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A normal form for beam and non-local nonlinear Schrödinger equations

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Abstract

We discuss normal forms of the completely resonant nonlinear beam equation and nonlinear Schrödinger equation. We work in $n > 1$ spatial dimensions and study both periodic and Dirichlet boundary conditions on cubes. We discuss the applications to the problem of finding quasi-periodic solutions. In the case of periodic boundary and the dimension $n = 2$, we apply KAM theory and prove the existence and stability of quasi-periodic solutions.

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1. Introduction

In this paper we discuss an approximate normal form for the completely resonant nonlinear beam and nonlinear Schrödinger equation (NLS for short):

$$v_{tt} + \Delta^2 v = \kappa v^3 \tag{1}$$

$$\mathrm{i}u_t - \Delta u = \kappa|\Phi_s(u)|^2\Phi_s(u), \tag{2}$$

either for $x \in \mathbb{T}^n$ (periodic boundary conditions) or with Dirichlet boundary conditions on the cube $D = [0, \pi]^n$.

In equation (2) the symbol Φ_s, $s \geqslant 0$, is a linear smoothing operator

$$\Phi_s\, \mathrm{e}^{\mathrm{i}k\cdot x} = |k|^{-s}\, \mathrm{e}^{\mathrm{i}k\cdot x},$$

so equation (2) reduces to the standard local NLS for $s = 0$.

It is well known that equations (1) and (2) are infinite-dimensional Hamiltonian systems with an elliptic fixed point at $v = 0$, so one wishes to apply in this infinite-dimensional setting the powerful tools of Birkhof normal form and KAM theory.

One must note that both (1) and (2) are completely resonant, namely in the linear part the frequencies are rational and all the bounded solutions are *periodic*. One expects small

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periodic, quasi-periodic (and hopefully almost periodic) solutions to occur due to the presence of the nonlinear term. To prove suitable 'non-degeneracy' assumptions one generally performs a step of resonant Birkhof normal form. There are essentially two problems.

(1) Hamiltonians (1) and (2) after the normal form step appear to have a very intricate behavior. Already in the case of finite-dimensional systems this may pose serious problems in proving the existence of invariant tori.
(2) Finding quasi-periodic solutions for nonlinear PDEs in $n > 1$ spatial dimension requires solving rather delicate small divisor problems related to the high multiplicity of the eigenvalues of the Laplacian Δ.

In this paper we focus on problem 1. As an application we prove the existence and stability of quasi-periodic solutions in the case of $n = 2$ and periodic boundary conditions.

1.1. Some background

There are two main methods for constructing quasi-periodic solutions: Newton methods—proposed by Craig and Wayne in [3] and extended by Bourgain (see [1, 2]) to PDEs in $n > 1$ dimensions—and KAM theory, first developed for one-dimensional Hamiltonian PDEs with Dirichlet boundary conditions (see [10, 13]).

For PDEs in a dimension higher than 1, most of the results are restricted to simplified non-resonant models, obtained by adding a Fourier multiplier to the Laplacian. For example in equation (1)

$$v_{tt} + (\Delta + M_\sigma)^2 v = f(x, v), \tag{3}$$

where M_σ is a linear operator, depending on a finite number—say m—of free parameters σ, which commutes with the Laplacian.

A KAM algorithm to prove the existence and stability of quasi-periodic solutions for the non-resonant beam (3) and non-local ($s > 0$) NLS on a torus was proposed in [5]. The authors use conservation of momentum and the fact that the nonlinearity is regularizing, to simplify the second-order Melnikov non-resonance condition.

The extension to the much more complex local NLS ($s = 0$) is due to Eliasson and Kuksin in [4].

To extend the results of [5] to the completely resonant cases (1) and (2), object of this paper, we perform one step of Birkhof normal form to introduce parameters. Consider a Hamiltonian

$$H = H^{(2)}(p, q) + H^{(4)}(p, q), \quad H^{(2)}(p, q) = \sum_k \lambda_k^2 \big(p_k^2 + q_k^2\big),$$

where $H^{(4)}$ is a polynomial of degree 4 and the λ_k are all rational numbers. A step of 'Birkhof normal form' is a symplectic change of variables which reduces the Hamiltonian H to

$$H_N = H^{(2)}(p, q) + H^{(4)}_{res}(p, q) + H^{(6)},$$

where $H^{(6)}$ is an analytic function of degree at least 6, $H^{(4)}_{res}$ is of degree 4 and Poisson commutes with $H^{(2)}$. Then one wishes to treat $H^{(2)}(p, q) + H^{(4)}_{res}(p, q)$ as the new unperturbed Hamiltonian and $H^{(6)}$ as a small perturbation. This can work provided that $H^{(2)}(p, q) + H^{(4)}_{res}(p, q)$ is simple enough (possibly completely integrable) and has quasi-periodic motions for large classes of initial data (which now play the role of the parameters σ). An ideal situation is when λ_k are non-resonant up to order 4 so that the normal form is integrable.

In the case of the beam equation with a positive 'mass term' (M_σ in (3) is substituted by a constant $\mu > 0$), Geng and You in [6] proved the existence of quasi-periodic solutions.

The main point in their paper is to show that the mass term simplifies the normal form so that they can apply the results in [5]. In the case of dimension $n = 2$ Geng, You and Xu proved the existence of quasi-periodic solutions for equation (2) with $s = 0$ by a combination of non-integrable normal form, momentum conservation (in the spirit of [5]) and the ideas of Eliasson and Kuksin; indeed they show that in the special dimension 2 and in the context of (2) (no explicit spacial dependence) many of the difficulties of [4] simplify. Their approach is based on a non-integrable normal form and hence they do not obtain the linear stability.

A related result appears in [8] and [9] where similar techniques allow us to construct wave packets of periodic solutions for the beam and NLS equations in any dimension. The method of these papers was a combination of Lyapunov–Schmidt decomposition and Lindstedt series. In those papers we were able to deal also with Dirichlet boundary conditions for which the normal form is much more complicated.

1.2. Results

Divide the oscillators into two suitable subsets, the *tangential* and the *normal* sites, with the strategy of analyzing the equations near the solutions in which the normal sites are at rest and the tangential sites move quasi-periodically on an m-torus. This is done by *doubling* the action variables in the tangential sites, so each action variable is written as $\xi_i + y_i$, where ξ_i are the free parameters and y_i are the symplectic dynamical variables coupled with the *angles* x_i.

In theorem 1 we prove that for infinitely many choices of the tangential sites, the Hamiltonians of (1) and (2), after a step of normal form, are

$$H = (\omega(\xi), y) + Q_M(\xi, x, \{z_k, \bar{z}_k\}) + P,$$

where Q is a block-diagonal quadratic form in the 'normal variables' $\{z_k, \bar{z}_k\}$ with coefficients trigonometric polynomials in x. A main point is that the dimensions of the blocks in Q are uniformly bounded. Finally P is a small perturbation.

This result is rather surprising for $n > 2$ especially in the case of Dirichlet boundary conditions. Indeed the 'normal form Hamiltonian' is very complicated, see section 3, and the block-diagonal structure is not easily recognized. The main idea is to transform theorem 1 into an abstract combinatorial setting, see lemmas 3.10 and 3.11.

In the case of *periodic boundary conditions* one can obtain much finer results. Namely one can prove that there exists a symplectic change of variables which reduces the quadratic normal form to an integrable one (no dependence of the angle variables x); this is discussed in [12].

We then specialize to the case $n = 2$ and periodic boundary conditions. With this restriction in theorem 2 we prove the existence and linear stability of quasi-periodic solutions of (1) and (2).

More precisely in proposition 4.1 we apply a KAM algorithm, following [7]; this implies the existence of quasi-periodic solutions. To prove the stability we reduce the normal form to constant coefficients and show that it is elliptic for appropriate choices of ξ.

2. Hamiltonian formalism

We sketch the Hamiltonian structure of the equations, see for instance [5] for a full presentation. We restrict our attention to the invariant subspace

$$\mathcal{U} := \left\{ u \in L^2(\mathbb{R} \times \mathbb{T}^n; \mathbb{C}) |\, u(t, x) = \sum_{k \in \mathbb{Z}^n_{\mathrm{odd}}} u_k(t)\, \mathrm{e}^{\mathrm{i}k\cdot x} \right\}$$

$$\mathbb{Z}^n_{\text{odd}} := \{k = (k_1, \dots, k_n) |\ k_1 = 2h_1, \dots, k_{n-1} = 2h_{n-1}, k_n = h_n,\ h_i \text{ odd}\}.$$

It is easily seen that on this subspace $u_0 = 0$ so $-\Delta$ is positive definite and $(-\Delta)^{-\frac{1}{2}} = \Phi_1$.

We write the beam equation in complex coordinates by setting

$$\sqrt{2}u(x,t) = (-\Delta)^{\frac{1}{2}} v(x,t) + \mathrm{i}(-\Delta)^{-\frac{1}{2}} v_t(x,t).$$

We obtain

$$\mathrm{i}u_t = \Delta u + \tfrac{1}{2}\Phi_1[(\Phi_1(u) + \Phi_1(\bar u))^3]. \tag{4}$$

We work on the scale of complex Hilbert spaces

$$\bar\ell^{(a,p)} := \left\{ u = \{u_k\}_{k\in\mathbb{Z}^n_{\text{odd}}} \,\middle|\, \sum_{k\in\mathbb{Z}^n} |u_k|^2\, \mathrm{e}^{2a|k|}|k|^{2p} := \|u\|^2_{a,p} \leqslant \infty \right\}, \tag{5}$$

where $a > 0$, $p > n/2$. This space is symplectic, with the conjugate variables $\{\mathrm{i}u_k, \bar u_k\}$, with respect to the complex symplectic form $\mathrm{i}\sum_k \mathrm{d}u_k \wedge \mathrm{d}\bar u_k$.

While studying the case of Dirichlet boundary conditions, we expand the solutions in sin Fourier series; this is equivalent to expanding in the exponentials and restricting to the subspace

$$\mathcal{D} := \left\{\{u_k\}_{k\in\mathbb{Z}^n_{\text{odd}}} |\ u_k = -u_{\sigma_i(k)}\right\} \sigma_i(k) = (k_1, \dots, -k_i, \dots, k_n),$$

which is invariant for the dynamics. Hence in this case the dynamical variables are $\{\mathrm{i}u_k, \bar u_k\}_{k\in\mathbb{Z}^n_{\text{odd}}(+)}$, where $\mathbb{Z}^n_{\text{odd}}(+) \equiv \mathbb{Z}^n_{\text{odd}} \cap \mathbb{Z}^n_+$. Let G be the group with 2^n elements generated by the reflections σ_i; by our assumption on the set $\mathbb{Z}^n_{\text{odd}}$ each orbit of this group has exactly 2^n points. We say that two elements h and k are equivalent $h \equiv k$ if they are in the same G orbit. $\mathbb{Z}^n_{\text{odd}}(+)$ can be identified with $\mathbb{Z}^n_{\text{odd}}$ with the equivalence relation $\equiv$. It is convenient to denote indifferently by $k \in \mathbb{Z}^n_{\text{odd}}(+)$ either the point in $\mathbb{Z}^n_{\text{odd}} \cap \mathbb{Z}^n(+)$ or the orbit

$$\left\{k^{(1)}, \dots, k^{(2^n)}\right\}, \quad k^{(i)} \in \mathbb{Z}^n_{\text{odd}}, \quad k^{(i)} \equiv k^{(j)}, \qquad \text{where} \quad k^{(1)} \in \mathbb{Z}^n(+).$$

2.1. *The Hamiltonians: periodic boundary conditions*

In (2) we can rescale u to avoid the constant κ and pass to the Fourier coefficients and obtain an equation for $\{u_k\}$: $\dot u_k = \mathrm{i}\partial_{\bar u_k} H$, where

$$H := \sum_{k\in\mathbb{Z}^n_{\text{odd}}} |k|^2 u_k \bar u_k + \sum_{\substack{k_i\in\mathbb{Z}^n_{\text{odd}}\\ k_1+k_3-k_2-k_4=0}} \frac{u_{k_1}\bar u_{k_2} u_{k_3}\bar u_{k_4}}{|k_1|^s|k_2|^s|k_3|^s|k_4|^s}, \tag{NLS}$$

$$H := \sum_{k\in\mathbb{Z}^n_{\text{odd}}} |k|^2 u_k \bar u_k + \sum_{\substack{s_i=\pm1,\, k_i\in\mathbb{Z}^n_{\text{odd}}\\ s_1k_1+s_2k_2+s_3k_3+s_4k_4=0}} \frac{u^{s_1}_{k_1} u^{s_2}_{k_2} u^{s_3}_{k_3} u^{s_4}_{k_4}}{|k_1|^s|k_2|^s|k_3|^s|k_4|^s}, \tag{Be}$$

where $u^+_k = u_k$ and $u^-_k = \bar u_k$.

Since both (2) and (1) do not depend explicitly on the spatial variable, the *total momentum* $M = \sum_{k\in\mathbb{Z}^n_{\text{odd}}} k|u_k|^2$ is a conserved quantity.

2.2. *The Hamiltonians: Dirichlet boundary conditions*

We restrict to the subspace $\mathcal{D}$ and use $\{\mathrm{i}u_k, \bar u_k\}$ as symplectic variables with $k \in \mathbb{Z}^n_{\text{odd}}(+)$; we obtain (as usual after rescaling)

$$H := \sum_{k\in\mathbb{Z}^n_{\text{odd}}(+)} |k|^2 u_k \bar u_k + 2^{-n} \sum_{\substack{k_i\in\mathbb{Z}^n_{\text{odd}}(+), a_i\\ k_1^{(a_1)}-k_2^{(a_2)}+k_3^{(a_3)}-k_4^{(a_4)}=0}} \frac{u_{k_1^{(a_1)}}\bar u_{k_2^{(a_2)}} u_{k_3^{(a_3)}}\bar u_{k_4^{(a_4)}}}{|k_1|^s|k_2|^s|k_3|^s|k_4|^s}, \tag{NLS}$$

$$H := \sum_{k\in\mathbb{Z}^n_{\mathrm{odd}}(+)} |k|^2 u_k \bar{u}_k + 2^{-n} \sum_{\substack{k_i\in\mathbb{Z}^n_{\mathrm{odd}}, s_i=\pm1, a_i \\ \sum_{i=1}^4 s_i k_i^{(a_i)}=0}} \frac{u^{s_1}_{k_1^{(a_1)}} u^{s_2}_{k_2^{(a_2)}} u^{s_3}_{k_3^{(a_3)}} u^{s_4}_{k_4^{(a_4)}}}{|k_1||k_2||k_3||k_4|}. \tag{Be}$$

Recall that $k^{(a)}$ with $a = 1, \dots, 2^n$ are the elements of the orbit of k and $u_{k^{(a)}} = (-1)^{\alpha(a)} u_k$. The factor 2^{-n} reminds us that if a quadruple appears in the quartic term of H, then all the 2^n elements of the orbit of the quadruple also appear and give the same contribution. We could remove the factor by restricting to the quadruples where say $a_4 = 1$ (or we could rescale the factor away!).

2.3. Analytic Hamiltonians

We consider real Hamiltonians on the space $\bar{\ell}^{(a,p)}$ which can be formally expanded in the Taylor series

$$F = \sum_i F^{(i)}(u, \bar{u}) = \sum_i \sum_{\substack{(\alpha,\beta)\in\mathbb{N}^\infty \\ |\alpha+\beta|_1=i}} F^{(\alpha,\beta)} \prod_{k\in\mathbb{Z}^n} u_k^{\alpha_k} \bar{u}_k^{\beta_k},$$

where $F^{(i)}$ is a multilinear form of i variables in $\bar{\ell}^{(a,p)}$.

We require that the series is totally convergent in some ball of positive radius, so that by definition the function is analytic. Clearly our Hamiltonians belong to this class and are convergent on any ball B_r.

For compactness we will write them as

$$H := \sum_{k\in A} |k|^2 u_k \bar{u}_k + 2^{-\delta n} \sideset{}{^\circ}\sum_{k_i\in A, s_i=\pm1, a_i\in O} \frac{u^{s_1}_{k_1^{(a_1)}} u^{s_2}_{k_2^{(a_2)}} u^{s_3}_{k_3^{(a_3)}} u^{s_4}_{k_4^{(a_4)}}}{|k_1|^s |k_2|^s |k_3|^s |k_4|^s}, \tag{6}$$

where, respectively, for periodic and Dirichlet boundary conditions A is either $\mathbb{Z}^n_{\mathrm{odd}}$ or $\mathbb{Z}^n_{\mathrm{odd}}(+)$, δ is either 0 or 1 and O is either $\{1\}$ or $\{1, \dots, 2^n\}$. The expression $\sum^\circ$ means that we restrict to $\sum_{i=1}^4 s_i k_i^{(a_i)} = 0$ and in the case of the NLS also to $\sum_i s_i = 0$.

We will also be interested in the Hamiltonian vector fields $X_F = \{\partial_{\bar{u}_k} F\}_{k\in\mathbb{Z}^n}$, where F is an analytic Hamiltonian in the ball B_r. With this notation the Hamilton equations are $\mathrm{i}\dot{u}_k = X_F$. We will restrict our attention to the vector fields which map $B_r \to \bar{\ell}_{a,\bar{p}}$ with $\bar{p} = p + s > p$. If

$$|X_F|_r := \sup_{u\in B_r} \|X_F\|_{a,\bar{p}}/r < \frac{1}{2}, \tag{7}$$

the symplectic change of variables generated by F is well defined and analytic say from $B_{r/2} \to B_r$. Note that condition (7) is *not* verified by the quadratic part of our Hamiltonians but only by the higher order terms.

3. Normal form

For small u (i.e. $\|u\|_{a,p} < \epsilon \ll 1$) we perform a standard step of the Birkhof normal form removing all the terms of order 4 of H which do not Poisson-commute with the quadratic part $H^{(2)} := \sum_k |k|^2 |u_k|^2$.

In fact the monomial $u_{k_1}^{s_1}u_{k_2}^{s_2}u_{k_3}^{s_3}u_{k_4}^{s_4}$ is an eigenvector with respect to $\{H^{(2)}, -\}$ with the eigenvalue $\mathrm{i}(s_1|k_1|^2 + s_2|k_2|^2 + s_3|k_3|^2 + s_4|k_4|^2)$. Thus, we perform the symplectic change of variables $H \mapsto \mathrm{e}^{\{F,-\}}(H)$, generated by the flow of

$$F := -\mathrm{i}2^{-\delta n} \sum^{\circ}_{\sum_i s_i|k_i|^2\neq 0} \frac{u^{s_1}_{k_1^{(a_1)}} u^{s_2}_{k_2^{(a_2)}} u^{s_3}_{k_3^{(a_3)}} u^{s_4}_{k_4^{(a_4)}}}{(|k_1|^2 - |k_2|^2 + |k_3|^2 - |k_4|^2)|k_1|^s|k_2|^s|k_3|^s|k_4|^s}.$$

For ϵ sufficiently small, this is a well-known analytic change of variables (cf [1, 5]) $\bar{\ell}^{(a,p)} \supset B_\epsilon \to B_{2\epsilon} \subset \bar{\ell}^{(a,p)}$ (where B_ϵ denotes as usual the open ball of radius ϵ) which brings (6) to the form

$$H_N := \sum_{k\in A} |k|^2 u_k \bar{u}_k + 2^{-\delta n} \sum^{\circ}_{\mathrm{RES}} \frac{u^{s_1}_{k_1^{(a_1)}} u^{s_2}_{k_2^{(a_2)}} u^{s_3}_{k_3^{(a_3)}} u^{s_4}_{k_4^{(a_4)}}}{|k_1|^s|k_2|^s|k_3|^s|k_4|^s} + P^{(6)}(u), \tag{8}$$

where $\sum^{\circ}_{\mathrm{RES}}$ is the sum restricted to

$$s_1|k_1|^2 + s_2|k_2|^2 + s_3|k_3|^2 + s_4|k_4|^2 = 0, \qquad \sum_{i=1}^{4} s_i k_i^{(a_i)} = 0, \tag{9}$$

with the further restriction $\sum_i s_i = 0$ for the NLS. $P^{(6)}(u)$ is analytic of degree at least 6 in $u_k|k|^{-s}$. In a small ball B_ϵ, $P^{(6)}(u)$ is perturbative with respect to the terms of degrees 2 and 4.

Lemma 3.1. *In the case of periodic boundary conditions F commutes with M so that H_N still satisfies momentum conservation: namely a monomial $\prod_k u_k^{\alpha_k}\bar{u}_k^{\beta_k}$ appears in H_N only if $\sum_k k(\alpha_k - \beta_k) = 0$.*

In the case of Dirichlet boundary conditions we obtain the much weaker restriction that $\sum_k k^{(a_k)}(\alpha_k - \beta_k) = 0$ for some choice of $a_k = 1, \ldots, 2^n$.

Lemma 3.2. *The resonance condition (9) can be verified for the quadruples $k_1, \ldots, k_4$ only if $\sum_i s_i = 0$.*

Proof. We restate (9) without loss of generality as

$$s_1|k_1|^2 + s_2|k_2|^2 + s_3|k_3|^2 + s_4|k_4|^2 = 0, \qquad k_1 + k_2 + k_3 + k_4 = 0,$$

$k_i \in \mathbb{Z}^n_{\mathrm{odd}}$.

If all s_i are positive or negative, then we have no solution, in the same way if $\sum_i s_i = \pm 2$, we have

$$|k_1|^2 + |k_2|^2 + |k_3|^2 = |k_1 + k_2 + k_3|^2 \leftrightarrow \langle k_1, k_2 + k_3\rangle + \langle k_2, k_3\rangle = 0$$

which is impossible since the restriction $k_i \in \mathbb{Z}^n_{\mathrm{odd}}$ implies that the left-hand side is an odd integer. □

The previous lemma shows that the resonance condition is the same for the beam and NLS equation. For $(k_1, k_2, k_3, k_4) \mid k_i \in \mathbb{Z}^n_{\mathrm{odd}}$, denote

$$\mathcal{P}' := \{(k_1, k_2, k_3, k_4) | k_1 + k_3 = k_2 + k_4, |k_1|^2 + |k_3|^2 = |k_2|^2 + |k_4|^2\}.$$

Our resonance condition states that $(k_1, k_2, k_3, k_4) \in \mathcal{P}'$ for periodic boundary conditions and that $\left(k_1^{(a_1)}, k_2^{(a_2)}, k_3^{(a_3)}, k_4^{(a_4)}\right) \in \mathcal{P}'$ for Dirichlet boundary conditions.

Trivial computations show that $(k_1, k_2, k_3, k_4) \in \mathcal{P}'$ is equivalent to

$$k_1 + k_3 = k_2 + k_4, \qquad (k_1 - k_2, k_3 - k_2) = 0, \tag{10}$$

so that the integer vectors k_1, k_2, k_3, k_4 form the vertices of a *rectangle*. In $\mathcal{P}'$ some of the k_i may coincide; we wish to evidence this structure.

Definition 3.3. ***P.*** *In the case of periodic boundary conditions we say that a rectangle is degenerate if it reduces to a segment or to a point. We call $\mathcal{P}$ the subset of $\mathcal{P}'$ of non-degenerate rectangles.* ***D.*** *In the case of Dirichlet boundary conditions we say that a rectangle is degenerate if either it degenerates to a segment or to a point or if the vertices of the rectangle are all equivalent. As before we call $\mathcal{P}$ the set of non-degenerate rectangles.*

We can rewrite our Hamiltonians as

$$H_N = \sum_{k\in A} |k|^2 u_k \bar{u}_k + 3^{\delta n} \sum_k \frac{|u_k|^4}{|k|^{4s}} + 2^{\delta n+1} \sum_{k_1\neq k_2\in A} \frac{|u_{k_1}|^2}{|k_1|^{2s}} \frac{|u_{k_2}|^2}{|k_2|^{2s}}$$
$$+ 2^{-\delta n} \sum_{\substack{k_i\in A, a_i\in O \\ (k_1^{(a_1)}, k_2^{(a_2)}, k_3^{(a_3)}, k_4^{(a_4)})\in\mathcal{P}}} \frac{u_{k_1^{(a_1)}} \bar{u}_{k_2^{(a_2)}} u_{k_3^{(a_3)}} \bar{u}_{k_4^{(a_4)}}}{|k_1|^s |k_2|^s |k_3|^s |k_4|^s} + P^{(6)}(u), \tag{11}$$

where $A = \mathbb{Z}^n_{\text{odd}}$, $\delta = 0$ and $O = \{1\}$ in the case of *periodic* boundary, while $A = \mathbb{Z}^n_{\text{odd}}(+)$, $\delta = 1$, $O = \{1, \ldots, 2^n\}$ in the case of Dirichlet boundary. Note that in the case of Dirichlet boundary the sum $\left(k_1^{(a_1)}, k_2^{(a_2)}, k_3^{(a_3)}, k_4^{(a_4)}\right) \in \mathcal{P}$ contains terms in which the k_i are pairwise equivalent.

3.4. *Elliptic-action angle variables*

We are interested in non-symmetric domains $D \subset B_\epsilon$ where some of the variables u_k (the tangential sites) are bounded away from zero—and hence can be passed in action angle variables—while all the other variables are much smaller.

We first partition our index spaces

$$\mathbb{Z}^n_{\text{odd}} = S \cup S^c, \qquad \mathbb{Z}^n_{\text{odd}}(+) = S \cup S^c, \tag{12}$$

respectively, for periodic and Dirichlet boundary conditions.

The set $S := (v_1, \ldots, v_m)$ is called *tangential sites* and S^c the *normal sites*. In the case of Dirichlet boundary conditions we will denote by $\left\{v_i^{(a)}\right\}_{a=1}^{2^n}$ the orbit of $v_i \in S$.

We set

$$u_k := z_k \qquad \text{for} \quad k \in S^c, \qquad u_{v_i} := \sqrt{\xi_i + y_i}\, \mathrm{e}^{\mathrm{i}x_i} \qquad \text{for} \quad i = 1, \ldots m;$$

$(y, x) \times (z, \bar{z}) \in \mathbb{R}^m \times \mathbb{T}^m \times \ell^{(a,p)}$ where we denote by $\ell^{(a,p)}$ the subspace of $\bar{\ell}^{(a,p)} \times \bar{\ell}^{(a,p)}$ generated by the indices in S^c and $w = (z, \bar{z})$ (considered as the row vectors) are the corresponding coordinates; the symplectic form is $\mathrm{d}y \wedge \mathrm{d}x + \mathrm{i}\mathrm{d}z \wedge \mathrm{d}\bar{z}$.

We consider the domain

$$A_\alpha \times D(\rho, r) := \left\{\xi : \tfrac{1}{2} r^\alpha \leqslant \xi_i \leqslant r^\alpha\right\} \times \left\{x, y, w : x \in \mathbb{T}^m_\rho, |y| \leqslant r^2, |w|_{a,p} \leqslant r\right\}$$
$$\subset \mathbb{R}^m \times \mathbb{T}^m_\rho \times \mathbb{C}^m \times \ell^{(a,p)}. \tag{13}$$

Here, $0 < \alpha < \frac{1}{2}$, $0 < r < 1$, $0 < \rho < \frac{1}{2}$ are the parameters. $\mathbb{T}^m_\rho$ denotes the open subset of the complex torus, where $\mathrm{Re}(x) \in \mathbb{T}^m$, $|\mathrm{Im}(x)| < \rho$.

We are interested in the functions $F(y, x, w)$ which are analytic in $D(\rho, r)$. Following [11] we use Lipschitz sup-norm for the dependence on ξ:

$$\|F\|^\lambda_{\rho,r} = \sup_{A_\alpha\times D(\rho,r)} |F| + \lambda \sup_{D(\rho,r)} \sup_{\substack{\xi,\eta\in A_\alpha \\ \xi\neq\eta}} \frac{|F(\xi) - F(\eta)|}{|\xi - \eta|},$$

where $\lambda \propto r^{\alpha}/\max_i(|v_i|^2)$. For the Hamiltonian vector fields

$$X_F = \{\partial_y F, -\partial_x F, J d_w F\},$$

we use the weighted Lipschitz norm (cf (7))

$$|X_F|_{\rho,r} := \sup_{A_\alpha \times D(\rho,r)} \left(|\partial_y F|^{\lambda} + r^{-2}|\partial_x F|^{\lambda} + r^{-1}\|d_w F\|^{\lambda}_{a,\bar{p}}\right).$$

3.5. Choice of the tangential sites

We impose the following constraint on the set S.

Constraint 1. *Any three elements $v_i, v_j, v_l \in S$ satisfy*

$$\left\langle v_i^{(a)} - v_j^{(b)}, v_l - v_j^{(b)} \right\rangle = 0,$$

only if they are the vertices of a degenerate rectangle, i.e. in the unavoidable cases $v_i^{(a)} = v_j^{(b)}$ or $v_j^{(b)} = v_l$ or $v_i \equiv v_j \equiv v_l$.

Constraint 2. *We choose v_i so that $|v_i| \neq |v_j|$ for all $i \neq j$ and $\sum_i \nu_i v_i \neq 0$ when $\sum_i |\nu_i| \leqslant 8$, $\nu \neq 0$.*

The Hamiltonian (8) can be written as

$$H_N = H_0 + P^{(3)}(z, y; \xi, x) + P^{(6)}(z, y; \xi, x), \quad \text{where} \tag{14}$$

(1) the term $P^{(3)}$ collects all the terms of $H_N - P^{(6)}$ in which at least three k indices are in S^c and it is of degree at least 3 (and at most 4) in $z, \bar{z}$;

(2) by constraint 2, $P^{(6)}$ at $z = 0$ does not contain any term of degree $\leqslant 9$ in the elements u_{v_i} which is non-constant in x;

(3) by constraint 1, no non-degenerate rectangles $\mathcal{P}$ contain more than two elements of S, hence setting $I_i = \xi_i + y_i$:

$$\begin{aligned} H_0 := &\sum_{i=1}^{m} \left(|v_i|^2 I_i + \frac{3^{\delta n}}{|v_i|^{4s}} I_i^2 \right) + 2^{\delta n+2} \sum_{i<j} \frac{I_i I_j}{|v_i|^{2s}|v_j|^{2s}} + 2^{\delta n+2} \sum_{i;k\in S^c} \frac{I_i |z_k|^2}{|v_i|^{2s}|k|^{2s}} \\ &+ \sum_{k \in S^c} |k|^2 |z_k|^2 + 4 \sum_{i \neq j; h,k \in S^c}^{*} \frac{\sqrt{I_i I_j}\, \mathrm{e}^{\mathrm{i}(x_i - x_j)}}{|v_i|^s |v_j|^s |h|^s |k|^s} z_h \bar{z}_k \\ &+ 2 \sum_{i<j;h,k\in S^c}^{**} \frac{\sqrt{I_i I_j}\, \mathrm{e}^{-\mathrm{i}(x_i + x_j)}}{|v_i|^s |v_j|^s |h|^s |k|^s} z_h z_k + 2 \sum_{i<j;h,k\in S^c}^{**} \frac{\sqrt{I_i I_j)}\, \mathrm{e}^{\mathrm{i}(x_i + x_j)}}{|v_i|^s |v_j|^s |h|^s |k|^s} \bar{z}_h \bar{z}_k. \end{aligned}$$

Here, $\sum^{*}$ denotes the constraint $\left(h^{(a_1)}, k^{(a_2)}, v_i^{(a_3)}, v_j^{(a_4)}\right) \in \mathcal{P}$ for some $a_1, \ldots, a_4 \in \mathcal{O}$. In the same way $\sum^{**}$ denotes the constraint $\left(h^{(a_1)}, v_i^{(a_2)}, k^{(a_3)}, v_j^{(a_4)}\right) \in \mathcal{P}$ for some $a_1, \ldots, a_4 \in \mathcal{O}$.

Remark 3.6. Let us give a geometric interpretation to the constraints:
P. In periodic boundary $*$ means that k belongs to the hyperplane $\mathcal{H}_{i,j}$ defined by $\langle x - v_i, v_i - v_j \rangle = 0$ and $h = k + v_j - v_i$ (hence $h \in \mathcal{H}_{j,i}$). In the same way $**$ means that k and h belong to the sphere $\mathcal{S}_{i,j}$ of the equation $\langle x - v_j, x - v_i \rangle = 0$.
D. In Dirichlet boundary $*$ means that k belongs to the hyperplane $\mathcal{H}_{(i,a),(j,b)}$ (with $a, b \in \{1, \ldots, 2^n\}$) defined by $\left\langle x - v_i^{(a)}, v_i^{(a)} - v_j^{(b)} \right\rangle = 0$ and $h^{(c)} = k + v_j^{(b)} - v_i^{(a)}$ for some c. In the same way $**$ means that k and h belong to the sphere $\mathcal{S}_{(i,a),(j,b)}$ of the equations $\left\langle x - v_j^{(b)}, x - v_i^{(a)} \right\rangle = 0$ and $h^{(c)} = -k + v_i^{(a)} + v_j^{(b)}$.

Conservation laws. The conservation of M in the new variables implies that the monomials appearing in H in the case of periodic boundary are of the form

$$z^{\alpha}\bar{z}^{\beta}y^{c}\mathrm{e}^{\mathrm{i}(x,\nu)}, \qquad \sum_i \nu_i v_i + \sum_{k\in S^c}(\alpha_k - \beta_k)k = 0, \tag{15}$$

where $\nu = (\nu_1,\dots,\nu_m)$, $\nu_i \in \mathbb{Z}$ and α,β are the multi-indices in $\mathbb{N}$ and $|\alpha|, |\beta|$ are the sum of their coordinates.

3.7. Final form of the Hamiltonian

We note that our Hamiltonian is analytic in the domain $A_\alpha \times D_{\rho,r}$; we define the degree of a monomial $y^i z^j \bar{z}^l$ by $2i + j + l$. We drop in formula (14) the constant part (depending only on the parameters ξ) and we separate $H = N + P$, where N is a ‘ quadratic normal form’ namely contains only the terms of H^0 which are of degree $\leqslant 2$:

$$N := (\omega(\xi), y) + \sum_k \Omega_k |z_k|^2 + Q_M(w), \tag{16}$$

where

$$\begin{aligned} \omega_i(\xi) &:= |v_i|^2 + 2\cdot 3^{\delta n}\frac{\xi_i}{|v_i|^{4s}} + 2^{\delta n+2}\sum_{j\neq i}\frac{\xi_j}{|v_j|^{2s}|v_i|^{2s}},\\ \Omega_k(\xi) &= |k|^2 + 2^{\delta n+2}\sum_j \frac{\xi_j}{|v_j|^{2s}|k|^{2s}}. \end{aligned} \tag{17}$$

Finally the quadratic form is

$$\begin{aligned} Q_M(\xi,x;w) = 4 &\sum_{1\leqslant i\neq j\leqslant m}\ \sum_{\substack{k\in\mathcal{H}_{i,j}\\ h=k+v_j-v_i}} \frac{\sqrt{\xi_i\xi_j}\,\mathrm{e}^{\mathrm{i}(x_i-x_j)}}{|v_i|^s|v_j|^s|h|^s|k|^s} z_h\bar{z}_k\\ &+4\sum_{i<j}\ \sum_{\substack{k\in\mathcal{S}_{i,j},h<k,\\ h=-k+v_j+v_i}} \frac{\sqrt{\xi_i\xi_j}}{|v_i|^s|v_j|^s|h|^s|k|^s}\left(\mathrm{e}^{-\mathrm{i}(x_i+x_j)}z_h z_k + \mathrm{e}^{\mathrm{i}(x_i+x_j)}\bar{z}_h\bar{z}_k\right), \end{aligned} \tag{18}$$

wherea couple $(h,k) \in \mathcal{S}_{i,j}$ is given; we decide arbitrarily an ordering $h < k$ with the convention that on $\mathcal{S}_{j,i}$ the ordering is reversed.

For the perturbation, due to constraint 2 we have the bounds

$$\|P\|_{\rho,r} \leqslant C r^{\min(5\alpha,1+\frac{5}{2}\alpha,4)}, \qquad |X_P|_{\rho,s} \leqslant C' r^{\min(\frac{5}{2}\alpha,2)}. \tag{19}$$

Theorem 1. *For all choices of tangential sites S satisfying constraints 1 and 2, the quadratic form $Q_M(w)$ is block diagonal with blocks of uniformly bounded dimension.*

Remark 3.8. In the case of $n = 2$, the block structure of Q_M is obvious since the lines $\mathcal{H}_{(i,a),(j,b)}$ may intersect only at a finite number of points.

In dimension $n > 2$ the intersection is an affine subspace so this simple proof fails and it is indeed very hard (especially in the case of Dirichlet boundary conditions) to have a geometric picture of which the couples (k,h) contribute non-trivially to Q_M. We therefore proceed in a more combinatorial way.

Proof. To understand the block structure of (18) it is useful to relate the quadratic form to a graph.

Definition 3.9. *Graph representation. Given h, k in $\mathbb{Z}^n_{\text{odd}}$ (resp. in $\mathbb{Z}^n_{\text{odd}}(+)$ for Dirichlet boundary conditions) if there exist $v_i, v_j \in S$ (resp. $v_i^{(a)}, v_j^{(b)} \in S$) such that $*$ holds, we connect the points with a black edge labeled as (v_i, v_j) (resp. $(v_i^{(a)}, v_j^{(b)})$).*

Given h, k in $\mathbb{Z}^n_{\text{odd}}$ (resp in $\mathbb{Z}^n_{\text{odd}}(+)$) if h, k satisfy $**$ for some $v_i, v_j \in S$ (resp. $v_i^{(a)}, v_j^{(b)} \in S$) we connect them with a red edge labeled as (v_i, v_j) (resp. $(v_i^{(a)}, v_j^{(b)})$). This defines a graph Γ_S in $\mathbb{Z}^n_{\text{odd}}$ (resp in $\mathbb{Z}^n_{\text{odd}}(+)$). We define a sequence of points $k_1, \ldots, k_N$ to be a path if all k_1, k_{i+1} are the edges of the graph.

It is obvious that the red edges are finite in number; indeed only the integer points on a finite number of spheres may be connected by a red edge.

The following is a standard lemma in graph theory.

Lemma 3.10. *The block-diagonal blocks of the quadratic form $Q_M(w)$ are given by the connected components of the above-described graph. Hence the quadratic form $Q_M(w)$ is block diagonal with blocks of uniformly bounded dimension if and only if the length of paths of black edges with no loops is uniformly bounded.*

To identify a path in Γ_S with no red edges, one may give a starting point $k \in \mathbb{Z}^n_{\text{odd}}$ and a list of the connected edges $\left(v_{i_1}^{(a_1)}, v_{j_1}^{(b_1)}\right), \ldots, \left(v_{i_N}^{(a_N)}, v_{j_N}^{(b_N)}\right)$.

We consider the set of distinct couples $\left(v_i^{(a)}, v_j^{(b)}\right)$ as an alphabet and we call a list of connected black edges in Γ_S a word. Finally we say that a word has no loops if the corresponding path in Γ_S has no loops.

We are now in the notations of appendix A4 of [8], we apply lemmas A4.1–A4.3 and we obtain the following.

Lemma 3.11. *There exists K such that if a word has a length $k \geqslant K$, then the word contains a loop. The value of K depends only on the number of letters of the alphabet.*

This completes the proof. □

4. $n = 2$ with periodic conditions

As we have stated in remark 3.8 that the case $n = 2$ is particularly simple. In the case of periodic boundary conditions Geng, You and Xu have shown in [7] that one may impose (rather complicated) arithmetic conditions on the set S so that

Constraint 3. *All the intersection points $x \notin S$ such that*

$$x \in \cup_{i,j,l,m}\mathcal{H}_{i,j} \cap \mathcal{H}_{l,m} \quad \text{or} \quad x \in \cup_{i,j,l,m}\mathcal{S}_{i,j} \cap \mathcal{H}_{l,m} \quad \text{or} \quad x \in \cup_{i,j,l,m}\mathcal{S}_{i,j} \cap \mathcal{S}_{l,m}$$

are not integer vectors.

Given any $m \in \mathbb{Z}$, there exist S satisfying constraints 1–3, with $|S| = m$, such that in the Hamiltonian (16), each $k \in S^c$ may belong to at most one element of the list $\{\mathcal{H}_{i,j}, \mathcal{S}_{i,j}\}$. We have obtained a normal form Hamiltonian which is essentially identical to that of [7].

A key point in [7] is the study of the block-diagonal homological equation associated with the normal form N. Let us define some spaces (by convention we denote $z_k = z_k^+, \bar{z}_k = z_k^-$)

$$\mathcal{F}^{0,1} := \text{Span}(\mathrm{e}^{\mathrm{i}\nu\cdot x} z_k, \mathrm{e}^{-\mathrm{i}\nu\cdot x}\bar{z}_k), \qquad \text{where} \quad \sum_i \nu_i v_i + k = 0;$$

$$\mathcal{F}^{0,2} := \text{Span}\left(\mathrm{e}^{\mathrm{i}\nu\cdot x} z_k^\sigma z_h^\tau\right), \qquad \text{where} \quad \sum_i \nu_i v_i + \sigma k - \tau h = 0.$$

We study the operator $ad(N) := \{N, \cdot\}$ on the spaces $\mathcal{F}^{0,1}$ and $\mathcal{F}^{0,2}$. A direct computation shows that i $ad(N)$ on $\mathcal{F}^{0,1}$ is block diagonal with blocks of dimension at most 2×2. If $k \notin \cup_{i,j}(\mathcal{S}_{i,j} \cup \mathcal{H}_{i,j})$, we have the usual diagonal term $(\omega, \nu) + \Omega_k$ for $\mathrm{e}^{\mathrm{i}\nu\cdot x} z_k$ such that $\sum_l \nu_l v_l = -k$.

Let $\bar{\omega}_i = \omega_i - |v_i|^2$ and $\bar{\Omega}_k = \Omega_k - |k|^2$. We have 2×2 blocks

$$((\omega, \nu) + |k|^2 - |v_i|^2)I + \begin{pmatrix} -\bar{\omega}_i + \bar{\Omega}_k & \frac{4\sigma\sqrt{\xi_i\xi_j}}{|v_i|^s|v_j|^s|h|^s|k|^s} \\ \frac{4\sqrt{\xi_i\xi_j}}{|v_i||v_j||h||k|} & \sigma(-\bar{\omega}_j + \bar{\Omega}_h) \end{pmatrix} \tag{20}$$

connecting $\mathrm{e}^{\mathrm{i}(\nu - e_i)\cdot x} z_k$ and $\mathrm{e}^{\mathrm{i}(\nu - \sigma e_j)\cdot x} z_h^\sigma$, where $i < j$, $h = \sigma(k - v_i) + v_j$ and if $\sigma = +$, $k \in \mathcal{H}_{ij}$, and if $\sigma = -$, $k \in \mathcal{S}_{ij}$ $h < k$.

Clearly there are only a finite number of non-symmetric matrices (since there are a finite number of $h, k \in \mathcal{S}_{i,j}$) and if $|k|$ is large enough, the off-diagonal terms are irrelevant.

By definition $|k|^2 - |v_i|^2 = \sigma(|h|^2 - |v_j|^2)$ and by momentum conservation

$$\sum_l \nu_l v_l = -(k - v_i) = -\sigma(h - v_j). \tag{21}$$

Note that, given ν, constraints 1–3 imply that there is at most one element in $\cup_{i,j}\mathcal{H}_{i,j} \cup \mathcal{S}_{i,j}$ and one couple (h, k) with $k \in \mathcal{H}_{i,j}$ (resp. $\mathcal{S}_{i,j}$) $i < j$, satisfying (21).

Let us denote by $\lambda_{\nu,k}, \lambda_{\nu,h}$ the two eigenvalues of (20); since $ad(N)$ is symplectic, the eigenvalues $-\lambda_{\nu,k}, -\lambda_{\nu,h}$ appear in the block $\mathrm{e}^{-\mathrm{i}(\nu-e_i)\cdot x} z_k^-, \mathrm{e}^{-\mathrm{i}(\nu-\sigma e_j)\cdot x} z_h^{-\sigma})$.

To study $ad(N)$ on $\mathcal{F}^{0,2}$, we note that $\mathcal{F}^{0,1} \otimes \mathcal{F}^{0,1} \to \mathcal{F}^{0,2}$ surjectively via the map $\mathrm{e}^{\mathrm{i}\nu\cdot x} z_k^\sigma z_h^\tau = \mathrm{e}^{\mathrm{i}\sigma\nu_1} z_k^\sigma \cdot \mathrm{e}^{\mathrm{i}\tau\nu_2} z_h^\tau$ for all ν_1, ν_2 such that $\nu_1 + k = \nu_2 + h = 0$ and $\sigma\nu_1 + \tau\nu_2 = \nu$. Then—by the Leibnitz rule—$ad(N)|_{\mathcal{F}^{0,2}}$ is induced by $ad(N)|_{\mathcal{F}^{0,1}} \otimes I + I \otimes ad(N)|_{\mathcal{F}^{0,1}}$, hence block diagonal with blocks of dimension at most 4×4. Finally, the eigenvalues of $ad(N)|_{\mathcal{F}^{0,2}}$ are the sum of two eigenvalues of $ad(N)|_{\mathcal{F}^{0,1}}$ and the eigenvectors are the product of two eigenvectors of $ad(N)|_{\mathcal{F}^{0,1}}$.

Proposition 4.1. *For all γ small enough and τ large enough, there exists a positive measure Cantor set $\mathcal{A}_{\gamma,\tau} \subset A_\alpha$ (with $|A_\alpha \backslash \mathcal{A}_{\gamma,\tau}| = O(\gamma r^\alpha)$) such that, for all $\xi \in \mathcal{A}_{\gamma,\tau}$ we have the following.*

(i) The operator $ad(N)|_{\mathcal{F}^{0,1}}$ is regular semi-simple, with the eigenvalues satisfying

$$|\lambda_{\nu,k}| > \frac{\gamma r^\alpha}{|\nu|^\tau}, \qquad |\lambda_{\nu_1,k_1} \pm \lambda_{\nu_2,k_2}| > \frac{\gamma r^\alpha}{1 + |\nu_1 \pm \nu_2|^\tau},$$

for all $(\nu, k) \neq (\nu', k')$, satisfying the usual momentum conservation if k or $k' \notin \cup_{i,j}(\mathcal{S}_{i,j} \cup \mathcal{H}_{i,j})$ and (21) otherwise.

(ii) The kernel of $ad(N)$ on $\mathcal{F}^{0,1} \times \mathcal{F}^{0,2}$ is the set of functions of the form

$$Q(\xi, x; w) = \sum_k O_k |z_k|^2 + \sum_{1\leqslant i\neq j\leqslant m} \sum_{\substack{k\in\mathcal{H}_{i,j} \\ h=k+v_j-v_i}} \mathrm{e}^{\mathrm{i}(x_i-x_j)} a_{ijkh} z_h \bar{z}_k$$
$$+ \sum_{i<j} \sum_{\substack{k\in\mathcal{S}_{i,j}, \\ h=-k+v_j+v_i}} b_{ijhk}(\mathrm{e}^{-\mathrm{i}(x_i+x_j)} z_h z_k + \mathrm{e}^{\mathrm{i}(x_i+x_j)} \bar{z}_h \bar{z}_k), \tag{22}$$

such that $ad(Q)$ is simultaneously diagonalizable with $ad(N)$ on $\mathcal{F}^{0,1}$.

Sketch of the Proof.

(i) A direct computation shows that $\lambda_{\nu,k}$ are all different functions of ξ; note that this holds true also for the two eigenvalues of a single block. The measure estimates on the Cantor set follow (recall that for $|k|$ large the off-diagonal terms in (20) are irrelevant).

(ii) Follows by standard linear algebra, see [12] for details. □

Proposition 4.2. *For $\alpha < 1/2$, γ and r small enough, there exists a positive measure Cantor set $\mathcal{A}^\infty \subset \mathcal{A}_{\gamma,\tau}$ and an analytic symplectic change of variables, defined for $\xi \in \mathcal{A}^\infty$, that transforms the Hamiltonian (16) in the form*

$$(\omega^\infty, y) + \sum_k \Omega_k |z_k|^2 + Q_{M^\infty}(\xi, x, w) + P^\infty = N^\infty + P^\infty, \tag{23}$$

where P^∞ is of degree at least 3 and $Q_{M^\infty}(\xi, x, w)$ is of the form (22) with $\sup_{k,i,j} |k|^{2s}(O_k, a_{ijkh}, b_{ijkh}) \ll r^\alpha$.

Proof. We can apply theorem 2 of [7]. Hypotheses (A1)–(A5) are directly verified. The non-degeneracy assumption (A3) follows from proposition 4.1. Finally, since the perturbation P is regularizing, in our case the *Töplitz–Lipschitz* hypothesis (A6) can be dropped. The smallness condition is in our setting $|X_P|_{\rho,r} r^{-2\alpha}\gamma^2 \ll 1$ and this is clearly verified since $\min(2-2\alpha, \frac{5}{2}\alpha) > 2\alpha$ provided $\alpha < \frac{1}{2}$. □

Consider the following symplectic (see [14]) and analytic change of variables:

$$z_k = \mathrm{e}^{\mathrm{i}x_i} z_k^+, \qquad \forall k \in \mathcal{H}_{i,j}, \qquad (z_h, z_k) = \left(\mathrm{e}^{\mathrm{i}x_i} z_h^+, \mathrm{e}^{\mathrm{i}x_j} z_k^+\right), \qquad \forall (h,k) \in \mathcal{S}_{i,j} h < k,$$
$$y_i = y_i^+ - \sum_{j\neq i} \sum_{k \in \mathcal{H}_{i,j}} \left|z_k^+\right|^2 - \sum_{j \neq i} \sum_{(h,k)\in\mathcal{S}_{i,j}, h<k} \left|z_h^+\right|^2. \tag{24}$$

It is easily seen that (24) reduces N^∞ to the constant coefficient normal form

$$N^+ = (\omega^\infty, y^+) + \sum_k \Omega_k \left|z_k^+\right|^2 + Q_M(\xi, x = 0, w^+)$$
$$- \sum_{i,j} \omega_i^\infty(\xi) \left(\sum_{k \in \mathcal{H}_{i,j}} \left|z_k^+\right|^2 + \sum_{(h,k)\in\mathcal{S}_{i,j}, h<k} |z_h^+|^2 \right).$$

Theorem 2. *Consider equations (1) and (2) with $s > 0$, in the dimension $n = 2$; for all $m \in \mathbb{N}$, there exist choices of the tangential sites S (see (12)) with $|S| = m$ such that the following holds. For $\alpha < 1/2$, γ and r small enough, there exists a positive measure 'Cantor-like' set $\mathcal{A}^\infty$ and an open domain $\tilde{A}_\alpha \subset A_\alpha$, with $|\tilde{A}_\alpha| \sim r^\alpha$, such that (i) for all $\xi \in \mathcal{A}^\infty$, there exist quasi-periodic solutions; (ii) for all $\xi \in \tilde{A}_\alpha \cap \mathcal{A}^\infty$, the solutions are linearly stable.*

Proof. The existence of the solutions is due to proposition 4.2. The stability is verified by showing that for some positive measure domain $\tilde{A}_\alpha \subset A_\alpha$, the normal form N^+ is elliptic, i.e. all the eigenvalues of $\mathrm{i}ad(N^+)|_{\mathcal{F}^{0,1}}$ are real. Since $\mathrm{i}\,ad(N^+)$ and $\mathrm{i}\,ad(N)$ are simultaneously diagonalizable and have real entries, we study the characteristic polynomials of the matrices in (20) with $\sigma = -$. The set $\tilde{A}_\alpha$ is defined by requiring that all the discriminants are positive. □

Remark 4.3. Note that in $A_\alpha \backslash \tilde{A}_\alpha$ one can still prove the existence of quasi-periodic solutions but these solutions are linearly unstable.

Remark 4.4. Note that the change of variables (24) does not preserve the property that X_P is regularizing. This is the main reason why we first prove the existence of solutions on the non-integrable normal form and then we show stability by reducing to constant coefficients.

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