)$ be an internal line. If ℓ is a p-line with $i_\ell = 0$, then $\boldsymbol{\nu}_\ell = \boldsymbol{\nu}'_\ell$. If ℓ is a p-line with $i_\ell = 1$, then $\{\boldsymbol{\nu}_\ell, \boldsymbol{\nu}'_\ell\}$ is a resonant pair. If ℓ is a q-line, then $\boldsymbol{\nu}_\ell, \boldsymbol{\nu}'_\ell \in \mathfrak{Q}$. If ℓ is an r-line, then $\boldsymbol{\nu}_\ell = \boldsymbol{\nu}'_\ell \in \mathfrak{R}$.*
13. *If ℓ exits an end-node $v \in E(\theta)$, then one sets $\boldsymbol{\nu}_\ell = \boldsymbol{\nu}_v$ and $\sigma_\ell = \sigma_v$.*
14. *If two p-lines ℓ and ℓ' have $i_\ell = i_{\ell'} = 1$ and are such that $\{\boldsymbol{\nu}_\ell, \boldsymbol{\nu}'_\ell, \boldsymbol{\nu}_{\ell'}, \boldsymbol{\nu}'_{\ell'}\}$ is a resonant set, then $|h_\ell - h_{\ell'}| \leq 1$.*

15. *If ℓ is the line exiting v and $\ell_1, \cdots, \ell_{p_v}$ are the lines entering v one has*

$$\boldsymbol{\nu}'_\ell = (0, m_v) + \sigma_v(\sigma_{\ell_1}\boldsymbol{\nu}_{\ell_1} + \cdots + \sigma_{\ell_{p_v}}\boldsymbol{\nu}_{\ell_{p_v}}) = (0, m_v) + \sigma_v \sum_{\ell' \in L(v)} \sigma_{\ell'}\boldsymbol{\nu}_{\ell'},$$

which represents a conservation rule for the momenta.
16. *Given an internal node $v \in V_0(\theta)$, if $p_v = 1$ one has $k_v \geq N$, while if $p_v \geq N$ one has $k_v = p_v - 1$.*
17. *With each end-node $v \in E(\theta)$ one associates the* **node factor** $\eta_v = u^{(0)\sigma_v}_{\boldsymbol{\nu}_v}$*; cf. Item 2 in Hypothesis 2 and (5.2) for notations.*
18. *Given an internal node $v \in V_0(\theta)$, if $p_v > 1$ one associates with v the* **node factor** $\eta_v = a^{\sigma_v}_{r_v,s_v,m_v}$*, where $a^\sigma_{r,s,m}$ satisfies Eq. (2.11), while if $p_v = 1$ one associates with v the* **node factor** $\eta_v = L^{(k_v)\sigma_v,\sigma_{\ell_1}}_{h_\ell,\boldsymbol{\nu}'_\ell,\boldsymbol{\nu}_{\ell_1}}$*, still to be defined (see Definition 5.21 below), where ℓ and ℓ_1 are the lines exiting and entering v, respectively.*
19. *One associates with each line $\ell \in L(\theta)$ a* **line propagator** $g_\ell \in \mathbb{C}$ *with the following rules. If ℓ is a p-line exiting the internal node v one sets $g_\ell := (G_{i_\ell,h_\ell})^{\sigma_\ell,\sigma_v}_{\boldsymbol{\nu}_\ell,\boldsymbol{\nu}'_\ell}$, if ℓ is an r-line one sets $g_\ell := 1/\delta_{\boldsymbol{\nu}_\ell}(\varepsilon)$, if ℓ is a q-line exiting the internal node v one sets $g_\ell := (J^{-1})^{\sigma_\ell,\sigma_v}_{\boldsymbol{\nu}_\ell,\boldsymbol{\nu}'_\ell}$, if ℓ exits an end-node one sets $g_\ell = 1$.*
20. *One defines the* **order** *of the tree θ as*

$$k(\theta) := \sum_{v \in V(\theta)} k_v,$$

the **momentum** *of θ as the momentum $\boldsymbol{\nu}_\ell$ of the root line ℓ, and the* **sign** *of θ as the sign σ_{v_0} of the node v_0 which the root line exits.*

Remark 5.8. The "line propagators" defined in Item 19 are not to be confused with the "propagators" *tout court* introduced in Definition 5.3. In fact, the line propagator coincides with the propagator when the latter is defined, but there are lines with which no propagator is associated.

Definition 5.9 (The sets of trees $\boldsymbol{\Theta}^{(k)\sigma}_{\boldsymbol{\nu}}$ and $\boldsymbol{\Theta}$). *We call $\Theta^{(k)\sigma}_{\boldsymbol{\nu}}$ the set of all the nonequivalent trees of order k, momentum $\boldsymbol{\nu}$ and sign σ, defined according to the diagrammatic rules of Definition 5.7. We call Θ the sets of trees belonging to $\Theta^{(k)\sigma}_{\boldsymbol{\nu}}$ for some $k \geq 1$, $\sigma = \pm$ and $\boldsymbol{\nu} \in \mathbb{Z}^{D+1}$.*

The reason for introducing these sets of trees is summed in the following result.

Lemma 5.10 *For any given counterterm $L \in \mathcal{B}_{\kappa,\rho}$ such that $L = \widehat{\chi}_1 L \widehat{\chi}_1$, the coefficients $u^{(k)\sigma}_{\boldsymbol{\nu}}$ can be written in terms of trees*

$$u^{(k)\sigma}_{\boldsymbol{\nu}} = \sum_{\theta \in \Theta^{(k)\sigma}_{\boldsymbol{\nu}}} \left(\prod_{\ell \in L(\theta)} g_\ell \right) \left(\prod_{v \in V(\theta)} \eta_v \right).$$

Proof. The proof is easily obtained by standard arguments in Taylor series expansions. For instance, one can proceed by induction, using the diagrammatic rules and definitions given in this section; we refer to Lemma 3.6 of [24] for details. □

However, in general we cannot prove the convergence of the series (5.2) for arbitrary L. This compels us to change the expansion, in such a way that suitably fixing L all the dangerous contributions disappear from the expansion.

5.4. Clusters and resonances.

Definition 5.11 (Clusters). *Given a tree $\theta \in \Theta_{\boldsymbol{\nu}}^{(k)\sigma}$ a* **cluster** *T on scale h is a connected maximal set of nodes and lines such that all the lines ℓ have a scale label $\le h$ and at least one of them has scale h; we shall call $h_T = h$ the scale of the cluster. We shall denote by $V(T)$, $V_0(T)$ and $E(T)$ the set of nodes, internal nodes and the set of end-nodes, respectively, which are contained inside the cluster T, and with $L(T)$ the set of lines connecting them. Finally $k(T) = \sum_{v \in V(T)} k_v$ will be called the order of T.*

An inclusion relation is established between clusters, in such a way that the innermost clusters are the clusters with lowest scale, and so on. A cluster T can have an arbitrary number of lines entering it (*entering lines*), but only one or zero line coming out from it (*exiting line* or *root line* of the cluster); we shall denote the latter (when it exists) with ℓ_T. Notice that, by definition, $|V(T)| > 1$ and all the entering and exiting lines have $i_\ell = 1$.

Next we introduce the notion of resonances. The resonances identify the clusters which, if not eliminated, would produce a runaway accumulation of small divisors (and hence the divergence of the algorithm to construct the solution). The idea will be to choose the matrices L in such a way to eliminate (iteratively) the resonances.

Definition 5.12 (Resonances). *We call* **resonance** *on scale h a cluster T on scale $h_T = h$ such that*

1. *the cluster has only one entering line ℓ_T^1 and one exiting line ℓ_T of scale $h_{\ell_T} \ge h+2$,*
2. *one has that $\{\boldsymbol{\nu}'_{\ell_T}, \boldsymbol{\nu}_{\ell_T^1}\}$ is a resonant pair and* $\min\{|\boldsymbol{\nu}_{\ell_T^1}|, |\boldsymbol{\nu}'_{\ell_T}|\} \ge 2^{(h-2)/\tau}$,
3. *for all $\ell \in \mathcal{P}(\ell_T^1, \ell_T)$ with $i_\ell = 1$ the pair $\{\boldsymbol{\nu}'_\ell, \boldsymbol{\nu}_{\ell_T^1}\}$ is not resonant,*
4. *for all $\ell \in L(T) \backslash \mathcal{P}(\ell_T^1, \ell_T)$ the pair $\{\boldsymbol{\nu}'_\ell, \boldsymbol{\nu}_{\ell_T^1}\}$ is not resonant.*

The line ℓ_T of a resonance will be called the root line of the resonance.

Definition 5.13 (The sets of trees $\boldsymbol{\mathcal{R}}_{\boldsymbol{h},\boldsymbol{\nu},\boldsymbol{\nu}'}^{\boldsymbol{(k)\sigma,\sigma'}}$ and $\boldsymbol{\mathcal{R}}$). *For $k \ge N$, $h \ge 1$ and a resonant pair $\{\boldsymbol{\nu}, \boldsymbol{\nu}'\}$ such that* $\min\{|\boldsymbol{\nu}|, |\boldsymbol{\nu}'|\} \ge 2^{(h-2)/\tau}$, *we define $\mathcal{R}_{h,\boldsymbol{\nu},\boldsymbol{\nu}'}^{(k)\sigma\sigma'}$ as the set of trees with the following differences with respect to $\Theta_{\boldsymbol{\nu}}^{(k)\sigma}$.*

1. *There is a single end-node, called e, with node factor $\eta_e = 1$ (but no label no labels $\boldsymbol{\nu}_e$ nor σ_e).*
2. *The line ℓ_e exiting e is a p-line. We associate with ℓ_e the labels $\boldsymbol{\nu}_{\ell_e} = \boldsymbol{\nu}'$, $\sigma_\ell = \sigma'$, and $i_{\ell_e} = 1$ (but no labels $\boldsymbol{\nu}'_\ell$ nor h_ℓ), and the corresponding line propagator is $g_{\ell_e} = \bar{\chi}_1(\delta_{\boldsymbol{\nu}'}(\varepsilon))$.*
3. *The root line ℓ_0 is a p-line. We associate with ℓ_0 the labels $i_{\ell_0} = 1$ and $\boldsymbol{\nu}'_{\ell_0} = \boldsymbol{\nu}$ (but no labels $\boldsymbol{\nu}_{\ell_0}$ nor h_{ℓ_0}), and the corresponding line propagator is $g_{\ell_0} = \bar{\chi}_1(\delta_{\boldsymbol{\nu}}(\varepsilon))$. Let v_0 be the node which the line ℓ_0 exits: we set $\sigma_{v_0} = \sigma$.*
4. *One has* $\max_{\ell \in L(\theta) \backslash \{\ell_0, \ell_e\}} h_\ell = h$.
5. *If $\ell \in \mathcal{P}(\ell_e, \ell_0)$ is such that $\{\boldsymbol{\nu}'_\ell, \boldsymbol{\nu}'\}$ is resonant, then $i_\ell = 0$.*
6. *For $\ell \notin \mathcal{P}(\ell_e, \ell_0)$ one has that $\{\boldsymbol{\nu}'_\ell, \boldsymbol{\nu}'\}$ is not a resonant pair.*

We call $\mathcal{R}$ the sets of trees belonging to $\mathcal{R}_{h,\boldsymbol{\nu},\boldsymbol{\nu}'}^{(k)\sigma\sigma'}$ for some $k \ge 1$, $h \ge 1$, $\sigma, \sigma' = \pm$, and $\boldsymbol{\nu}, \boldsymbol{\nu} \in \mathfrak{S}$ such that $\{\boldsymbol{\nu}, \boldsymbol{\nu}'\}$ is resonant and $\min\{|\boldsymbol{\nu}|, |\boldsymbol{\nu}'|\} \ge 2^{(h-2)/\tau}$.

Definition 5.14 (Clusters for trees in $\boldsymbol{\mathcal{R}}$). *Given a tree $\theta \in \mathcal{R}$, a cluster T on scale $h_T \le h$ is a connected maximal set of nodes $v \in V(\theta)$ and lines $\ell \in L(\theta)\backslash\{\ell_0, \ell_e\}$ such that all the lines ℓ have a scale label $\le h_T$ and at least one of them has scale h_T.*

Note that if $\theta \in \mathcal{R}^{(k)\sigma,\sigma'}_{h,\boldsymbol{\nu},\boldsymbol{\nu}'}$, then for any cluster T in θ one necessarily has $h_T \le h$.

Definition 5.15 (Resonances for trees in $\boldsymbol{\mathcal{R}}$). *Given a tree $\theta \in \mathcal{R}$, a cluster T is a resonance if the four items of Definition 5.12 are satisfied.*

Remark 5.16. There is a one-to-one correspondence between resonances T of order k and scale h with $\boldsymbol{\nu}_{\ell_T^1} = \boldsymbol{\nu}'$, $\boldsymbol{\nu}'_{\ell_T} = \boldsymbol{\nu}$, $\sigma_{v_0} = \sigma$, $\sigma_{\ell_T^1} = \sigma'$ (here v_0 is the node which ℓ_T exits) and trees $\theta \in \mathcal{R}^{(k)\sigma,\sigma'}_{h,\boldsymbol{\nu},\boldsymbol{\nu}'}$; cf. [24], Sect. 3.4 and Fig. 7.

Definition 5.17 (The sets of renormalised trees $\boldsymbol{\Theta}^{(k)\sigma}_{\boldsymbol{R},\boldsymbol{\nu}}$, $\boldsymbol{\mathcal{R}}^{(k)\sigma,\sigma'}_{\boldsymbol{R},\boldsymbol{h},\boldsymbol{\nu},\boldsymbol{\nu}'}$, $\boldsymbol{\Theta_R}$ and $\boldsymbol{\mathcal{R}_R}$). *We define the set of* **renormalised trees** *$\Theta^{(k)\sigma}_{R,\boldsymbol{\nu}}$ and $\mathcal{R}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$ as the set of trees defined as $\Theta^{(k)\sigma}_{\boldsymbol{\nu}}$ and $\mathcal{R}^{(k)\sigma,\sigma'}_{h,\boldsymbol{\nu},\boldsymbol{\nu}'}$, respectively, but with no resonances and no nodes v with $p_v = 1$. Analogously we define the sets Θ_R and $\mathcal{R}_R$.*

In the following it will turn out to be convenient to introduce also the following set of trees.

Definition 5.18 (The set of renormalised trees $\boldsymbol{\mathcal{S}}^{(k)\sigma,\sigma'}_{\boldsymbol{R},\boldsymbol{h},\boldsymbol{\nu},\boldsymbol{\nu}'}$ and $\boldsymbol{\mathcal{S}_R}$). *For $k \ge N$, $h \ge 1$ and $\boldsymbol{\nu}, \boldsymbol{\nu}' \in \mathfrak{S}$ such that $|\boldsymbol{\nu}'| \ge 2^{(h-2)/\tau}$ we define the set of renormalised trees $\mathcal{S}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$ as the set of trees with the following differences with respect to $\mathcal{R}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$ (see Definition 5.13).*
Items 1 *and* 2 *are unchanged.*
3′ One assigns to the line ℓ_0 the further label $h_{\ell_0} \le h$, and requires $|\boldsymbol{\nu}| \ge 2^{(h_{\ell_0}-2)/\tau}$.
4′ One has $\max_{\ell\in L(\theta)\backslash\{\ell_e\}} h_\ell = h$
Items 5 *and* 6 *are unchanged.*
The set $\mathcal{S}_R$ is defined analogously as $\mathcal{R}_R$.

Remark 5.19 Note that if $\theta \in \mathcal{R}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$ then $\mathrm{Val}(\theta) = \mathrm{Val}(\theta')$ with $\theta' \in \mathcal{S}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$ such that $h_{\ell_0} = h - 1$. Thus, it is enough to study the set $\mathcal{S}_R$ in order to obtain bounds for trees in $\mathcal{R}_R$.

Definition 5.20 (Tree values). *For any tree or renormalised tree θ call*

$$\mathrm{Val}(\theta) = \left(\prod_{\ell\in L(\theta)} g_\ell\right)\left(\prod_{v\in V(\theta)} \eta_v\right)$$

the **value** *of the tree θ. To make explicit the dependence of the tree value on ε and M, sometimes we shall write* $\mathrm{Val}(\theta) = \mathrm{Val}(\theta; \varepsilon, M)$.

Definition 5.21 (Counterterms). *We define the node factors $L^{(k)\sigma,\sigma'}_{h,\boldsymbol{\nu},\boldsymbol{\nu}'}$ (cf. item 21 in Definition 5.7) by setting*

$$L^{(k)\sigma,\sigma'}_{h,\boldsymbol{\nu},\boldsymbol{\nu}'} = \sum_{h'<h-1} \sum_{\theta\in\mathcal{R}^{(k)\sigma,\sigma'}_{R,h',\boldsymbol{\nu},\boldsymbol{\nu}'}} \mathrm{Val}(\theta), \qquad \sigma,\sigma' = \pm, \tag{5.9}$$

for all $k \ge N$, all $h \ge 1$, and all resonant pairs $\{\boldsymbol{\nu}, \boldsymbol{\nu}'\}$. The counterterms L are then expressed in terms of (5.9) through (5.1) and (5.6).

Lemma 5.22 *For any tree* $\theta \in \mathcal{R}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$ *there exists a tree* $\theta' \in \mathcal{R}^{(k)-\sigma',-\sigma}_{R,h,\boldsymbol{\nu}',\boldsymbol{\nu}}$ *such that* $\mathrm{Val}(\theta) = \mathrm{Val}(\theta')$.

Proof. Given a tree $\theta \in \mathcal{R}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$, consider the path $\mathcal{P} = \mathcal{P}(\ell_e, \ell_0)$, and set $\mathcal{P} = \{\ell_1, \cdots, \ell_N\}$, with $\ell_0 \succ \ell_1 \succ \ldots \succ \ell_N \succ \ell_{N+1} = \ell_e$ (if $\mathcal{P} = \emptyset$, set $N = 0$ in the forthcoming discussion). For $k = 0, \ldots, N$, denote by v_k the node which the line ℓ_k exits and by $L_0(v_k)$ the set $L(v_k)\backslash\{\ell_{k+1}\}$ (cf. Item 1 in Definition 5.7).

We construct a tree $\theta' \in \mathcal{R}^{(k)-\sigma',-\sigma}_{R,h,\boldsymbol{\nu}',\boldsymbol{\nu}}$ in the following way:

1. We shift the sign labels down the path $\mathcal{P}$ and change their sign, so that $\sigma_{\ell_k} \to -\sigma_{v_k}$ and $\sigma_{v_k} \to -\sigma_{\ell_{k+1}}$ for $k = 0, \ldots, N$. In particular ℓ_0 acquires the label $-\sigma_{v_0}$, while ℓ_e loses its label σ_{ℓ_e} (which with the opposite sign becomes associated with the node v_N).
2. The end-node e becomes the root, and the root becomes the end-node e. In particular the line ℓ_e becomes the root line, and the line ℓ_0 becomes the entering line, so that the arrows of all the lines $\ell \in \mathcal{P}$ are reverted, while the ordering of all the lines and nodes outside $\mathcal{P}$ is not changed.
3. For all the lines $\ell \in \mathcal{P}$ we exchange the labels $\boldsymbol{\nu}_\ell, \boldsymbol{\nu}'_\ell$, so that $\boldsymbol{\nu}_{\ell_k} \to \boldsymbol{\nu}'_{\ell_k}$ and $\boldsymbol{\nu}'_{\ell_k} \to \boldsymbol{\nu}_{\ell_k}$ for $k = 1, \ldots, N$, and we set $\boldsymbol{\nu}'_{\ell_e} = \boldsymbol{\nu}'$ and $\boldsymbol{\nu}_{\ell_0} = \boldsymbol{\nu}$.
4. For all $k = 0, \ldots, N$ we replace $m_{v_k} \to -\sigma_{v_k}\sigma_{\ell_{k+1}} m_{v_k}$.

By construction, the tree θ' belongs to $\mathcal{R}^{(k)-\sigma',-\sigma}_{R,h,\boldsymbol{\nu}',\boldsymbol{\nu}}$, and all line propagators and node factors of the lines and nodes, respectively, which do not belong to $\mathcal{P}$ remain the same.

Moreover, the line propagator of each $\ell_k \in \mathcal{P}$ in θ' is $(G_{i_{\ell_k},h_{\ell_k}})^{-\sigma_{v_k},-\sigma_{\ell_k}}_{\boldsymbol{\nu}_\ell,\boldsymbol{\nu}'_{\ell'_k}}$ $= (G_{i_{\ell_k},h_{\ell_k}})^{\sigma_{\ell_k},\sigma_{v_k}}_{\boldsymbol{\nu}'_{\ell'_k},\boldsymbol{\nu}_\ell}$, hence it does not change with respect to the line propagator of the corresponding line in θ. For each node v_k, the conservation law

$$\boldsymbol{\nu}_{\ell_{k+1}} = (0, -\sigma_{v_k}\sigma_{\ell_{k+1}} m_{v_k}) - \sigma_{\ell_{k+1}} \left(-\sigma_{v_k}\boldsymbol{\nu}'_{\ell_k} + \sum_{\ell' \in L_0(v_k)} \sigma_{\ell'}\boldsymbol{\nu}_{\ell'} \right) \tag{5.10}$$

is assured by the conservation law (cf. Item 15 in Definition 5.7)

$$\boldsymbol{\nu}'_{\ell_k} = (0, m_{v_k}) + \sigma_{v_k} \left(\sigma_{\ell_{k+1}}\boldsymbol{\nu}_{\ell_{k+1}} + \sum_{\ell' \in L_0(v_k)} \sigma_{\ell'}\boldsymbol{\nu}_{\ell'} \right) \tag{5.11}$$

for the corresponding node v_k in θ: simply multiply (5.11) times $\sigma_{v_k}\sigma_{\ell_{k+1}}$ in order to obtain (5.10).

Finally we want to show that the product of the combinatorial factors times the node factors of the nodes $v_0, \ldots, v_N$ do not change. Take a node $v = v_k$, for $k = 0, \ldots, N$, and call r'_v and s'_v the number of lines $\ell' \in L_0(v)$ with $\sigma_{\ell'} = \sigma_v$ and $\sigma_{\ell'} = -\sigma_v$, respectively. Set $\sigma_v = \sigma$ and $\sigma_{\ell_{k+1}} = \sigma'$.

Consider first the case $\sigma' = \sigma$. In that case in θ one has $r_v = r'_v + 1$ and $s_v = s'_v$, and the combinatorial factor contains a factor r_v because there are r_v lines ℓ entering v with $\sigma_\ell = \sigma$. In θ' one has $\sigma_v \to -\sigma$, $r_v \to s'_v + 1$, $s_v \to r'_v$ and $m_v \to -m_v$ (because $\sigma\sigma' = 1$). Moreover the corresponding combinatorial factor contains a factor $(s_v + 1)$ because there are $s_v + 1$ lines ℓ entering v with $\sigma_\ell = -\sigma$. Therefore, taking into account also the combinatorics, the node factor associated with the node v in θ is $(s_v+1)a^{-\sigma}_{s_v+1,r_v-1,-m_v} = r_v\, a^{\sigma}_{r_v,s_v,m_v}$, i.e. the same as in θ, by the condition (2.11).

Now, we pass to the case $\sigma = -\sigma'$. In that case in θ one has $r_v = r'_v$, $s_v = s'_v + 1$. In θ' one has the same values for r_v, s_v and σ_v, so that, by using also that $-\sigma\sigma' m_v = m_v$ in such a case, the node factors $a^{\sigma_v}_{r_v,s_v,m_v}$ do not change. Of course the combinatorial factors do not change either.

In conclusion, one has $\mathrm{Val}(\theta) = \mathrm{Val}(\theta')$, which yields the assertion. □

Remark 5.23. By Lemma 5.22 we have that the matrix $L_h^{(k)}$ is self-adjoint, and Definition 5.21 together with (5.6) implies that we can write

$$L^{(k)\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'} = \sum_{h=-1}^{\infty} C_h(x_{\boldsymbol{\nu}}(\varepsilon)) \sum_{\theta\in\mathcal{R}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}} \mathrm{Val}(\theta), \quad C_h(x) = \sum_{h'=h+2}^{\infty} \chi_h(x), \quad \sigma = \pm,$$

for all $k \ge N$, all $h \ge 1$, and all resonant pairs $\{\boldsymbol{\nu},\boldsymbol{\nu}'\}$. By construction $x_{\boldsymbol{\nu}}(\varepsilon) = x_{\boldsymbol{\nu}'}(\varepsilon)$ whenever $L^{(k)\sigma,\sigma'}_{h,\boldsymbol{\nu},\boldsymbol{\nu}'} \ne 0$, so that also $L^{(k)}$ is self-adjoint. Finally we have that $L^{(k)} = \widehat{\chi}_1 L^{(k)} \widehat{\chi}_1$ (cf. the definition of the line propagators g_{ℓ_0} and g_{ℓ_e} for trees $\theta \in \mathcal{R}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$ in Definition 5.13).

Lemma 5.24. *One has*

$$u^{(k)\sigma}_{\boldsymbol{\nu}} = \sum_{\theta\in\Theta^{(k)\sigma}_{R,\boldsymbol{\nu}}} \mathrm{Val}(\theta), \quad \sigma = \pm, \tag{5.12}$$

for all $k \ge 1$ and all $\boldsymbol{\nu} \in \mathbb{Z}^{D+1}$.

Proof. For any given counterterm L, the coefficients $u^{(k)\sigma}_{\boldsymbol{\nu}}$ can be written as sums over tree values

$$u^{(k)\sigma}_{\boldsymbol{\nu}} = \sum_{\theta\in\Theta^{(k)\sigma}_{\boldsymbol{\nu}}} \mathrm{Val}(\theta).$$

This can be easily proved by induction, using the diagrammatic rules and definitions given in this section; we refer to Lemma 3.6 of [24] for details. Then, defining the counterterms according to Definition 5.21, all contributions arising from trees belonging to the set $\Theta^{(k)\sigma}_{\boldsymbol{\nu}}$ but not to the set $\Theta^{(k)\sigma}_{R,\boldsymbol{\nu}}$ cancel out exactly — see Lemma 3.13 of [24] for further details — and hence the assertion follows. □

6. Bryuno Lemmas and Bounds

Given a tree $\theta \in \Theta_R$, call $\mathfrak{Z}(\theta,\gamma)$ the set of $(\varepsilon, M) \in \mathfrak{D}_0$ such that for all $\ell \in L_p(\theta)$ with $i_\ell = 1$ one has

$$\begin{cases} 2^{-h_\ell-1}\gamma \le |x_{\boldsymbol{\nu}_\ell}(\varepsilon)| \le 2^{-h_\ell+1}\gamma, & h_\ell \ne -1, \\ |x_{\boldsymbol{\nu}_\ell}(\varepsilon)| \ge \gamma, & h_\ell = -1, \end{cases} \tag{6.1}$$

and for all $\ell \in L_p(\theta)$ one has

$$\begin{cases} |\delta_{\boldsymbol{\nu}_\ell}(\varepsilon)| \leq \bar{\gamma}, \quad |\delta_{\boldsymbol{\nu}'_\ell}(\varepsilon)| \leq \bar{\gamma}, & i_\ell = 1, \\ \bar{\gamma} \leq |\delta_{\boldsymbol{\nu}_\ell}(\varepsilon)|, & i_\ell = 0. \end{cases} \tag{6.2}$$

Here γ is the constant introduced in Definition 4.12.

Define also $\mathfrak{D}(\theta, \gamma) \subset \mathfrak{D}_0$ as the set of $(\varepsilon, M) \in \mathfrak{D}_0$ such that for all $\ell \in L_p(\theta)$ with $i_\ell = 0$ one has $|\delta_{\boldsymbol{\nu}_\ell}(\varepsilon) \pm \bar{\gamma}| \geq \gamma/|\boldsymbol{\nu}_\ell|^{\tau_1}$, while for all $\ell \in L_p(\theta)$ with $i_\ell = 1$ one has

$$x_{\boldsymbol{\nu}_\ell}(\varepsilon) \geq \frac{\gamma}{p^{\tau}_{\boldsymbol{\nu}_\ell}(\varepsilon)}, \qquad |\delta_{\boldsymbol{\nu}}(\varepsilon) \pm \bar{\gamma}| \geq \frac{\gamma}{|\boldsymbol{\nu}|^{\tau_1}} \;\; \forall \boldsymbol{\nu} \in \mathcal{C}_{\boldsymbol{\nu}_\ell} \cup \mathcal{C}_{\boldsymbol{\nu}'_\ell}, \tag{6.3}$$

for some $\tau, \tau_1 > 0$. Note that the second condition in (6.3) does not depend on M.

Analogously, given a tree $\theta \in \mathcal{S}_R$, we call $\widetilde{\mathfrak{Z}}(\theta, \gamma)$ the set of $(\varepsilon, M) \in \mathfrak{D}_0$ such that (6.1) holds for all $\ell \in L_p(\theta) \backslash \{\ell_e, \ell_0\}$ with $i_\ell = 1$ and (6.2) holds for all $\ell \in L_p(\theta)$, and we call $\widetilde{\mathfrak{D}}(\theta, \gamma)$ the set of $(\varepsilon, M) \in \mathfrak{D}_0$ such that (6.3) holds for all $\ell \in L_p(\theta) \backslash \{\ell_e, \ell_0\}$ with $i_\ell = 1$, while for all $\ell \in L_p(\theta)$ with $i_\ell = 0$ one has $|\delta_{\boldsymbol{\nu}_\ell}(\varepsilon) \pm \bar{\gamma}| \geq \gamma/|\boldsymbol{\nu}_\ell|^{\tau_1}$.

Remark 6.1. If $(\varepsilon, M) \in \mathfrak{Z}(\theta, \gamma)$ then $\mathrm{Val}(\theta; \varepsilon, M) \neq 0$, while $(\varepsilon, M) \in \mathfrak{D}(\theta, \gamma)$ means that we can use the bounds (6.3) to estimate $\mathrm{Val}(\theta; \varepsilon, M)$. Analogous considerations hold for trees $\theta \in \mathcal{S}_R$.

Remark 6.2. If for some ε one has $\mathrm{Val}(\theta; \varepsilon, M) \neq 0$ and for two comparable lines $\ell, \ell' \in L(\theta)$ the pair $\{\boldsymbol{\nu}'_\ell, \boldsymbol{\nu}_{\ell'}\}$ is resonant, then all the set $\{\boldsymbol{\nu}_\ell, \boldsymbol{\nu}'_\ell, \boldsymbol{\nu}_{\ell'}, \boldsymbol{\nu}'_{\ell'}\}$ is resonant. This motivates the condition in Item 14 in Definition 5.7.

Remark 6.3. If $\theta \in \mathcal{R}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$ is such that $\mathrm{Val}(\theta; \varepsilon, M) \neq 0$, then $\boldsymbol{\nu}, \boldsymbol{\nu}' \in \Delta_j(\varepsilon)$ for some j, so that $p_{\boldsymbol{\nu}}(\varepsilon) = p_{\boldsymbol{\nu}'}(\varepsilon)$ and $|\boldsymbol{\nu} - \boldsymbol{\nu}'| \leq C_1 C_2 p^{\alpha+\beta}_{\boldsymbol{\nu}}(\varepsilon) \leq C_1 C_2 p^{2\alpha}_{\boldsymbol{\nu}}(\varepsilon)$. Moreover $p_{\boldsymbol{\nu}}(\varepsilon) \leq |\boldsymbol{\nu}|, |\boldsymbol{\nu}'| \leq 2 p_{\boldsymbol{\nu}}(\varepsilon)$. Such properties follow from Hypothesis 3 — cf. also Lemma 2.9.

Definition 6.4 (The quantity $\boldsymbol{N_h(\theta)}$). *Define $N_h(\theta)$ as the number of lines $\ell \in L(\theta)$ with $i_\ell = 1$ and scale $h_\ell \geq h$.*

Definition 6.5 (The quantity $\boldsymbol{K(\theta)}$). *Define*

$$K(\theta) = k(\theta) + \sum_{v \in V_0(\theta)} |m_v| + \sum_{\ell \in L_q(\theta)} |\boldsymbol{\nu}_\ell - \boldsymbol{\nu}'_\ell| + \sum_{v \in E(\theta)} |\boldsymbol{\nu}_v|,$$

where $k(\theta)$ is the order of θ.

Lemma 6.6 *There exists a constant B such that the following holds:*

1. *For all $\theta \in \Theta_R$ and all lines $\ell \in L(\theta)$ one has $|\boldsymbol{\nu}_\ell| \leq B(K(\theta))^{1+4\alpha}$.*
2. *If $\theta \in \mathcal{S}_R$, for all lines $\ell \in L(\theta) \backslash (\mathcal{P}(\ell_e, \ell_0) \cup \{\ell_0, \ell_e\})$ one has $|\boldsymbol{\nu}_\ell| \leq B(K(\theta))^{1+4\alpha}$, while for all lines $\ell \in \mathcal{P}(\ell_e, \ell_0) \cup \{\ell_0\}$ one has $|\boldsymbol{\nu}'_\ell| \leq B(|\boldsymbol{\nu}_{\ell_e}| + K(\theta))^{1+4\alpha}$.*
3. *Given a tree θ let $\ell, \ell' \in L(\theta)$ be two comparable lines, with $\ell \prec \ell'$, such that $i_\ell = i_{\ell'} = 1$ and $i_{\ell''} = 0$ for all the lines $\ell'' \in \mathcal{P}(\ell, \ell')$. If $|\boldsymbol{\nu}'_\ell - \boldsymbol{\nu}_{\ell'}| \geq B K(\theta)^{1+4\alpha}$, then one has $\mathrm{Val}(\theta) = 0$ for all ε.*
4. *If $\theta \in \mathcal{S}_R$, $\ell \in \mathcal{P}(\ell_e, \ell_0) \cup \{\ell_0\}$ and, moreover, $i_{\ell'} = 0$ for all lines $\ell' \in \mathcal{P}(\ell_e, \ell)$, then $|\boldsymbol{\nu}'_\ell| \leq |\boldsymbol{\nu}_{\ell_e}| + B(K(\theta))^{1+4\alpha}$.*

Proof. Let us consider first trees $\theta \in \Theta_R$. The proof is by induction on the order of the tree $k = k(\theta)$. For $k = 1$ the bound is trivial. If the root line ℓ_0 is either a q-line or an r-line or a p-line with $i_{\ell_0} = 0$, again the bound follows trivially from the inductive bound. If ℓ_0 is a p-line with $i_{\ell_0} = 1$, call v_0 the node such that $\ell_0 = \ell_{v_0}$ and $\theta_1, \ldots, \theta_s$ the subtrees with root in v_0. By the inductive hypothesis and Hypothesis 3 one obtains, for a suitable constant C and taking B large enough,

$$|\boldsymbol{\nu}_\ell| \leq |m_{v_0}| + B\left(K(\theta) - 1 - |m_{v_0}|\right)^{1+4\alpha} + C\left(|m_{v_0}| + B(K(\theta) - 1 - |m_{v_0}|)\right)^{2\alpha(1+4\alpha)} \leq B(K(\theta))^{1+4\alpha},$$

which proves the assertion for Θ_R in Item 1.

As a byproduct also the bound for $\mathcal{S}_R$ is obtained, as far as lines $\ell \notin \mathcal{P}(\ell_e, \ell_0) \cup \{\ell_0, \ell_e\}$ are concerned. The bound $|\boldsymbol{\nu}'_\ell| \leq B(|\boldsymbol{\nu}_{\ell_e}| + K(\theta))^{1+4\alpha}$ for the lines $\ell \in \mathcal{P}(\ell_e, \ell_0) \cup \{\ell_0\}$ can be proved similarly by induction. Thus, also Item 2 is proved.

Given two comparable lines ℓ, ℓ' such that $i_{\ell''} = 0$ for all lines $\ell'' \in \mathcal{P}(\ell, \ell')$, then by momentum conservation one has $\min\{|\boldsymbol{\nu}'_\ell - \boldsymbol{\nu}_{\ell'}|, |\boldsymbol{\nu}'_\ell + \boldsymbol{\nu}_{\ell'}|\} \leq B(K(\theta))^{1+4\alpha}$ in Case (I) and $|\boldsymbol{\nu}'_\ell - \boldsymbol{\nu}_{\ell'}| \leq B(K(\theta))^{1+4\alpha}$ in Case (II). This proves the bounds in Item 3 in Case (II) and in Item 4 for both Cases (I) and (II).

In Case (I), if $i_\ell = i_{\ell'} = 1$ and $\max\{|\delta_{\boldsymbol{\nu}'_\ell}(\varepsilon)|, |\delta_{\boldsymbol{\nu}_{\ell'}}(\varepsilon)|\} < 1/2$, then $|\boldsymbol{\nu}'_\ell - \boldsymbol{\nu}_{\ell'}| \leq |\boldsymbol{\nu}'_\ell + \boldsymbol{\nu}_{\ell'}|$ by Item 5 in Hypothesis 1. On the other hand if $i_\ell = i_{\ell'} = 1$ and $\max\{|\delta_{\boldsymbol{\nu}'_\ell}(\varepsilon)|, |\delta_{\boldsymbol{\nu}_{\ell'}}(\varepsilon)|\} \geq 1/2$, one has $\mathrm{Val}(\theta; \varepsilon, M) = 0$. Hence Item 3 follows also in Case (I). □

The following bound will allow us to bound the tree values. This bound, and the analogous bound in Lemma 6.12, will be called a Bryuno Lemma, by analogy with the kind of bounds used in the Siegel-Bryuno arguments in the case of Siegel's problem; see [12,18,33].

Lemma 6.7. *Given a tree $\theta \in \Theta_R$ such that $\mathfrak{D}(\theta, \gamma) \cap \mathfrak{Z}(\theta, \gamma) \neq \emptyset$, for all $h \geq 1$ one has*

$$N_h(\theta) \leq \max\{0, c\, K(\theta) 2^{(2-h)\beta/2\tau} - 1\},$$

where c is a suitable constant.

Proof. Define $E_h := c^{-1} 2^{(h-2)\beta/2\tau}$. So, we have to prove that $N_h(\theta) \leq \max\{0, K(\theta) E_h^{-1} - 1\}$.

If a line ℓ is on scale $h \geq 0$ then $\gamma / p^\tau_{\boldsymbol{\nu}_\ell}(\varepsilon) < x_{\boldsymbol{\nu}_\ell}(\varepsilon) \leq 2^{-h+1}\gamma$ by (6.1) and (6.3). Hence

$$B(K(\theta))^2 \geq B(K(\theta))^{1+4\alpha} \geq |\boldsymbol{\nu}_\ell| \geq p_{\boldsymbol{\nu}_\ell}(\varepsilon) > 2^{(h-1)/\tau},$$

by Lemma 6.6, so that

$$K(\theta) E_h^{-1} \geq c B^{-1/2} 2^{(h-1)/2\tau} 2^{(2-h)\beta/2\tau} \geq 2$$

for c suitably large. Therefore if a tree θ contains a line ℓ on scale h one has $\max\{0, K(\theta) E_h^{-1} - 1\} = K(\theta) E_h^{-1} - 1 \geq 1$.

The bound $N_h(\theta) \leq \max\{0, K(\theta) E_h^{-1} - 1\}$ will be proved by induction on the order of the tree. Let ℓ_0 be the root line of θ and call $\theta_1, \ldots, \theta_m$ the subtrees of θ whose root lines $\ell_1, \ldots, \ell_m$ are the lines on scale $h_{\ell_i} \geq h - 1$ and $i_{\ell_i} = 1$ which are the closest to ℓ_0.

If $h_{\ell_0} < h$ we can write $N_h(\theta) = N_h(\theta_1) + \cdots + N_h(\theta_m)$, and the bound follows by induction. If $h_{\ell_0} \geq h$ then $\ell_1, \cdots, \ell_m$ are the entering lines of a cluster T with exiting line ℓ_0; in that case we have $N_h(\theta) = 1 + N_h(\theta_1) + \ldots + N_h(\theta_m)$. Again the bound follows by induction for $m = 0$ and $m \geq 2$. The case $m = 1$ can be dealt with as follows.

If $\{\boldsymbol{\nu}'_{\ell_0}, \boldsymbol{\nu}_{\ell_1}\}$ is a resonant pair, then either there exists a line $\ell \in \mathcal{P}(\ell_1, \ell_0)$ with $i_\ell = 1$ such that $\{\boldsymbol{\nu}'_\ell, \boldsymbol{\nu}_{\ell_1}\}$ is a resonant pair or there must be a line $\ell \in L(T)\backslash\mathcal{P}(\ell_1, \ell_0)$ with $\{\boldsymbol{\nu}'_\ell, \boldsymbol{\nu}_{\ell_1}\}$ a resonant pair. In fact, the first case is not possible: indeed, also $\{\boldsymbol{\nu}'_{\ell_0}, \boldsymbol{\nu}'_\ell\}$ would be resonant (cf. Remark 6.2), so that $|h_\ell - h_{\ell_0}| \leq 1$ (cf. Item 14 in Definition 5.7), and hence the contradiction $h-2 \geq h_\ell \geq h_{\ell_0} - 1 \geq h-1$ would follow. In the second case, one has $|\boldsymbol{\nu}'_\ell| \geq p_{\boldsymbol{\nu}_{\ell_1}}(\varepsilon) > 2^{(h-2)/\tau}$, hence if θ' is the subtree with root line ℓ, then one has $K(\theta) - K(\theta_1) > K(\theta') > 2E_h$, and the bound follows once more by the inductive hypothesis.

If $\{\boldsymbol{\nu}'_{\ell_0}, \boldsymbol{\nu}_{\ell_1}\}$ is not a resonant pair, call $\bar{\ell}$ the line along the path $\mathcal{P}(\ell_1, \ell_0) \cup \{\ell_1\}$ with $i_{\bar{\ell}} = 1$ closest to ℓ_0. Since $i_{\bar{\ell}} = 1$ and by hypothesis $h_{\bar{\ell}} < h - 1$ then $\{\boldsymbol{\nu}_{\bar{\ell}}, \boldsymbol{\nu}_{\ell_0}\}$ is not a resonant pair (see Item 14 in Definition 5.7). Call $\tilde{T}$ the set of nodes and lines preceding ℓ_0 and following $\bar{\ell}$, and define $K(T) = K(\theta) - K(\theta_1)$ and $K(\tilde{T}) = K(\theta) - K(\bar{\theta})$, where $\bar{\theta}$ is the tree with root line $\bar{\ell}$. Set also $\bar{\boldsymbol{\nu}} = \boldsymbol{\nu}_{\bar{\ell}}$ and $\boldsymbol{\nu}_0 = \boldsymbol{\nu}'_{\ell_0}$. One has $2|\bar{\boldsymbol{\nu}} - \boldsymbol{\nu}_0| \geq C_2(p_{\bar{\boldsymbol{\nu}}}(\varepsilon) + p_{\boldsymbol{\nu}_0}(\varepsilon))^\beta \geq C_2 p^\beta_{\boldsymbol{\nu}_0}(\varepsilon)$ (see Lemma 2.9), so that by Lemma 6.6 one finds

$$B(K(\theta) - K(\theta_1)^2 \geq B(K(\tilde{T}))^2 \geq |\bar{\boldsymbol{\nu}} - \boldsymbol{\nu}_0| \geq \frac{1}{2} C_2\, p^\beta_{\boldsymbol{\nu}_0}(\varepsilon) \geq \frac{1}{2} C_2 2^{(h-1)\beta/\tau}.$$

Hence $(K(\theta) - K(\theta_1))E_h^{-1} \geq K(T)E_h^{-1} \geq K(\tilde{T})E_h^{-1} \geq 2$, provided c is large enough. This proves the bound. □

Lemma 6.8. *There exists positive constants ξ_0 and D_0 such that, if $\xi > \xi_0$ in Definition 4.10, then for all trees $\theta \in \Theta_R$ and for all $(\varepsilon, M) \in \mathfrak{D}(\theta, \gamma) \cap \mathfrak{Z}(\theta, \gamma)$ one has*

$$|\mathrm{Val}(\theta)| \leq D_0^k \mathrm{e}^{-\kappa K(\theta)} \prod_{\substack{\ell \in L(\theta) \\ i_\ell = 1}} p_{\boldsymbol{\nu}_\ell}^{-(\xi - \xi_0)}(\varepsilon), \tag{6.4a}$$

$$|\partial_\varepsilon \mathrm{Val}(\theta)| \leq D_0^k \mathrm{e}^{-\kappa K(\theta)} \prod_{\substack{\ell \in L(\theta) \\ i_\ell = 1}} p_{\boldsymbol{\nu}_\ell}^{-(\xi - \xi_0)}(\varepsilon), \tag{6.4b}$$

$$\sum_{\boldsymbol{\nu} \in \mathfrak{S}} \sum_{\boldsymbol{\nu}' \in \mathcal{C}_{\boldsymbol{\nu}}} \sum_{\sigma, \sigma' = \pm} \left| \partial_{M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}} \mathrm{Val}(\theta) \right| \leq D_0^k \mathrm{e}^{-\kappa K(\theta)} \prod_{\substack{\ell \in L(\theta) \\ i_\ell = 1}} p_{\boldsymbol{\nu}_\ell}^{-(\xi - \xi_0)}(\varepsilon). \tag{6.4c}$$

Proof. The propagators are bounded according to (5.5), so that for all trees $\theta \in \Theta^{(k)}_{R,\boldsymbol{\nu}}$ one has

$$|\mathrm{Val}(\theta)| \leq C^k \left(\prod_{v \in V_0(\theta)} \mathrm{e}^{-A_2|m_v|} \right) \left(\prod_{\ell \in L_q(\theta)} \mathrm{e}^{-\lambda_0|\boldsymbol{\nu}_\ell - \boldsymbol{\nu}'_\ell|} \right) \times \left(\prod_{v \in E(\theta)} \mathrm{e}^{-\lambda_0|\boldsymbol{\nu}_v|} \right) 2^{kh_0} \left(\prod_{h=h_0+1}^{\infty} 2^{hN_h(\theta)} \right) \prod_{\substack{\ell \in L(\theta) \\ i_\ell = 1}} p_{\boldsymbol{\nu}_\ell}^{-\xi}(\varepsilon)\, p_{\boldsymbol{\nu}_\ell}^{a_0}(\varepsilon),$$

for arbitrary h_0 and for suitable constants C and a_0. For $(\varepsilon, M) \in \mathfrak{D}(\theta,\gamma) \cap \mathfrak{Z}(\theta,\gamma)$ one can bound $N_h(\theta)$ through Lemma 6.7. Therefore, by choosing h_0 large enough the bound (6.4a) follows, provided $\xi - a_0 > 0$ and κ is suitably chosen.

When bounding $\partial_\varepsilon \mathrm{Val}(\theta)$, one has to consider derivatives of the line propagators, i.e. $\partial_\varepsilon g_\ell$. If ℓ is an r-line then $|\partial_\varepsilon g_\ell|$ is bounded proportionally to $|\boldsymbol{\nu}_\ell|^{c_0}$, whereas if ℓ is a p-line, then the derivative produces factors which admit bounds of the form

$$C p_{\boldsymbol{\nu}_\ell}^{a_1}(\varepsilon)\, 2^{2h_\ell} p_{\boldsymbol{\nu}_\ell}^{c_0}(\varepsilon) p_{\boldsymbol{\nu}_\ell}^{-\xi}(\varepsilon), \tag{6.5}$$

for suitable constants C and a_1; see the proof of Lemma 4.2 in [24] for details (and use Item 3 in Hypothesis 1).

The extra factor 2^{h_ℓ} can be taken into account by bounding the product of line propagators with

$$2^{2h_0 k} \prod_{h=h_0+1}^{\infty} 2^{2hN_h(\theta)}.$$

One can bound $|\boldsymbol{\nu}_\ell| \le B(K(\theta))^2$, and use part of the exponential decaying factors $\mathrm{e}^{-A_2|m_v|}$, $\mathrm{e}^{-\lambda_0|\boldsymbol{\nu}_\ell-\boldsymbol{\nu}'_\ell|}$, and $\mathrm{e}^{-\lambda_0|\boldsymbol{\nu}_v|}$, to control the contribution $\sum_{v\in V_0(\theta)}|m_v|$ + $\sum_{\ell\in L_q(\theta)}|\boldsymbol{\nu}_\ell-\boldsymbol{\nu}'_\ell|$ + $\sum_{v\in E(\theta)}|\boldsymbol{\nu}_v|$ to $K(\theta)$ (cf. Definition 6.5). Then, if ξ is large enough, so that $\xi - a_1 > 0$ for all possible values of a_1 in (6.5), the bound (6.4b) follows.

Also the bound (6.4c) can be discussed in the same way. We refer again to [24] for the details. □

Remark 6.9. Note that for $(\varepsilon, M) \in \mathfrak{D}(\theta,\gamma)$ the singularities of the functions $\bar\chi_1$ are avoided, so that $\partial_\varepsilon \bar\chi_1(\delta_{\boldsymbol{\nu}_\ell}(\varepsilon)) = 0$ for all $\ell \in L(\theta)$. Note also that the bound (6.4c) is not really needed in the following.

Lemma 6.10. *There are two positive constants B_2 and B_3 such that the following holds:*

1. *Given a tree $\theta \in \mathcal{S}_R$ such that* $\mathrm{Val}(\theta;\varepsilon,M) \neq 0$, *if $K(\theta) \le B_2 p_{\boldsymbol{\nu}_{\ell_e}}^{\beta/2}(\varepsilon)$ then for all lines $\ell \in \mathcal{P}(\ell_e,\ell_0)$ one has $i_\ell = 0$. Moreover for all such lines ℓ, if $\{\boldsymbol{\nu}'_\ell, \boldsymbol{\nu}_{\ell_e}\}$ is not a resonant pair, then one has $|\delta_{\boldsymbol{\nu}_\ell}(\varepsilon)| \ge 1/2$.*
2. *Given a tree $\theta \in \mathcal{R}_R$ such that* $\mathrm{Val}(\theta;\varepsilon,M) \neq 0$, *one has $\left|\boldsymbol{\nu}'_{\ell_0} - \boldsymbol{\nu}_{\ell_e}\right| \le B_3(K(\theta))^{1/\rho}$, with ρ depending on α and β.*

Proof. Suppose that $\theta \in \mathcal{S}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$ and $\mathcal{P}(\ell_e,\ell_0)$ contains lines ℓ with $i_\ell = 1$ and consequently with $\{\boldsymbol{\nu}'_\ell, \boldsymbol{\nu}'\}$ not resonant (cf. Definition 5.18). Let $\bar\ell$ be the one closest to ℓ_e; thus, one has $|\boldsymbol{\nu}'_{\bar\ell} - \boldsymbol{\nu}'| \ge C_3(|\boldsymbol{\nu}'_{\bar\ell}| + |\boldsymbol{\nu}'|)^\beta \ge C_3 p_{\boldsymbol{\nu}'}^\beta(\varepsilon) = C_3 p_{\boldsymbol{\nu}}^\beta(\varepsilon)$, so that we can apply Item 3 in Lemma 6.6 to obtain $B(K(\theta))^2 \ge C p_{\boldsymbol{\nu}}^\beta(\varepsilon)$, for some positive constant C. This proves the first statement in Item 1. The proof of the second statement is identical, since $|\delta_{\boldsymbol{\nu}_\ell}(\varepsilon)| < 1/2$ implies that $\boldsymbol{\nu}_\ell \in \Delta_{j_1}(\varepsilon)$ for some j_1, so that if $\{\boldsymbol{\nu}'_\ell, \boldsymbol{\nu}'\}$ is not a resonant pair then $\boldsymbol{\nu}' \notin \Delta_{j_1}(\varepsilon)$, and therefore $|\boldsymbol{\nu}'_\ell - \boldsymbol{\nu}'| \ge C_3 p_{\boldsymbol{\nu}'}^\beta(\varepsilon)$.

To prove Item 2, notice that $|\boldsymbol{\nu} - \boldsymbol{\nu}'| \le C_1 C_2 p_{\boldsymbol{\nu}}^{\alpha+\beta}(\varepsilon)$ (cf. Remark 6.3). If $K(\theta) > B_2 p_{\boldsymbol{\nu}}^{\beta/2}(\varepsilon)$ then $K(\theta) \ge C|\boldsymbol{\nu} - \boldsymbol{\nu}'|^{\beta/2(\alpha+\beta)}$. If $K(\theta) \le B_2 p_{\boldsymbol{\nu}}^{\beta/2}(\varepsilon)$ then $\mathcal{P}(\ell_e,\ell_0)$ has only lines with $i_\ell = 0$, so that by Item 3 in Lemma 6.6 one finds $|\boldsymbol{\nu} - \boldsymbol{\nu}'| \le BK(\theta)^2$. □

Lemma 6.11. *Given a tree $\theta \in \mathcal{S}_R$ such that $\widetilde{\mathfrak{D}}(\theta,\gamma) \cap \widetilde{\mathfrak{Z}}(\theta,\gamma) \neq \emptyset$, if $N_h(\theta) \geq 1$ for some $h \geq 1$, then $cK(\theta)2^{(2-h)\beta/2\tau} \geq 1$, with c the same constant as in Lemma 6.7.*

Proof. Consider a tree $\theta \in \mathcal{S}^{(k)\sigma,\sigma'}_{R,\bar{h},\boldsymbol{\nu},\boldsymbol{\nu}'}$ for some $k \geq 1$, $\bar{h} \geq 1$, $\sigma, \sigma' = \pm$ and $\boldsymbol{\nu}, \boldsymbol{\nu}' \in \mathfrak{S}$ such that $|\boldsymbol{\nu}'| \geq 2^{(\bar{h}-2)/\tau}$. Assume $N_h(\theta) \geq 1$ for some $\bar{h} \geq h \geq 1$.

If there is a line $\ell \in L(\theta)$, which does not belong to $\mathcal{P} := \mathcal{P}(\ell_e, \ell_0)$, such that $h_\ell \geq h$, then one can reason as at the beginning of the proof of Lemma 6.7 to obtain $K(\theta)E_h^{-1} \geq 2$, with $E_h = c^{-1}2^{(h-2)\beta/2\tau} \geq 1$.

Otherwise, there are lines $\ell \in \mathcal{P}$ on scale $h_\ell \geq h$, and hence such that $i_\ell = 1$ and, consequently, $\{\boldsymbol{\nu}'_\ell, \boldsymbol{\nu}'\}$ is not a resonant pair. Let $\bar{\ell}$ be the one closest to ℓ_e among such lines; thus, one has $|\boldsymbol{\nu}'_{\bar{\ell}} - \boldsymbol{\nu}'| \geq C_3 p^{\beta}_{\boldsymbol{\nu}'}(\varepsilon)$, so that one obtains $B(K(\theta))^2 \geq C p^{\beta}_{\boldsymbol{\nu}'}(\varepsilon) \geq C2^{(\bar{h}-2)\beta/\tau}$, for some positive constant C. So, the desired bound follows once more. □

Lemma 6.12. *Given a tree $\theta \in \mathcal{S}_R$ such that $\widetilde{\mathfrak{D}}(\theta,\gamma) \cap \widetilde{\mathfrak{Z}}(\theta,\gamma) \neq \emptyset$, for all $h \geq 1$ one has*

$$N_h(\theta) \leq c\,K(\theta)2^{(2-h)\beta/2\tau},$$

where c is the same constant as in Lemma 6.7.

Proof. Consider a tree $\theta \in \mathcal{S}^{(k)\sigma,\sigma'}_{R,\bar{h},\boldsymbol{\nu},\boldsymbol{\nu}'}$ for some $k \geq 1$, $\bar{h} \geq 1$, $\sigma, \sigma' = \pm$ and $\boldsymbol{\nu}, \boldsymbol{\nu}' \in \mathfrak{S}$ such that $|\boldsymbol{\nu}'| \geq 2^{(\bar{h}-2)/\tau}$.

For $k(\theta) = 1$ one has $N_h(\theta) \leq 1$, so that the bound follows from Lemma 6.11.

For $k(\theta) > 1$ one can proceed as follows. Let ℓ_0 be the root line of θ and call $\theta_1, \ldots, \theta_m$ the subtrees of θ whose root lines $\ell_1, \ldots, \ell_m$ are the lines on scale $h_{\ell_i} \geq h-1$ and $i_{\ell_i} = 1$ which are the closest to ℓ_0. All the trees θ_i such that $\ell_i \notin \mathcal{P}(\ell_e, \ell_0)$ belong to some $\Theta^{(k_i)\pm}_{R,\boldsymbol{\nu}_i}$ with $k_i < k$. If $K(\theta) \geq B_2 p^{\beta/2}_{\boldsymbol{\nu}'}(\varepsilon)$ (cf. Lemma 6.10) it may be possible that a line, say ℓ_1, belongs to $\mathcal{P}(\ell_e, \ell_0)$, so that $\mathrm{Val}(\theta_1) = g_{\ell_1}\,\mathrm{Val}(\theta'_1)$, with $\theta'_1 \in \mathcal{S}^{(k_1),\sigma_1,\sigma'}_{R,h_1,\boldsymbol{\nu}_1,\boldsymbol{\nu}'}$ with $h_1 \leq \bar{h}$, $\sigma_1 = \pm$ and $k_1 < k$.

If $h_{\ell_0} < h$ one has $N_h(\theta) = N_h(\theta_1) + \cdots + N_h(\theta_m)$, so that the bound $N_h(\theta) \leq K(\theta)E_h^{-1}$ follows by the inductive hypothesis.

If $h_{\ell_0} \geq h$ one has $N_h(\theta) = 1 + N_h(\theta_1) + \cdots + N_h(\theta_m)$. For $m = 0$ the bound can be obtained once more from Lemma 6.11, while for $m \geq 2$ at least one tree, say θ_m, belongs to $\Theta^{(k')\pm}_{R,\boldsymbol{\nu}'}$ for some k' and $\boldsymbol{\nu}'$ so that we can apply Lemma 6.7 and the inductive hypothesis to obtain

$$\begin{aligned} N_h(\theta) &\leq 1 + (K(\theta_1) + \cdots + K(\theta_{m-1}))\,E_h^{-1} + \left(K(\theta_m)E_h^{-1} - 1\right) \\ &\leq (K(\theta_1) + \cdots + K(\theta_{m-1}))\,E_h^{-1} + K(\theta_m)E_h^{-1} \leq K(\theta)E_h^{-1}, \end{aligned}$$

which yields the bound.

Finally if $m = 1$ one has $N_h(\theta) = 1 + N_h(\theta_1)$. Hence, if $\ell_1 \notin \mathcal{P}(\ell_e, \ell_0)$, again the bound follows from Lemma 6.7. If on the contrary $\ell_1 \in \mathcal{P}(\ell_e, \ell_0)$, one can adapt the discussion of the case $m = 1$ in the proof of Lemma 6.7. □

Lemma 6.13. *There exist positive constants κ, ξ_1 and D_1 such that, if $\xi > \xi_1$ in Definition 4.10, then for all trees $\theta \in \mathcal{R}_R$ and for all $(\varepsilon, M) \in \widetilde{\mathfrak{D}}(\theta,\gamma) \cap \mathfrak{Z}(\theta,\gamma)$, by setting $\boldsymbol{\nu} = \boldsymbol{\nu}'_{\ell_0}$ and $\boldsymbol{\nu}' = \boldsymbol{\nu}_{\ell_e}$, one has*

$$|\mathrm{Val}(\theta)| \le D_1^k 2^{-h} \mathrm{e}^{-\kappa|\boldsymbol{\nu}-\boldsymbol{\nu}'|^\rho} \prod_{\substack{\ell\in L(\theta)\\ i_\ell=1}} p_{\boldsymbol{\nu}_\ell}^{-(\xi-\xi_1)}(\varepsilon), \tag{6.6a}$$

$$|\partial_\varepsilon \mathrm{Val}(\theta)| \le D_1^k 2^{-h} p_{\boldsymbol{\nu}}^{c_0}(\varepsilon)\, \mathrm{e}^{-\kappa|\boldsymbol{\nu}-\boldsymbol{\nu}'|^\rho} \prod_{\substack{\ell\in L(\theta)\\ i_\ell=1}} p_{\boldsymbol{\nu}_\ell}^{-(\xi-\xi_1)}(\varepsilon), \tag{6.6b}$$

$$\sum_{\boldsymbol{\nu}_1\in\mathfrak{S}} \sum_{\boldsymbol{\nu}_2\in\mathcal{C}_{\boldsymbol{\nu}_1}} \sum_{\sigma_1,\sigma_2=\pm} \left|\partial_{M^{\sigma_1,\sigma_2}_{\boldsymbol{\nu}_1,\boldsymbol{\nu}_2}} \mathrm{Val}(\theta)\right| \le D_1^k 2^{-h} \mathrm{e}^{-\kappa|\boldsymbol{\nu}-\boldsymbol{\nu}'|^\rho} \prod_{\substack{\ell\in L(\theta)\\ i_\ell=1}} p_{\boldsymbol{\nu}_\ell}^{-(\xi-\xi_1)}(\varepsilon), \tag{6.6c}$$

with ρ as in Lemma 6.10.

Proof. Set for simplicity $\mathcal{P} = \mathcal{P}(\ell_e, \ell_0)$ and

$$\Sigma(\theta) = \sum_{v\in V_0(\theta)} |m_v| + \sum_{\ell\in L_q(\theta)} |\boldsymbol{\nu}_\ell - \boldsymbol{\nu}'_\ell| + \sum_{v\in E(\theta)} |\boldsymbol{\nu}_v|,$$

$$\Pi(\theta) = \left(\prod_{v\in V_0(\theta)} \mathrm{e}^{A_2|m_v|/8}\right)\left(\prod_{\ell\in L_q(\theta)} \mathrm{e}^{\lambda_0|\boldsymbol{\nu}_\ell-\boldsymbol{\nu}'_\ell|}\right)\left(\prod_{v\in E(\theta)} \mathrm{e}^{\lambda_0|\boldsymbol{\nu}_v|}\right).$$

If $\theta \in \mathcal{R}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$ for some $k \ge 1$, $h \ge 1$, $\sigma, \sigma' = \pm$ and $\{\boldsymbol{\nu}, \boldsymbol{\nu}'\}$ resonant, then $N_h(\theta) \ge 1$, so that $K(\theta) = k + \Sigma(\theta) > C 2^{h\beta/2\tau}$, for some constant C, which imply $1 \le 2^{-h} C^k \Pi(\theta)$, for some constant C. This produces the extra factor 2^{-h}.

By Item 2 in Lemma 6.10 one has $(B_3^{-1}|\boldsymbol{\nu}-\boldsymbol{\nu}'|)^\rho \le K(\theta)$, so that $1 \le \mathrm{e}^{-|\boldsymbol{\nu}-\boldsymbol{\nu}'|^\rho} C^k \Pi(\theta)$, for some constant C. The factor $\Pi(\theta)$ can be bounded by using part of the factors $\mathrm{e}^{-A_2|m_v|}$, $\mathrm{e}^{-\lambda_0|\boldsymbol{\nu}_v|}$, and $\mathrm{e}^{-\lambda_0|\boldsymbol{\nu}_\ell-\boldsymbol{\nu}'_\ell|}$, associated with the nodes and with the q-lines. This proves the bound (6.6a),

To prove the bound (6.6b) one has to take into account the further ε-derivative acting on the line propagator g_ℓ, for some $\ell \in L(\theta)$. If the line ℓ does not belong to $\mathcal{P}$ then one can reason as in the proof of (6.4b) in Lemma 6.8. If $\ell \in \mathcal{P}$ one has to distinguish between two cases. If there exists a line $\bar\ell \in \mathcal{P}$ such that $i_{\bar\ell} = 1$, then $K(\theta) > B_2 p_{\boldsymbol{\nu}}^{\beta/2}(\varepsilon)$ by Item 1 in Lemma 6.10, so that, by Item 2 in Lemma 6.6, one has $p_{\boldsymbol{\nu}_\ell}(\varepsilon) \le |\boldsymbol{\nu}'_\ell| \le B(|\boldsymbol{\nu}_{\ell_e}| + K(\theta))^{1+4\alpha} \le B(2p_{\boldsymbol{\nu}}(\varepsilon) + K(\theta))^{1+4\alpha} \le C(K(\theta))^{4/\beta}$, for some constant C. If $i_\ell = 0$ for all lines $\ell \in \mathcal{P}$ then, by Item 3 in Lemma 6.6, one has $p_{\boldsymbol{\nu}_\ell}(\varepsilon) \le |\boldsymbol{\nu}'_\ell| \le |\boldsymbol{\nu}_{\ell_e}| + B(K(\theta))^2$. Then Item 3 in Hypothesis 1 implies the bound (6.6b).

To prove (6.6c) one has to study a sum of terms each containing a derivative $\partial_{M^{\sigma_1,\sigma_2}_{\boldsymbol{\nu}_1,\boldsymbol{\nu}_2}} g_\ell$, for some $\ell \in L(\theta)$. If $\ell \in \mathcal{P}$ we distinguish between the two cases. If $K(\theta) > B_2 p_{\boldsymbol{\nu}}^{\beta/2}(\varepsilon)$, the sum over $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2$ has the limitations $|\boldsymbol{\nu}_1 - \boldsymbol{\nu}_2| \le C p_{\boldsymbol{\nu}_1}^{\alpha+\beta}(\varepsilon)$, $|\boldsymbol{\nu}_1 - \boldsymbol{\nu}_\ell| \le C p_{\boldsymbol{\nu}_1}^{\alpha+\beta}(\varepsilon)$ and $|\boldsymbol{\nu}_\ell| \le (|\boldsymbol{\nu}_{\ell_e}| + BK(\theta))^{1+4\alpha} \le C(K(\theta))^{4/\beta}$, for some constant C: hence the sum over $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2$ produces a factor $C(K(\theta))^{C'}$ for suitable constants C and C', and one has $(K(\theta))^{C'} \le C^k \Pi(\theta)$, for some constant C. If $K(\theta) \le B_2 p_{\boldsymbol{\nu}}^{\beta/2}(\varepsilon)$, then $i_\ell = 0$ for all lines $\ell \in \mathcal{P}$, so that the line propagators g_ℓ do not depend on M. Finally if $\ell \notin \mathcal{P}$ then one has $|\boldsymbol{\nu}_\ell| \le B(K(\theta))^{1+4\alpha}$, so that the sum over $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2$ is bounded once more proportionally to $(K(\theta))^{C'}$, for some constant C', and again one can bound $(K(\theta))^{C'} \le C^k \Pi(\theta)$, for some constant C. □

Remark 6.14. Both Lemmas 6.12 and 8.2 deal with the first derivatives of $\mathrm{Val}(\theta)$. One can easily extend the analysis so to include derivatives of arbitrary order, at the price of allowing larger constants ξ_1 and D_1 — and a factor $p_{\boldsymbol{\nu}}^{c_0}(\varepsilon)$ for any further ε-derivative. Therefore, one can prove that the function $\mathrm{Val}(\theta)$ is C^r for any integer r, in particular for $r = 1$, which we shall need in the following — cf. in particular the forthcoming Lemma 7.2.

7. Proof of Proposition 1

Definition 7.1. (The extended tree values). *Let the function χ_{-1} be as in Definition 5.1. Define*

$$\mathrm{Val}^E(\theta) = \left(\prod_{\substack{\ell\in L(\theta)\\ i_\ell=1}} \chi_{-1}(|x_{\boldsymbol{\nu}_\ell}(\varepsilon)|\, p^{\tau}_{\boldsymbol{\nu}_\ell}(\varepsilon)) \right) \left(\prod_{\substack{\ell\in L(\theta)\\ i_\ell=1}} \prod_{\boldsymbol{\nu}\in\mathcal{C}_{\{\boldsymbol{\nu}_\ell,\boldsymbol{\nu}'_\ell\}}} \chi_{-1}(||\delta_{\boldsymbol{\nu}}(\varepsilon)| - \bar{\gamma}|\, |\boldsymbol{\nu}|^{\tau_1}) \right)$$
$$\times \left(\prod_{\substack{\ell\in L_p(\theta)\\ i_\ell=0}} \chi_{-1}(||\delta_{\boldsymbol{\nu}_\ell}(\varepsilon)| - \bar{\gamma}|\, |\boldsymbol{\nu}_\ell|^{\tau_1}) \right) \mathrm{Val}(\theta) \tag{7.1}$$

for $\theta \in \Theta^{(k)}_{R,\boldsymbol{\nu}}$, and

$$\mathrm{Val}^E(\theta) = \left(\prod_{\substack{\ell\in L(\theta)\setminus\{\ell_0,\ell_e\}\\ i_\ell=1}} \chi_{-1}(|x_{\boldsymbol{\nu}_\ell}(\varepsilon)|\, p^{\tau}_{\boldsymbol{\nu}_\ell}(\varepsilon)) \right)$$
$$\times \left(\prod_{\substack{\ell\in L(\theta)\setminus\{\ell_0,\ell_e\}\\ i_\ell=1}} \prod_{\boldsymbol{\nu}\in\mathcal{C}_{\{\boldsymbol{\nu}_\ell,\boldsymbol{\nu}'_\ell\}}} \chi_{-1}(||\delta_{\boldsymbol{\nu}}(\varepsilon)| - \bar{\gamma}|\, p^{\tau_1}_{\boldsymbol{\nu}_\ell}(\varepsilon)) \right)$$
$$\times \left(\prod_{\substack{\ell\in L_p(\theta)\\ i_\ell=0,\ \boldsymbol{\nu}_\ell\notin\mathcal{C}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}}} \chi_{-1}(||\delta_{\boldsymbol{\nu}_\ell}(\varepsilon)| - \bar{\gamma}|\, |\boldsymbol{\nu}_\ell|^{\tau_1}) \right) \mathrm{Val}(\theta) \tag{7.2}$$

for $\theta \in \mathcal{R}^{(k)}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$. We call $\mathrm{Val}^E(\theta)$ *the* **extended value** *of the tree θ.*

The following result proves Proposition 1.

Lemma 7.2. *Given $\theta \in \mathcal{R}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$, the function* $\mathrm{Val}(\theta)$ *can be extended to the function (7.1) defined and C^1 in $\mathfrak{D}_0\backslash\mathcal{I}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}(\gamma)$, such that, defining the "extended" counterterm $L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}$ according to Definition 5.21, with* $\mathrm{Val}(\theta)$ *replaced with* $\mathrm{Val}^E(\theta)$*, the following holds:*

1. *Possibly with different constants* ξ_1 *and* K_0, $\mathrm{Val}^E(\theta)$ *satisfies for all* $(\varepsilon, M) \in \mathfrak{D}_0 \backslash \mathcal{I}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}(\gamma)$ *the same bounds in Lemma* 6.13 *as* $\mathrm{Val}(\theta)$ *in* $\mathfrak{D}(\gamma)$.
2. *There exist constants* ξ_1, K_1, κ, ρ *and* η_0, *such that, if* $\xi > \xi_1$ *in Definition* 4.10, $L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}$ *satisfies, for all* $(\varepsilon, M) \in \mathfrak{D}_0 \backslash \mathcal{I}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}(\gamma)$ *and* $|\eta| \le \eta_0$, *the bounds*

$$\left| L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'} \right| \le |\eta|^N K_1 \mathrm{e}^{-\kappa|\boldsymbol{\nu}-\boldsymbol{\nu}'|^{\rho}}, \qquad \left| \partial_{\varepsilon} L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'} \right| \le |\eta|^N K_1 p_{\boldsymbol{\nu}}^{c_0} \mathrm{e}^{-\kappa|\boldsymbol{\nu}-\boldsymbol{\nu}'|^{\rho}},$$
$$\left| \partial_{\eta} L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'} \right| \le N\,|\eta|^{N-1} K_1 \mathrm{e}^{-\kappa|\boldsymbol{\nu}-\boldsymbol{\nu}'|^{\rho}},$$
$$\sum_{\boldsymbol{\nu}_1 \in \mathfrak{S}, \sigma_1=\pm} \ \sum_{\boldsymbol{\nu}_2 \in \mathcal{C}_{\boldsymbol{\nu}_1}, \sigma_2=\pm} \left| \partial_{M^{\sigma_1,\sigma_2}_{\boldsymbol{\nu}_1,\boldsymbol{\nu}_2}} L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'} \right| \mathrm{e}^{\kappa|\boldsymbol{\nu}-\boldsymbol{\nu}'|^{\rho}} \le |\eta|^N K_1.$$

3. $\mathrm{Val}^E(\theta) = \mathrm{Val}(\theta)$ *for* $(\varepsilon, M) \in \mathfrak{D}(2\gamma)$ *and* $\mathrm{Val}^E(\theta) = 0$ *for* $(\varepsilon, M) \in \mathfrak{D}_0 \backslash \mathfrak{D}(\gamma)$.

Analogously, given $\theta \in \Theta^{(k)\sigma}_{R,\boldsymbol{\nu}}$, *the function* $\mathrm{Val}(\theta)$ *can be extended to the function* (7.2) *defined and* C^1 *in* $\mathfrak{D}_0$, *such that, defining* $u^{E\,(k)}_{\boldsymbol{\nu}}$ *as in Lemma* 5.24 *with* $\mathrm{Val}(\theta)$ *replaced with* $\mathrm{Val}^E(\theta)$, *the following holds:*

1. *Possibly with different constants* ξ_1 *and* K_0, $\mathrm{Val}^E(\theta)$ *satisfies for all* $(\varepsilon, M) \in \mathfrak{D}_0$ *the same bounds in Lemma* 6.8 *as* $\mathrm{Val}(\theta)$ *in* $\mathfrak{D}(\gamma)$.
2. *There exist constants* ξ_1, K_1, κ *and* η_0 *such that, if* $\xi > \xi_1$ *in Definition* 4.10, $u^{E\,\sigma}_{\boldsymbol{\nu}}$ *satisfies, for all* $(\varepsilon, M) \in \mathfrak{D}_0$ *and* $|\eta| \le \eta_0$, *the bounds*

$$\left| u^{E\,\sigma}_{\boldsymbol{\nu}} \right| \le |\eta|^N K_1 \mathrm{e}^{-\kappa|\boldsymbol{\nu}|^{1/2}}$$

for all $\boldsymbol{\nu} \in \mathbb{Z}^{D+1}$.

3. $\mathrm{Val}^E(\theta) = \mathrm{Val}(\theta)$ *for* $(\varepsilon, M) \in \mathfrak{D}(2\gamma)$ *and* $\mathrm{Val}^E(\theta) = 0$ *for* $(\varepsilon, M) \in \mathfrak{D}_0 \backslash \mathfrak{D}(\gamma)$.

Proof. We shall consider explicitly the case of trees $\theta \in \mathcal{R}^{(k)\sigma,\sigma'}_{R,h,\boldsymbol{\nu},\boldsymbol{\nu}'}$. The case of trees $\theta \in \Theta^{(k)\sigma}_{R,\boldsymbol{\nu}}$ can be discussed in the same way.

Item 3 follows from the very definition. The bounds of Item 1 can be proved by reasoning as in Sect. 6, by taking into account the further derivatives which arise because of the compact support functions χ_{-1} in (7.2). On the other hand all such derivatives produce factors proportional to $p^{a_2}_{\boldsymbol{\nu}_\ell}(\varepsilon)$ for some constant a_2 (again we refer to [24] for details); in particular we are using Item 2 in Hypothesis 1 to bound the derivatives of $\delta_{\boldsymbol{\nu}_\ell}(\varepsilon)$ with respect to ε. Therefore by using Lemma 6.7 and possibly taking larger constants ξ_1 and K_0 the bounds of Lemma 6.13 follow also for the extended function (7.2).

Finally the bounds on L^E in Item 2 come directly from the definition. Indeed, the counterterms $L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}$ are expressed in terms of the values $\mathrm{Val}(\theta)$ according to Remark 5.23, and the factor 2^{-h} is used to perform the summation over the scale labels. Hence we have to control the sum over the trees.

Let us fix ε. For each $v \in E(\theta)$ the sum over $|\boldsymbol{\nu}_v|$ is controlled by using the exponential factors $\mathrm{e}^{-\lambda_0|\boldsymbol{\nu}_v|}$. For each line $\ell \in L(\theta)$ the labels $\boldsymbol{\nu}'_\ell$ are fixed by the conservation rule of Item 12 in Definition 5.7, while the sum over $\boldsymbol{\nu}_\ell$ gives a factor $C_1 p^{\alpha}_{\boldsymbol{\nu}_\ell}(\varepsilon)$ for the p-lines (see Item 2 in Hypothesis 3), and it is controlled by using the exponential factors $\mathrm{e}^{-\lambda_0|\boldsymbol{\nu}_\ell-\boldsymbol{\nu}'_\ell|}$ for the q-lines. The sums over i_ℓ and h_ℓ can be bounded by a factor 4. Finally the sum over all the unlabelled trees of order k is bounded by C^k for some constant C. Thus, the bounds on $L^E_{\boldsymbol{\nu},\boldsymbol{\nu}'}$ are proved.

Finally, the C^1 smoothness follows from Remark 6.14. □

8. Proof of Proposition 2

The following result proves Items 1 and 2 in Proposition 2. Here and henceforth we write $L = L(\eta, \varepsilon, M)$ and $L^E = L^E(\eta, \varepsilon, M)$, and we fix $\eta = \varepsilon^{1/N}$.

Lemma 8.1. *There exists constants $\varepsilon_0 > 0$ such that there exist functions $M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon) = M^{\sigma,\sigma'}_{\boldsymbol{\nu}',\boldsymbol{\nu}}(\varepsilon)$ well defined and C^1 for $\varepsilon \in [0, \varepsilon_0]\backslash\overline{\mathcal{I}}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}(\gamma)$, such that the "extended" compatibility equation*

$$M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon) = L^{E\,\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon^{1/N}, \varepsilon, M(\varepsilon))$$

holds for all $\varepsilon \in (0, \varepsilon_0)\backslash\overline{\mathcal{I}}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}(\gamma)$. The functions $u^{E\,\sigma}_{\boldsymbol{\nu}}(\varepsilon^{1/N}, \varepsilon, M(\varepsilon))$ are C^1 in $[0, \varepsilon_0]$.

Proof. By definition we set $M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon) = 0$ for all ε such that $\bar{\chi}_1(\delta_{\boldsymbol{\nu}}(\varepsilon))\bar{\chi}_1(\delta_{\boldsymbol{\nu}'}(\varepsilon)) = 0$. Consider the Banach space $\overline{\mathcal{B}}$ of lists $\{M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon)\}$, with $\{\boldsymbol{\nu}, \boldsymbol{\nu}'\}$ a resonant pair, such that each $M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon)$ is well defined and C^1 in $\varepsilon \in [0, \varepsilon_0]\backslash\overline{\mathcal{I}}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}(\gamma)$, and $M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon) = 0$ for $\varepsilon \in \overline{\mathcal{I}}_{\{\boldsymbol{\nu},\boldsymbol{\nu}'\}}(\gamma)$.

By definition $\{L^{E\,\sigma_1,\sigma_2}_{\boldsymbol{\nu}_1,\boldsymbol{\nu}_2}(\varepsilon^{1/N}, \varepsilon, \{M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}(\varepsilon)\})\}$ is well defined as a continuously differentiable application from $\overline{\mathcal{B}}$ in itself, since, for each tree $\theta \in \mathcal{R}^{(k)\sigma_1,\sigma_2}_{R,h,\boldsymbol{\nu}_1,\boldsymbol{\nu}_2}$, the value $\mathrm{Val}^E(\theta)$ by definition smooths out to zero the value of each line propagator g_ℓ in the corresponding intervals $\overline{\mathcal{I}}_{\{\boldsymbol{\nu}_\ell,\boldsymbol{\nu}'_\ell\}}(2\gamma)\backslash\overline{\mathcal{I}}_{\{\boldsymbol{\nu}_\ell,\boldsymbol{\nu}'_\ell\}}(\gamma)$. Again by definition $L^E(0,0,0) = 0$ and $|\partial_M L(0,0,0)|_{\mathrm{op}} = 0$, so that we can apply the implicit function theorem. Analogously one discusses the smoothness of the functions $u^{E\,\sigma}_{\boldsymbol{\nu}}(\varepsilon^{1/N}, \varepsilon, M(\varepsilon))$. □

Lemma 8.2. *Let $A = A(\varepsilon)$ be a self-adjoint matrix piecewise differentiable in the parameter ε. Then, if $\lambda^{(i)}(A)$ and $\phi^{(i)}(A)$ denote the eigenvalues and the (normalised) eigenvectors of A, respectively, the following holds:*

1. *One has $|\lambda^{(i)}(A(\varepsilon))| \le \|A(\varepsilon)\|_2$.*
2. *The eigenvalues $\lambda^{(i)}(A(\varepsilon))$ are piecewise differentiable in ε.*
3. *One has $|\partial_\varepsilon\lambda^{(i)}(A(\varepsilon))| \le \|\partial_\varepsilon A(\varepsilon)\|_2$.*

Proof. See [26] for Items 1 and 2. Moreover, for each interval in which A is differentiable, let A_n be an analytic approximation of A in such an interval, with $A_n \to A$ as $n \to \infty$: then the eigenvalues $\phi^{(i)}(A_n)$ are piecewise differentiable [26], and one has

$$\begin{aligned}\partial_\varepsilon\lambda^{(i)}(A_n) &= \partial_\varepsilon\left(\phi^{(i)}, A_n\phi^{(i)}\right) = \lambda^{(i)}(A_n)\partial_\varepsilon\left(\phi^{(i)}, \phi^{(i)}\right) + \left(\phi^{(i)}, \partial_\varepsilon A_n\phi^{(i)}\right) \\ &= \left(\phi^{(i)}, \partial_\varepsilon A_n\phi^{(i)}\right),\end{aligned}$$

which yields Item 3 when the limit $n \to \infty$ is taken. □

For $M \in \mathcal{B}_{\kappa,\rho}$ we can write $\mathscr{M} = \widehat{\chi}_1 M \widehat{\chi}_1 = \bigoplus_j \mathscr{M}_j$, where $\mathscr{M}_j$ are block matrices, so that we can define $\|\mathscr{M}\|_2 = \sup_j \|\mathscr{M}_j\|_2$, with $\|\mathscr{M}_j\|_2$ given as in Definition 4.8.

Lemma 8.3. *For $M \in \mathcal{B}_{\kappa,\rho}$ one has $\|\mathscr{M}\|_2 \le C\varepsilon_0$ for some constant C depending on κ and ρ.*

Proof. If $M \in \mathcal{B}_{\kappa,\rho}$ then $\mathscr{M} = \bigoplus_j \mathscr{M}_j$, with $\mathscr{M}_j$ a block matrix with dimension d_j depending on j, and $\mathscr{M}_j(i,i') = M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}$, for suitable $\boldsymbol{\nu},\boldsymbol{\nu}',\sigma,\sigma'$ such that $|M^{\sigma,\sigma'}_{\boldsymbol{\nu},\boldsymbol{\nu}'}| \le D\varepsilon_0 \mathrm{e}^{-\kappa|\boldsymbol{\nu}-\boldsymbol{\nu}'|^\rho}$ for some constant D. Therefore

$$
\begin{aligned}
\|\mathscr{M}_j\|_2^2 =& \max_{|x|_2\le 1} \left|\mathscr{M}_j x\right|_2^2 \le \max_{|x|_2\le 1} \sum_{i,i',i''=1}^{d_j} \left|\mathscr{M}_j(i,i')\right| \left|x(i')\right| \left|\mathscr{M}_j(i,i'')\right| \left|x(i'')\right| \\
\le& \frac{1}{2} \max_{|x|_2\le 1} \sum_{i,i',i''=1}^{d_j} \left|\mathscr{M}_j(i,i')\right| \left|\mathscr{M}_j(i,i'')\right| \left(\left|x(i')\right|^2 + \left|x(i'')\right|^2\right) \\
\le& \max_{|x|_2\le 1} \sum_{i=1}^{d_j} \left|\mathscr{M}_j(i,i')\right| \sum_{i''=1}^{d_j} \left|\mathscr{M}_j(i,i'')\right| \sum_{i'=1}^{d_j} \left|x(i')\right|^2 \le \left(\sum_{i=1}^{d_j} \left|\mathscr{M}_j(i,i'')\right|\right)^2,
\end{aligned}
$$

which yields the assertion. □

Lemma 8.4. *Let A, B be two self-adjoint $d \times d$ matrices. Then*

$$
\left|\lambda^{(i)}(A+B) - \lambda^{(i)}(A)\right| \le \sum_{j=1}^{d} \left|\lambda^{(j)}(B)\right|
$$

for all $i = 1, \dots, d$.

Proof. The result follows from Lidskii's Lemma; cf. [26]. □

Define $\mathfrak{E}_1 = \{\varepsilon \in [0,\varepsilon_0] : x_{\boldsymbol{\nu}}(\varepsilon) \ge 2\gamma/p^\tau_{\boldsymbol{\nu}}(\varepsilon)\ \forall \boldsymbol{\nu} \in \mathfrak{S}\}$ and $\mathfrak{E}_2 = \{\varepsilon \in [0,\varepsilon_0] : ||\delta_{\boldsymbol{\nu}}(\varepsilon)| - \bar{\gamma}| \ge 2\gamma/|\boldsymbol{\nu}|^{\tau_1}\ \forall \boldsymbol{\nu} \in \mathfrak{S}\}$, and set $\mathfrak{E} = \mathfrak{E}_1 \cap \mathfrak{E}_2$.

We can denote by $\lambda^\sigma_{\boldsymbol{\nu}}(A)$, with $\boldsymbol{\nu} \in \mathfrak{S}$ and $\sigma = \pm$, the eigenvalues of the block matrix $A = \mathscr{D} + \mathscr{M}$. If $|\delta_{\boldsymbol{\nu}}(\varepsilon)| \ge \bar{\gamma}$, then $\lambda^\sigma_{\boldsymbol{\nu}}(\varepsilon) = \delta_{\boldsymbol{\nu}}(\varepsilon)$. Moreover for each $\varepsilon \in [0,\varepsilon_0]$ and each $\boldsymbol{\nu} \in \mathfrak{S}$ such that $|\delta_{\boldsymbol{\nu}}(\varepsilon)| < \bar{\gamma}$, there exists a block $A^{\boldsymbol{\nu}}(\varepsilon)$ of the matrix A, of size $d_{\boldsymbol{\nu}}(\varepsilon) \le 2C_1 p^\alpha_{\boldsymbol{\nu}}(\varepsilon)$ such that $\lambda^\pm_{\boldsymbol{\nu}}(A)$ depends only on the entries of such a block. This follows from Lemma 5.24 and Remark 5.4.

Therefore we have to discard from $[0,\varepsilon_0]$ only values of ε such that $|\delta_{\boldsymbol{\nu}}(\varepsilon)| < \bar{\gamma}$ for some $\boldsymbol{\nu} \in \mathfrak{S}$: for all such $\boldsymbol{\nu}$ the matrix $A^{\boldsymbol{\nu}}(\varepsilon)$ is well defined, and one has $\lambda^\sigma_{\boldsymbol{\nu}}(A) = \lambda^\sigma_{\boldsymbol{\nu}}(A_{\boldsymbol{\nu}}(\varepsilon))$.

One has, by Item 3 in Lemma 4.9,

$$
x_{\boldsymbol{\nu}}(\varepsilon) \ge \frac{1}{p^\xi_{\boldsymbol{\nu}}(\varepsilon)} \min_{i=1,\dots,d_{\boldsymbol{\nu}}(\varepsilon)} \left|\lambda^{(i)}(A^{\boldsymbol{\nu}}(\varepsilon))\right| \ge \frac{1}{p^\xi_{\boldsymbol{\nu}}(\varepsilon)} \min_{\boldsymbol{\nu}'\in\overline{\mathcal{C}}_{\boldsymbol{\nu}}(\varepsilon)} \min_{\sigma=\pm} \left|\lambda^\sigma_{\boldsymbol{\nu}'}(A^{\boldsymbol{\nu}}(\varepsilon))\right|, \tag{8.1}
$$

so that, by using that $\lambda^\sigma_{\boldsymbol{\nu}'}(A^{\boldsymbol{\nu}}(\varepsilon)) = \lambda^\sigma_{\boldsymbol{\nu}'}(A^{\boldsymbol{\nu}'}(\varepsilon)) = \lambda^\sigma_{\boldsymbol{\nu}'}(A)$ for all $\boldsymbol{\nu}' \in \overline{\mathcal{C}}_{\boldsymbol{\nu}}(\varepsilon)$, we shall impose the conditions

$$
\left|\lambda^\sigma_{\boldsymbol{\nu}}(A^{\boldsymbol{\nu}}(\varepsilon))\right| \ge \frac{\gamma_2}{|\boldsymbol{\nu}|^{\tau_2}}, \qquad \boldsymbol{\nu} \in \mathfrak{S}, \qquad \sigma = \pm, \tag{8.2}
$$

for suitable $\gamma_2 > 2\gamma$. Thus, the conditions (8.2), together with the bound $|\boldsymbol{\nu}| \le 2p_{\boldsymbol{\nu}}(\varepsilon)$ (cf. Remark 6.3), will imply through (8.1) the bounds (6.3) for $x_{\boldsymbol{\nu}}(\varepsilon)$.

Define

$$\mathfrak{K}_{\boldsymbol{\nu}}^{\sigma} = \left\{ \varepsilon \in [0, \varepsilon_0] : \left| \lambda_{\boldsymbol{\nu}}^{\sigma}(A) \right| \le \frac{\gamma_2}{|\boldsymbol{\nu}|^{\tau_2}} \right\}, \qquad \boldsymbol{\nu} \in \mathfrak{S}, \qquad \sigma = \pm, \tag{8.3}$$

with $\tau_2 = \tau - \xi$, so that we can estimate

$$\text{meas}([0, \varepsilon_0] \backslash \mathfrak{E}_1) \le \sum_{\boldsymbol{\nu} \in \mathfrak{S}} \sum_{\sigma = \pm} \text{meas}(\mathfrak{K}_{\boldsymbol{\nu}}^{\sigma}). \tag{8.4}$$

Moreover, by defining

$$\mathfrak{H}_{\boldsymbol{\nu},\sigma} = \left\{ \varepsilon \in [0, \varepsilon_0] : |\delta_{\boldsymbol{\nu}}(\varepsilon) - \sigma \bar{\gamma}| \le \frac{2\gamma}{|\boldsymbol{\nu}|^{\tau_1}} \right\}, \qquad \boldsymbol{\nu} \in \mathcal{C}_j, \quad j \in \mathbb{N}, \quad \sigma = \pm, \tag{8.5}$$

with τ_1 to be determined, one has

$$\text{meas}([0, \varepsilon_0] \backslash \mathfrak{E}_2) \le \sum_{\boldsymbol{\nu} \in \mathbb{Z}^{D+1}} \sum_{\sigma = \pm} \text{meas}(\mathfrak{H}_{\boldsymbol{\nu},\sigma}). \tag{8.6}$$

Lemma 8.5. *There exist constants w_0 and w_1 such that $\mathfrak{K}_{\boldsymbol{\nu}}^{\pm} = \emptyset$ for all $\boldsymbol{\nu}$ such that $|\boldsymbol{\nu}| \le w_0/\varepsilon_0^{w_1}$. There exist constants y_0 and y_1 such that $\mathfrak{H}_{\boldsymbol{\nu},\pm} = \emptyset$ for all $\boldsymbol{\nu}$ such that $|\boldsymbol{\nu}| \le y_0/\varepsilon_0^{y_1}$.*

Proof. We start by considering the sets $\mathfrak{K}_{\boldsymbol{\nu}}^{\sigma}$ for $\boldsymbol{\nu} \in \mathfrak{S}$ and $\sigma = \pm$. If $|\delta_{\boldsymbol{\nu}}(\varepsilon)| < \bar{\gamma}$ one can write $A^{\boldsymbol{\nu}}(\varepsilon) = \text{diag}\{\delta_{\boldsymbol{\nu}'}(0), \delta_{\boldsymbol{\nu}'}(0)\}_{\boldsymbol{\nu}' \in \overline{\mathcal{C}}_{\boldsymbol{\nu}}(\varepsilon)} + B^{\boldsymbol{\nu}}(\varepsilon)$, which defines the matrix $B^{\boldsymbol{\nu}}(\varepsilon)$ as

$$B^{\boldsymbol{\nu}}(\varepsilon) = \text{diag}\{\delta_{\boldsymbol{\nu}'}(\varepsilon) - \delta_{\boldsymbol{\nu}'}(0)\}_{\boldsymbol{\nu}' \in \overline{\mathcal{C}}_{\boldsymbol{\nu}}(\varepsilon)}^{\sigma = \pm} + M^{\boldsymbol{\nu}}(\varepsilon),$$

where $M^{\boldsymbol{\nu}}(\varepsilon)$ is the block of $M(\varepsilon)$ with entries $M_{\boldsymbol{\nu}_1,\boldsymbol{\nu}_2}^{\sigma_1,\sigma_2}(\varepsilon)$ such that $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2 \in \overline{\mathcal{C}}_{\boldsymbol{\nu}}(\varepsilon)$. By Lemma 8.4, one has

$$\left| \lambda_{\boldsymbol{\nu}}^{\sigma}(A) - \delta_{\boldsymbol{\nu}}(0) \right| \le \sum_{i=1}^{d_{\boldsymbol{\nu}}(\varepsilon)} \left| \lambda^{(i)}(B^{\boldsymbol{\nu}}(\varepsilon)) \right| \le 2C_1 p_{\boldsymbol{\nu}}^{\alpha}(\varepsilon) \| B^{\boldsymbol{\nu}}(\varepsilon) \|_2, \qquad \boldsymbol{\nu} \in \mathfrak{S}, \qquad \sigma = \pm, \tag{8.7}$$

where we have used Remark 4.11 to bound $d_{\boldsymbol{\nu}}(\varepsilon)$.

One has $|\delta_{\boldsymbol{\nu}}(0)| \ge \gamma_0/|\boldsymbol{\nu}|^{\tau_0} \ge \gamma_0/(2p_{\boldsymbol{\nu}}(\varepsilon))^{\tau_0}$ by Item 2 in Hypothesis 1, whereas $\|B^{\boldsymbol{\nu}}(\varepsilon)\|_2 \le c_2(2p_{\boldsymbol{\nu}}(\varepsilon))^{c_0}\varepsilon_0 + \|M^{\boldsymbol{\nu}}(\varepsilon)\|_2$, by Items 1 and 2 in Hypothesis 1, and $\|M^{\boldsymbol{\nu}}(\varepsilon)\|_2 \le \|M(\varepsilon)\|_2 \le C_0\varepsilon_0$ by Lemma 8.3. Therefore (8.7) implies

$$\left| \lambda_{\boldsymbol{\nu}}^{\sigma}(A) \right| \ge \frac{\gamma_0}{(2p_{\boldsymbol{\nu}}(\varepsilon))^{\tau_0}} - C p_{\boldsymbol{\nu}}^{c_0+1}(\varepsilon)\, \varepsilon_0,$$

for a suitable constant C, so that, by setting $w_1 = c_0 + 1 + \tau_0$ and choosing suitably the constants γ_2, τ and w_0, one has $|\lambda_{\boldsymbol{\nu}}^{\pm}(A)| \ge \gamma_0/2(2p_{\boldsymbol{\nu}}(\varepsilon))^{\tau_0} \ge \gamma_2/p_{\boldsymbol{\nu}}^{\tau_2}(\varepsilon)$ for all $\boldsymbol{\nu}$ such that $|\boldsymbol{\nu}| \le w_0/\varepsilon_0^{w_1}$.

For the sets $\mathfrak{H}_{\boldsymbol{\nu},\sigma}$, one can reason in the same way, by using that $\bar{\gamma} \in \mathfrak{G}$ (cf. Definition 4.1). □

Lemma 8.6. *Let $\xi > \xi_1$ and $\varepsilon_0 = \eta_0^N$ be fixed as in Lemma 7.2. There exist constants γ, τ and τ_1 such that* $\text{meas}([0, \varepsilon_0] \backslash \mathfrak{E}) = o(\varepsilon_0)$.

Proof. First of all we have to discard from $[0, \varepsilon_0]$ the sets $\mathfrak{H}_{\boldsymbol{\nu},\sigma}$. It is easy to see that one has

$$\mathrm{meas}(\mathfrak{H}_{\boldsymbol{\nu},\sigma}) \le \frac{2\gamma}{|\boldsymbol{\nu}|^{\tau_1}} \frac{2}{c_1|\boldsymbol{\nu}|^{c_0}},$$

for some positive constant C, so that, by using the second assertion in Lemma 8.5, we find

$$\sum_{\boldsymbol{\nu}\in\mathfrak{S}} \sum_{\sigma=\pm 1} \mathrm{meas}(\mathfrak{H}_{\boldsymbol{\nu},\sigma}) \le \sum_{\substack{\boldsymbol{\nu}\in\mathfrak{S} \\ |\boldsymbol{\nu}|\ge y_0/\varepsilon_0^{y_1}}} \sum_{\sigma=\pm 1} \mathrm{meas}(\mathfrak{H}_{\boldsymbol{\nu},\sigma}) \le C\varepsilon_0^{y_1(\tau_1+c_0-D-1)},$$

for some constant C, provided $\tau_1 + c_0 - D > 1$, so that we shall require for τ_1 to be such that $\tau_1 + c_0 - D > 1$ and $y_1(\tau_1 + c_0 - D - 1) > 1$.

Next, we consider the sets $\mathfrak{K}_{\boldsymbol{\nu}}^{\pm}$. For all $\boldsymbol{\nu} \in \mathfrak{S}$ consider $A^{\boldsymbol{\nu}}(\varepsilon)$ and write $A^{\boldsymbol{\nu}}(\varepsilon) = \delta_{\boldsymbol{\nu}}(\varepsilon)I + B^{\boldsymbol{\nu}}(\varepsilon)$, which defines the matrix $B^{\boldsymbol{\nu}}(\varepsilon)$ as

$$B^{\boldsymbol{\nu}}(\varepsilon) = \mathrm{diag}\{\delta_{\boldsymbol{\nu}'}(\varepsilon) - \delta_{\boldsymbol{\nu}}(\varepsilon)\}^{\sigma=\pm}_{\boldsymbol{\nu}\in\overline{\mathcal{C}}_{\boldsymbol{\nu}}(\varepsilon)} + M^{\boldsymbol{\nu}}(\varepsilon),$$

with $M^{\boldsymbol{\nu}}(\varepsilon)$ defined as in the proof of Lemma 8.5.

Then the eigenvalues of $A^{\boldsymbol{\nu}}(\varepsilon)$ are of the form $\lambda^{(i)}(A^{\boldsymbol{\nu}}(\varepsilon)) = \delta_{\boldsymbol{\nu}}(\varepsilon) + \lambda^{(i)}(B^{\boldsymbol{\nu}}(\varepsilon))$, so that for all $\varepsilon \in [0, \varepsilon_0]\backslash\overline{\mathcal{I}}_{\boldsymbol{\nu}}(\gamma)$ one has

$$\left|\partial_\varepsilon \lambda^{(i)}(A^{\boldsymbol{\nu}})\right| \ge |\partial_\varepsilon \delta_{\boldsymbol{\nu}}(\varepsilon)| - \left\|\partial_\varepsilon B^{\boldsymbol{\nu}}(\varepsilon)\right\|_2,$$

where Item 3 in Lemma 8.2 has been used. One has $|\partial_\varepsilon\delta_{\boldsymbol{\nu}}(\varepsilon)| \ge c_1|\boldsymbol{\nu}|^{c_0}$, by Item 2 in Hypothesis 1, and $\|\partial_\varepsilon B^{\boldsymbol{\nu}}(\varepsilon)\|_2 \le \max_{\boldsymbol{\nu}'\in\overline{\mathcal{C}}_{\boldsymbol{\nu}}(\varepsilon)} |\partial_\varepsilon(\delta_{\boldsymbol{\nu}'}(\varepsilon) - \delta_{\boldsymbol{\nu}}(\varepsilon))| + \|\partial_\varepsilon M^{\boldsymbol{\nu}}(\varepsilon)\|_2 \le \zeta c_3 p_{\boldsymbol{\nu}(\varepsilon)}\, p_{\boldsymbol{\nu}}^{c_0-1}(\varepsilon) + \varepsilon_0 C p_{\boldsymbol{\nu}}^{c_0}(\varepsilon)$, for a suitable constant C, as follows from Item 4 in Hypothesis 1, from Hypothesis 3 (see Lemma 2.9 for the definition of ζ), from Lemma 7.2, and from Lemma 8.1. Hence we can bound $|\partial_\varepsilon\lambda^{(i)}(A^{\boldsymbol{\nu}})| \ge c_1|\boldsymbol{\nu}_0|^{c_0}/2$ for ε_0 small enough.

Therefore one has

$$\mathrm{meas}(\mathfrak{K}_{\boldsymbol{\nu}}^{\sigma}) \le \frac{2\gamma_2}{|\boldsymbol{\nu}|^{\tau_2}} \frac{2}{c_1|\boldsymbol{\nu}|^{c_0}} \left(\overline{C}|\boldsymbol{\nu}|^{(\alpha+\beta)(D+1)}\right), \tag{8.8}$$

for some constant $\overline{C}$, where the last factor $\overline{C}|\boldsymbol{\nu}|^{(\alpha+\beta)(D+1)}$ arises for the following reason. The eigenvalues $\lambda_{\boldsymbol{\nu}}^{\sigma}(A)$ are differentiable in ε except for those values ε such that for some $\boldsymbol{\nu}' \in \mathcal{C}_{\boldsymbol{\nu}}$ one has $|\delta_{\boldsymbol{\nu}'}(\varepsilon)| = \bar{\gamma}$ and $|\delta_{\boldsymbol{\nu}}(\varepsilon)| < \bar{\gamma}$. Because of Item 3 in Hypothesis 1 all functions $\delta_{\boldsymbol{\nu}'}(\varepsilon)$ are monotone in ε as far as $|\delta_{\boldsymbol{\nu}'}(\varepsilon)| < 1/2$, so that for each $\boldsymbol{\nu}' \in \mathcal{C}_{\boldsymbol{\nu}}$ the condition $|\delta_{\boldsymbol{\nu}'}(\varepsilon)| = \bar{\gamma}$ can occur at most twice. The number of $\boldsymbol{\nu}' \in \mathcal{C}_{\boldsymbol{\nu}}$ such that the conditions $|\delta_{\boldsymbol{\nu}'}(\varepsilon)| = \bar{\gamma}$ and $|\delta_{\boldsymbol{\nu}}(\varepsilon)| < \bar{\gamma}$ can occur for some $\varepsilon \in [0, \varepsilon_0]$ is bounded by the volume of a sphere of centre $\boldsymbol{\nu}$ and radius proportional to $|\boldsymbol{\nu}|^{\alpha+\beta}$ (cf. Lemma 5.24). Hence $\overline{C}|\boldsymbol{\nu}|^{(\alpha+\beta)(D+1)}$ counts the number of intervals in $[0, \varepsilon_0]\backslash\overline{\mathcal{I}}_{\boldsymbol{\nu}}(\gamma)$.

Thus, (8.8) yields, by making use of the first assertion of Lemma 8.5,

$$\sum_{\boldsymbol{\nu}\in\mathfrak{S}} \sum_{\sigma=\pm} \mathrm{meas}(\mathfrak{K}_{\boldsymbol{\nu}}^{\sigma}) \le \sum_{\substack{\boldsymbol{\nu}\in\mathbb{Z}^{D+1} \\ |\boldsymbol{\nu}|\ge w_0/\varepsilon_0^{w_1}}} \frac{8\gamma}{c_1}|\boldsymbol{\nu}|^{-\tau_2-c_0}\left(\overline{C}|\boldsymbol{\nu}|^{2\alpha}\right) \le C\varepsilon_0^{w_1(\tau_2+c_0-2\alpha-D-1)},$$

for some positive constant C, provided $\tau_2 + c_0 - 2\alpha - D = \tau + c_0 - 2\alpha - D - \xi > 1$, so that (8.4) implies that $\mathrm{meas}([0, \varepsilon_0]\backslash\mathfrak{E}_1) \le C\varepsilon_0^{w_1(\tau_2+c_0-D-1)}$.

Therefore, the assertion follows provided $\min\{\tau_1, \tau_2 - 2\alpha\} > D - c_0 + 1$, $y_1(\tau_1 + c_0 - D - 1) > 1$ and $w_1(\tau_2 + c_0 - 2\alpha - D - 1) > 1$. $\square$

A. Proof of Lemma 3.4

Lemma 3.4 is a consequence of the following elementary proposition in Galois theory.

Proposition. *If $p_1, \ldots, p_k$ are distinct primes then the field*

$$F := \mathbb{Q}[\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_k}]$$

obtained from the rational numbers $\mathbb{Q}$ by adding the k square roots $\sqrt{p_i}$ has dimension 2^k over $\mathbb{Q}$ with basis the elements $\prod_{i \in I} \sqrt{p_i}$ as I varies on the 2^k subsets of $\{1, 2, \ldots, k\}$.

The group of automorphisms[2] *of F which fix $\mathbb{Q}$ (i.e. the Galois group of $F/\mathbb{Q}$) is an Abelian group generated by the automorphisms τ_i defined by $\tau_i(\sqrt{p_j}) = (-1)^{\delta(i,j)}\sqrt{p_j}$.*

Proof. We prove by induction both statements. Let us assume the statements valid for $p_1, \ldots, p_{k-1}$ and let $F' := \mathbb{Q}[\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_{k-1}}]$ so that $F = F'[\sqrt{p_k}]$. We first prove that $\sqrt{p_k} \notin F'$. Assume it to be false. Since $(\sqrt{p_k})^2$ is integer, each element – say τ – of the Galois group of $F'/\mathbb{Q}$ must either fix $\sqrt{p_k}$ or transform it into $-\sqrt{p_k}$ (by definition $\tau(p_k) = \tau(\sqrt{p_k})^2 = p_k$).

Now any element $b \in F'$ is by induction uniquely expressed as

$$b = \sum_{I \subset \{1,2,\ldots,k-1\}} a_I \prod_{i \in I} \sqrt{p_i}, \qquad a_I \in \mathbb{Q}.$$

If h of the numbers a_I are non-zero, it is easily seen that b has 2^h transforms (changing the signs of each of the a_I) under the Galois group of F'. Therefore $b = \sqrt{p_k}$ if and only if $h = 1$, that is one should have $\sqrt{p_k} = m/n \prod_{i \in I} \sqrt{p_i}$, $I \subset \{1, 2, \ldots, k-1\}$ for m, n integers. This implies that $p_k n^2 = m^2 \prod_{i \in I} p_i$ which is impossible by the unique factorisation of integers. This proves the first statement.

To construct the Galois group of $F/\mathbb{Q}$ we extend the action of τ_i for $i = 1, \ldots, k-1$ by setting $\tau_i(\sqrt{p_k}) = \sqrt{p_k}$. Finally we define the automorphism τ_k as $\tau_k(\sqrt{p_j}) = (-1)^{\delta(k,j)}\sqrt{p_j}$ for $j = 1, \ldots, k$. □

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References

1. Ablowitz, M., Prinari, B.: Nonlinear Schrodinger systems: continuous and discrete. Scholarpedia **3**(8), 5561 (2008)
2. Artin, M.: *Algebra*. Englewood Cliffs, NJ: Prentice Hall, 1991
3. Baldi, P., Berti, M.: Periodic solutions of nonlinear wave equations for asymptotically full measure sets of frequencies. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei Mat. Appl. **17**(3), 257–277 (2006)
4. Berti, M., Bolle, Ph.: Cantor families of periodic solutions for completely resonant nonlinear wave equations. Duke Math. J. **134**(2), 359–419 (2006)
5. Berti, M., Bolle, Ph.: Cantor families of periodic solutions of wave equations with C^k nonlinearities. Nonlinear Diff. Eqs. Appl. **15**(1–2), 247–276 (2008)
6. Berti, M., Bolle, Ph.: Cantor families of periodic solutions for wave equations via a variational principle. Adv. Math. **217**(4), 1671–1727 (2008)
7. Berti, M., Bolle, Ph.: *Sobolev periodic solutions of nonlinear wave equations in higher spatial dimensions*. Preprint, 2008

[2] I.e. the linear transformations τ such that $\tau(uv) = \tau(u)\tau(v)$.

8. Bourgain, J.: Construction of periodic solutions of nonlinear wave equations in higher dimension. Geom. Funct. Anal. **5**, 629–639 (1995)
9. Bourgain, J.: Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equations. Ann. of Math. **148**(2), 363–439 (1998)
10. Bourgain, J.: Periodic solutions of nonlinear wave equations. In: *Harmonic analysis and partial differential equations (Chicago, IL, 1996)*, Chicago Lectures in Mathematics, Chicago, IL: University Chicago Press, 1999, pp. 69–97
11. Bourgain, J.: *Green's Function Estimates for Lattice Schrödinger Operators and Applications*. Ann. Math. Studies **158**, Princeton, NJ: Princeton University Press, 2005
12. Bruno, A.D.: Analytic form of differential equations. I, II. Trudy Moskov. Mat. Obšč. **25**, 119–262 (1971); ibid. **26**, 199–239 (1972)
13. Buslaev, V.S., Perelman, G.S.: Nonlinear scattering: the states which are close to a soliton. J. Math. Sci. **77**(3), 3161–3169 (1995)
14. Cheng, Ch.-Q.: Birkhoff-Kolmogorov-Arnold-Moser tori in convex Hamiltonian systems. Commun. Math. Phys. **177**(3), 529–559 (1996)
15. Chierchia, L., You, J.: KAM tori for 1D nonlinear wave equations with periodic boundary conditions. Commun. Math. Phys. **211**(2), 497–525 (2000)
16. Craig, W., Wayne, C.E.: Newton's method and periodic solutions of nonlinear wave equations. Comm. Pure Appl. Math. **46**, 1409–1498 (1993)
17. Eliasson, L.H., Kuksin, S.: *KAM for non-linear Schrödinger equation*. Ann. of Math., to appear, available at http://pjm.math.berkeley.edu/editorial/uploads/annals/accepted/080510-Eliasson/080510-Eliasson-v2.pdf
18. Gallavotti, G.: Twistless KAM tori. Commun. Math. Phys. **164**(1), 145–156 (1994)
19. Geng, J., You, J.: A KAM theorem for Hamiltonian partial differential equations in higher dimensional spaces. Commun. Math. Phys. **262**(2), 343–372 (2006)
20. Geng, J., You, J.: KAM tori for higher dimensional beam equations with constant potentials. Nonlinearity **19**(10), 2405–2423 (2006)
21. Gentile, G., Mastropietro, V.: Construction of periodic solutions of nonlinear wave equations with Dirichlet boundary conditions by the Lindstedt series method. J. Math. Pures Appl. (9) **83**(8), 1019–1065 (2004)
22. Gentile, G., Mastropietro, V., Procesi, M.: Periodic solutions for completely resonant nonlinear wave equations with Dirichlet boundary conditions. Commun. Math. Phys. **256**(2), 437–490 (2005)
23. Gentile, G., Procesi, M.: Conservation of resonant periodic solutio.ns for the one-dimensional nonlinear Schrödinger equation. Commun. Math. Phys. **262**(3), 533–553 (2006)
24. Gentile, G., Procesi, M.: Periodic solutions for the Schrödinger equation with nonlocal smoothing nonlinearities in higher dimension. J. Diff. Eqs. **245**(11), 3095–3544 (2008)
25. Gang, Zh., Sigal, I.M.: Relaxation of solitons in nonlinear Schrödinger equations with potential. Adv. Math. **216**(2), 443–490 (2007)
26. Kato, T.: *Perturbation Theory for Linear Operators*. Berlin-New York: Springer-Verlag, 1976
27. Kuksin, S.B.: *Nearly Integrable Infinite-Dimensional Hamiltonian Systems*. Lecture Notes in Mathematics **1556**, Berlin: Springer-Verlag, 1993
28. Kuksin, S.B., Pöschel, J.: Invariant Cantor manifolds of quasi-periodic oscillations for a nonlinear Schrödinger equation. Ann. of Math. **143**(1), 149–179 (1996)
29. Moser, J.: Convergent series expansions for quasi-periodic motions. Math. Ann. **169**, 136–176 (1967)
30. Pöschel, J.: Quasi-periodic solutions for a nonlinear wave equation. Comment. Math. Helv. **71**(2), 269–296 (1996)
31. Pöschel, J.: On the construction of almost periodic solutions for a nonlinear Schrödinger equation. Erg. Th. Dynam. Syst. **22**(5), 1537–1549 (2002)
32. Procesi, M.: Quasi-periodic solutions for completely resonant non-linear wave equations in 1D and 2D. Discrete Contin. Dyn. Syst. **13**(3), 541–552 (2005)
33. Siegel, C.L.: Iteration of analytic functions. Ann. of Math. **43**, 607–612 (1942)
34. Soffer, A., Weinstein, M.I.: Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations. Invent. Math. **136**, 9–74 (1999)
35. Wayne, C.E.: Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory. Commun. Math. Phys. **127**(3), 479–528 (1990)
36. Yuan, X.: Quasi-periodic solutions of completely resonant nonlinear wave equations. J. Diff. Eqs. **230**(1), 213–274 (2006)

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