Partial Differential Equations – Prof. N. Fusco
Programma del corso

Testo consigliato:
L.C. Evans, Partial differential equations. Appunti in rete.

1) The divergence theorem and applications. Convolutions. Multi-indices and the multinomial theorem. The functions Gamma and Beta. Measure of the unit ball in R^n.

2) The Laplace equation, physical and probabilistic interpretation. The fundamental solution of Laplacian equation. The Newtonian potential. The mean value theorem. The maximum principle for harmonic functions. Uniqueness of solutions of Poisson equation. Mollifiers. The fundamental lemma of Calculus of Variations. Weyl's lemma. Estimates of the derivatives of harmonic functions. Analyticity of harmonic functions. Harnack inequality. Liouville theorem. Consequences of Liouville theorem. The Green function. The Green function in a half space and in a ball. Symmetry of the Green function of a general domain. The Dirichlet principle.

3) The heat equation: physical interpretation. The fundamental solution of the heat equation. Solution of the homogeneous equation in the whole space. Solution of the inhomogeneous heat equation. The heat ball. The mean value theorem for the solutions of the heat equation. The maximum principle for the heat equation in bounded domains. Uniqueness of the Cauchy problem for the heat equation in bounded domains. The maximum principle for the controlled solutions of the heat equation in the whole space. Regularity of the soutions of the heat equation. Uniqueness and backward uniqueness of the solutions of the heat equations via energy methods.

4) The transport equation. Solution of the transport equation in the whole space. Solutions of the Cauchy problem for the homogeneous wave equation in dimensions 1,2,3. Formulae of D'Alembert, Kirchhoff and Poisson. Solution of the Cauchy problem for the inhomogeneous wave equation. Uniqueness of the solutions of the Cauchy problem. The characteristic cone.

5) Separation of variables: solution of the Poisson equation in a rectangle and in an annulus. Separation of variables: solutions of the heat equation in a bounded domain, of the porous media equation and of the Hamilton-Jacobi equation. Travelling waves solutions of the heat equation, of the wave equation, of the Schrödinger equation, of the Airy equation and of the Korteweg-de Vries equation. Use of Fourier transform methods to solve the Poisson equation, the heat equation, the Schrödinger equation, the wave equation and the telegraph equation. Definition of Laplace transform and uniqueness of Laplace transform. Application of Laplace transform for solving the Poisson equation and the wave equation in odd dimension.

6) Definition of weak derivative and elementary properties of weak derivatives. Sobolev spaces. Examples. Local approximation of Sobolev functions by smooth functions. Global approximation: the theorem H=W. Two lemmas on partition of unity. Translations of L^p functions. Approximation of Sobolev functions with smooth functions up to the boundary. Extension theorem for Sobolev functions. The space of Sobolev functions with zero boundary trace. Properties of Sobolev functions with zero boundary trace. The Poincaré inequality for zero trace functions.The Dirichlet principle for weak solutions of the Poisson equation. Existence of weak solutions for the Dirichlet problem for the Laplacian. Gagliardo-Nirenberg imbedding theorem. Morrey imbedding theorem. Sobolev imbedding theorem in bounded domains. The Rellich-Kondrakov compact imbedding theorem. The Poincaré inequality for zero mean value functions. The imbedding theorem for Sobolev spaces with higher order derivatives. General elliptic equations in divergence form.

7) The Lax Milgram theorem and its application to the existence of solutions for second order elliptic equation in divergence form with principal part only. Compact operators on Hilbert spaces. Fredholm theorem for compact operators in Hilbert spaces. The Fredholm alternative theorem for existence of solutions of general elliptic PDE's in divergence form. The spectrum of a linear and continuous operator. Eigenvalues of a symmetric and compact operator. Properties of the eigenfunctions. Orthonormal bases of eigenfunctions. Eigenvalues of an elliptic operator in

divergence form. Variational characterization of eigenvalues. Interior regularity of weak solutions of linear elliptic operators.